Cosmological dynamics on the brane

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In Randall-Sundrum-type brane-world cosmologies, the dynamical equations on the three-brane differ from the general relativity equations by terms that carry the effects of embedding and of the free gravitational field in the five-dimensional bulk. Instead of starting from an ansatz for the metric, we derive the covariant nonlinear dynamical equations for the gravitational and matter fields on the brane, and then linearize to find the perturbation equations on the brane. The local energy-momentum corrections are significant only at very high energies. The imprint on the brane of the nonlocal gravitational field in the bulk is more subtle, and we provide a careful decomposition of this effect into nonlocal energy density, flux and anisotropic stress. The nonlocal energy density determines the tidal acceleration in the off-brane direction, and can oppose singularity formation via the generalized Raychaudhuri equation. Unlike the nonlocal energy density and flux, the nonlocal anisotropic stress is not determined by an evolution equation on the brane, reflecting the fact that brane observers cannot in general make predictions from initial data. In particular, isotropy of the cosmic microwave background may no longer guarantee a Friedmann geometry. Adiabatic density perturbations are coupled to perturbations in the nonlocal bulk field, and in general the system is not closed on the brane. But on super-Hubble scales, density perturbations satisfy a decoupled third-order equation, and can be evaluated by brane observers. Tensor perturbations on the brane are suppressed by local bulk effects during inflation, while nonlocal effects can serve as a source or a sink. Vorticity on the brane decays as in general relativity, but nonlocal bulk effects can source the gravito-magnetic field, so that vector perturbations can be generated in the absence of vorticity.

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I. INTRODUCTION

Einstein's theory of general relativity breaks down at high enough energies, and is likely to be the limit of a more general theory. Recent developments in string theory indicate that gravity may be a truly higher-dimensional theory, becoming effectively 4-dimensional at lower energies. These exciting theoretical developments need to be accompanied by efforts to test such higher-dimensional theories against their cosmological and astrophysical implications. In that spirit, we investigate here a particular class of models, showing how their dynamical properties generalize those of Einstein's theory, and discussing the broad implications of these generalizations for cosmological dynamics.

In many higher-dimensional gravity theories inspired by string theory, the matter fields are confined to a 3-brane in $1+3+d$ dimensions, while the gravitational field can propagate also in the d extra dimensions (see, e.g., [1]). It is not necessary for the d extra dimensions to be small, or even finite: recently Randall and Sundrum [2] have shown that for $d=1$, gravity can be localized on a single 3-brane even when the fifth dimension is infinite (see also [3]). An elegant geometric formulation and generalization of the Randall-Sundrum scenario has been given by Shiromizu, Maeda and Sasaki [4]. The Friedmann equation on the brane in these models is modified by both high-energy matter terms and a term carrying nonlocal bulk effects onto the brane. The Friedmann brane models have been extensively investigated (see, e.g., [5–7]), and inflationary scalar perturbations in these models have also been considered [8]. The models are compatible with observations subject to reasonable constraints on the parameters. A broader study of cosmological dynamics, i.e., for induced metrics more general than the simple Friedmann case, has not been done. In particular, the analysis of perturbed Friedmann models also remains to be done. (Considerable work has been done on perturbations of flat brane metrics; see, e.g., [2,9,10].)

In this paper, we initiate a study of nonlinear and perturbed cosmological dynamics in Randall-Sundrum-type brane-world models, generalizing some important results in general relativity. We find the bulk corrections to the propagation and constraint equations, using the covariant Lagrangian approach [11,12]. This approach is well suited to identifying the geometric and physical quantities that determine inhomogeneity and anisotropy on the brane, and it is also the basis for a gauge-invariant perturbation theory [13]. Our first task is to identify and interpret the covariant physical content of the bulk effects on the brane. Local effects lead to quadratic corrections of the density, pressure and energy flux. The nonlocal effects of the free gravitational field in the bulk are transmitted by a Weyl projection term, which we decompose into energy density, energy flux and anisotropic stress parts. We calculate the gravitational (tidal) and non-gravitational acceleration of fluid world lines, finding the role of the nonlocal energy density in localization of gravity, and showing how the world lines have a non-gravitational acceleration off the brane at high energies. During inflation, the acceleration is directed towards the brane.

We derive the propagation (''conservation'') equations

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governing the nonlocal energy density and flux parts; the evolution of the anisotropic stress part is *not* determined on the brane. These nonlocal terms also enter into crucial dynamical equations, such as the Raychaudhuri equation and the shear propagation equation, and can lead to important changes from the general relativistic case. For example, it is possible via the nonlocal term to avoid the initial singularity in a nonrotating model without cosmological constant. Nonlocal effects also mean that isotropy of the cosmic microwave background (CMB) may no longer guarantee a Friedmann geometry.

The covariant nonlinear equations lead to a covariant and gauge-invariant description of perturbations on the brane. We derive the equations governing adiabatic scalar perturbations, which are not in general closed on the brane, because of nonlocal effects. However, on super-Hubble scales, the density perturbations satisfy a decoupled third-order equation, with an additional nonlocal degree of freedom, and can therefore be evaluated by brane observers. Tensor perturbations cannot be determined by brane observers on any scales. The local bulk effects tend to enhance tensor perturbations during non-inflationary expansion and suppress them during inflation. Nonlocal bulk effects can in principle act either way. Vorticity on the brane decays as in general relativity, but bulk effects act as a source for the gravito-magnetic field, and hence vector perturbations, on the brane.

Our results remain incomplete in one fundamental aspect; i.e., we do not provide a description of the gravitational field in the bulk, but confine our investigations to effects that can be measured by brane observers. In order to fill this gap, one would need to study the off-brane derivatives of the curvature, which are given in general in [4,10]. This is an important topic for further research.

The 5-dimensional field equations are Einstein's equations, with a (negative) bulk cosmological constant $\tilde{\Lambda}$ and brane energy-momentum as source:

$$\tilde{G}_{AB}=\tilde{\kappa}^2[-\tilde{\Lambda}\tilde{g}_{AB}+\delta(\chi)\{-\lambda g_{AB}+T_{AB}\}]. \tag{1}$$

The tildes denote the bulk (5-dimensional) generalization of standard general relativity quantities, and $\tilde{\kappa}^2=8\pi/\tilde{M}_{\rm p}^3$, where $\tilde{M}_{\rm p}$ is the fundamental 5-dimensional Planck mass, which is typically much less than the effective Planck mass on the brane, $M_{\rm p}=1.2\times10^{19}$ GeV. The brane is given by $\chi=0$, so that a natural choice of coordinates is $x^A=(x^\mu,\chi)$, where $x^\mu=(t,x^i)$ are spacetime coordinates on the brane. The brane tension is λ, and $g_{AB}=\tilde{g}_{AB}-n_An_B$ is the induced metric on the brane, with n_A the spacelike unit normal to the brane. Matter fields confined to the brane make up the brane energy-momentum tensor T_{AB} (with $T_{AB}n^B=0$).

Although it is usually assumed that the spacetime is exactly anti–de Sitter in the absence of a brane ($\lambda=0=T_{AB}$), this is not necessarily the case. The brane-free bulk metric can be any solution of $\tilde{G}_{AB}=-\tilde{\kappa}^2\tilde{\Lambda}\tilde{g}_{AB}$, including non-conformally flat solutions. When the induced metric on the brane is Friedmann, then the 5-dimensional Schwarzschild–anti–de Sitter metric is a solution of Eq. (1) [7]. However, more general bulk metrics are in principle possible.

The field equations induced on the brane are derived via an elegant geometric approach by Shiromizu *et al.* [4], using the Gauss-Codazzi equations, matching conditions and Z_2 symmetry. The result is a modification of the standard Einstein equations, with the new terms carrying bulk effects onto the brane:[1]

$$G_{\mu\nu}=-\Lambda g_{\mu\nu}+\kappa^2T_{\mu\nu}+\tilde{\kappa}^4S_{\mu\nu}-\mathcal{E}_{\mu\nu}, \tag{2}$$

where $\kappa^2=8\pi/M_{\rm p}^2$. The energy scales are related to each other via

$$\lambda=6\frac{\kappa^2}{\tilde{\kappa}^4},\quad \Lambda=\frac{4\pi}{\tilde{M}_{\rm p}^3}\left[\tilde{\Lambda}+\left(\frac{4\pi}{3\tilde{M}_{\rm p}^3}\right)\lambda^2\right]. \tag{3}$$

The bulk corrections to the Einstein equations on the brane are of two forms: first, the matter fields contribute local quadratic energy-momentum corrections via the tensor $S_{\mu\nu}$, and second, there are nonlocal effects from the free gravitational field in the bulk, transmitted via the projection $\mathcal{E}_{\mu\nu}$ of the bulk Weyl tensor. The matter corrections are given by

$$S_{\mu\nu}=\tfrac{1}{12}T_\alpha{}^\alpha T_{\mu\nu}-\tfrac{1}{4}T_{\mu\alpha}T^\alpha{}_\nu+\tfrac{1}{24}g_{\mu\nu}[3T_{\alpha\beta}T^{\alpha\beta}-(T_\alpha{}^\alpha)^2]. \tag{4}$$

The projection of the bulk Weyl tensor is[2]

$$\mathcal{E}_{AB}=\tilde{C}_{ACBD}n^Cn^D, \tag{5}$$

which is symmetric[3] and traceless ($\mathcal{E}_{[AB]}=0=\mathcal{E}_A{}^A$) and without components orthogonal to the brane, so that $\mathcal{E}_{AB}n^B=0$ and $\mathcal{E}_{AB}\to\mathcal{E}_{\mu\nu}\delta_A{}^\mu\delta_B{}^\nu$ as $\chi\to0$. The Weyl tensor $\tilde{C}_{ABCD}$ represents the free, nonlocal gravitational field in the bulk. The local part of the bulk gravitational field is the Einstein tensor $\tilde{G}_{AB}$, which is determined via the bulk field equations (1). Thus *$\mathcal{E}_{\mu\nu}$ transmits nonlocal gravitational degrees of freedom from the bulk to the brane, including tidal (or Coulomb), gravito-magnetic and transverse traceless (gravitational wave) effects.*

II. COVARIANT DECOMPOSITION OF BULK EFFECTS

We now provide a covariant decomposition of the bulk correction tensors given by Shiromizu *et al.* [4].

A. Local bulk effects

For *any* matter fields (scalar fields, perfect fluids, kinetic gases, dissipative fluids, etc.), including a combination of

[1]For clarity and consistency, we have changed the notation of [4].

[2]This projection is called the ''electric'' part of the Weyl tensor in [4], but the term is potentially misleading, since the electric part is associated with projection on a timelike vector [11,14], and n^A is spacelike. $\mathcal{E}_{\mu\nu}$ should not be confused with the electric part of the *brane* Weyl tensor, $E_{\mu\nu}$, defined below.

