

Proof of classical versions of the Bousso entropy bound and of the generalized second law

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Bousso has conjectured that in any spacetime satisfying Einstein's equation and satisfying the dominant energy condition, the "entropy flux" S through any null hypersurface L generated by geodesics with non-positive expansion starting from some spacelike 2 surface of area A must satisfy $S \leq A/4G\hbar$. This conjecture reformulates earlier conjectured entropy bounds of Bekenstein and also of Fischler and Susskind, and can be interpreted as a statement of the so-called holographic principle. We show that Bousso's entropy bound can be derived from either of two sets of hypotheses. The first set of hypotheses is (i) associated with each null surface L in spacetime there is an entropy flux 4-vector s_L^a whose integral over L is the entropy flux through L , and (ii) along each null geodesic generator of L , we have $|s_L^a k_a| \leq \pi(\lambda_\infty - \lambda) T_{ab} k^a k^b / \hbar$, where T_{ab} is the stress-energy tensor, λ is an affine parameter, $k^a = (d/d\lambda)^a$, and λ_∞ is the value of affine parameter at the endpoint of the geodesic. The second (purely local) set of hypotheses is (i) there exists an absolute entropy flux 4-vector s^a such that the entropy flux through any null surface L is the integral of s^a over L , and (ii) this entropy flux 4-vector obeys the pointwise inequalities $(s_a k^a)^2 \leq T_{ab} k^a k^b / (16\pi\hbar^2 G)$ and $|k^a k^b \nabla_a s_b| \leq \pi T_{ab} k^a k^b / (4\hbar)$ for any null vector k^a . Under the first set of hypotheses, we also show that a stronger entropy bound can be derived, which directly implies the generalized second law of thermodynamics.

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I. INTRODUCTION AND SUMMARY**A. Background and motivation**

In recent years, a number of independent universal entropy bounds have been postulated to hold for arbitrary systems. The first such bound was conjectured by Bekenstein, who proposed that the entropy S and energy E of any matter put into a box must obey [1]

$$S/E \leq 2\pi R, \quad (1.1)$$

where R denotes some suitable measure of the size of the box. [Throughout this paper, we use units with $G=c=\hbar=k=1$.] The original motivation for the bound (1.1) was the belief that it is necessary for the validity of the generalized second law (GSL) of thermodynamics, which states that in all physical processes the generalized entropy

$$S' = S + S_{\text{bh}} \quad (1.2)$$

must always increase, where S is the entropy of matter outside of black holes, $S_{\text{bh}} = A_H/4$, and A_H denotes the total surface area of all black hole horizons. Subsequently it was shown [2–4] that the bound (1.1) is not necessary for the

validity of the generalized second law.¹ In addition, the bound fails when the number of species of particles is sufficiently large.² Finally, it is far from clear what the precise meaning of "R" in the conjecture is supposed to be, particularly in curved spacetime; in curved spacetime, it is also far from clear what "E" means. Nevertheless, a case can be made that the bound (1.1) may hold for all physically realistic systems found in nature; see Ref. [6] for further discussion.

More recently, an alternative entropy bound has been con-

¹Very recently, Bekenstein [5] has used the fact that the buoyancy formulas must be modified due to finite box size effects to again argue that a bound of the form (1.1) is needed for the validity of the GSL. However, we believe that an analysis of the type given in [3] could be used to show that no such entropy bound is needed. Indeed, if a violation of the GSL could be obtained in any process involving the quasi-static lowering of a box toward a black hole, then we expect that it should be possible to obtain a violation of the ordinary second law by a similar quasi-static lowering of a box into a real star composed of unconstrained thermal matter.

²In the canonical ensemble, it is easy to show that the bound (1.1) also fails at sufficiently low temperatures for all systems whose ground state energy vanishes. However, a detailed analysis of a variety of systems given in Ref. [6] provides strong evidence that this failure does not occur in the microcanonical ensemble.

sidered: the entropy S inside any region whose boundary has area A must satisfy [7]

$$S \leq A/4. \tag{1.3}$$

An argument given in Ref. [8] suggests that the bound (1.3) should follow from the GSL together with the assumption that the entropy of a black hole counts the number of possible internal states of the black hole.³ In addition, when $E \leq R$, this bound would follow from the original Bekenstein bound (1.1). The inequality (1.3), like the bound (1.1), can be violated if the number of massless particle species is allowed to be arbitrarily large.⁴ The inequality (1.3) is related to the hypothesis known as the holographic principle, which states that the physics in any spatial region can be fully described in terms of degrees of freedom living on the boundary of that region, with one degree of freedom per Planck area [12,8]. If the holographic principle is correct, then since the entropy in any region should be bounded above by the number of fundamental degrees of freedom in that region, a bound of the form (1.3) should be valid for all systems, including those with strong self-gravity.

As it stands, the bound (1.3) is ambiguous, since the precise meaning of the ‘‘bounding area,’’ A , has not been spelled out. In particular, note that any world tube can always be ‘‘enclosed’’ by a two-surface of arbitrarily small area, since given any two-surface in spacetime, there exists a two-surface of arbitrarily small area arbitrarily close to the original two-surface (obtained by ‘‘wiggling’’ the original two-surface suitably in spacetime). However, very recently, a specific conjecture of the form (1.3) was suggested by Bousso [13,14], who improved an earlier suggestion of Fischler and Susskind [15,16]. Bousso showed that several ex-

ample spacetimes, including cosmological models and gravitational collapse spacetimes, are consistent with his conjecture.

Bousso’s conjecture is as follows. Let (M, g_{ab}) be a spacetime satisfying Einstein’s equation and also the dominant energy condition [17]. Let B be a connected 2 dimensional spacelike surface in M . Suppose that k^a is a smooth null vector field on B which is everywhere orthogonal to B . Then the expansion

$$\theta = \nabla_a k^a \tag{1.4}$$

of k^a is well defined and is independent of how k^a is extended off B . Suppose that $\theta \leq 0$ everywhere on B . Let L denote the null hypersurface generated by the null geodesics starting at B with initial tangent k^a , where each null geodesic is terminated if and only if a caustic is reached (where $\theta \rightarrow -\infty$), and otherwise is extended as far as possible. Then the entropy flux, S_L , through L satisfies

$$S_L \leq A_B/4, \tag{1.5}$$

where A_B is the area of B .

There is a close relationship between Bousso’s conjecture and the generalized second law. Consider a foliation of the horizon of a black hole by spacelike two-surfaces $B(\alpha)$, where α is a continuous label that increases in the future direction (with respect to the time orientation used to define the black hole). Let $A(\alpha)$ be the area of the two surface $B(\alpha)$, and let $S(\alpha)$ be the total entropy that crosses the horizon before the 2-surface $B(\alpha)$. Then if one assumes the ordinary second law, the GSL is equivalent to the statement that for any $\alpha_1 < \alpha_2$ we have

$$S(\alpha_2) - S(\alpha_1) \leq \frac{1}{4}[A(\alpha_2) - A(\alpha_1)]. \tag{1.6}$$

On the other hand, Bousso’s entropy bound applied to the 2-surface $B(\alpha)$ —with k^a taken to be the past directed normal to the horizon, so that we have $\theta \leq 0$ on $B(\alpha)$ when the null energy condition is satisfied—demands merely that

$$S(\alpha) \leq \frac{1}{4}A(\alpha) \tag{1.7}$$

for all α . Thus, Bousso’s bound implies that the GSL holds for the case when the initial time, α_1 , is taken to be the time when the black hole is first formed [so that $S(\alpha_1) = A(\alpha_1) = 0$]. In general, however, it is clear that the statement (1.7) is weaker than the statement (1.6).

