Renormalized stress tensor in Kerr space-time: General results

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(Received 7 April 2000; published 22 September 2000)

We derive constraints on the form of the renormalized stress tensor for states on Kerr space-time based on general physical principles: symmetry, the conservation equations, the trace anomaly and regularity on (sections of) the event horizon. This is then applied to the physical vacua of interest. We introduce the concept of past and future Boulware vacua and discuss the non-existence of a state empty at both $\mathfrak{I}^-$ and $\mathfrak{I}^+$. By calculating the stress tensor for the Unruh vacuum at the event horizon and at infinity, we are able to check our earlier conditions. We also discuss the difficulties of defining a state equivalent to the Hartle-Hawking vacuum and comment on the properties of two candidates for this state.

PACS number(s): 04.62.+v, 04.70.Dy

I. INTRODUCTION

One of the central quantities of physical interest in a study of quantum field theory in curved space-time is the renormalized expectation value of the stress-energy tensor (RSET), since it is this quantity which couples, via the semi-classical Einstein equations, to the background geometry and thus yields the one-loop correction dynamics of the geometry. This paper is devoted to the properties of the RSET in the states of greatest physical interest on Kerr space-times. Any assault on it by direct computation in black hole geometries is invariably a long and complex process, requiring much algebraic dexterity and ingenuity, and usually resorting to numerical analysis via computer. The aim in this paper is to present what information can be gathered from more physical principles and general considerations. The most important of these are the symmetries of the space-time and states together with the conservation equations. In addition, various restrictions on the form of the RSET follow from its behavior at the event horizon and far from the black hole. In a subsequent paper we shall present numerical results for the RSET in the states appropriate to a Kerr black hole with and without a bounding ''box.''

The contents of this paper are as follows. In Sec. II we briefly review the solution of the wave equation in Kerr space-time, concentrating for simplicity on the case of a conformally coupled, massless scalar field. We also introduce the standard definitions of the Boulware and Unruh vacua, and discuss the subtleties of defining the Hartle-Hawking vacuum in Kerr. In the absence of a true Hartle-Hawking state, we define two possible candidates. Next, in Sec. IV we investigate how much information can be gathered about the stress tensor using the conservation equations, symmetries of the geometry, and regularity conditions on sections of the event horizon. This greatly reduces the number of unknown functions in the stress tensor. The analysis of this section is applicable to any quantum field, and any of the physical vacua. In Sec. V we consider the properties of the physical vacua in the asymptotic regions, at the event horizon and at infinity, again concentrating on the massless scalar field. We calculate the differences in expectation values of the stress tensor in the Unruh vacuum and other states, which can be calculated without renormalization. These calculations are in exact agreement with our earlier analysis. We also discuss the properties of the candidate Hartle-Hawking states, in particular their symmetry and regularity on the event horizon.

We follow the space-time conventions of Misner, Thorne and Wheeler [1] and work in geometric units throughout.

II. THE WAVE EQUATION IN KERR SPACE-TIME

The Kerr line element in Boyer-Lindquist co-ordinates has the form

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)d\phi - a\,dt)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2\,d\theta^2 \tag{2.1}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Here M is the mass of the black hole and a its angular momentum per unit mass as viewed from infinity.

The metric possesses two coordinate singularities at the roots of the equation $\Delta = 0$, which we label $r = r_+ = M + (M^2 - a^2)^{1/2}$, defining the outer event horizon and $r = r_- = M - (M^2 - a^2)^{1/2}$, defining the inner Cauchy horizon. In addition, there is a curvature singularity on the ring defined by the equation $\rho^2 = 0$ (corresponding to $r = 0$ and $\theta = \pi/2$).

The space-time is stationary and axisymmetric, possessing two Killing vectors, $\zeta = \partial/\partial t$ and $\eta = \partial/\partial\phi$. The former is timelike at infinity but becomes null when $r = r_s = M + \sqrt{M^2 - a^2\cos^2\theta}$. This surface is known as the stationary limit surface and between it and the event horizon is a region called the ergosphere. Within the ergosphere, ζ is spacelike and it is impossible for observers to remain at rest with re-

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spect to infinity. The stationary limit surface is timelike except on the axis of symmetry $\theta=0$, where it joins the event horizon and becomes null. The Killing vector $\zeta+\Omega_+\eta$, where $\Omega_+=a/(r_+^2+a^2)=a/2Mr_+$ is the angular velocity of the event horizon, generates the Killing horizon at $r=r_+$. This Killing vector is null on the event horizon, and timelike outside it up to the velocity of light surface, at which point it becomes null again. The velocity of light surface is the surface at which an observer with angular velocity Ω_+ must move with the speed of light. It is not the same as the stationary limit surface. In addition, the space-time possesses a Killing-Yano tensor which we shall discuss later.