[3]Round (square) brackets enclosing indices denote (anti-) symmetrization.

different fields, the general form of the brane energy-momentum tensor can be covariantly given as

$$T_{\mu\nu}=\rho u_\mu u_\nu+p h_{\mu\nu}+\pi_{\mu\nu}+2q_{(\mu}u_{\nu)}. \tag{6}$$

The decomposition is irreducible for any chosen 4-velocity u^μ. Here ρ and p are the energy density and isotropic pressure, and $h_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu$ projects orthogonal to u^μ. The energy flux obeys $q_\mu=q_{\langle\mu\rangle}$, and the anisotropic stress obeys $\pi_{\mu\nu}=\pi_{\langle\mu\nu\rangle}$, where angular brackets denote the projected, symmetric and tracefree part:

$$V_{\langle\mu\rangle}=h_\mu{}^\nu V_\nu, \qquad W_{\langle\mu\nu\rangle}=[h_{(\mu}{}^\alpha h_{\nu)}{}^\beta-\tfrac{1}{3}h^{\alpha\beta}h_{\mu\nu}]W_{\alpha\beta}.$$

Equations (4) and (6) imply the irreducible decomposition

$$\begin{aligned}S_{\mu\nu}=&\tfrac{1}{24}[2\rho^2-3\pi_{\alpha\beta}\pi^{\alpha\beta}]u_\mu u_\nu+\tfrac{1}{24}[2\rho^2+4\rho p\\&+\pi_{\alpha\beta}\pi^{\alpha\beta}-4q_\alpha q^\alpha]h_{\mu\nu}-\tfrac{1}{12}(\rho+2p)\pi_{\mu\nu}\\&+\pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^\alpha+q_{\langle\mu}q_{\nu\rangle}+\tfrac{1}{3}\rho q_{(\mu}u_{\nu)}-\tfrac{1}{12}q^\alpha\pi_{\alpha(\mu}u_{\nu)}.\end{aligned} \tag{7}$$

For a perfect fluid or minimally coupled scalar field,

$$S_{\mu\nu}=\tfrac{1}{12}\rho^2 u_\mu u_\nu+\tfrac{1}{12}\rho(\rho+2p)h_{\mu\nu},$$

in agreement with [4]. The quadratic energy-momentum corrections to standard general relativity will be significant for $\tilde{\kappa}^4\rho^2\gtrsim\kappa^2\rho$, i.e., in the high-energy regime

$$\rho\gtrsim\lambda\sim\left(\frac{\tilde{M}_{\rm p}}{M_{\rm p}}\right)^2\tilde{M}_{\rm p}^4.$$

B. Nonlocal bulk effects

The symmetry properties of $\mathcal{E}_{\mu\nu}$ imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field u^μ as

$$\mathcal{E}_{\mu\nu}=-\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\mathcal{U}(u_\mu u_\nu+\tfrac{1}{3}h_{\mu\nu})+\mathcal{P}_{\mu\nu}+2\mathcal{Q}_{(\mu}u_{\nu)}]. \tag{8}$$

The factor $(\tilde{\kappa}/\kappa)^4$ is introduced for dimensional reasons. Here

$$\mathcal{U}=-\left(\frac{\kappa}{\tilde{\kappa}}\right)^4\mathcal{E}_{\mu\nu}u^\mu u^\nu$$

is an effective nonlocal energy density on the brane, arising from the free gravitational field in the bulk. This nonlocal energy density need not be positive (see below). There is an effective nonlocal anisotropic stress

$$\mathcal{P}_{\mu\nu}=-\left(\frac{\kappa}{\tilde{\kappa}}\right)^4\mathcal{E}_{\langle\mu\nu\rangle}$$

on the brane, arising from the free gravitational field in the bulk, and

$$\mathcal{Q}_\mu=\left(\frac{\kappa}{\tilde{\kappa}}\right)^4\mathcal{E}_{\langle\mu\rangle\nu}u^\nu$$

is an effective nonlocal energy flux on the brane, arising from the free gravitational field in the bulk.

If the induced metric on the brane is flat, and the bulk is anti–de Sitter, as in the original Randall-Sundrum scenario [2], then $\mathcal{E}_{\mu\nu}=0$. Treating this as a background, it follows that tensor (transverse traceless) perturbations on the brane arising from nonlocal bulk degrees of freedom are given by

$$\mathcal{U}=0=\mathcal{Q}_\mu, \quad {\rm D}^\nu\mathcal{P}_{\mu\nu}=0, \tag{9}$$

where ${\rm D}_\mu$ is the totally projected part of the brane covariant derivative:

$${\rm D}_\mu F^{\alpha\cdots}{}_{\cdots\beta}=h_\mu{}^\nu h^\alpha{}_\gamma\cdots h_\beta{}^\delta\nabla_\nu F^{\gamma\cdots}{}_{\cdots\delta}.$$

Equation (9) provides a covariant characterization of brane tensor perturbations on an anti–de Sitter background.

In cosmology, the background induced metric is not flat, but a spatially homogeneous and isotropic Friedmann model, for which

$${\rm D}_\mu\mathcal{U}=\mathcal{Q}_\mu=\mathcal{P}_{\mu\nu}=0. \tag{10}$$

Thus for a perturbed Friedmann model, the nonlocal bulk effects are covariantly and gauge invariantly described by the first-order quantities ${\rm D}_\mu\mathcal{U}$, $\mathcal{Q}_\mu$, $\mathcal{P}_{\mu\nu}$. Since $\mathcal{U}\neq 0$ in general in the Friedmann background [5,6], it follows that, for tensor perturbations on the brane,

$${\rm D}_\mu\mathcal{U}=0=\mathcal{Q}_\mu, \quad {\rm D}^\nu\mathcal{P}_{\mu\nu}=0. \tag{11}$$

Scalar perturbations (Coulomb-like bulk effects) will be characterized by

$$\mathcal{Q}_\mu={\rm D}_\mu\mathcal{Q}, \quad \mathcal{P}_{\mu\nu}={\rm D}_{\langle\mu}{\rm D}_{\nu\rangle}\mathcal{P}, \tag{12}$$

for some scalars $\mathcal{Q}$ and $\mathcal{P}$, while for vector perturbations (gravito-magnetic-like bulk effects)

$${\rm D}^\mu\mathcal{Q}_\mu=0, \quad \mathcal{P}_{\mu\nu}={\rm D}_{\langle\mu}\mathcal{P}_{\nu\rangle}, \quad {\rm D}^\mu\mathcal{P}_\mu=0. \tag{13}$$

C. Gravitational and non-gravitational acceleration

In order to find the role of bulk effects in tidal acceleration on the brane, we start from the relation

$$\mathcal{E}_{\mu\nu}u^\mu u^\nu=\lim_{\chi\to 0}\tilde{\mathcal{C}}_{ABCD}\tilde{u}^A n^B\tilde{u}^C n^D,$$

where $\tilde{u}^A$ is an extension off the brane of the 4-velocity (with $\tilde{u}^A n_A=0$). The tidal acceleration in the n^A direction measured by comoving observers is $-n_A\tilde{R}^A{}_{BCD}\tilde{u}^B n^C\tilde{u}^D$. Now

$$\begin{aligned}\tilde{R}_{ABCD}=&\tilde{C}_{ABCD}+\tfrac{2}{3}\{\tilde{g}_{A[C}\tilde{R}_{D]B}+\tilde{g}_{B[D}\tilde{R}_{C]A}\}\\&-\tfrac{1}{6}\tilde{R}\tilde{g}_{A[C}\tilde{g}_{D]B},\end{aligned}$$

so that by the field equation (1) (and recalling that $T_{AB}n^B = 0$),

$$-\tilde{R}_{ABCD}n^A\tilde{u}^B n^C\tilde{u}^D = -\mathcal{E}_{AB}\tilde{u}^A\tilde{u}^B + \tfrac{1}{6}\tilde{\kappa}^2\tilde{\Lambda}.$$

Thus *the comoving tidal acceleration on the brane, in the off-brane direction, is*

$$\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{U} + \frac{1}{6}\tilde{\kappa}^2\tilde{\Lambda}. \tag{14}$$

Since $\tilde{\Lambda}$ is negative, it contributes to acceleration towards the brane. This reflects the confining role of the negative bulk cosmological constant on the gravitational field in the warped metric models of Randall-Sundrum type. Equation (14) shows that *localization of the gravitational field near the brane is enhanced by negative $\mathcal{U}$, while positive $\mathcal{U}$ acts against confinement*, by contributing to tidal acceleration away from the brane. This picture is consistent with a Newtonian interpretation, where the gravitational field carries negative energy density.

The non-gravitational acceleration of fluid world lines on the brane is

$$\tilde{A}^A = \tilde{u}^B\tilde{\nabla}_B\tilde{u}^A \quad \text{at} \quad \chi = 0.$$

Locally, near the brane, the metric may be written as [4]

$$d\tilde{s}^{\,2} = d\chi^2 + g_{\mu\nu}(x^\alpha,\chi)dx^\mu dx^\nu,$$

so that

$$\tilde{\Gamma}^A{}_{\mu\nu}(x,0) = \Gamma^\alpha{}_{\mu\nu}(x)\,\delta_\alpha{}^A - \tfrac{1}{2}\,g_{\mu\nu,\chi}(x,0)\,\delta_\chi{}^A.$$

This allows us to express the 5-dimensional acceleration $\tilde{A}^A$ in terms of the 4-dimensional acceleration $A^\mu = u^\nu\nabla_\nu u^\mu$ on the brane. First, we use the covariant form $g_{\mu\nu,\chi} = \mathcal{L}_{\mathbf{n}}g_{\mu\nu}$, where $\mathcal{L}_{\mathbf{n}}$ is the Lie derivative along n^A. Then we use the expression for the extrinsic curvature of the brane [4],

$$K^+_{\mu\nu} = \frac{1}{2}\lim_{\chi\to 0^+}\mathcal{L}_{\mathbf{n}}g_{\mu\nu},$$

which leads to

$$\tilde{A}^A(x,0^+) = A^\mu(x)\,\delta_\mu{}^A - K^+_{\mu\nu}(x)u^\mu(x)u^\nu(x)n^A(x,0^+), \tag{15}$$

on the brane. The extrinsic curvature is given in terms of the brane tension and energy-momentum by [4]

$$K^+_{\mu\nu} = -\tfrac{1}{6}\tilde{\kappa}^2[\lambda g_{\mu\nu} + 3T_{\mu\nu} + (\rho - 3p)g_{\mu\nu}].$$

Substituting in Eq. (15), we find

$$\tilde{A}^A(x,0^+) = A^\mu(x)\,\delta_\mu{}^A + \tfrac{1}{6}\tilde{\kappa}^2[2\rho(x) + 3p(x) - \lambda]n^A(x,0^+). \tag{16}$$

It follows that $n_A\tilde{A}^A$ is nonzero on the brane; i.e., there is *a non-gravitational acceleration of fluid world lines orthogonal to the brane*. The direction depends on the sign of $2\rho + 3p - \lambda$: *for $2\rho + 3p - \lambda > 0$, the transverse acceleration is away from the brane, while for $2\rho + 3p - \lambda < 0$, it is towards the brane.* If the pressure is positive, then at high energies $2\rho + 3p - \lambda > 0$, so that either the brane must accelerate, or there must be a non-gravitational mechanism for keeping matter on the brane.