This observation motivates a generalization of Bousso’s conjecture. Namely, if one allows the geodesics generating the hypersurface L to terminate at some spacelike 2-surface

³The argument is attributed to Bekenstein in Ref. [8] (see also Ref. [9]) and goes as follows. If black hole entropy counts the number of internal states of a black hole, then any system having $S \geq A/4$ is not a black hole. Then, one would expect to be able to make that system into a black hole with area A by collapsing a sufficiently massive spherical shell of matter around it. In this process, it appears that no entropy escapes, but this means that we convert an $S \geq A/4$ system into a black hole of area A , violating the generalized second law. An antecedent to this argument can be found in Ref. [11]. For a counterargument, see Ref. [10]

⁴For example, consider N free massless scalar fields in flat spacetime, in a cube of edge length L with Dirichlet boundary conditions. In the canonical ensemble, the thermal state with temperature T with $T \ll 1/L$ has energy E which scales as $E \sim N \exp[-\pi/(LT)]/L$ and entropy S which scales like $S \sim N \exp[-\pi/(LT)]/(LT)$. For the system to be weakly self-gravitating (a necessary condition for the flat-spacetime analysis to be a good approximation) we must have $E = \varepsilon L$ for some $\varepsilon \ll 1$. Using this restriction to solve for the maximum allowed value of N yields $S/L^2 \sim \varepsilon/(LT)$, which can be arbitrarily large. In the microcanonical ensemble, the violation of Eq. (1.3) for sufficiently large N follows immediately from the fact that the density of states at a fixed total energy E grows unboundedly with N at fixed L . (Note, however, that Casimir energy has been ignored here.)

B' before coming to a caustic or singularity, one can replace the conjectured inequality (1.5) by the condition

$$S_L \leq \frac{1}{4}[A_B - A_{B'}]. \quad (1.8)$$

It is clear from the above discussion that this more general bound implies both the original Bousso entropy bound and the GSL (assuming of course the validity of the ordinary second law).

In this paper we shall prove Bousso's entropy bound (1.5) under two independent sets of hypotheses concerning the local entropy content of matter. Furthermore, under the first set of hypotheses, we will prove the more general entropy bound (1.8). We note that proofs of the GSL that are more general than the proof of this paper have previously been given [18]; however, the previous proofs used specific properties of black-hole spacetimes, unlike our analysis.

Finally, we note that, as discussed further at the end of Sec. III, our results can be generalized straightforwardly to arbitrary spacetime dimensions greater than 2.

B. Derivations of entropy bound and of generalized second law: framework, viewpoint and assumptions

The starting point for our derivation of the entropy bounds (1.5) and (1.8) is a postulated phenomenological description of entropy, which differs from assumptions that have been used in the past to derive the GSL [18]. In this section we describe our phenomenological description of entropy and its motivation.

First, note that one of the hypotheses of Bousso's conjecture is the dominant energy condition, which is often violated by the expected stress energy tensor of matter in semiclassical gravity. Hence the conjecture cannot have the status of a fundamental law as it is currently stated, but rather can only be relevant in "classical regimes" where the dominant energy condition is satisfied [19]. It may be possible to replace the dominant energy condition by a quantum inequality of the type invented by Ford and Roman [20–24] to overcome this difficulty.⁵ In this paper we will assume the null convergence condition, that $T_{ab}k^ak^b \geq 0$ for all null vectors k^a [see Eqs. (1.9) and (1.10), (1.11) below], which is weaker than the dominant energy condition. Thus, our proof of

Bousso's conjecture, like the conjecture itself, is limited to "classical regimes" in which local energy conditions are satisfied.

Clearly, in order to derive the bounds (1.5) and (1.8), we must make some assumptions about entropy. The entropy that the conjecture refers to presumably should include gravitational contributions. It seems plausible that any gravitational entropy flux through the null hypersurface L will be associated with a shearing of that hypersurface, which has the same qualitative effect in the Raychaudhuri equation [see Eq. (2.13) below] as a matter stress-energy flux. Thus, it may be possible to treat gravitational contributions to entropy in a manner similar to the matter contributions. However, our present understanding of quantum gravity is not sufficient to attempt to meaningfully quantify the gravitational contributions to entropy. Consequently, in our analysis below, we shall consider only the matter contributions to entropy.

With regard to the matter contribution to entropy, for both the GSL and the Bousso bound, there is an apparent tension between the fact that these statements are supposed to have the status of fundamental laws and the fact that entropy is a quantity whose definition is coarse-graining dependent. However, this tension is resolved by noting that the number of degrees of freedom should be an upper bound for the entropy S , irrespective of choice of coarse-graining [14]. Equivalently, we may restrict attention to the case where the matter is locally in thermal equilibrium (i.e., maximum entropy density for its given energy density); if the bound holds in this case, it must hold in all cases.

We shall proceed by assuming that a phenomenological description of matter entropy can be given in terms of an entropy flux 4-vector s^a . We shall then postulate some properties of s^a . In fact, we shall postulate two independent sets of hypotheses on s^a , each of which will be sufficient to prove the bound (1.5); the first set of hypotheses also will suffice to prove the bound (1.8). Note that it is not a central goal of this paper to justify our hypotheses, although we do discuss some motivations below. Instead we shall merely observe that they appear to hold in certain regimes. Note also that, at a fundamental level, entropy is a non-local quantity and so can be well described by an entropy flux 4-vector only in certain regimes and over certain scales. This fact is reflected in our hypotheses below.

The first of our two sets of hypotheses is very much in the spirit of the original Bekenstein bound (1.1). Suppose that one has a null hypersurface, L , the generators of which terminate at a finite value λ_∞ of affine parameter λ . Suppose that one puts matter in a box and drops it through L in such a way that the back end of the box crosses L at λ_∞ . Then, if a bound of the nature of Eq. (1.1) holds, the amount of entropy crossing L should be limited by the energy within the box and the box "size." The box size, in turn, would be related to the affine parameter at which the front end of the box crossed L . On the other hand, suppose that matter flowing through L near λ_∞ were not confined by a box. Then there would be no "box size restriction" on the entropy flux near λ_∞ . However, in order to have a larger entropy flux

⁵Lowe [19] argues that the Bousso conjecture must fail for a system consisting of an evaporating black hole accreting at just the right rate to balance the Hawking radiation mass loss. For such a system, it would seem that the black hole can accrete an arbitrary amount of entropy without changing its area, and in addition it is hard to see how a modified Bousso conjecture incorporating a quantum inequality rather than a local energy condition could be satisfied. However, this counterexample might be resolved by the fact that it may be appropriate to assign a negative entropy flux at the horizon to states with an outgoing Hawking flux, or it might be resolved by making adjustments to the Bousso conjecture.

than one could achieve when using a box, it clearly would be necessary to put the matter in a state where the ‘‘modes’’ carrying the entropy ‘‘spill over’’ beyond λ_∞ . In that case, it is far from clear that the entropy carried by these modes should be credited as arriving prior to λ_∞ , so that they would count in the entropy flux through L . In other words, it seems reasonable to postulate that the entropy flux through L cannot be higher than the case where the matter is placed in a box whose back end crosses L at λ_∞ , and to consider a bound on this entropy flux of the general form of Eq. (1.1).