Consider a conformally coupled massless scalar field satisfying the equation $\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu)\Phi=0$ (the scalar curvature R being zero in Kerr space-time). This equation is separable in the Kerr metric [2] and the basis functions may be taken to be

$$u_{\omega lm}(x)=\frac{N_{\omega lm}}{(r^2+a^2)^{1/2}}e^{-i\omega t+im\phi}S_{\omega lm}(\cos\theta)R_{\omega lm}(r) \tag{2.2}$$

where $N_{\omega lm}$ is a normalization constant, l and m are integers with $|m|\le l$. $N_{\omega lm}$ is determined so that our mode functions are orthonormal with respect to the standard inner product

$$\langle u_1,u_2\rangle=\frac{1}{2}i\int_\Sigma\sqrt{-g}\,(u^*_{2,\mu}u_1-u^*_2u_{1,\mu})\,d\Sigma^\mu \tag{2.3}$$

where Σ is any Cauchy hypersurface.

$S_{\omega lm}(\xi)$ is a spheroidal harmonic satisfying the eigenvalue equation

$$\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}-\frac{m^2}{1-\xi^2}+2ma\omega-(a\omega)^2(1-\xi^2)+\lambda_{lm}(a\omega)\right]S_{\omega lm}(\xi)=0 \tag{2.4}$$

subject to regularity at $\xi=\pm1$. The eigenvalue $\lambda_{lm}(a\omega)$ depends on the integers l and m and has the known value $\lambda_{lm}(0)=l(l+1)$, with $S_{0lm}(\xi)$ simply an associated Legendre function. We may normalize the spheroidal harmonics so that

$$\int_{-1}^{1}S_{\omega lm}(\xi)S_{\omega l'm}(\xi)\,d\xi=\delta_{ll'}. \tag{2.5}$$

The radial equation may be written in the form of a 1-dimensional time-independent Schrödinger equation

$$\left[\frac{d^2}{dr_*^2}-V_{\omega lm}(r)\right]R_{\omega lm}(r)=0 \tag{2.6}$$

where

$$V_{\omega lm}(r)=-\left(\omega-\frac{ma}{r^2+a^2}\right)^2+\lambda_{lm}(a\omega)\frac{\Delta}{(r^2+a^2)^2}+\frac{2(Mr-a^2)\Delta}{(r^2+a^2)^3}+\frac{3a^2\Delta^2}{(r^2+a^2)^4}, \tag{2.7}$$

and the ''tortoise'' co-ordinate r_* is defined as

$$r_*=\int\frac{r^2+a^2}{\Delta}dr=r+\frac{1}{2\kappa_+}\log|r-r_+|+\frac{1}{2\kappa_-}\log|r-r_-|, \tag{2.8}$$

with

$$\kappa_\pm=\frac{r_\pm-r_\mp}{2(r_\pm^2+a^2)}, \tag{2.9}$$

being the surface gravity on the inner and outer horizons.

In the asymptotic regions $r\to r_+$ ($r_*\to-\infty$) and $r\to\infty$ ($r_*\to\infty$) the potential (2.7) reduces to

$$V_{\omega lm}(r)\sim\begin{cases}-(\omega-m\Omega_+)^2 & \text{as } r_*\to-\infty,\\ -\omega^2 & \text{as } r_*\to\infty\end{cases}. \tag{2.10}$$

We may thus choose as a basis of solutions to Eq. (2.6), two classes of solutions with the asymptotic forms

$$R^-_{\omega lm}(r)\sim\begin{cases}e^{i\tilde{\omega}r_*}+A^-_{\omega lm}e^{-i\tilde{\omega}r_*} & r_*\to-\infty\\ B^-_{\omega lm}e^{i\omega r_*} & r_*\to\infty\end{cases}$$
$$R^+_{\omega lm}(r)\sim\begin{cases}B^+_{\omega lm}e^{-i\tilde{\omega}r_*} & r_*\to-\infty\\ e^{-i\omega r_*}+A^+_{\omega lm}e^{i\omega r_*} & r_*\to\infty\end{cases} \tag{2.11}$$

where $\tilde{\omega}=\omega-m\Omega_+$. In the language of the Schrödinger equation analogy it is natural to speak of A and B as the ''reflection'' and ''transmission'' coefficients, respectively.

The eigenvalues λ_{lm} are real and hence if R is a solution of Eq. (2.6) then so too is R^*. Using this and the constancy of the Wronskian for solutions to Eq. (2.6) for various combinations of the radial wave functions, it can be shown that following relations hold [3]:

$$1-|A^+_{\omega lm}|^2=\frac{\omega-m\Omega_+}{\omega}|B^+_{\omega lm}|^2 \tag{2.12a}$$

$$1-|A^-_{\omega lm}|^2=\frac{\omega}{\omega-m\Omega_+}|B^-_{\omega lm}|^2 \tag{2.12b}$$

$$\omega B^-_{\omega lm}{}^*A^+_{\omega lm}=-(\omega-m\Omega_+)B^+_{\omega lm}A^-_{\omega lm}{}^* \tag{2.12c}$$

$$\omega B^-_{\omega lm}=(\omega-m\Omega_+)B^+_{\omega lm}. \tag{2.12d}$$

The first two of these relations show that for $\omega>0$, $\omega-m\Omega_+=\tilde{\omega}<0$, both $|A^-|^2$ and $|A^+|^2$ are greater than 1.