Inflationary expansion, when pressure is negative, can change this situation. Inflation on the brane is characterized by [8]

$$p < -\left(\frac{\lambda + 2\rho}{\lambda + \rho}\right)\frac{\rho}{3}, \tag{17}$$

and this condition implies that $2\rho + 3p - \lambda < 0$. Thus during inflation on the brane, the transverse acceleration is towards the brane; *inflation acts as a non-gravitational mechanism keeping matter on the brane.*

D. Effective total energy-momentum tensor

All the bulk corrections may be consolidated into effective total energy density, pressure, anisotropic stress and energy flux, as follows. The modified Einstein equations take the standard Einstein form with a redefined energy-momentum tensor:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T^{\rm tot}_{\mu\nu}, \tag{18}$$

where

$$T^{\rm tot}_{\mu\nu} = T_{\mu\nu} + \frac{\tilde{\kappa}^4}{\kappa^2}S_{\mu\nu} - \frac{1}{\kappa^2}\mathcal{E}_{\mu\nu}. \tag{19}$$

Then

$$\rho^{\rm tot} = \rho + \frac{\tilde{\kappa}^4}{\kappa^6}\left[\frac{\kappa^4}{24}(2\rho^2 - 3\pi_{\mu\nu}\pi^{\mu\nu}) + \mathcal{U}\right] \tag{20}$$

$$p^{\rm tot} = p + \frac{\tilde{\kappa}^4}{\kappa^6}\left[\frac{\kappa^4}{24}(2\rho^2 + 4\rho p + \pi_{\mu\nu}\pi^{\mu\nu} - 4q_\mu q^\mu) + \tfrac{1}{3}\mathcal{U}\right] \tag{21}$$

$$\pi^{\rm tot}_{\mu\nu} = \pi_{\mu\nu} + \frac{\tilde{\kappa}^4}{\kappa^6}\left[\frac{\kappa^4}{12}\{-(\rho + 3p)\pi_{\mu\nu} + \pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^\alpha + q_{\langle\mu}q_{\nu\rangle}\} + \mathcal{P}_{\mu\nu}\right] \tag{22}$$

$$q^{\rm tot}_\mu = q_\mu + \frac{\tilde{\kappa}^4}{\kappa^6}\left[\frac{\kappa^4}{24}(4\rho q_\mu - \pi_{\mu\nu}q^\nu) + \mathcal{Q}_\mu\right]. \tag{23}$$

(Note that $\tilde{\kappa}^4/\kappa^6$ is dimensionless.)

These general expressions simplify in the case of a perfect fluid (or minimally coupled scalar field or isotropic one-particle distribution function), i.e., for $q_\mu = 0 = \pi_{\mu\nu}$. However, the total energy flux and anisotropic stress do not vanish in this case in general:

$$q_\mu^{\rm tot}=\frac{\tilde\kappa^4}{\kappa^6}\mathcal{Q}_\mu\,,\quad \pi_{\mu\nu}^{\rm tot}=\frac{\tilde\kappa^4}{\kappa^6}\mathcal{P}_{\mu\nu}\,.$$

Thus *nonlocal bulk effects can contribute to effective imperfect fluid terms even when the matter on the brane has perfect fluid form.*

III. LOCAL AND NONLOCAL CONSERVATION EQUATIONS

As a consequence of the form of the bulk energy-momentum tensor in Eq. (1) and of Z_2 symmetry, it follows [4] that the brane energy-momentum tensor separately satisfies the conservation equations, i.e.,

$$\nabla^\nu T_{\mu\nu}=0. \tag{24}$$

Then the Bianchi identities on the brane imply that the projected Weyl tensor obeys the constraint

$$\nabla^\mu \mathcal{E}_{\mu\nu}=\tilde\kappa^4\nabla^\mu S_{\mu\nu}, \tag{25}$$

which shows that its longitudinal part is sourced by quadratic energy-momentum terms, including spatial gradients and time derivatives. Thus evolution and inhomogeneity in the matter fields generates nonlocal Coulomb-like gravitational effects in the bulk, which ''back react'' on the brane.

The brane energy-momentum tensor *and* the consolidated effective energy-momentum tensor are *both* conserved separately. Conservation of $T_{\mu\nu}$ gives the standard general relativity energy and momentum conservation equations [15]

$$\dot\rho+\Theta(\rho+p)+\mathrm{D}^\mu q_\mu+2A^\mu q_\mu+\sigma^{\mu\nu}\pi_{\mu\nu}=0, \tag{26}$$

$$\dot q_{\langle\mu\rangle}+\tfrac{4}{3}\Theta q_\mu+\mathrm{D}_\mu p+(\rho+p)A_\mu +\mathrm{D}^\nu\pi_{\mu\nu}+A^\nu\pi_{\mu\nu}+\sigma_{\mu\nu}q^\nu-[\omega,q]_\mu=0. \tag{27}$$

An overdot denotes $u^\nu\nabla_\nu$, $\Theta=\mathrm{D}^\mu u_\mu$ is the volume expansion rate of the u^μ congruence, $A_\mu=\dot u_\mu=A_{\langle\mu\rangle}$ is its 4-acceleration, $\sigma_{\mu\nu}=\mathrm{D}_{\langle\mu}u_{\nu\rangle}$ is its shear rate, and $\omega_\mu=-\frac{1}{2}\mathrm{curl}\,u_\mu=\omega_{\langle\mu\rangle}$ is its vorticity rate. The covariant spatial curl is given by [12]

$$\mathrm{curl}\,V_\mu=\varepsilon_{\mu\alpha\beta}\mathrm{D}^\alpha V^\beta,\quad \mathrm{curl}\,W_{\mu\nu}=\varepsilon_{\alpha\beta(\mu}\mathrm{D}^\alpha W^\beta{}_{\nu)},$$

where $\varepsilon_{\mu\nu\sigma}$ is the projected alternating tensor. The covariant cross-product is

$$[V,Y]_\mu=\varepsilon_{\mu\alpha\beta}V^\alpha Y^\beta.$$

The conservation of $T^{\rm tot}_{\mu\nu}$ gives, upon using Eqs. (20)–(27), a propagation equation for the nonlocal energy density $\mathcal{U}$,

$$\dot{\mathcal{U}}+\tfrac{4}{3}\Theta\mathcal{U}+\mathrm{D}^\mu\mathcal{Q}_\mu+2A^\mu\mathcal{Q}_\mu+\sigma^{\mu\nu}\mathcal{P}_{\mu\nu}=\tfrac{1}{24}\kappa^4[6\pi^{\mu\nu}\dot\pi_{\mu\nu}+6(\rho+p)\sigma^{\mu\nu}\pi_{\mu\nu}+2\Theta(2q^\mu q_\mu-\pi^{\mu\nu}\pi_{\mu\nu})+2A^\mu q^\nu\pi_{\mu\nu} +4q^\mu\mathrm{D}_\mu\rho+q^\mu\mathrm{D}^\nu\pi_{\mu\nu}+\pi^{\mu\nu}\mathrm{D}_\mu q_\nu-2\sigma^{\mu\nu}\pi_{\alpha\mu}\pi_\nu{}^\alpha-2\sigma^{\mu\nu}q_\mu q_\nu], \tag{28}$$

and a propagation equation for the nonlocal energy flux $\mathcal{Q}_\mu$:

$$\begin{aligned}
&\dot{\mathcal{Q}}_{\langle\mu\rangle}+\tfrac{4}{3}\Theta\mathcal{Q}_\mu+\tfrac{1}{3}\mathrm{D}_\mu\mathcal{U}+\tfrac{4}{3}\mathcal{U}A_\mu+\mathrm{D}^\nu\mathcal{P}_{\mu\nu}+A^\nu\mathcal{P}_{\mu\nu}+\sigma_{\mu\nu}\mathcal{Q}^\nu-[\omega,\mathcal{Q}]_\mu\\
&\quad=\tfrac{1}{24}\kappa^4[-4(\rho+p)\mathrm{D}_\mu\rho+q^\nu\dot\pi_{\langle\mu\nu\rangle}+\pi_\mu{}^\nu\mathrm{D}_\nu(2\rho+5p)\\
&\qquad+6(\rho+p)\mathrm{D}^\nu\pi_{\mu\nu}-\tfrac{2}{3}\pi^{\alpha\beta}(\mathrm{D}_\mu\pi_{\alpha\beta}+3\mathrm{D}_\alpha\pi_{\beta\mu})\\
&\qquad-3\pi_{\mu\alpha}\mathrm{D}_\beta\pi^{\alpha\beta}+\tfrac{28}{3}q^\nu\mathrm{D}_\mu q_\nu+4\rho A^\nu\pi_{\mu\nu}-3\pi_{\mu\alpha}A_\beta\pi^{\alpha\beta}+\tfrac{8}{3}A_\mu\pi^{\alpha\beta}\pi_{\alpha\beta}\\
&\qquad-\pi_{\mu\alpha}\sigma^{\alpha\beta}q_\beta+\sigma_{\mu\alpha}\pi^{\alpha\beta}q_\beta\\
&\qquad+\pi_{\mu\nu}[\omega,q]^\nu-\varepsilon_{\mu\alpha\beta}\omega^\alpha\pi^{\beta\nu}q_\nu+4(\rho+p)\Theta q_\mu+6q_\mu A^\nu q_\nu+\tfrac{14}{3}A_\mu q^\nu q_\nu+4q_\nu\sigma^{\alpha\beta}\pi_{\alpha\beta}].
\end{aligned} \tag{29}$$

These equations may be thought of as conservation equations on the brane for the nonlocal energy density and energy flux due to the free gravitational field in the bulk. In general, the 4 independent equations determine 4 of the 9 independent components of $\mathcal{E}_{\mu\nu}$ on the brane. What is missing is an evolution equation for $\mathcal{P}_{\mu\nu}$ (which has up to 5 independent components). Thus, in general, the projection of the 5-dimensional field equations onto the brane does not lead to a closed system. Nor could we expect this to be the case, since there are bulk degrees of freedom whose impact on the brane cannot be predicted by brane observers. These degrees of freedom could arise from propagating gravity waves in the bulk, possibly generated by inhomogeneity on the brane itself. The point is that waves which penetrate the 5th dimension are governed by off-brane bulk dynamical equations. Our decomposition of $\mathcal{E}_{\mu\nu}$ has shown that *the evolution of the nonlocal energy density and flux (associated with the scalar and vector parts of $\mathcal{E}_{\mu\nu}$) is determined on the brane, while the evolution of the nonlocal anisotropic stress (associated with the tensor part of $\mathcal{E}_{\mu\nu}$) is not.*

If the nonlocal anisotropic stress contribution from the bulk field vanishes, i.e., if

$$\mathcal{P}_{\mu\nu}=0,$$

then the evolution of $\mathcal{E}_{\mu\nu}$ is determined by Eqs. (28) and (29). A special case of this arises when the induced metric on the brane is Friedmann, i.e., when Eq. (10) holds. Then $\mathcal{E}_{\mu\nu}$ has only 1 independent component, $\mathcal{U}$, and it is determined by Eq. (28) (see below), with Eq. (29) reducing to $0=0$. Another case when the equations close on the brane is when the brane is static (see [16]).