The above considerations motivate the following hypothesis concerning the entropy flux. We assume that associated with every null surface L there is an entropy flux 4-vector s_L^a from which one can compute the entropy flux through L . Let γ be a null geodesic generator of L , with affine parameter λ and tangent $k^a = (d/d\lambda)^a$. If γ is of infinite affine parameter length, then $T_{ab}k^ak^b = 0$ along γ by the focusing theorem [17], and we assume that $s_L^a = 0$ along γ . On the other hand, if γ ends at a finite value, λ_∞ , of affine parameter, then we assume that⁶

$$|s_L^ak_a| \leq \pi(\lambda_\infty - \lambda)T_{ab}k^ak^b. \quad (1.9)$$

The inequality (1.9) is a direct analog of the original Bekenstein bound (1.1), with $|s_L^ak_a|$ playing the role of S , $T_{ab}k^ak^b$ playing the role of E , and $\lambda_\infty - \lambda$ playing the role of R . As discussed above, the motivation for the bound (1.9) is essentially the same as that for the bound (1.1). Note that Eq. (1.9) is independent of the choice of affine parameterization of γ ; i.e., both sides of this equation scale the same way under a change of affine parameter.

The above set of hypotheses has the property that the entropy flux, $-s_L^ak_a$, depends upon L in the sense (described above) that modes that only partially pass through L prior to λ_∞ do not contribute to the entropy flux. In our second set of hypotheses, we assume the existence of an absolute entropy flux 4-vector s^a , which is independent of the choice of L . We assume that this s^a obeys the following purely local, pointwise inequalities for any null vector k^a :

$$(s_ak^a)^2 \leq \alpha_1 T_{ab}k^ak^b \quad (1.10)$$

and

⁶From Eq. (2.13), our proof also works if we replace the hypothesis (1.9) with the weaker hypotheses that

$$|s_L^ak_a| \leq (\lambda_\infty - \lambda)[\pi T_{ab}k^ak^b + \hat{\sigma}_{ab}\hat{\sigma}^{ab}/8],$$

where $\hat{\sigma}_{ab}$ is the shear tensor [Eq. (9.2.28) of Ref. [17]] associated with the generators of L . In this context, we can interpret s_L^a to be the combined matter and gravitational entropy flux, rather than just the matter entropy flux.

$$|k^ak^b\nabla_a s_b| \leq \alpha_2 T_{ab}k^ak^b, \quad (1.11)$$

where T_{ab} is the stress-energy tensor.⁷ Here α_1 and α_2 can be any positive constants that satisfy

$$(\pi\alpha_1)^{1/4} + (\alpha_2/\pi)^{1/2} = 1. \quad (1.12)$$

(Recall that we are using Planck units with $G = c = \hbar = k = 1$.) A specific simple choice of α_1 and α_2 that satisfy the condition (1.12) is $\alpha_1 = 1/(16\pi)$ and $\alpha_2 = \pi/4$, which are the values quoted in the abstract above. Note that, like Eq. (1.9), Eqs. (1.10) and (1.11) are independent of the choice of scaling of k^a . Also note that both of our sets of hypotheses (1.9) and (1.10), and (1.11) imply the null convergence condition $T_{ab}k^ak^b \geq 0$, as mentioned above.

We now turn to a discussion of the physical regimes in which we expect the pointwise assumptions (1.10) and (1.11) of our second set of hypotheses to be valid. The first assumption (1.10) of our second set of hypotheses says, roughly speaking, that the entropy density is bounded above by the square root of the energy density. One can check that the condition is satisfied for thermal equilibrium states of Bose and Fermi gases except at temperatures above a critical temperature of order the Planck temperature.⁸ One can also

⁷The stress energy tensor appearing in these inequalities should be interpreted as a macroscopic or averaged stress energy tensor \bar{T}_{ab} , rather than a microscopic stress energy tensor T_{ab} . For example, for an atomic gas, the fundamental microscopic stress-energy tensor T_{ab} will vary rapidly over atomic and nuclear scales, while a suitable averaged macroscopic stress tensor \bar{T}_{ab} can be taken to vary only over macroscopic scales (like the conventional entropy current s^a). Thus our results apply to null surfaces L of an averaged, macroscopic metric \bar{g}_{ab} rather than the physical metric g_{ab} [25]. Note that null surfaces of \bar{g}_{ab} can differ significantly from the null surfaces of g_{ab} , since with suitable microscopic sources (for example cosmic strings) a null surface of g_{ab} can be made to intersect itself very frequently without the occurrence of caustics. However, the boundary of the future (or past) of the 2-surface B with respect to \bar{g}_{ab} should be close to the boundary of the future (respectively, past) of B with respect to g_{ab} . Thus, if one wishes to work with the exact metric g_{ab} , one should presumably replace the null hypersurface, L , in the Bousso conjecture and our generalization (1.8) with a suitable portion of the boundary of the future (or past) of B . This new formulation of the conjectures should hold whenever Eq. (1.9) or Eqs. (1.10) and (1.11) hold for the macroscopically averaged entropy current and macroscopically averaged stress energy tensor. An alternative interpretative framework would be to assume the existence of an ‘‘entropy current’’ which varies rapidly on the smallest scales that are compatible with our gradient assumption (1.11) (atomic and nuclear scales in our example), in which case our result would apply directly to the microscopic metric.

⁸Specifically, for a free massless boson gas at temperature T the stress energy tensor has the form $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$ and the entropy flux vector is $s^a = \sigma u^a$, where $p = \rho/3$ and $\sigma = 4\rho/(3T)$. It follows that for any null vector k^a we have $(s_ak^a)^2/T_{ab}k^ak^b = 4\rho/(3T^2) = 2\pi^2 g N_s T^2/45$, where g is the number of polarization components and N_s is the number of species.

check that for quantum fields in a box at low temperatures (the example discussed in Sec. I A above), the condition (1.10) is violated only if the box is Planck size or smaller, or if the number of species is allowed to be very large. Thus, it seems plausible that the bound (1.10) will be universally valid if one assumes a Planck scale cutoff for physics and if one also assumes a limit to the number of species. Also one can argue as follows that a bound of the form of Eq. (1.10) should follow from the Bekenstein bound (1.1). Consider a region of space of that is sufficiently small that (i) the entropy density and energy density are approximately uniform over the region, and (ii) the region is weakly self-gravitating so that its total energy E satisfies $E \lesssim R$, where R is the size of the region. Then, if S is the total entropy in the region, the ratio of entropy density squared to energy density is $\sim S^2/ER^3 \leq 4\pi^2 E/R$ by Eq. (1.1), which is ≤ 1 as $E \lesssim R$.

The second assumption (1.11) states roughly that the gradient of the entropy density is bounded above by the energy density. For a free, massless boson or fermion gas in local thermal equilibrium, this condition reduces to the condition that the temperature gradient, $|\nabla T|$, be small compared with T^2 , i.e., that the fractional change in T over a distance $1/T$ be smaller than unity. This condition must be satisfied in order for the notion of local thermal equilibrium to make sense.