III. QUANTUM FIELD THEORY IN KERR SPACE-TIME

A. The mode functions

We start by considering two natural complete, orthonormal sets of solutions to the Klein-Gordon equation. It is then straightforward to construct states with particular properties along a given Cauchy surface, for example $\mathfrak{I}^- \cup \mathfrak{H}^-$. Later, we shall address the much more difficult question of constructing states characterized on surfaces which do not form a Cauchy surface, for example $\mathfrak{H}^- \cup \mathfrak{H}^+$.

With the understanding that $\omega > 0$, we take as the ''past'' basis the following [4]:

$$u^{in}_{\omega lm} = \frac{1}{\sqrt{8\pi^2 \omega (r^2+a^2)}} e^{-i\omega t} e^{im\phi} S_{\omega lm}(\cos\theta) R^+_{\omega lm}(r) \quad \tilde{\omega} > -m\Omega_+ \tag{3.1a}$$

$$u^{up}_{\omega lm} = \frac{1}{\sqrt{8\pi^2 \tilde{\omega} (r^2+a^2)}} e^{-i\omega t} e^{im\phi} S_{\omega lm}(\cos\theta) R^-_{\omega lm}(r) \quad \tilde{\omega} > 0 \tag{3.1b}$$

$$u^{up}_{-\omega l -m} = \frac{1}{\sqrt{8\pi^2 (-\tilde{\omega}) (r^2+a^2)}} e^{i\omega t} e^{-im\phi} S_{\omega lm}(\cos\theta) R^-_{-\omega l -m}(r) \quad 0 > \tilde{\omega} > -m\Omega_+ \tag{3.1c}$$

where we have used the property $S_{-\omega l -m}(\cos\theta) = S_{\omega lm}(\cos\theta)$. These modes are orthonormal in the sense that

$$(u^{in}_{\omega lm}, u^{in}_{\omega' l' m'}) = \delta(\omega - \omega')\,\delta_{ll'}\,\delta_{mm'} \quad \tilde{\omega} > -m\Omega_+ \quad [\omega > 0] \tag{3.2a}$$

$$(u^{up}_{\omega lm}, u^{up}_{\omega' l' m'}) = \delta(\omega - \omega')\,\delta_{ll'}\,\delta_{mm'} \quad \tilde{\omega} > 0 \quad [\omega > m\Omega_+] \tag{3.2b}$$

$$(u^{up}_{-\omega l -m}, u^{up}_{-\omega' l' -m'}) = \delta(\omega - \omega')\,\delta_{ll'}\,\delta_{mm'} \quad 0 > \tilde{\omega} > -m\Omega_+ \quad [m\Omega_+ > \omega > 0] \tag{3.2c}$$

with all other inner products vanishing. Our conventions here adhere to those of the ''distant observer viewpoint'' of Frolov and Thorne [5] which we will follow consistently throughout this series of papers.

From Eq. (2.11),

$$u^{in}_{\omega lm} \sim \frac{S_{\omega lm}(\cos\theta)}{\sqrt{8\pi^2 \omega (r^2+a^2)}} \times \begin{cases} 0 & \text{at } \mathfrak{H}^-, \\ \exp(-i\omega v + im\phi) & \text{at } \mathfrak{I}^- \quad \tilde{\omega} > -m\Omega_+, \\ B^+_{\omega lm}\exp(-i\tilde{\omega} v + im\phi_+) & \text{at } \mathfrak{H}^+ \quad [\omega > 0], \\ A^+_{\omega lm}\exp(-i\omega u + im\phi) & \text{at } \mathfrak{I}^+, \end{cases} \tag{3.3a}$$

$$u^{up}_{\omega lm} \sim \frac{S_{\omega lm}(\cos\theta)}{\sqrt{8\pi^2 \tilde{\omega} (r^2+a^2)}} \times \begin{cases} \exp(-i\tilde{\omega} u + im\phi_+) & \text{at } \mathfrak{H}^-, \\ 0 & \text{at } \mathfrak{I}^- \quad \tilde{\omega} > 0, \\ A^-_{\omega lm}\exp(-i\tilde{\omega} v + im\phi_+) & \text{at } \mathfrak{H}^+ \quad [\omega > m\Omega_+], \\ B^-_{\omega lm}\exp(-i\omega u + im\phi) & \text{at } \mathfrak{I}^+, \end{cases} \tag{3.3b}$$