In the perfect fluid case, the conservation equations (26)–(29) reduce to

$$\dot{\rho}+\Theta(\rho+p)=0, \tag{30}$$

$$\mathrm{D}_\mu p+(\rho+p)A_\mu=0. \tag{31}$$

For a minimally coupled scalar field, $\rho=\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi+V(\varphi)$ and $p=\frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi-V(\varphi)$. In the adiabatic case, Eq. (31) gives

$$A_\mu=-\frac{c_\mathrm{s}^2}{\rho+p}\mathrm{D}_\mu\rho,\quad c_\mathrm{s}^2=\frac{\dot{p}}{\dot{\rho}}. \tag{32}$$

The nonlocal conservation equations (28) and (29) reduce to

$$\dot{\mathcal{U}}+\tfrac{4}{3}\Theta\mathcal{U}+\mathrm{D}^\mu\mathcal{Q}_\mu+2A^\mu\mathcal{Q}_\mu+\sigma^{\mu\nu}\mathcal{P}_{\mu\nu}=0, \tag{33}$$

$$\begin{aligned}\dot{\mathcal{Q}}_{\langle\mu\rangle}&+\tfrac{4}{3}\Theta\mathcal{Q}_\mu+\tfrac{1}{3}\mathrm{D}_\mu\mathcal{U}+\tfrac{4}{3}\mathcal{U}A_\mu+\mathrm{D}^\nu\mathcal{P}_{\mu\nu}\\&+A^\nu\mathcal{P}_{\mu\nu}+\sigma_{\mu\nu}\mathcal{Q}^\nu-[\omega,\mathcal{Q}]_\mu\\&=-\tfrac{1}{6}\kappa^4(\rho+p)\mathrm{D}_\mu\rho.\end{aligned} \tag{34}$$

Equation (34) shows that *if $\mathcal{E}_{\mu\nu}=0$ and the brane energy-momentum tensor has perfect fluid form, then the density ρ must be homogeneous* [4]. The converse does not hold; i.e., homogeneous density does *not* in general imply a vanishing $\mathcal{E}_{\mu\nu}$. This is readily apparent from Eq. (34). A simple example is provided by the Friedmann case: Equation (34) is trivially satisfied, while Eq. (33) gives the ''dark radiation'' solution

$$\mathcal{U}=\mathcal{U}_0\left(\frac{a_0}{a}\right)^4. \tag{35}$$

A simple generalization of the Friedmann case is the purely Coulomb-like case, $\mathcal{Q}_\mu=0=\mathcal{P}_{\mu\nu}$. Equation (35) still holds, but with a an average scale factor, which is in general inhomogeneous. Equation (34) reduces to a constraint on the acceleration. Local momentum conservation already provides the constraint in Eq. (32). It follows that in the purely Coulomb-like case, the spatial gradient of the nonlocal energy density must be proportional to that of the local energy density:

$$\mathrm{D}_\mu\mathcal{U}=\left[\frac{8c_\mathrm{s}^2\mathcal{U}-\kappa^4(\rho+p)^2}{2(\rho+p)}\right]\mathrm{D}_\mu\rho.$$

Linearization about a Friedmann background does not change Eqs. (30) and (31), but Eqs. (33) and (34) lead to

$$\dot{\mathcal{U}}+\tfrac{4}{3}\Theta\mathcal{U}+\mathrm{D}^\mu\mathcal{Q}_\mu=0, \tag{36}$$

$$\dot{\mathcal{Q}}_\mu+4H\mathcal{Q}_\mu+\tfrac{1}{3}\mathrm{D}_\mu\mathcal{U}+\tfrac{4}{3}\mathcal{U}A_\mu+\mathrm{D}^\nu\mathcal{P}_{\mu\nu}=-\tfrac{1}{6}\kappa^4(\rho+p)\mathrm{D}_\mu\rho, \tag{37}$$

where H is the Hubble rate in the background. The nonlocal tensor mode, which satisfies $\mathrm{D}^\nu\mathcal{P}_{\mu\nu}=0\neq\mathcal{P}_{\mu\nu}$, does not enter the nonlocal conservation equations.

IV. PROPAGATION AND CONSTRAINT EQUATIONS

Equations (26)–(29) are propagation equations for the local and nonlocal energy density and flux. The remaining covariant equations on the brane are the propagation and constraint equations for the kinematic quantities and the free gravitational field on the brane. The kinematic quantities govern the relative motion of neighboring fundamental world lines, and describe the universal expansion and its local anisotropies. The locally free gravitational field *on the brane* is given by the brane Weyl tensor $C_{\mu\nu\alpha\beta}$. This splits irreducibly for a given u^μ into the gravito-electric and gravito-magnetic fields on the brane:

$$E_{\mu\nu}=C_{\mu\alpha\nu\beta}u^\alpha u^\beta=E_{\langle\mu\nu\rangle},$$

$$H_{\mu\nu}=\tfrac{1}{2}\varepsilon_{\mu\alpha\beta}C^{\alpha\beta}{}_{\nu\gamma}u^\gamma=H_{\langle\mu\nu\rangle},$$

where $E_{\mu\nu}$ must not be confused with $\mathcal{E}_{\mu\nu}$. The Ricci identity for u^μ and the Bianchi identities $\nabla^\beta C_{\mu\nu\alpha\beta}=\nabla_{[\mu}(-R_{\nu]\alpha}+\frac{1}{6}Rg_{\nu]\alpha})$ produce the fundamental evolution and constraint equations governing the above covariant quantities [11]. Einstein's equations are incorporated via the algebraic replacement of the Ricci tensor $R_{\mu\nu}$ by the effective total energy-momentum tensor, according to Eq. (18). These are derived directly from the standard general relativity versions (see, e.g., [15]) by simply replacing the energy-momentum tensor terms $\rho,\ldots$ by $\rho^{\mathrm{tot}},\ldots$. The result for a general imperfect fluid is given in Appendix A.

A. Nonlinear equations

For a perfect fluid or minimally coupled scalar field, the equations in Appendix A reduce to the following:

Expansion propagation (generalized Raychaudhuri equation):

$$\begin{aligned}\dot{\Theta}&+\tfrac{1}{3}\Theta^2+\sigma_{\mu\nu}\sigma^{\mu\nu}-2\omega_\mu\omega^\mu-\mathrm{D}^\mu A_\mu+A_\mu A^\mu\\&+\tfrac{1}{2}\kappa^2(\rho+3p)-\Lambda\\&=-\frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4\rho(2\rho+3p)+12\mathcal{U}].\end{aligned} \tag{38}$$

Vorticity propagation:

$$\dot{\omega}_{\langle\mu\rangle}+\tfrac{2}{3}\Theta\,\omega_{\mu}+\tfrac{1}{2}\mathrm{curl}\,A_{\mu}-\sigma_{\mu\nu}\omega^{\nu}=0. \qquad (39)$$

Shear propagation:

$$\dot{\sigma}_{\langle\mu\nu\rangle}+\tfrac{2}{3}\Theta\,\sigma_{\mu\nu}+E_{\mu\nu}-\mathrm{D}_{\langle\mu}A_{\nu\rangle}+\sigma_{\alpha\langle\mu}\sigma_{\nu\rangle}{}^{\alpha}+\omega_{\langle\mu}\omega_{\nu\rangle}-A_{\langle\mu}A_{\nu\rangle}=\frac{1}{2}\left(\frac{\tilde{\kappa}}{\kappa}\right)^{4}\mathcal{P}_{\mu\nu}. \qquad (40)$$

Gravito-electric propagation:

$$\dot{E}_{\langle\mu\nu\rangle}+\Theta E_{\mu\nu}-\mathrm{curl}\,H_{\mu\nu}+\tfrac{1}{2}\kappa^{2}(\rho+p)\sigma_{\mu\nu}-2A^{\alpha}\varepsilon_{\alpha\beta(\mu}H_{\nu)}{}^{\beta}-3\sigma_{\alpha\langle\mu}E_{\nu\rangle}{}^{\alpha}+\omega^{\alpha}\varepsilon_{\alpha\beta(\mu}E_{\nu)}{}^{\beta}$$
$$=-\frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^{4}[\{\kappa^{4}\rho(\rho+p)+8\mathcal{U}\}\sigma_{\mu\nu}+6\dot{\mathcal{P}}_{\langle\mu\nu\rangle}+2\Theta\mathcal{P}_{\mu\nu}+6\mathrm{D}_{\langle\mu}\mathcal{Q}_{\nu\rangle}+12A_{\langle\mu}\mathcal{Q}_{\nu\rangle}+6\sigma^{\alpha}{}_{\langle\mu}\mathcal{P}_{\nu\rangle\alpha}+6\omega^{\alpha}\varepsilon_{\alpha\beta(\mu}\mathcal{P}_{\nu)}{}^{\beta}]. \qquad (41)$$

Gravito-magnetic propagation:

$$\dot{H}_{\langle\mu\nu\rangle}+\Theta H_{\mu\nu}+\mathrm{curl}\,E_{\mu\nu}-3\sigma_{\alpha\langle\mu}H_{\nu\rangle}{}^{\alpha}+\omega^{\alpha}\varepsilon_{\alpha\beta(\mu}H_{\nu)}{}^{\beta}+2A^{\alpha}\varepsilon_{\alpha\beta(\mu}E_{\nu)}{}^{\beta}=\frac{1}{2}\left(\frac{\tilde{\kappa}}{\kappa}\right)^{4}[\mathrm{curl}\,\mathcal{P}_{\mu\nu}-3\omega_{\langle\mu}\mathcal{Q}_{\nu\rangle}+\sigma^{\alpha}{}_{(\mu}\varepsilon_{\nu)\alpha\beta}\mathcal{Q}^{\beta}]. \qquad (42)$$