In addition, it would appear that condition (1.11) is necessary for our entire phenomenological description of entropy as represented by an 4-current s^a to be valid. To see this, consider the following illustrative example. Consider a wavepacket mode of a quantum field, where the wavelength is λ and where the volume occupied by the wavepacket is $f\lambda^3$ for some dimensionless factor $f \gtrsim 1$. Consider a state where this wavepacket mode is occupied by N particles. Such a system has a well defined expected stress energy tensor $\langle \hat{T}_{ab} \rangle$, whose corresponding energy density will be of order

$$\rho \sim \frac{N}{f\lambda^4}. \tag{1.13}$$

We now imagine that we are to somehow model such a system with a smooth entropy flux vector s^a . We expect that the total entropy carried by the system should be of order N , so that the entropy density s should be approximately

$$s \sim \frac{N}{f\lambda^3}. \tag{1.14}$$

Clearly the concept of local entropy flux here cannot make sense on scales short compared to the wavelength λ ; only in an averaged sense, on scales comparable to λ or larger, does the concept of entropy flux make sense. Thus, the lengthscale $\mathcal{L} = s/|\nabla s|$ over which the entropy density varies should be greater than or of the order of λ . From the estimates (1.13) and (1.14), the condition $\mathcal{L} \gtrsim \lambda$ is equivalent to $|\nabla s| \leq \rho$, which is essentially our assumption (1.11). Hence, our second condition (1.11) rules out the class of entropy currents s^a which vary significantly over scales shorter than λ , allowing only the more appropriate s^a that vary over scales of a wavelength or longer.

In summary, we expect our second set of hypotheses to be valid in regimes where the following conditions are satisfied: (i) Spacetime structure can be accurately described by a classical metric, g_{ab} , and the gravitational contributions to entropy, other than that from black holes, are negligible. (ii) The matter entropy can be accurately be described by an entropy current s^a . In particular, this condition will be valid in familiar hydrodynamic regimes. (iii) The vacuum energy contributions to the stress-energy tensor are negligible, so that the stress-energy tensor satisfies classical energy conditions.

We shall refer to regimes satisfying the above three conditions as ‘‘classical,’’ even though, in such regimes, quantum physics may play an essential role in accounting for the entropy of matter. In classical regimes, our hypotheses (1.10) and (1.11) should be valid. We have argued above that hypothesis (1.9) also should hold. Hence our arguments show that the Bousso bound (1.5) and its generalization (1.8) should hold in classical regimes. While our arguments do not show that the entropy bounds (1.5) and (1.8) hold at any fundamental level, they do show that any counterexample either must involve quantum phenomena in an essential way (in the sense of failure to be in a classical regime), or must violate Eq. (1.9) and/or Eq. (1.10) or Eq. (1.11).

II. DERIVATION OF ENTROPY BOUNDS

In this section we derive the generalized entropy bound (1.8) from the assumption (1.9) and the Bousso bound (1.5) from the assumptions (1.10), (1.11).

We start with some definitions and constructions. First, we can without loss of generality take the vector field k^a on the 2-surface B to be future directed, since the conjecture is time reversal invariant. Let l^a be the unique vector field on B which is null, future directed, orthogonal to B and which satisfies $l^a k_a = -1$. We extend both k^a and l^a to L by parallel transport along the null geodesic generators of L . Thus, k^a is tangent to each geodesic. Then the expansion $\theta = \nabla_a k^a$ is well defined on L and independent of how k^a is extended off of L , since

$$\theta = (g^{ab} + k^a l^b + k^b l^a) \nabla_a k_b. \tag{2.1}$$

By the hypotheses of Bousso’s conjecture and of its generalization (1.8), θ is nonpositive everywhere on L . Let $\{x^\Gamma\} = (x^1, x^2)$ be any coordinate system on B . Then one obtains a natural coordinate system (λ, x^1, x^2) on L in the obvious way, where $k^a = (d/d\lambda)^a$ and we take $\lambda = 0$ on B .

For the generator γ which starts at the point x^Γ on B , let $\lambda_\infty(x^\Gamma)$ be the value of affine parameter (possibly $\lambda_\infty = \infty$) at the end point of the generator. This endpoint can either be a caustic ($\theta = -\infty$) or have a finite expansion θ . We can without loss of generality exclude the case $\lambda_\infty = \infty$, since otherwise we must have $T_{ab} k^a k^b = 0$ and $\theta = 0$ along γ by the focusing theorem, and then either version of our hypotheses implies that $s^a = 0$ along γ , so that there is no contribution to the entropy flux. Thus, generators of infinite affine parameter length make no contribution to the LHS of the inequalities (1.5) and (1.8) while making a non-negative contribution to

the RHS, and so can be ignored. For the generators of finite affine parameter length, we can without loss of generality rescale the affine parameter along each generator in order to make the end point occur at $\lambda_\infty = 1$.

A. Reducing the conjecture to each null geodesic generator

Next, we show that it is sufficient to focus attention on each individual generator of L , one at a time. More specifically we have the following lemma.

Lemma. *A sufficient condition for the generalized entropy bound (1.8) is that for each null geodesic generator γ of L of finite affine parameter length, we have*

$$\int_0^1 d\lambda (-s_a k^a) \mathcal{A}(\lambda) \leq \frac{1}{4} [1 - \mathcal{A}(1)], \quad (2.2)$$

where

$$\mathcal{A}(\lambda) \equiv \exp \left[\int_0^\lambda d\bar{\lambda} \theta(\bar{\lambda}) \right] \quad (2.3)$$

is an area-decrease factor associated with the given generator. Similarly, a sufficient condition for the Bousso bound (1.5) is that

$$\int_0^1 d\lambda (-s_a k^a) \mathcal{A}(\lambda) \leq \frac{1}{4} \quad (2.4)$$

along each generator.

To prove the lemma, first note that by assumption the entropy flux through L is given by

$$S_L = \int_L s^a \epsilon_{abcd}, \quad (2.5)$$

where the orientation on the hypersurface L is that determined by the 3-form

$$\hat{\epsilon}_{bcd} \equiv l^a \epsilon_{abcd}. \quad (2.6)$$

Here we are using the notation of Appendix B of Ref. [17] for integrals of differential forms, and we also use the abstract index notation of Ref. [17] for all Roman indices throughout the paper. The formula (2.5) applies for either version of our phenomenological description of entropy; i.e., we can use either s^a or s_L^a in the integrand. In Appendix A we derive the following formula for the integral (2.5) in the coordinate system (λ, x^Γ) :

$$S_L = \int_B d^2x \sqrt{\det h_{\Gamma\Lambda}(x)} \int_0^{\lambda_\infty(x)} d\lambda s(\lambda) \mathcal{A}(\lambda). \quad (2.7)$$

Here $x \equiv (x^1, x^2) = x^\Gamma$, $h_{\Gamma\Lambda}(x)$ is the induced 2-metric on the 2-surface B , $\lambda_\infty(x)$ is the value of affine parameter at the endpoint of the generator which starts at x , and

$$s \equiv -s_a k^a. \quad (2.8)$$

Note that s is non-negative for future directed, timelike or null s^a , which we expect to be the case. (However, our proof does not require the assumptions that s^a be timelike or null and future directed.) Now as discussed above, $s(\lambda) = 0$ for those generators of infinite affine parameter length, so it follows that

$$S_L = \int_B' d^2x \sqrt{\det h_{\Gamma\Lambda}(x)} \int_0^1 d\lambda s(\lambda) \mathcal{A}(\lambda), \quad (2.9)$$

where $\int_B' d^2x$ denotes an integral only over those generators of finite affine parameter length. Now we see that if the condition (2.2) is satisfied, then we obtain from Eq. (2.9) that

$$S_L \leq \frac{1}{4} \int_B' d^2x \sqrt{\det h_{\Gamma\Lambda}(x)} [1 - \mathcal{A}(1, x)]. \quad (2.10)$$

The generalized entropy bound (1.8) now follows from Eq. (2.10), using the fact that the area A_B of the 2-surface B is given by

$$A_B = \int_B d^2x \sqrt{\det h_{\Gamma\Lambda}(x)}, \quad (2.11)$$

while the area $A_{B'}$ of the 2-surface B' composed of the end points of the generators is

$$A_{B'} = \int_B' d^2x \sqrt{\det h_{\Gamma\Lambda}(x)} \mathcal{A}(1, x). \quad (2.12)$$

Similar arguments show that the Bousso bound (1.5) follows from the assumption (2.4).