$$u^{up}_{-\omega l -m} \sim \frac{S_{\omega lm}(\cos\theta)}{\sqrt{8\pi^2 |\tilde{\omega}| (r^2+a^2)}} \times \begin{cases} \exp(-i|\tilde{\omega}|u - im\phi_+) & \text{at } \mathfrak{H}^- \\ 0 & \text{at } \mathfrak{I}^- \quad 0 > \tilde{\omega} > -m\Omega_+, \\ A^-_{\omega lm}\exp(-i|\tilde{\omega}|v - im\phi_+) & \text{at } \mathfrak{H}^+ \quad [m\Omega_+ > \omega > 0], \\ B^-_{\omega lm}\exp(i\omega u - im\phi) & \text{at } \mathfrak{I}^+, \end{cases} \tag{3.3c}$$

where

$$u = t - r_*, \quad v = t + r_*, \quad \phi_+ = \phi - \Omega_+ t. \tag{3.4}$$

These modes are natural to the initial surfaces $\mathfrak{H}^-$ and $\mathfrak{I}^-$ in the sense that u^{in} describes unit flux coming in from $\mathfrak{I}^-$ and zero flux coming up from $\mathfrak{H}^-$, whereas u^{up} describes unit flux coming up from $\mathfrak{H}^-$ and zero incoming flux coming in from $\mathfrak{I}^-$. For modes with $\tilde{\omega} < 0$ (but $\omega > 0$), $|A^-|^2 > 1$, so that they are reflected back to $\mathfrak{H}^+$ with an amplitude greater than that they had originally at $\mathfrak{H}^-$. This is the classical phenomenon of *superradiance*. Of course, as $\omega>0$ and $\Omega_+>0$ it is only possible for $\tilde{\omega}=\omega-m\Omega_+$ to be negative if $m>0$, that is for co-rotating waves. Corresponding comments apply to in modes with $\tilde{\omega}<0$: they are reflected back to $\mathfrak{I}^+$ with an amplitude greater than that they had originally at $\mathfrak{I}^-$.

Another aspect of superradiance is important to our discussion. From Eq. (3.3c), one sees that the up modes (3.1c) with $\tilde{\omega}<0$ have a negative energy wave propagating to $\mathfrak{I}^+$ (conservation of energy). This is a consequence of ∂_t not being a globally time-like Killing vector. ∂_t is space-like in the ergosphere, however the combination $\partial_t+\Omega\partial_\phi$, where $\Omega=-g_{t\phi}/g_{\phi\phi}$, is time-like down to the horizon upon which it becomes null. Observers following integral curves of this time-like vector field are locally non-rotating observers (LNRO). A LNRO near the horizon would measure the frequency of the superradiant up modes in (3.1c) to be $|\tilde{\omega}|=-\tilde{\omega}=-\omega+m\Omega_+$, in particular, the LNRO would see positive frequency waves for all modes. For $u^{in}_{\omega lm}$ all modes are positive frequency at $\mathfrak{I}^+$ and $\mathfrak{I}^-$. A LNRO near the horizon measures $\tilde{\omega}$ for the frequency and thus sees negative frequency modes in the superradiant regime. An up mode having positive frequency with respect to u at $\mathfrak{H}^-$ will have negative frequency with respect to u at $\mathfrak{I}^+$ if $\tilde{\omega}<0$ but $\omega>0$.

B. The physical vacua

We now turn to the delicate issue of defining analogs of the standard three vacuum states in Schwarzschild space-time (Boulware, Hartle-Hawking and Unruh) in Kerr space-time. (Our discussion here concerns states on the full exterior region of Kerr, in later papers we shall also talk about the case when the black hole is contained within a ‘‘box.’’) The construction of vacuum states in Kerr is a more subtle problem than for Schwarzschild black holes, for the following reasons:

(1) The existence of superradiant modes makes the definition of positive frequency more complicated. For example, in Schwarzschild, an outgoing mode which has positive frequency with respect to the retarded null co-ordinate u at the past horizon $\mathfrak{H}^-$ will also have positive frequency with respect to u at $\mathfrak{I}^+$, so it does not matter if we define positive frequency with respect to u at $\mathfrak{H}^-$ or at $\mathfrak{I}^+$. This is no longer the case in Kerr: a superradiant mode can have positive frequency with respect to u at $\mathfrak{I}^+$ but negative frequency at $\mathfrak{H}^-$. This is why our definition of the basis of mode functions (3.1) had to be so carefully done.

(2) As a consequence of this, it is *only* straightforward to define states with particular properties along a given Cauchy surface, such as $\mathfrak{I}^-\cup\mathfrak{H}^-$. By contrast, it has become conventional in Schwarzschild space-time to consider the Boulware vacuum in terms of its properties on $\mathfrak{I}^-\cup\mathfrak{I}^+$ and the Hartle-Hawking vacuum in terms of its properties on $\mathfrak{H}^-\cup\mathfrak{H}^+$.