Vorticity constraint:

$$\mathrm{D}^{\mu}\omega_{\mu}-A^{\mu}\omega_{\mu}=0. \qquad (43)$$

Shear constraint:

$$\mathrm{D}^{\nu}\sigma_{\mu\nu}-\mathrm{curl}\,\omega_{\mu}-\tfrac{2}{3}\mathrm{D}_{\mu}\Theta+2[\omega,A]_{\mu}=-\left(\frac{\tilde{\kappa}}{\kappa}\right)^{4}\mathcal{Q}_{\mu}. \qquad (44)$$

Gravito-magnetic constraint:

$$\mathrm{curl}\,\sigma_{\mu\nu}+\mathrm{D}_{\langle\mu}\omega_{\nu\rangle}-H_{\mu\nu}+2A_{\langle\mu}\omega_{\nu\rangle}=0. \qquad (45)$$

Gravito-electric divergence:

$$\mathrm{D}^{\nu}E_{\mu\nu}-\tfrac{1}{3}\kappa^{2}\mathrm{D}_{\mu}\rho-[\sigma,H]_{\mu}+3H_{\mu\nu}\omega^{\nu}=\frac{1}{18}\left(\frac{\tilde{\kappa}}{\kappa}\right)^{4}\{\kappa^{4}\rho\mathrm{D}_{\mu}\rho+6\mathrm{D}_{\mu}\mathcal{U}-6\Theta\,\mathcal{Q}_{\mu}-9\mathrm{D}^{\nu}\mathcal{P}_{\mu\nu}+9\sigma_{\mu\nu}\mathcal{Q}^{\nu}-27[\omega,\mathcal{Q}]_{\mu}\}. \qquad (46)$$

Gravito-magnetic divergence:

$$\mathrm{D}^{\nu}H_{\mu\nu}-\kappa^{2}(\rho+p)\omega_{\mu}+[\sigma,E]_{\mu}-3E_{\mu\nu}\omega^{\nu}=\frac{1}{6}\left(\frac{\tilde{\kappa}}{\kappa}\right)^{4}\{\kappa^{4}\rho(\rho+p)\omega_{\mu}-3\,\mathrm{curl}\,\mathcal{Q}_{\mu}+8\mathcal{U}\omega_{\mu}-3[\sigma,\mathcal{P}]_{\mu}-3\mathcal{P}_{\mu\nu}\omega^{\nu}\}. \qquad (47)$$

Here the covariant tensor commutator is

$$[W,Z]_{\mu}=\varepsilon_{\mu\alpha\beta}W^{\alpha}{}_{\gamma}Z^{\beta\gamma}.$$

The standard 4-dimensional general relativity results are regained by setting all right hand sides to zero in Eqs. (38)–(47).

Together with Eqs. (30)–(34), these equations govern the dynamics of the matter and gravitational fields on the brane, incorporating both the local (quadratic energy-momentum) and nonlocal (projected Weyl) effects from the bulk. These effects give rise to important new driving and source terms in the propagation and constraint equations. The vorticity propagation and constraint, and the gravito-magnetic constraint have no explicit bulk effects, but all other equations do. Local and nonlocal energy densities are driving terms in the expansion propagation, and note that these are the terms that determine the gravitational and non-gravitational acceleration transverse to the brane. The spatial gradients of local and nonlocal energy densities provide sources for the gravito-electric field. The nonlocal anisotropic stress is a driving term in the propagation of shear and the gravito-electric or -magnetic fields, and the nonlocal energy flux is a source for shear and the gravito-magnetic field.

In general the system of equations is not closed: there is no evolution equation for the nonlocal anisotropic stress $\mathcal{P}_{\mu\nu}$, which carries the tensor modes in perturbed solutions. If we set $\mathcal{E}_{\mu\nu}=0$ to close the system, i.e., if we allow only local matter effects from the fifth dimension, then, as noted above, the density is forced to be homogeneous. This is clearly too restrictive. A less restrictive way of closing the system is to assume that the nonlocal anisotropic stress vanishes, i.e., $\mathcal{P}_{\mu\nu}=0$. However, this rules out tensor modes arising from the free field in the bulk, and also limits the scalar and vector modes, which will in general also be present in $\mathcal{P}_{\mu\nu}$.

B. Linearized equations

We have derived the exact nonlinear equations that govern gravitational dynamics on the brane as seen by brane observers. These equations hold for any geometry of the brane, and they are fully covariant on the brane. In particular, this means that we can linearize the equations by taking a suitable limit, and without starting from a given background solution. In this way we avoid the need for choosing coordinates, and we deal directly with covariant physical and geometric quantities, rather than metric components.

The limiting case of the background Friedmann brane is characterized by the vanishing of all inhomogeneous and anisotropic quantities. These quantities are then first order of smallness in the linearization scheme, and since they vanish in the background, they are gauge invariant [13]. The standard general relativity scheme is modified by the additional degrees of freedom arising from bulk effects. In particular, the generalized Friedmann equation on the brane is [6]

$$H^2=\tfrac{1}{3}\Lambda+\tfrac{1}{3}\kappa^2\rho-\frac{K}{a^2}+\tfrac{1}{36}\tilde{\kappa}^4\rho^2+\frac{1}{3}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{U}_0\left(\frac{a_0}{a}\right)^4, \tag{48}$$

where $K=0,\pm 1$.

Linearization about a Friedmann brane model of the propagation and constraint equations leads to the reduced system

$$\dot{\Theta}+\tfrac{1}{3}\Theta^2-\mathrm{D}^\mu A_\mu+\tfrac{1}{2}\kappa^2(\rho+3p)-\Lambda =-\frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4\rho(2\rho+3p)+12\mathcal{U}], \tag{49}$$

$$\dot{\omega}_\mu+2H\omega_\mu+\tfrac{1}{2}\mathrm{curl}\,A_\mu=0, \tag{50}$$

$$\dot{\sigma}_{\mu\nu}+2H\sigma_{\mu\nu}+E_{\mu\nu}-\mathrm{D}_{\langle\mu}A_{\nu\rangle}=\frac{1}{2}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{P}_{\mu\nu}, \tag{51}$$

$$\dot{E}_{\mu\nu}+3HE_{\mu\nu}-\mathrm{curl}\,H_{\mu\nu}+\tfrac{1}{2}\kappa^2(\rho+p)\sigma_{\mu\nu} =-\frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\{\kappa^4\rho(\rho+p)+8\mathcal{U}\}\sigma_{\mu\nu} +6\mathrm{D}_{\langle\mu}\mathcal{Q}_{\nu\rangle}+6\dot{\mathcal{P}}_{\mu\nu}+6H\mathcal{P}_{\mu\nu}], \tag{52}$$

$$\dot{H}_{\mu\nu}+3HH_{\mu\nu}+\mathrm{curl}\,E_{\mu\nu}=\frac{1}{2}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathrm{curl}\,\mathcal{P}_{\mu\nu}, \tag{53}$$

$$\mathrm{D}^\mu\omega_\mu=0, \tag{54}$$

$$\mathrm{D}^\nu\sigma_{\mu\nu}-\mathrm{curl}\,\omega_\mu-\tfrac{2}{3}\mathrm{D}_\mu\Theta=-\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{Q}_\mu, \tag{55}$$

$$\mathrm{curl}\,\sigma_{\mu\nu}+\mathrm{D}_{\langle\mu}\omega_{\nu\rangle}-H_{\mu\nu}=0, \tag{56}$$

$$\mathrm{D}^\nu E_{\mu\nu}-\tfrac{1}{3}\kappa^2\mathrm{D}_\mu\rho=\frac{1}{18}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4\rho\mathrm{D}_\mu\rho+6\mathrm{D}_\mu\mathcal{U} -18H\mathcal{Q}_\mu-9\mathrm{D}^\nu\mathcal{P}_{\mu\nu}], \tag{57}$$

$$\mathrm{D}^\nu H_{\mu\nu}-\kappa^2(\rho+p)\omega_\mu=\frac{1}{6}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\{\kappa^4\rho(\rho+p)+8\mathcal{U}\} \times\omega_\mu-3\,\mathrm{curl}\,\mathcal{Q}_\mu]. \tag{58}$$

These equations, together with the linearized conservation equations (30), (31), (36) and (37), are the basis for a covariant and gauge-invariant description of perturbations on the brane. The local bulk effects (quadratic energy-momentum effects) are purely scalar, as is the nonlocal energy density. The nonlocal energy flux has in general scalar and vector modes

$$\mathcal{Q}_\mu=\mathrm{D}_\mu\mathcal{Q}+\overline{\mathcal{Q}}_\mu, \tag{59}$$

and scalar, vector and tensor modes enter the nonlocal anisotropic stress:

$$\mathcal{P}_{\mu\nu}=\mathrm{D}_{\langle\mu}\mathrm{D}_{\nu\rangle}\mathcal{P}+\mathrm{D}_{\langle\mu}\overline{\mathcal{P}}_{\nu\rangle}+\overline{\mathcal{P}}_{\mu\nu}. \tag{60}$$

In these equations, an overbar denotes a transverse (divergence-free) quantity:

$$\mathrm{D}^\mu\overline{\mathcal{Q}}_\mu=0=\mathrm{D}^\mu\overline{\mathcal{P}}_\mu, \quad \mathrm{D}^\nu\overline{\mathcal{P}}_{\mu\nu}=0.$$

V. NONLINEAR AND PERTURBATIVE DYNAMICS

Bulk effects introduce new degrees of freedom into the dynamics on the brane, subject to the additional nonlocal ''conservation'' equations (33) and (34). Standard results in general relativity may or may not continue to hold under this higher-dimensional modification. We now use the conservation, propagation and constraint equations to generalize some standard results of 4-dimensional general relativity.

A. CMB isotropy and brane homogeneity

In standard general relativity, the isotropy of the CMB radiation has crucial implications for the homogeneity of the universe. If all fundamental observers after last scattering observe an isotropic CMB, then it follows from a theorem of Ehlers, Geren and Sachs [17] that the universe must have a homogeneous Friedmann geometry. This has been generalized to the almost-isotropic case [18], providing a foundation for the perturbative analysis of CMB anisotropies. The Ehlers-Geren-Sachs theorem is based on the collisionless Boltzmann equation and on the kinematic-dynamic characterization:

$$A_\mu=\omega_\mu=\sigma_{\mu\nu}=0\neq\Theta \quad \text{and}$$

$$q_\mu=\pi_{\mu\nu}=0\Rightarrow \text{Friedmann geometry}. \tag{61}$$

Bulk effects do not change the Boltzmann equation, but they do mean that the Friedmann characterization is no longer in general true on the brane. While the gravito-magnetic constraint, Eq. (45), still leads to $H_{\mu\nu}=0$, the shear propagation equation (40) no longer forces $E_{\mu\nu}=0$, because of the nonlocal term $\mathcal{P}_{\mu\nu}$, so that the intrinsic metric need not be conformally flat.