B. Preliminaries

From the lemma, it is sufficient now to prove the condition (2.2) or, respectively, the condition (2.4) for each finite-affine-parameter-length null generator γ of L . For ease of notation we henceforth drop the dependence on the x^Σ coordinates in all quantities. Now the twist along each of the null generators will vanish, since it is vanishing initially on the two surface B , and the evolution equation for the twist [17] then implies that it always vanishes. The Raychaudhuri equation in the relevant case of vanishing twist can thus be written as

$$-\frac{d\theta}{d\lambda} = \frac{1}{2} \theta^2 + f(\lambda), \quad (2.13)$$

where $f = 8\pi T_{ab} k^a k^b + \hat{\sigma}_{ab} \hat{\sigma}^{ab}$ and $\hat{\sigma}_{ab}$ is the shear tensor. The function f is non-negative by the null convergence condition [which follows from either Eq. (1.9) or Eqs. (1.10), (1.11)] since $\hat{\sigma}_{ab} \hat{\sigma}^{ab} \geq 0$ always. The assumption (1.9) of our first set of hypotheses now implies that

$$|s(\lambda)| \leq (1 - \lambda) f(\lambda) / 8. \quad (2.14)$$

Similarly, our second set of hypotheses (1.10) and (1.11) implies that

$$s(\lambda)^2 \leq \bar{\alpha}_1 f(\lambda) \quad (2.15)$$

and

$$|s'(\lambda)| \leq \bar{\alpha}_2 f(\lambda), \quad (2.16)$$

where

$$\bar{\alpha}_1 = 8\pi\alpha_1, \quad \bar{\alpha}_2 = 8\pi\alpha_2. \quad (2.17)$$

We define the quantity

$$I_\gamma \equiv \int_0^1 d\lambda s(\lambda) \mathcal{A}(\lambda). \quad (2.18)$$

Our tasks now are to show that $I_\gamma \leq [1 - \mathcal{A}(1)]/4$ when Eq. (2.14) holds, and that $I_\gamma \leq 1/4$ when Eqs. (2.15) and (2.16) hold, using only the definition (2.3) of the area-decrease factor and the Raychaudhuri equation (2.13).

Now by assumption any geodesic generator must terminate no later than the point (if it exists) at which $\mathcal{A}(\lambda) \rightarrow 0$. Hence we have $\mathcal{A}(\lambda) \geq 0$ everywhere on L . It is convenient to define $G(\lambda) = \sqrt{\mathcal{A}(\lambda)}$, from which it follows from the definition (2.3) of the area-decrease factor and from the Raychaudhuri equation (2.13) that

$$f(\lambda) = -2 \frac{G''(\lambda)}{G(\lambda)}. \quad (2.19)$$

It follows that G'' is negative. Also the expansion θ is always negative, and hence G' is always negative, so that G is monotonically decreasing, starting at the value $G(0) = 1$, and ending at some value $G(1)$ with $0 \leq G(1) \leq 1$. In particular, we have $0 \leq G(\lambda) \leq 1$ for all λ . For those generators which terminate at caustics we have $G(1) = \mathcal{A}(1) = 0$, but not all generators will terminate at caustics; some might terminate at the auxiliary spacelike 2-surface B' .

C. Proof of the generalized Bousso bound under the first set of hypotheses

Using the formula (2.14) and the definition $G = \sqrt{\mathcal{A}}$ we find that the integral (2.18) satisfies

$$I_\gamma \leq \frac{1}{8} \int_0^1 d\lambda (1-\lambda) f(\lambda) G(\lambda)^2. \quad (2.20)$$

From the formula (2.19) for $f(\lambda)$, this can be written as

$$I_\gamma \leq - \int_0^1 d\lambda (1-\lambda) G''(\lambda) G(\lambda) / 4. \quad (2.21)$$

Now we have $0 \leq G(\lambda) \leq 1$, so we can drop the factor of $G(\lambda)$ in the integrand of Eq. (2.21). Integrating by parts and using the fundamental theorem of calculus now gives

$$I_\gamma \leq \frac{1}{4} [G(0) - G(1) + G'(0)]. \quad (2.22)$$

Now

$$G(0) - G(1) = 1 - G(1) \leq 1 - \mathcal{A}(1), \quad (2.23)$$

since $G(0) = 1$ and $G(1) = \sqrt{\mathcal{A}(1)} \geq \mathcal{A}(1)$. Also the third term in Eq. (2.22) is negative. It follows that

$$I_\gamma \leq \frac{1}{4} [1 - \mathcal{A}(1)], \quad (2.24)$$

as required.

D. Proof of the original Bousso bound under the second set of hypotheses

First, note that without loss of generality we can assume that the function $s(\lambda)$ is nonnegative. This is because we can replace $s(\lambda)$ by $|s(\lambda)|$ in the integral (2.18) without decreasing the value of the integral, and the assumptions (2.15) and (2.16) are satisfied by the function $|s|$ if they are satisfied by s , since $||s'| \leq |s'|$.

We start by fixing a λ_1 in $(0,1)$, the value of which we will pick later. We then choose a λ_0 in $[0, \lambda_1]$ which minimizes f in the interval $[0, \lambda_1]$; i.e., we choose a λ_0 which satisfies

$$f(\lambda_0) = \min_{0 \leq \lambda \leq \lambda_1} f(\lambda). \quad (2.25)$$

[We assume that the function $f(\lambda)$ is continuous so that this minimum is attained].⁹ We now show that

$$f(\lambda_0) \leq \frac{\pi^2}{2\lambda_1^2} [1 - \mathcal{A}(1)]. \quad (2.26)$$

To see this, let $\theta_0(\lambda)$ and $\mathcal{A}_0(\lambda)$ be the expansion and area-decrease factor that would be obtained by solving the Raychaudhuri equation (2.13) with $f(\lambda)$ replaced by $f(\lambda_0)$ and using the same initial condition $\theta_0(0) = \theta(0)$. Since $f(\lambda) \geq f(\lambda_0)$ for $0 \leq \lambda \leq \lambda_1$, it is clear that we must have

$$\mathcal{A}(\lambda) \leq \mathcal{A}_0(\lambda) \quad (2.27)$$

for $0 \leq \lambda \leq \lambda_1$. But the explicit solution of the Raychaudhuri equation for $\theta_0(\lambda)$ and $\mathcal{A}_0(\lambda)$ is

$$\theta_0(\lambda) = -\sqrt{2f(\lambda_0)} \tan \left[\sqrt{\frac{f(\lambda_0)}{2}} (\lambda + \hat{\lambda}) \right] \quad (2.28)$$

and

⁹The proof extends easily to the non-continuous case. If we choose $\epsilon > 0$ and choose λ_0 so that

$$f(\lambda_0) = (1 + \epsilon) \text{g.l.b.} \{f(\lambda) | 0 \leq \lambda \leq \lambda_1\},$$

then we can come as close as we please to satisfying the inequality (2.25). Hence the inequality (2.25) is satisfied.