To be explicit, we may expand the scalar field $\Phi(x)$ in terms of the mode functions we introduced above

$$\begin{aligned}\Phi(x)=&\sum_{l,m}\left(\int_0^\infty d\omega(a^{in}_{\omega lm}u^{in}_{\omega lm}+a^{in\dagger}_{\omega lm}u^{in*}_{\omega lm})+\int_{\omega_{min}}^\infty d\omega(a^{up}_{\omega lm}u^{up}_{\omega lm}+a^{up\dagger}_{\omega lm}u^{up*}_{\omega lm})\right)\\&+\sum_{l,m}\int_0^{\omega_{min}}d\omega(a^{up}_{-\omega l-m}u^{up}_{-\omega l-m}+a^{up\dagger}_{-\omega l-m}u^{up*}_{-\omega l-m})\\=&\sum_{l,m}\left(\int_0^\infty d\omega(a^{in}_{\omega lm}u^{in}_{\omega lm}+a^{in\dagger}_{\omega lm}u^{in*}_{\omega lm})+\int_0^\infty d\tilde{\omega}(a^{up}_{\omega lm}u^{up}_{\omega lm}+a^{up\dagger}_{\omega lm}u^{up*}_{\omega lm})\right)\end{aligned}\tag{3.5}$$

where $\omega_{min}=\max\{0,m\Omega_+\}$, so $\omega_{min}=0$ for counter-rotating waves ($m\leqslant0$) and $\omega_{min}=m\Omega_+$ for co-rotating waves ($m>0$). Given this expansion, the natural way to quantize the field is for the coefficients to become operators satisfying the commutation relations

$$[\hat{a}^{in}_{\omega lm},\hat{a}^{in\dagger}_{\omega' l' m'}]=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\qquad \tilde{\omega}>-m\Omega_+\tag{3.6a}$$

$$[\hat{a}^{up}_{\omega lm},\hat{a}^{up\dagger}_{\omega' l' m'}]=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\qquad \tilde{\omega}>0\tag{3.6b}$$

$$[\hat{a}^{up}_{-\omega l-m},\hat{a}^{up\dagger}_{-\omega' l'-m'}]=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\qquad 0>\tilde{\omega}>-m\Omega_+\tag{3.6c}$$

with all other commutators vanishing. From Eq.(3.3), the operators $\hat{a}^{in\dagger}$ and $\hat{a}^{up\dagger}$ have the natural interpretation that they will, respectively, create particles incident from $\mathfrak{I}^-$ and $\mathfrak{H}^-$. With this in mind, we define a ‘‘past Boulware’’ vacuum state by

$$\hat{a}^{in}_{\omega lm}|B^-\rangle=0\qquad \tilde{\omega}>-m\Omega_+\tag{3.7a}$$

$$\hat{a}^{up}_{\omega lm}|B^-\rangle=0\qquad \tilde{\omega}>0\tag{3.7b}$$

$$\hat{a}^{up}_{-\omega l-m}|B^-\rangle=0\qquad 0>\tilde{\omega}>-m\Omega_+\tag{3.7c}$$

corresponding to an absence of particles from $\mathfrak{H}^-$ and $\mathfrak{I}^-$.

This state does not precisely correspond to the idea of a Boulware state in Schwarzschild as that state which is most empty at infinity. The state $|B^-\rangle$ contains, at $\mathfrak{I}^+$, an outward flux of particles in the superradiant modes; this is the Unruh-Starobinskii effect [6].

One might suppose that a more appropriate definition for the Boulware vacuum would be to define a state which is empty at $\mathfrak{I}^-$ and $\mathfrak{I}^+$. However, it is straightforward to see that such a state cannot exist within conventional quantum field theory by introducing the mode functions natural for defining the ''future Boulware'' vacuum. (We shall discuss later the non-conventional ''η-formalism'' construction proposed by Frolov and Thorne [5].)

The mode functions relevant to the ''future Boulware'' vacuum are those representing a unit (locally-positive frequency) flux out to $\mathfrak{I}^+$ and down $\mathfrak{H}^+$. From the asymptotic forms for the radial functions Eq. (2.11), it is clear that we should take as our ''future'' basis [4]:

$$u^{out}_{\omega lm}=\frac{1}{\sqrt{8\pi^2\omega(r^2+a^2)}}e^{-i\omega t}e^{im\phi}S_{\omega lm}(\cos\theta)R^{+*}_{\omega lm}(r)\quad \tilde{\omega}>-m\Omega_+, \tag{3.8a}$$

$$u^{down}_{\omega lm}=\frac{1}{\sqrt{8\pi^2\tilde{\omega}(r^2+a^2)}}e^{-i\omega t}e^{im\phi}S_{\omega lm}(\cos\theta)R^{-*}_{\omega lm}(r)\quad \tilde{\omega}>0, \tag{3.8b}$$

$$u^{down}_{-\omega l-m}=\frac{1}{\sqrt{8\pi^2|\tilde{\omega}|(r^2+a^2)}}e^{i\omega t}e^{-im\phi}S_{\omega lm}(\cos\theta)R^{-*}_{-\omega l-m}(r)\quad 0>\tilde{\omega}>-m\Omega_+. \tag{3.8c}$$