A consistent solution on the brane of the system of (nonlinear) conservation, propagation and constraint equations can be given as follows. We take

$$\mathrm{D}_\mu\rho=\mathrm{D}_\mu\rho_{\mathrm{r}}=\mathrm{D}_\mu\mathcal{U}=\mathrm{D}_\mu\Theta=0, \quad \mathcal{Q}_\mu=0=\mathrm{D}^\nu\mathcal{P}_{\mu\nu}.$$

Then the system of equations reduces to the consistent set

$$\dot{\rho}+\Theta\rho=0,$$

$$\dot{\rho}_{\mathrm{r}}+\tfrac{4}{3}\Theta\rho_{\mathrm{r}}=0,$$

$$\dot{\mathcal{U}}+\tfrac{4}{3}\Theta\mathcal{U}=0,$$

$$\dot{\Theta}+\tfrac{1}{3}\Theta^2+\tfrac{1}{2}\kappa^2(\rho+2\rho_{\rm r})-\Lambda=-\frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4(\rho+\rho_{\rm r})\times(2\rho+3\rho_{\rm r})+12\mathcal{U}],$$

$$\dot{\mathcal{P}}_{\mu\nu}+\tfrac{2}{3}\Theta\mathcal{P}_{\mu\nu}=0,$$

$$E_{\mu\nu}=\frac{1}{2}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{P}_{\mu\nu}.$$

In general, provided that the propagation equation for $\mathcal{P}_{\mu\nu}$ is consistent with the bulk geometry, $E_{\mu\nu}$ need not be zero, so that the brane geometry need not be Friedmann, although it is asymptotically Friedmann.

Thus it is in principle possible via nonlocal bulk effects that isotropic CMB does *not* force the brane metric to be Friedmann.

B. Generalized gravitational collapse

Another important question is how the higher-dimensional bulk effects modify the picture of gravitational collapse and singularities, which depends on Raychaudhuri's equation [11].

The generalized Raychaudhuri equation (38) governs gravitational collapse and initial singularity behavior on the brane. The local energy density and pressure corrections

$$\tfrac{1}{12}\tilde{\kappa}^4\rho(2\rho+3p)$$

further enhance the tendency to collapse, if $2\rho+3p>0$. This condition will be satisfied in thermal collapse (or time-reversed expansion), but it is violated during very high-energy inflation ($\rho\gg\lambda$), by Eq. (17), and in that case the local bulk term acts to further accelerate expansion. This is consistent with the results given in [8]. Thus *local bulk effects at high energy reinforce the formation of singularities during thermal collapse, as predicted in general relativity, while further accelerating expansion during high-energy inflation.*

The nonlocal term

$$\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{U}$$

can act either way, depending on its sign. As shown from Eq. (14), a negative $\mathcal{U}$ enhances the localization of the gravitational field on the brane. In this case, the effect of $\mathcal{U}$ is to counteract gravitational collapse. A positive $\mathcal{U}$ acts against localization, and also reinforces the tendency to collapse.

If higher-dimensional corrections to Einstein's theory tend to *prevent* singularities, then the effective energy density $\mathcal{U}$ on the brane of the free gravitational field in the bulk should be *negative*. In this case, $\mathcal{U}$ also acts to reinforce confinement of the gravitational field to the brane.

C. Cosmological scalar perturbations

The linearized equations on the brane derived in the previous two sections encompass scalar, vector and tensor modes. In order to covariantly (and locally) separate out the scalar modes, we impose the condition that all perturbative quantities be spatial gradients of scalars, i.e.,

$$V_\mu=\mathrm{D}_\mu V,\quad W_{\mu\nu}=\mathrm{D}_{\langle\mu}\mathrm{D}_{\nu\rangle}W.$$

The identities in Appendix B, the vorticity constraint equation (54) and the gravito-magnetic constraint equation (56) then show that

$$\mathrm{curl}\,V_\mu=0=\mathrm{curl}\,W_{\mu\nu},\quad \mathrm{D}^\nu W_{\mu\nu}=\tfrac{2}{3}\mathrm{D}^2(\mathrm{D}_\mu W),$$
$$\omega_\mu=0=H_{\mu\nu},\tag{62}$$

as in standard general relativity.

If we choose the fundamental 4-velocity u^μ such that $\sigma_{\mu\nu}=0$, which is the covariant analogue of the longitudinal or conformal Newtonian gauge in metric-based perturbation theory (see [19,20] for further discussion), then the shear propagation equation (51) becomes a constraint determining the brane tidal tensor:

$$E_{\mu\nu}=\mathrm{D}_{\langle\mu}\mathrm{D}_{\nu\rangle}\left[\Phi+\frac{1}{2}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{P}\right].$$

Here Φ is the relativistic generalization of the gravitational potential, defined by $A_\mu=\mathrm{D}_\mu\Phi$, and $\mathcal{P}$ is the potential for the nonlocal anisotropic stress, defined in Eq. (60). It follows that *nonlocal bulk effects lead to a change in the gravitational tidal potential* (in longitudinal-like gauge):

$$\Phi\to\Phi+\frac{1}{2}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{P}.\tag{63}$$

In the general case, i.e., when u^μ is not chosen to give $\sigma_{\mu\nu}=0$, this simple relation does not hold.

In order to derive the equations governing density perturbations in the general case, we define the density and expansion gradients (as in [13])

$$\Delta_\mu=\frac{a}{\rho}\mathrm{D}_\mu\rho,\quad Z_\mu=a\mathrm{D}_\mu\Theta,\tag{64}$$

and the (dimensionless) gradients describing inhomogeneity in the nonlocal quantities:

$$U_\mu=\frac{a}{\rho}\mathrm{D}_\mu\mathcal{U},\quad Q_\mu=\frac{1}{\rho}\mathrm{D}_\mu\mathcal{Q},\quad P_\mu=\frac{1}{a\rho}\mathrm{D}_\mu\mathcal{P},\tag{65}$$

where $\mathcal{Q}$ is defined in Eq. (59). Then we take the spatial gradient of the energy conservation equations (30) and (36) and the generalized Raychaudhuri equation (49), using the identities in Appendix B and the adiabatic equation (32). We arrive at the following system of equations:

$$\dot{\Delta}_\mu=3wH\Delta_\mu-(1+w)Z_\mu,\tag{66}$$

$$\dot{Z}_\mu = -2HZ_\mu - \left(\frac{c_s^2}{1+w}\right) D^2\Delta_\mu - \left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \rho U_\mu$$
$$- \tfrac{1}{2}\rho\left[\kappa^2 + \tfrac{1}{6}\tilde{\kappa}^4(4+3w)\rho - \left(\frac{\tilde{\kappa}}{\kappa}\right)^4\left(\frac{2c_s^2}{1+w}\right)\frac{\mathcal{U}}{\rho}\right]\Delta_\mu, \quad (67)$$

$$\dot{U}_\mu = (3w-1)HU_\mu - \left(\frac{4c_s^2}{1+w}\right)\left(\frac{\mathcal{U}}{\rho}\right)H\Delta_\mu$$
$$-\left(\frac{4\mathcal{U}}{3\rho}\right)Z_\mu - aD^2Q_\mu, \quad (68)$$

$$\dot{Q}_\mu = (1-3w)HQ_\mu - \frac{1}{3a}U_\mu - \frac{2}{3}aD^2P_\mu$$
$$+\frac{1}{6a}\left[\left(\frac{8c_s^2}{1+w}\right)\frac{\mathcal{U}}{\rho} - \kappa^4\rho^2(1+w)\right]\Delta_\mu, \quad (69)$$

where $w=p/\rho$. The background Friedmann equation (48) relates H and ρ, with $\mathcal{U}$ in the background given by Eq. (35).

In standard general relativity, only the first two equations apply, with $\tilde{\kappa}$ set to zero in Eq. (67). In this case we can decouple the density perturbations via a second-order equation for Δ_μ, whose independent solutions are adiabatic growing and decaying modes. Nonlocal bulk effects introduce important changes to this simple picture. First, we note that there is no equation for $\dot{P}_\mu$, so that *in general, scalar perturbations on the brane cannot be predicted by brane observers without additional information from the unobservable bulk.*

However, there is a very important exception to this, arising from the fact that Q_μ and P_μ only occur in Eqs. (66)–(68) via the Laplacian terms D^2Q_μ and D^2P_μ, and the latter term is the only occurrence of P_μ in the system. It follows that *on super-Hubble scales, the system does close, and brane observers can predict scalar perturbations from initial conditions intrinsic to the brane.* The system reduces to 3 coupled equations in Δ_μ, Z_μ, and U_μ, plus an equation for Q_μ, which is determined once the other 3 quantities are solved for. *One can decouple the density perturbations via a third-order equation for* Δ_μ. The nonlocal energy density plays the role of a non-interacting radiation fluid with the same velocity as the ordinary fluid, and inhomogeneity in $\mathcal{U}$ introduces an additional entropy-like scalar mode. As may have been expected, this additional mode is absent during radiation domination; in this case Eqs. (66) and (68) show that

$$\dot{U}_\mu = \frac{\mathcal{U}_0}{\rho_0}\dot{\Delta}_\mu.$$

In principle, it is straightforward to solve the coupled system on super-Hubble scales, although numerical techniques will be necessary. We do not attempt particular solutions here. However, it may be instructive to see the decoupled third-order equation for density perturbations during matter domination, on a flat background:

$$\dddot{\Delta}_\mu + 2H\ddot{\Delta}_\mu + \left[\frac{4}{3}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\mathcal{U} - \tfrac{7}{6}\kappa^2\rho - \tfrac{2}{9}\tilde{\kappa}^4\rho^2 - \tfrac{8}{3}\Lambda\right]\dot{\Delta}_\mu$$
$$+\left[\tfrac{1}{2}\kappa^2 - \tfrac{2}{3}\tilde{\kappa}^4\rho\right]\rho H\Delta_\mu = 0. \quad (70)$$

D. Cosmological tensor perturbations

Tensor perturbations are covariantly characterized by

$$D_\mu f = 0, \quad A_\mu = \omega_\mu = \mathcal{Q}_\mu = 0, \quad D^\nu W_{\mu\nu} = 0,$$

where $f=\rho, p, \Theta, \mathcal{U}$, and $W_{\mu\nu} = \sigma_{\mu\nu}, E_{\mu\nu}, H_{\mu\nu}, \mathcal{P}_{\mu\nu}$. Then all the conservation equations reduce to background equations, and the system of linearized propagation and constraint equations in the previous section reduces to

$$\dot{\bar{\sigma}}_{\mu\nu} + 2H\bar{\sigma}_{\mu\nu} + E^*_{\mu\nu} = 0, \quad (71)$$

$$\dot{E}^*_{\mu\nu} + 3HE^*_{\mu\nu} - \mathrm{curl}\,\bar{H}_{\mu\nu} + \tfrac{1}{2}\kappa^2(\rho+p)\bar{\sigma}_{\mu\nu}$$
$$= -\frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\left[\{\kappa^4\rho(\rho+p) + 8\mathcal{U}\}\bar{\sigma}_{\mu\nu}\right.$$
$$\left. + 12(\dot{\bar{\mathcal{P}}}_{\mu\nu} + 2H\bar{\mathcal{P}}_{\mu\nu})\right], \quad (72)$$

$$\dot{\bar{H}}_{\mu\nu} + 3H\bar{H}_{\mu\nu} + \mathrm{curl}\,E^*_{\mu\nu} = 0, \quad (73)$$

$$\mathrm{curl}\,\bar{\sigma}_{\mu\nu} - \bar{H}_{\mu\nu} = 0, \quad (74)$$

where the overbars denote transverse tensors, and

$$E^*_{\mu\nu} = \bar{E}_{\mu\nu} - \frac{1}{2}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\bar{\mathcal{P}}_{\mu\nu}.$$

Since there is no equation for $\dot{\bar{\mathcal{P}}}_{\mu\nu}$, the system of equations does not close on the brane: *brane observers cannot evaluate tensor perturbations on the brane without additional information from the unobservable bulk.* This remains true on super-Hubble scales, unlike the scalar perturbation case.