$$\mathcal{A}_0(\lambda) = \frac{\cos^2 \left[\sqrt{\frac{f(\lambda_0)}{2}} (\lambda + \hat{\lambda}) \right]}{\cos^2 \left[\sqrt{\frac{f(\lambda_0)}{2}} \hat{\lambda} \right]}, \quad (2.29)$$

where $\hat{\lambda}$ is a constant in $[0,1]$. Applying the inequality (2.27) at $\lambda = \lambda_1$ now yields

$$\mathcal{A}(\lambda_1) \leq \mathcal{A}_0(\lambda_1) \leq \cos^2 \left[\sqrt{\frac{f(\lambda_0)}{2}} \lambda_1 \right]. \quad (2.30)$$

Using the inequality $\sin \chi \geq 2\chi/\pi$ which is valid for $0 \leq \chi \leq \pi/2$, and the inequality $\mathcal{A}(1) \leq \mathcal{A}(\lambda_1)$, one can obtain the upper bound (2.26) from Eq. (2.30).

Next, we split the integral (2.18) into a contribution I_1 from the interval $[0, \lambda_0]$ and a contribution I_2 from the interval $[\lambda_0, 1]$:

$$I_\gamma = \int_0^{\lambda_0} s \mathcal{A} + \int_{\lambda_0}^1 s \mathcal{A} = I_1 + I_2. \quad (2.31)$$

In the formula for I_1 , we drop the factor of \mathcal{A} which is ≤ 1 , insert a factor of $1 = d\lambda/d\lambda$, and integrate by parts to obtain

$$I_1 \leq I_{1b} + I'_1. \quad (2.32)$$

Here I_{1b} is the boundary term that is generated, given by

$$I_{1b} = s(\lambda_0)\lambda_0, \quad (2.33)$$

and

$$I'_1 = - \int_0^{\lambda_0} d\lambda s'(\lambda)\lambda. \quad (2.34)$$

Similarly we insert a factor of $1 = d(\lambda - 1)/d\lambda$ into the formula for I_2 and integrate by parts, which yields

$$I_2 = I_{2b} + I'_2, \quad (2.35)$$

where the boundary term is

$$I_{2b} = s(\lambda_0)\mathcal{A}(\lambda_0)(1 - \lambda_0) \quad (2.36)$$

and where

$$I'_2 = \int_{\lambda_0}^1 d\lambda [s' \mathcal{A}(1 - \lambda) + s \mathcal{A}'(1 - \lambda)]. \quad (2.37)$$

An upper bound on the total integral is now given by the relation

$$I_\gamma \leq I_{1b} + I_{2b} + I'_1 + I'_2. \quad (2.38)$$

We now proceed to derive bounds on the integrals I'_1 , I'_2 and on the total boundary term $I_{1b} + I_{2b}$.

Consider first the total boundary term, which from Eqs. (2.33) and (2.36) is given by

$$I_{1b} + I_{2b} = s(\lambda_0)[\lambda_0 + (1 - \lambda_0)\mathcal{A}(\lambda_0)]. \quad (2.39)$$

Since $\mathcal{A}(\lambda_0)$ and λ_0 both lie in $[0,1]$, this is bounded above by $s(\lambda_0)$. If we now use our assumption (2.15), we find

$$I_{1a} + I_{1b} \leq \sqrt{\bar{\alpha}_1} f(\lambda_0), \quad (2.40)$$

and using the bound (2.26) on $f(\lambda_0)$ finally yields

$$I_{1a} + I_{1b} \leq \frac{\sqrt{\bar{\alpha}_1} \pi}{\sqrt{2\lambda_1}} [1 - \mathcal{A}(1)]^{1/2}. \quad (2.41)$$

Turn now to the integral I'_1 . Inserting the assumption (2.16) into the formula (2.34) for I'_1 and using the formula (2.19) for $f(\lambda)$ yields

$$I'_1 \leq -2\bar{\alpha}_2 \int_0^{\lambda_0} \frac{G''}{G} \lambda d\lambda. \quad (2.42)$$

Since G is a decreasing function, we can replace the $1/G(\lambda)$ in the integrand by $1/G(\lambda_0)$. If we then integrate by parts and use the fundamental theorem of calculus, we obtain

$$I'_1 \leq \frac{2\bar{\alpha}_2}{G(\lambda_0)} [G(\lambda_0) - 1 - G'(\lambda_0)\lambda_0]. \quad (2.43)$$

Since $G(\lambda_0) \leq 1$ this yields

$$I'_1 \leq - \frac{2\bar{\alpha}_2 G'(\lambda_0)\lambda_0}{G(\lambda_0)}. \quad (2.44)$$

We now show that for all λ in $[0,1]$,

$$- \frac{G'(\lambda)}{G(\lambda)} \leq \frac{1}{1-\lambda} [1 - \mathcal{A}(1)]. \quad (2.45)$$

To see this, apply the mean value theorem to the function G over the interval $[\lambda, 1]$, which yields

$$G(1) - G(\lambda) = (1 - \lambda)G'(\lambda_*), \quad (2.46)$$

for some λ_* in $[\lambda, 1]$. But since G'' is negative by Eq. (2.19), we have $G'(\lambda_*) \leq G'(\lambda)$, and it follows that

$$\begin{aligned} - \frac{G'(\lambda)}{G(\lambda)} &\leq \frac{1}{1-\lambda} \left[1 - \frac{G(1)}{G(\lambda)} \right] \\ &\leq \frac{1}{1-\lambda} [1 - G(1)] \leq \frac{1}{1-\lambda} [1 - \mathcal{A}(1)]. \end{aligned} \quad (2.47)$$

Here the last inequality follows from $G = \sqrt{\mathcal{A}}$ and $0 \leq G \leq 1$. Using the relation (2.45), our upper bound (2.44) for I'_1 now yields

$$\frac{I'_1}{1 - \mathcal{A}(1)} \leq 2\bar{\alpha}_2 \frac{\lambda_0}{1 - \lambda_0} \leq 2\bar{\alpha}_2 \frac{\lambda_1}{1 - \lambda_1}. \quad (2.48)$$

Finally, we turn to the integral I'_2 . The second term in the formula (2.37) for I'_1 is negative and so can be dropped. In the first term, we use the formula $G = \sqrt{\mathcal{A}}$, the formula (2.19) for $f(\lambda)$ and our gradient assumption in the form (2.16) to obtain

$$I'_2 \leq -2\bar{\alpha}_2 \int_{\lambda_0}^1 d\lambda G''G(1-\lambda). \quad (2.49)$$

Now since $0 \leq G(\lambda) \leq 1$ for all λ , we can drop the factor of $G(\lambda)$ in the integrand. If we then integrate by parts and use the fundamental theorem of calculus we obtain

$$I'_2 \leq 2\bar{\alpha}_2 [G(\lambda_0) - G(1) + (1 - \lambda_0)G'(\lambda_0)]. \quad (2.50)$$