These modes are orthonormal in the sense that

$$(u^{out}_{\omega lm},u^{out}_{\omega' l' m'})=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\quad \tilde{\omega}>-m\Omega_+\quad [\omega>0] \tag{3.9a}$$

$$(u^{down}_{\omega lm},u^{down}_{\omega' l' m'})=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\quad \tilde{\omega}>0\quad [\omega>m\Omega_+] \tag{3.9b}$$

$$(u^{down}_{-\omega l-m},u^{down}_{-\omega' l'-m'})=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\quad 0>\tilde{\omega}>-m\Omega_+\quad [m\Omega_+>\omega>0] \tag{3.9c}$$

with all other inner products vanishing. Their asymptotic properties are given by

$$u^{out}_{\omega lm}\sim\frac{S_{\omega lm}(\cos\theta)}{\sqrt{8\pi^2\omega(r^2+a^2)}}\times\begin{cases}B^{+*}_{\omega lm}\exp(-i\tilde{\omega}u+im\phi_+) & \text{at } \mathfrak{H}^-\\ A^{+*}_{\omega lm}\exp(-i\omega v+im\phi) & \text{at } \mathfrak{I}^-\ \tilde{\omega}>-m\Omega_+\\ 0 & \text{at } \mathfrak{H}^+\,[\omega>0]\\ \exp(-i\omega u+im\phi) & \text{at } \mathfrak{I}^+\end{cases} \tag{3.10a}$$

$$u^{down}_{\omega lm}\sim\frac{S_{\omega lm}(\cos\theta)}{\sqrt{8\pi^2\tilde{\omega}(r^2+a^2)}}\times\begin{cases}A^{-*}_{\omega lm}\exp(-i\tilde{\omega}u+im\phi_+) & \text{at } \mathfrak{H}^-\\ B^{-*}_{\omega lm}\exp(-i\omega v+im\phi) & \text{at } \mathfrak{I}^-\ \tilde{\omega}>0\\ \exp(-i\tilde{\omega}v+im\phi_+) & \text{at } \mathfrak{H}^+\,[\omega>m\Omega_+]\\ 0 & \text{at } \mathfrak{I}^+\end{cases} \tag{3.10b}$$

$$u^{down}_{-\omega l-m}\sim\frac{S_{\omega lm}(\cos\theta)}{\sqrt{8\pi^2|\tilde{\omega}|(r^2+a^2)}}\times\begin{cases}A^{-*}_{-\omega l-m}\exp(-i|\tilde{\omega}|u-im\phi_+) & \text{at } \mathfrak{H}^-\\ B^{-*}_{-\omega l-m}\exp(i\omega v-im\phi) & \text{at } \mathfrak{I}^-\ 0>\tilde{\omega}>-m\Omega_+\\ \exp(-i|\tilde{\omega}|v-im\phi_+) & \text{at } \mathfrak{H}^+\ [m\Omega_+>\omega>0]\\ 0 & \text{at } \mathfrak{I}^+\end{cases} \tag{3.10c}$$

We may expand the scalar field $\Phi(x)$ in terms of these mode functions we introduced above

$$\Phi(x)=\sum_{l,m}\left(\int_0^\infty d\omega(a^{out}_{\omega lm}u^{out}_{\omega lm}+a^{out\dagger}_{\omega lm}u^{out*}_{\omega lm})+\int_{\omega_{min}}^\infty d\omega(a^{down}_{\omega lm}u^{down}_{\omega lm}+a^{down\dagger}_{\omega lm}u^{down*}_{\omega lm})\right)$$
$$+\sum_{l,m}\int_0^{\omega_{min}}d\omega(a^{down}_{-\omega l-m}u^{down}_{-\omega l-m}+a^{down\dagger}_{-\omega l-m}u^{down*}_{-\omega l-m})$$
$$=\sum_{l,m}\left(\int_0^\infty d\omega(a^{out}_{\omega lm}u^{out}_{\omega lm}+a^{out\dagger}_{\omega lm}u^{out*}_{\omega lm})+\int_0^\infty d\tilde{\omega}(a^{down}_{\omega lm}u^{down}_{\omega lm}+a^{down\dagger}_{\omega lm}u^{down*}_{\omega lm})\right) \tag{3.11}$$