Equations (71) and (74) show that the shear is a gravitopotential for $E^*_{\mu\nu}$ and $\bar{H}_{\mu\nu}$. Using the identities in Appendix B, we can derive the following covariant wave equation for the shear:

$$D^2\bar{\sigma}_{\mu\nu} - \ddot{\bar{\sigma}}_{\mu\nu} - 5H\dot{\bar{\sigma}}_{\mu\nu} - \left[2\Lambda + \tfrac{1}{2}\kappa^2(\rho-3p)\right.$$
$$\left. - \frac{1}{12}\tilde{\kappa}^4\rho(\rho+3p)\right]\bar{\sigma}_{\mu\nu}$$
$$= -\left(\frac{\tilde{\kappa}}{\kappa}\right)^4\left[\dot{\bar{\mathcal{P}}}_{\mu\nu} + 2H\bar{\mathcal{P}}_{\mu\nu}\right]. \quad (75)$$

For adiabatic tensor perturbations in standard general relativity, the right hand side falls away. Nonlocal bulk effects provide driving terms that are like anisotropic stress terms in general relativity [21]. In the latter case, however, the evolution of anisotropic stress is determined by the Boltzmann equation or other intrinsic physics.

What we can conclude from Eq. (75) is that the *local bulk effects enhance tensor perturbations for non-inflationary expansion, and suppress them during inflation*, since inflation implies $\rho+3p<0$ (at all energy scales) by Eq. (17). The nature of the *nonlocal* bulk effects carried by $\overline{\mathcal{P}}_{\mu\nu}$ requires knowledge of the off-brane dynamics of $\mathcal{E}_{AB}$, which we have not considered.

E. Cosmological vector perturbations

The linearized vorticity propagation equation (50) does not carry any bulk effects, and vorticity decays as in standard general relativity. However, the gravito-magnetic divergence equation (58) shows that the local and nonlocal energy density and the nonlocal energy flux provide additional sources for the brane gravito-magnetic field:

$$\mathrm{D}^{\nu}H_{\mu\nu}=\kappa^2(\rho+p)\omega_\mu+\frac{1}{6}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4 \times[\{\kappa^4\rho(\rho+p)+8\mathcal{U}\}\omega_\mu-3\,\mathrm{curl}\,\mathcal{Q}_\mu].$$

In standard general relativity, it is necessary to increase the angular momentum density $\kappa^2(\rho+p)\omega_\mu$ in order to increase the gravito-magnetic field, but bulk effects allow an increased gravito-magnetic field without this. In particular, unlike in general relativity, *it is possible to source vector perturbations even when the vorticity vanishes*, since curl $\mathcal{Q}_\mu$ may be nonzero.

VI. CONCLUSION

By adopting a covariant approach based on physical and geometrical quantities that are in principle measurable by brane observers, we have given a comprehensive analysis of intrinsic cosmological dynamics in Randall-Sundrum-type brane worlds. This has allowed us to carefully delineate what can and cannot be predicted by brane observers without additional information from the unobservable bulk.

Our main result is probably that *scalar perturbations on super-Hubble scales can be evaluated intrinsically on the brane*. This is a generalization of the result given in [8], i.e., the evaluation of adiabatic scalar perturbations on super-Hubble scales during inflation on the brane, done under the assumption that $\mathcal{E}_{\mu\nu}$ may be neglected.

We showed that tensor perturbations cannot be evaluated intrinsically on any scales. This is not surprising, since the ability of gravitational waves to penetrate the 5th dimension inevitably introduces unpredictability from the standpoint of observers confined to the brane. Vector perturbations on the brane can be generated even in the absence of vorticity, via the curl of the nonlocal energy flux.

Our nonperturbative results include the following:

(i) Calculating the gravitational and non-gravitational acceleration felt by brane observers, allowing us to provide covariant characterizations of gravity and matter localization, and showing in particular that the non-gravitational off-brane acceleration of fundamental world-lines is towards the brane during inflation.

(ii) Showing how bulk effects can disrupt the relation between isotropy of the CMB and spatial isotropy and homogeneity on the brane.

(iii) Showing how bulk effects modify the dynamics of gravitational collapse and singularity formation.

We have derived the exact nonlinear equations governing cosmological dynamics on the brane and the corresponding covariant and gauge-invariant linearized equations. Further implications of these equations could usefully be pursued. In particular, an important topic for further research is the calculation of scalar perturbations on very large scales and the investigation of limits imposed by observations.

However, the major further step required, and not undertaken here, is to complete the picture by investigating the dynamical equations of the gravitational field off the brane. A starting point is provided by the general equations given in [4,10], which determine

$$\mathcal{L}_{\mathbf{n}}\mathcal{E}_{AB}\,,\quad \mathcal{L}_{\mathbf{n}}\mathcal{B}_{ABC}\,,\quad \mathcal{L}_{\mathbf{n}}R_{ABCD}\,,$$

where R_{ABCD} is the 4-dimensional Riemann tensor, and

$$\mathcal{B}_{ABC}=g_A{}^{D}g_B{}^{E}\tilde{C}_{DECF}n^F.$$

However, it may turn out to be more useful to develop an alternative decomposition of the bulk Weyl tensor, along a timelike direction $\tilde{u}^A$ rather than a spatial direction n^A. Intuitively, this may provide a more direct and transparent route to the evolution equation for $\mathcal{P}_{\mu\nu}$, whose absence leads to unpredictability on the brane. Such a timelike decomposition requires a generalization to higher dimensions of the 4-dimensional decomposition of the Weyl tensor [11]. The generalization has been given in [14].

The complete and closed system of dynamical equations would allow us to develop more systematic and probing tests of the Randall-Sundrum-type models against observational constraints. Despite the appealing geometric and particle-physics properties of such models, it is their confrontation with cosmological observational tests that will provide a more decisive arbiter as to whether they are viable generalizations of Einstein's theory.

Note added. Since this work was completed, a number of papers have appeared, setting up the 5-dimensional formalism (metric based) of bulk perturbations [22], and gravitational waves produced during inflation on the brane have also been studied [23].

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APPENDIX A: GENERAL PROPAGATION AND CONSTRAINT EQUATIONS

For a general, imperfect energy-momentum tensor, as in Eq. (6), the propagation and constraint equations (38)–(47) are generalized to the following:

Propagation:

$$\dot{\Theta}+\tfrac{1}{3}\Theta^2+\sigma_{\mu\nu}\sigma^{\mu\nu}-2\,\omega_\mu\omega^\mu-\mathrm{D}^\mu A_\mu+A_\mu A^\mu+\tfrac{1}{2}\kappa^2(\rho+3p)-\Lambda=-\frac{1}{12}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4(2\rho^2+3\rho p-3q_\mu q^\mu)+12\mathcal{U}], \tag{A1}$$

$$\dot{\omega}_{\langle\mu\rangle}+\tfrac{2}{3}\Theta\,\omega_\mu+\tfrac{1}{2}\mathrm{curl}\,A_\mu-\sigma_{\mu\nu}\omega^\nu=0, \tag{A2}$$

$$\dot{\sigma}_{\langle\mu\nu\rangle}+\tfrac{2}{3}\Theta\,\sigma_{\mu\nu}+E_{\mu\nu}-\tfrac{1}{2}\kappa^2\pi_{\mu\nu}-\mathrm{D}_{\langle\mu}A_{\nu\rangle}+\sigma_{\alpha\langle\mu}\sigma_{\nu\rangle}{}^\alpha+\omega_{\langle\mu}\omega_{\nu\rangle}-A_{\langle\mu}A_{\nu\rangle}$$
$$=\frac{1}{24}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4\{-(\rho+3p)\pi_{\mu\nu}+\pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^\alpha+q_{\langle\mu}q_{\nu\rangle}\}+12\mathcal{P}_{\mu\nu}], \tag{A3}$$

$$\dot{E}_{\langle\mu\nu\rangle}+\Theta E_{\mu\nu}-\mathrm{curl}\,H_{\mu\nu}+\tfrac{1}{2}\kappa^2(\rho+p)\sigma_{\mu\nu}+\tfrac{1}{2}\kappa^2\dot{\pi}_{\langle\mu\nu\rangle}+\tfrac{1}{2}\kappa^2\mathrm{D}_{\langle\mu}q_{\nu\rangle}+\tfrac{1}{6}\kappa^2\Theta\,\pi_{\mu\nu}+\kappa^2A_{\langle\mu}q_{\nu\rangle}$$
$$-2A^\alpha\varepsilon_{\alpha\beta(\mu}H_{\nu)}{}^\beta-3\sigma_{\alpha\langle\mu}E_{\nu\rangle}{}^\alpha+\omega^\alpha\varepsilon_{\alpha\beta(\mu}E_{\nu)}{}^\beta+\tfrac{1}{2}\kappa^2\sigma^\alpha{}_{\langle\mu}\pi_{\nu\rangle\alpha}+\tfrac{1}{2}\kappa^2\omega^\alpha\varepsilon_{\alpha\beta(\mu}\pi_{\nu)}{}^\beta$$
$$=\frac{1}{72}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[-\kappa^4\{3(2\rho^2+2\rho p-\pi_{\alpha\beta}\pi^{\alpha\beta}-2q_\alpha q^\alpha)\sigma_{\mu\nu}+3(\dot{\rho}+\dot{p})\pi_{\mu\nu}+3(\rho+3p)\dot{\pi}_{\langle\mu\nu\rangle}-6\pi_{\alpha\langle\mu}\dot{\pi}_{\nu\rangle}{}^\alpha-6q_{\langle\mu}\dot{q}_{\nu\rangle}$$
$$+\tfrac{3}{2}\mathrm{D}_{\langle\mu}(\pi_{\nu\rangle\alpha}q^\alpha-4\rho q_{\nu\rangle})+\Theta([\rho+3p]\pi_{\mu\nu}-\pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^\alpha-q_{\langle\mu}q_{\nu\rangle})-3A_{\langle\mu}(4\rho q_{\nu\rangle}-\pi_{\nu\rangle\alpha}q^\alpha)+3(\rho+3p)\sigma^\alpha{}_{\langle\mu}\pi_{\nu\rangle\alpha}$$
$$-3\sigma^\alpha{}_{\langle\mu}(\pi^\beta{}_{\langle\nu\rangle}\pi_{\alpha\rangle\beta}+q_{\nu\rangle}q_\alpha)+3(\rho+3p)\omega^\alpha\varepsilon_{\alpha\beta(\mu}\pi_{\nu)}{}^\beta-3\,\omega_\alpha\varepsilon^{\alpha\beta}{}_{(\mu}(\pi^\gamma{}_{\langle\nu)}\pi_{\beta\rangle\gamma}+q_{\nu)}q_\beta)\}-48\mathcal{U}\sigma_{\mu\nu}-36\dot{\mathcal{P}}_{\langle\mu\nu\rangle}$$
$$-36\mathrm{D}_{\langle\mu}\mathcal{Q}_{\nu\rangle}-12\Theta\mathcal{P}_{\mu\nu}-72A_{\langle\mu}\mathcal{Q}_{\nu\rangle}-36\sigma^\alpha{}_{\langle\mu}\mathcal{P}_{\nu\rangle\alpha}-36\omega^\alpha\varepsilon_{\alpha\beta(\mu}\mathcal{P}_{\nu)}{}^\beta], \tag{A4}$$