Now since $G(\lambda_0) \leq 1$ and $G = \sqrt{A}$ we have

$$G(\lambda_0) - G(1) \leq 1 - G(1) \leq 1 - \mathcal{A}(1). \quad (2.51)$$

Also, the last term in Eq. (2.50) is negative. Hence we obtain the upper bound

$$I'_2 \leq 2\bar{\alpha}_2 [1 - \mathcal{A}(1)]. \quad (2.52)$$

Finally we combine Eq. (2.38) with the upper bounds (2.41), (2.48), and (2.52) for the boundary term $I_{1b} + I_{2b}$ and for the integrals I'_1 and I'_2 to yield

$$\begin{aligned} I_\gamma &\leq \sqrt{\bar{\alpha}_1} \frac{\pi}{\sqrt{2\lambda_1}} [1 - \mathcal{A}(1)]^{1/2} + 2\bar{\alpha}_2 \frac{1}{1 - \lambda_1} [1 - \mathcal{A}(1)] \\ &\leq \left[\sqrt{\bar{\alpha}_1} \frac{\pi}{\sqrt{2\lambda_1}} + 2\bar{\alpha}_2 \frac{1}{1 - \lambda_1} \right] [1 - \mathcal{A}(1)]^{1/2}. \end{aligned} \quad (2.53)$$

Choosing the value of λ_1 that minimizes this upper bound yields

$$I_\gamma \leq \left[\sqrt{\pi} \sqrt{\frac{\bar{\alpha}_1}{2}} + \sqrt{2\bar{\alpha}_2} \right]^2 [1 - \mathcal{A}(1)]^{1/2}. \quad (2.54)$$

Using the definition (2.17) of the parameters $\bar{\alpha}_1$ and $\bar{\alpha}_2$ together with the assumption (1.12) yields

$$I_\gamma \leq \frac{1}{4} [1 - \mathcal{A}(1)]^{1/2} \leq \frac{1}{4}, \quad (2.55)$$

as required. Note that our proof actually implies the inequality

$$S_L \leq \frac{1}{4} A_B^{1/2} [A_B - A_{B'}]^{1/2}. \quad (2.56)$$

This inequality is stronger than the Bousso bound (1.5) but weaker than the generalized Bousso bound (1.8).

III. CONCLUSION

We have shown that the generalization (1.8) of Bousso's entropy bound is satisfied under the hypothesis (1.9), and that the original Bousso bound (1.5) holds under the hypotheses (1.10), (1.11). While these hypotheses are unlikely to represent relations in any fundamental theory, they appear to be satisfied for matter in a certain semi-classical regime below the Planck scale. As such, our results rule out a large class of possible counterexamples to Bousso's conjecture,

including cases involving gravitational collapse or other strong gravitational interactions. As with Bousso's bound, if the holographic principle is indeed part of a fundamental theory, it may be that the hypotheses discussed here will provide clues to its formulation.

Note that we do not show in this paper that the entropy bounds (1.5) and (1.8) can be saturated by entropy 4-currents s^a satisfying our assumptions. However, consideration of simple examples shows that the bound (1.5) comes within a factor of order unity of being saturated by currents satisfying our second set of hypotheses (1.10) and (1.11). Also, a simple scaling argument shows that the least upper bound on the ratio S_L/A_B for currents satisfying our assumptions is of the form $\alpha_2 F(\alpha_1 \alpha_2^{-2})$ for some function F . As a result, at fixed $\alpha_1 \alpha_2^{-2}$ the least upper bound depends continuously on α_2 . This guarantees that there exist some values of α_1 and α_2 , of order unity, such the entropy bound (1.5) is both satisfied and can be saturated by entropy currents satisfying the inequalities (1.10) and (1.11). A similar statement is true for the bound (1.8) and the hypothesis (1.9).

In our analysis above, we have taken the dimension of spacetime to be 4. However, in an n -dimensional spacetime with $n > 2$, the Raychaudhuri equation continues to take the form (2.13), except that the coefficient of θ^2 on the right side is now $1/(n-2)$. Consequently, if we define $G = \mathcal{A}^{1/(n-2)}$ in the n -dimensional case, an equation of the form (2.19) will continue to hold with the factor of 2 on the right side replaced by $(n-2)$. The remainder of our analysis can then be carried out in direct parallel with the 4-dimensional case. Thus, with suitable adjustments to the numerical factors appearing in Eqs. (1.9), (1.10), and (1.11), all of our results continue to hold for all spacetime dimensions greater than 2.

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APPENDIX: FORMULA FOR INTEGRAL OVER NULL HYPERSURFACE

In this Appendix we derive the formula (2.7) for the integral

$$S_L = \int_L s^a \epsilon_{abcd} \quad (A1)$$

of the entropy current over the null hypersurface L [cf. Eq. (2.5) above], using the coordinate system $(\lambda, x^\Gamma) = (\lambda, x^1, x^2)$ defined in Sec. II.

We start by discussing the relation between tensors on the spacetime M and tensors on the null surface L . We introduce the notation that capital roman indices A, B, C, \dots denote tensors on L , in the sense of the abstract index convention of Ref. [17]. For any 1-form w_a defined on L , we will denote the pullback of w_a to L as

$$w_A = P_A^a w_a; \quad (\text{A2})$$

this defines the operator P_A^a . Since k_a is normal to L , the pullback of k_a vanishes, so $P_A^a k_a = 0$. Using the null tetrad introduced at the beginning of Sec. II, we can define a similar mapping between vectors on M and vectors on L . At any point \mathcal{P} in L , the projection operation

$$v^a \rightarrow (\delta_b^a + l^a k_b) v^b \quad (\text{A3})$$

maps the 4-dimensional tangent space $T_{\mathcal{P}}(M)$ into the 3-dimensional tangent space $T_{\mathcal{P}}(L)$. Thus one can write the mapping (A3) as $v^a \rightarrow v^A = Q_A^a v^a$, which defines the operator Q_A^a . Note that the vector l^a is annihilated by the projection operation (A3), while k^a and vectors perpendicular to k^a and l^a are unchanged. We define $k^A = Q_A^a k^a = (d/d\lambda)^A$, which is the tangent vector in L the generators of L .

Consider now the integrand in the integral (A1). It is proportional to

$$s^a k_{[a} l_b e_c f_{d]}, \quad (\text{A4})$$

where e^a and f^a are spacelike vector fields such that $\{k^a, l^a, e^a, f^a\}$ is an orthonormal basis. When we pullback the 3-form (A4) to L , all the terms where the index on k_a is free and not contracted with s^a will be annihilated. Hence without loss of generality we can replace s^a with $-(s_b k^b) l^a$. Thus we obtain from Eqs. (A1) and (2.6) that

$$S_L = \int_L s \hat{\epsilon}_{abc}, \quad (\text{A5})$$

where $s = -s_a k^a$ and where the 3-form $\hat{\epsilon}_{abc}$ is defined in Eq. (2.6) above.