where $\omega_{min}=\max\{0,m\Omega_+\}$, as before. Given this expansion, the natural way to quantize the field is for the coefficients to become operators satisfying the commutation relations

$$[\hat{a}^{out}_{\omega lm},\hat{a}^{out\dagger}_{\omega' l' m'}]=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'} \quad \tilde{\omega}>-m\Omega_+ \tag{3.12a}$$

$$[\hat{a}^{down}_{\omega lm},\hat{a}^{down\dagger}_{\omega' l' m'}]=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'} \quad \tilde{\omega}>0 \tag{3.12b}$$

$$[\hat{a}^{down}_{-\omega l-m},\hat{a}^{down\dagger}_{-\omega' l'-m'}]=\delta(\omega-\omega')\delta_{ll'}\delta_{mm'} \quad 0>\tilde{\omega}>-m\Omega_+ \tag{3.12c}$$

with all other commutators vanishing. From Eq. (3.10), the operators $\hat{a}^{out\dagger}$ and $\hat{a}^{down\dagger}$ have the natural interpretation that they will, respectively, create particles incident from $\mathfrak{I}^+$ and $\mathfrak{H}^+$. Thus, we define the ''future Boulware'' vacuum state by

$$\hat{a}^{out}_{\omega lm}|B^+\rangle=0 \quad \tilde{\omega}>-m\Omega_+ \tag{3.13a}$$

$$\hat{a}^{down}_{\omega lm}|B^+\rangle=0 \quad \tilde{\omega}>0 \tag{3.13b}$$

$$\hat{a}^{down}_{-\omega l-m}|B^+\rangle=0 \quad 0>\tilde{\omega}>-m\Omega_+ \tag{3.13c}$$

corresponding to an absence of particles from $\mathfrak{H}^+$ and $\mathfrak{I}^+$. In this language, the Unruh-Starobinskii effect is a statement about the behavior of

$$\langle B^-|\hat{T}_{\mu\nu}|B^-\rangle-\langle B^+|\hat{T}_{\mu\nu}|B^+\rangle \tag{3.14}$$

as $r\to\infty$.

A vacuum state empty at $\mathfrak{I}^-$ and $\mathfrak{I}^+$ must be constructed from modes $u^{in}_{\omega lm}$ and $u^{out}_{\omega lm}$ up to a trivial Bogoliubov transformation (i.e., one with all β-coefficients vanishing). However, $u^{in}_{\omega lm}$ and $u^{out}_{\omega lm}$ are not orthogonal and the fact that they cannot be made so by any trivial Bogoliubov transformation is seen most easily by writing $u^{out}_{\omega lm}$ in terms of the basis given by $u^{in}_{\omega lm}$ and $u^{up}_{\omega lm}$. For non-superradiant modes the transformation does correspond to a trivial Bogoliubov transformation:

$$u^{out}_{\omega lm}=A^{+*}_{\omega lm}u^{in}_{\omega lm}+\sqrt{\frac{\tilde{\omega}}{\omega}}B^{+*}_{\omega lm}u^{up}_{\omega lm}, \quad \tilde{\omega}>0, \tag{3.15a}$$

$$u^{down}_{\omega lm}=\sqrt{\frac{\omega}{\tilde{\omega}}}B^{-*}_{\omega lm}u^{in}_{\omega lm}+A^{-*}_{\omega lm}u^{up}_{\omega lm}, \quad [\omega>m\Omega_+], \tag{3.15b}$$

but for superradiant modes

$$u^{out}_{\omega lm}=A^{+*}_{\omega lm}u^{in}_{\omega lm}-\sqrt{\frac{-\tilde{\omega}}{\omega}}B^{+*}_{\omega lm}u^{up*}_{-\omega l-m}, \quad 0>\tilde{\omega}>-m\Omega_+[m\Omega_+>\omega>0], \tag{3.16a}$$

$$u^{down}_{-\omega l-m}=\sqrt{\frac{\omega}{-\tilde{\omega}}}B^{-*}_{-\omega l-m}u^{in*}_{\omega lm}+A^{-*}_{-\omega l-m}u^{up}_{\omega lm}, \quad 0>\tilde{\omega}>-m\Omega_+[m\Omega_+>\omega>0]. \tag{3.16b}$$

As no trivial Bogoliubov transformation can affect the total number of ''particles'' produced, $\Sigma_{i,r}|\beta_{ir}|^2$, it is impossible to define a vacuum state empty with respect to in modes at $\mathfrak{I}^-$ and out modes at $\mathfrak{I}^+$.