$$\dot{H}_{\langle\mu\nu\rangle}+\Theta H_{\mu\nu}+\mathrm{curl}\,E_{\mu\nu}-\tfrac{1}{2}\kappa^2\,\mathrm{curl}\,\pi_{\mu\nu}-3\sigma_{\alpha\langle\mu}H_{\nu\rangle}{}^\alpha+\omega^\alpha\varepsilon_{\alpha\beta(\mu}H_{\nu)}{}^\beta+2A^\alpha\varepsilon_{\alpha\beta(\mu}E_{\nu)}{}^\beta+\tfrac{3}{2}\kappa^2\omega_{\langle\mu}q_{\nu\rangle}-\tfrac{1}{2}\kappa^2\sigma^\alpha{}_{(\mu}\varepsilon_{\nu)\alpha\beta}q^\beta$$
$$=\frac{1}{48}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4\{-2\,\mathrm{curl}((\rho+3p)\pi_{\mu\nu}-\pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^\alpha-q_{\langle\mu}q_{\nu\rangle})-12\rho\,\omega_{\langle\mu}q_{\nu\rangle}+4\,\omega_{\langle\mu}\pi_{\nu\rangle\alpha}q^\alpha+\sigma^\alpha{}_{(\mu}\varepsilon_{\nu)\alpha\beta}(4\rho q^\beta-\pi^{\beta\gamma}q_\gamma)\}$$
$$+24\,\mathrm{curl}\,\mathcal{P}_{\mu\nu}-72\omega_{\langle\mu}\mathcal{Q}_{\nu\rangle}+24\sigma^\alpha{}_{(\mu}\varepsilon_{\nu)\alpha\beta}\mathcal{Q}^\beta]. \tag{A5}$$

Constraint:

$$\mathrm{D}^\mu\omega_\mu-A^\mu\omega_\mu=0, \tag{A6}$$

$$\mathrm{D}^\nu\sigma_{\mu\nu}-\mathrm{curl}\,\omega_\mu-\tfrac{2}{3}\mathrm{D}_\mu\Theta+\kappa^2q_\mu+2[\omega,A]_\mu=-\frac{1}{24}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4(4\rho q_\mu-\pi_{\mu\nu}q^\nu)+24\mathcal{Q}_\mu], \tag{A7}$$

$$\mathrm{curl}\,\sigma_{\mu\nu}+\mathrm{D}_{\langle\mu}\omega_{\nu\rangle}-H_{\mu\nu}+2A_{\langle\mu}\omega_{\nu\rangle}=0, \tag{A8}$$

$$\mathrm{D}^\nu E_{\mu\nu}+\tfrac{1}{2}\kappa^2\mathrm{D}^\nu\pi_{\mu\nu}-\tfrac{1}{3}\kappa^2\mathrm{D}_\mu\rho+\tfrac{1}{3}\kappa^2\Theta q_\mu-[\sigma,H]_\mu+3H_{\mu\nu}\omega^\nu-\tfrac{1}{2}\kappa^2\sigma_{\mu\nu}q^\nu+\tfrac{3}{2}\kappa^2[\omega,q]_\mu$$
$$=\frac{1}{48}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4[\kappa^4\{\tfrac{2}{3}\Theta(\pi_{\mu\nu}q^\nu-4\rho q_\mu)+2\mathrm{D}^\nu[(\rho+3p)\pi_{\mu\nu}-\pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^\alpha-q_{\langle\mu}q_{\nu\rangle}]+\tfrac{8}{3}\rho\mathrm{D}_\mu\rho-4\pi^{\alpha\beta}\mathrm{D}_\mu\pi_{\alpha\beta}$$
$$+\sigma_{\mu\nu}(4\rho q^\nu-\pi^{\nu\alpha}q_\alpha)+3\varepsilon_{\mu\alpha\beta}\omega^\alpha\pi^{\beta\gamma}q_\gamma-4\rho[\omega,q]_\mu\}+16\mathrm{D}_\mu\mathcal{U}-16\Theta\,\mathcal{Q}_\mu-24\mathrm{D}^\nu\mathcal{P}_{\mu\nu}+24\sigma_{\mu\nu}\mathcal{Q}^\nu-72[\omega,\mathcal{Q}]_\mu], \tag{A9}$$

$$\mathrm{D}^\nu H_{\mu\nu}+\tfrac{1}{2}\kappa^2\,\mathrm{curl}\,q_\mu-\kappa^2(\rho+p)\omega_\mu+[\sigma,E]_\mu+\tfrac{1}{2}\kappa^2[\sigma,\pi]_\mu-3E_{\mu\nu}\omega^\nu+\tfrac{1}{2}\kappa^2\pi_{\mu\nu}\omega^\nu$$
$$=\frac{1}{48}\left(\frac{\tilde{\kappa}}{\kappa}\right)^4(\kappa^4\{\mathrm{curl}(\pi_{\mu\nu}q^\nu-4\rho q_\mu)+4(2\rho^2+2\rho p-\pi_{\alpha\beta}\pi^{\alpha\beta}-2q_\alpha q^\alpha)\omega_\mu-2\varepsilon_{\mu\alpha\beta}\sigma^\alpha{}_\gamma(\pi_\nu{}^{\langle\beta}\pi^{\gamma\rangle\nu}+q^{\langle\beta}q^{\gamma\rangle})$$
$$+2[(\rho+3p)\pi_{\mu\nu}-\pi_{\alpha\langle\mu}\pi_{\nu\rangle}{}^\alpha-q_{\langle\mu}q_{\nu\rangle}]\omega^\nu+2(\rho+3p)[\sigma,\pi]_\mu\}-24\,\mathrm{curl}\,\mathcal{Q}_\mu+64\mathcal{U}\omega_\mu-24[\sigma,\mathcal{P}]_\mu-24\mathcal{P}_{\mu\nu}\omega^\nu). \tag{A10}$$

The 4-dimensional general relativistic results are regained by setting all the right hand sides of these equations to 0.

APPENDIX B: DIFFERENTIAL IDENTITIES

On a flat Friedmann background, the following covariant linearized identities hold [20]:

$$\mathrm{D}_\mu \dot{f} = (\mathrm{D}_\mu f)^\bullet + H\mathrm{D}_\mu f - \dot{f} A_\mu , \tag{B1}$$

$$\mathrm{D}^2(\mathrm{D}_\mu f) = \mathrm{D}_\mu(\mathrm{D}^2 f) + 2\dot{f}\omega_\mu , \tag{B2}$$

$$(\mathrm{D}^2 f)^\bullet = \mathrm{D}^2\dot{f} - 2H\mathrm{D}^2 f + \dot{f}\mathrm{D}^\mu A_\mu , \tag{B3}$$

$$\mathrm{curl}\, \mathrm{D}_\mu f = -2\dot{f}\omega_\mu , \tag{B4}$$

$$\mathrm{curl}\, \mathrm{D}_{\langle\mu}\mathrm{D}_{\nu\rangle} f = 0, \tag{B5}$$

$$(\mathrm{D}_\mu V_\nu)^\bullet = \mathrm{D}_\mu \dot{V}_\nu - H\mathrm{D}_\mu V_\nu , \tag{B6}$$

$$\mathrm{D}_{[\mu}\mathrm{D}_{\nu]} V_\alpha = 0 = \mathrm{D}_{[\mu}\mathrm{D}_{\nu]} W_{\alpha\beta} , \tag{B7}$$

$$\mathrm{D}^\mu\, \mathrm{curl}\, V_\mu = 0, \tag{B8}$$

$$\mathrm{D}^\nu \mathrm{D}_{\langle\mu} V_{\nu\rangle} = \tfrac{1}{2}\mathrm{D}^2 V_\mu + \tfrac{1}{6}\mathrm{D}_\mu(\mathrm{D}^\nu V_\nu), \tag{B9}$$

$$\mathrm{curl}\, \mathrm{D}_{\langle\mu} V_{\nu\rangle} = \tfrac{1}{2}\mathrm{D}_{\langle\mu}\mathrm{curl}\, V_{\nu\rangle} , \tag{B10}$$

$$\mathrm{curl}\,\mathrm{curl}\, V_\mu = -\mathrm{D}^2 V_\mu + \mathrm{D}_\mu(\mathrm{D}^\nu V_\nu), \tag{B11}$$

$$(\mathrm{D}_\mu W_{\alpha\beta})^\bullet = \mathrm{D}_\mu \dot{W}_{\alpha\beta} - H\mathrm{D}_\mu W_{\alpha\beta} , \tag{B12}$$

$$\mathrm{D}^\nu\, \mathrm{curl}\, W_{\mu\nu} = \tfrac{1}{2}\,\mathrm{curl}(\mathrm{D}^\nu W_{\mu\nu}), \tag{B13}$$

$$\mathrm{curl}\,\mathrm{curl}\, W_{\mu\nu} = -\mathrm{D}^2 W_{\mu\nu} + \tfrac{3}{2}\mathrm{D}_{\langle\mu}\mathrm{D}^\alpha W_{\nu\rangle\alpha} , \tag{B14}$$

where $V_\mu = V_{\langle\mu\rangle}$ and $W_{\mu\nu} = W_{\langle\mu\nu\rangle}$ vanish in the background.

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