As a tool for evaluating the integral (A5), we define an induced connection D_A on L by

$$D_A v^B = P_A^c Q_d^B \nabla_c v^d \quad (\text{A6})$$

where v^d is any vector field on M with $v^A = Q_A^a v^a$. One can check that this formula defines a derivative operator on L . Next we note that the pullback $\hat{\epsilon}_{ABC}$ of the 3-form (2.6) is parallel transported along each null generator of L with respect to the connection (A6):

$$k^A D_A \hat{\epsilon}_{BCD} = 0. \quad (\text{A7})$$

This follows from the fact that k^a , l^a and ϵ_{abcd} are parallel transported along each generator with respect to the

4-dimensional connection.¹⁰ Next, consider the Lie derivative $\mathcal{L}_{\tilde{k}} \hat{\epsilon}$ of $\hat{\epsilon}_{ABC}$ with respect to k^A . Since the result is a 3-form we must have

$$(\mathcal{L}_{\tilde{k}} \hat{\epsilon})_{ABC} = \eta \hat{\epsilon}_{ABC}, \quad (\text{A8})$$

for some scalar field η . We can define an upper index volume form $\hat{\epsilon}^{ABC}$ by the requirement that

$$\hat{\epsilon}^{ABC} \hat{\epsilon}_{ABC} = 3!. \quad (\text{A9})$$

[A definition in terms of raising indices is inapplicable here since there is no natural non-degenerate metric on L .] Now contracting both sides of Eq. (A8) with $\hat{\epsilon}^{ABC}$ and using Eq. (A7) yields

$$\eta = D_A k^A = \nabla_a k^a, \quad (\text{A10})$$

which is just the usual expansion θ .

Next, we define a 3-form $\tilde{\epsilon}_{ABC}$ on L by demanding that it coincide with $\hat{\epsilon}_{ABC}$ on the 2-surface B , and that it be Lie transported along the generators of L . If we write $\hat{\epsilon}_{ABC} = \zeta \tilde{\epsilon}_{ABC}$, where ζ is a scalar field on L , it follows from Eq. (A8) with $\eta = \theta$ that

$$\mathcal{L}_{\tilde{k}} \zeta = \theta \zeta. \quad (\text{A11})$$

Solving this equation using the definition (2.3) of the area-decrease factor yields $\zeta = \mathcal{A}$. Thus we see that the geometrical meaning of the factor \mathcal{A} is that it is the ratio between the Lie-transported 3-volume form $\tilde{\epsilon}_{ABC}$ and the parallel transported 3-volume form $\hat{\epsilon}_{ABC}$, where in both cases one starts from the 2-surface B .

Consider now the specific coordinate system $(\lambda, x^\Gamma) = (\lambda, x^1, x^2)$. In this coordinate system the fact that $\tilde{\epsilon}_{ABC}$ is Lie transported along the generators translates into

$$\frac{\partial}{\partial \lambda} \tilde{\epsilon}_{\lambda x^1 x^2}(\lambda, x) = 0, \quad (\text{A12})$$

so that

$$\tilde{\epsilon}_{\lambda x^1 x^2}(\lambda, x) = \tilde{\epsilon}_{\lambda x^1 x^2}(0, x) = \sqrt{\det h_{\Gamma\Lambda}(x)}, \quad (\text{A13})$$

where $\tilde{\epsilon}_{\lambda x^1 x^2}(\lambda, x)$ denotes one of the coordinate components of the tensor $\tilde{\epsilon}_{ABC}$ in the coordinate system (λ, x^Γ) , $x \equiv (x^1, x^2)$ as before, and $h_{\Gamma\Lambda}$ is the induced 2-metric on B . It follows that

$$\hat{\epsilon}_{\lambda x^1 x^2}(\lambda, x) = \mathcal{A}(\lambda, x) \sqrt{\det h_{\Gamma\Lambda}(x)}. \quad (\text{A14})$$

Combining this with the formula (A5) for the entropy flux finally yields the formula (2.7).

¹⁰Note however that unlike the situation in the four dimensional setting, in general $D_A \hat{\epsilon}_{BCD} \neq 0$; i.e., $\hat{\epsilon}_{ABC}$ is not covariantly constant with respect to D_A .

- [1] J.D. Bekenstein, Phys. Rev. D **7**, 2333 (1973); **23**, 287 (1981); **49**, 1912 (1994).
- [2] W.G. Unruh and R.M. Wald, Phys. Rev. D **25**, 942 (1982).
- [3] W.G. Unruh and R.M. Wald, Phys. Rev. D **27**, 2271 (1983).
- [4] M.A. Pelath and R.M. Wald, Phys. Rev. D **60**, 104009 (1999).
- [5] J.D. Bekenstein, Phys. Rev. D **60**, 124010 (1999).
- [6] J.D. Bekenstein and M. Schiffer, Int. J. Mod. Phys. C **1**, 355 (1990); J.D. Bekenstein, Phys. Rev. D **30**, 1669 (1984).
- [7] G. 't Hooft, in *Quantum Gravity: proceedings*, edited by M.A. Markov, V.A. Berezin, and V.P. Frolov (World Scientific, Singapore, 1988); J.D. Bekenstein, in *Proceedings of the VII Marcel Grossman Meeting on General Relativity V*, edited by R.T. Jantsen and G.M. Keiser (World Scientific, Singapore, 1996), p. 39; gr-qc/9409015; L. Smolin, gr-qc/9508064; S. Corley and T. Jacobson, Phys. Rev. D **53**, 6720 (1996).
- [8] L. Susskind, J. Math. Phys. **36**, 6377 (1995).
- [9] J.D. Bekenstein, Phys. Lett. B **481**, 339 (2000).
- [10] R.M. Wald, "The Thermodynamics of Black Holes," gr-qc/9912119.
- [11] J.D. Bekenstein, Phys. Rev. D **9**, 3292 (1974).
- [12] G. 't Hooft, "Dimensional reduction in quantum gravity," gr-qc/9310026.
- [13] R. Bousso, J. High Energy Phys. **07**, 004 (1999).
- [14] R. Bousso, J. High Energy Phys. **06**, 028 (1999).
- [15] W. Fischler and L. Susskind, "Holography and cosmology," hep-th/9806039.
- [16] For other comments on extensions of the original Fischler-Susskind bound [15], see R. Dawid, "Holographic Cosmology and its Relative Degrees of Freedom," hep-th/9907115.
- [17] R.M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).
- [18] W. Zurek, Phys. Rev. Lett. **49**, 1683 (1982); W.H. Zurek and K.S. Thorne, *ibid.* **54**, 2171 (1985); K.S. Thorne, W.H. Zurek, and R.H. Price, in *Black Holes: The Membrane Paradigm*, edited by K.S. Thorne, R.H. Price and D.A. MacDonald (Yale University Press, New Haven, CT, 1986) p. 280; W.G. Unruh and R.M. Wald, Phys. Rev. D **25**, 942 (1982); **27**, 2271 (1983); correction: M.J. Radzikowski and W.G. Unruh, *ibid.* **37**, 3059(E) (1988); R.M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994); R. Sorkin, Phys. Rev. Lett. **56**, 1885 (1986); V.P. Frolov and D.N. Page, *ibid.* **71**, 3902 (1993).
- [19] David A. Lowe, J. High Energy Phys. **010**, 026 (1999).
- [20] L.H. Ford and T.A. Roman, Phys. Rev. D **51**, 4277 (1995).
- [21] L.H. Ford and T.A. Roman, Phys. Rev. D **55**, 2082 (1997).
- [22] É.É. Flanagan, Phys. Rev. D **56**, 4922 (1997).
- [23] C.J. Fewster and S.P. Eveson, Phys. Rev. D **58**, 084010 (1998).
- [24] L.H. Ford and T.A. Roman, Phys. Rev. D **60**, 104018 (1999).
- [25] R.M. Zalaletdinov, Bull. Astron. Soc. India **25**, 401 (1997).