The non-existence of a ''true Boulware'' state is intimately linked with the non-existence of a ''true Hartle-Hawking'' state (defined as being a Hadamard state which respects the symmetries of the space-time and is regular everywhere, in particular, on both future and past event horizons) on Kerr space-time [7]. In the former case, one wishes to define the state on $\mathfrak{I}^-\cup\mathfrak{I}^+$, in the latter on $\mathfrak{H}^-\cup\mathfrak{H}^+$. Indeed, one can make the analogy quite precise by, in the language of Frolov and Thorne, switching from a ''distant'' to a ''near horizon'' viewpoint.

The (past) Unruh state $|U^-\rangle$ is easily defined as that state empty at $\mathfrak{I}^-$ but with the ''up'' modes (natural modes on $\mathfrak{H}^-$) thermally populated. For a proof that this is equivalent to using modes which are positive frequency with respect to a future-increasing affine parameter on $\mathfrak{H}^-$ see Ref. [5]. As before, we use the notation $|U^-\rangle$ in order to emphasize that this state is naturally defined by considerations on $\mathfrak{H}^- \cup \mathfrak{I}^-$. One can, of course also define a state $|U^+\rangle$ empty at $\mathfrak{I}^+$ but with the ''down'' modes (natural modes on $\mathfrak{H}^+$) thermally populated. Indeed, one can also make such a distinction in the Schwarzschild case for the Unruh vacuum. However, one rarely considers $|U^+\rangle$ as it is $|U^-\rangle$ that mimics the state arising at late times from the collapse of a star to a black hole. For this reason we shall usually drop the term ''past'' but we will retain the terminology $|U^-\rangle$ to make clear that this state is naturally defined in terms of ''in'' and ''up'' modes. In this language, the (Kruskal space-time model of the) Hawking effect is a statement about the behavior of

$$\langle U^-|\hat{T}_{\mu\nu}|U^-\rangle - \langle B^+|\hat{T}_{\mu\nu}|B^+\rangle \tag{3.17}$$

as $r\to\infty$.

With these definitions, it is straightforward to write down mode sum expressions for the two-point functions of the field in the past and future Boulware and (past) Unruh vacuum states:

$$\begin{aligned} G_{B^-}(x,x') &= \langle B^-|\hat{\Phi}(x)\hat{\Phi}(x')|B^-\rangle \\ &= \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\, u^{up}_{\omega lm}(x)u^{up*}_{\omega lm}(x') + \int_0^\infty d\omega\, u^{in}_{\omega lm}(x)u^{in*}_{\omega lm}(x')\right) \end{aligned} \tag{3.18a}$$

$$\begin{aligned} G_{B^+}(x,x') &= \langle B^+|\hat{\Phi}(x)\hat{\Phi}(x')|B^+\rangle \\ &= \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\, u^{down}_{\omega lm}(x)u^{down*}_{\omega lm}(x') + \int_0^\infty d\omega\, u^{out}_{\omega lm}(x)u^{out*}_{\omega lm}(x')\right) \end{aligned} \tag{3.18b}$$

$$\begin{aligned} G_{U^-}(x,x') &= \langle U^-|\hat{\Phi}(x)\hat{\Phi}(x')|U^-\rangle \\ &= \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\coth\left(\frac{\pi\tilde{\omega}}{\kappa}\right) u^{up}_{\omega lm}(x)u^{up*}_{\omega lm}(x) + \int_0^\infty d\omega\, u^{in}_{\omega lm}(x)u^{in*}_{\omega lm}(x')\right). \end{aligned} \tag{3.18c}$$

The corresponding expressions for the unrenormalized expectation values of the stress tensor in the past and future Boulware and (past) Unruh vacuum states are

$$\langle B^-|\hat{T}_{\mu\nu}|B^-\rangle = \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\, T_{\mu\nu}[u^{up}_{\omega lm},u^{up*}_{\omega lm}] + \int_0^\infty d\omega\, T_{\mu\nu}[u^{in}_{\omega lm},u^{in*}_{\omega lm}]\right) \tag{3.19a}$$

$$\langle B^+|\hat{T}_{\mu\nu}|B^+\rangle = \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\, T_{\mu\nu}[u^{down}_{\omega lm},u^{down*}_{\omega lm}] + \int_0^\infty d\omega\, T_{\mu\nu}[u^{out}_{\omega lm},u^{out*}_{\omega lm}]\right) \tag{3.19b}$$

$$\langle U^-|\hat{T}_{\mu\nu}|U^-\rangle = \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\coth\left(\frac{\pi\tilde{\omega}}{\kappa}\right) T_{\mu\nu}[u^{up}_{\omega lm},u^{up*}_{\omega lm}] + \int_0^\infty d\omega\, T_{\mu\nu}[u^{in}_{\omega lm},u^{in*}_{\omega lm}]\right) \tag{3.19c}$$

where the contribution to the stress-energy tensor, for a massless scalar field mode in Ricci-flat Kerr space-time, assuming conformal coupling, is

$$\begin{aligned} T_{\mu\nu}[u,u^*] &= \frac{1}{3}(u_{;\mu}u^*_{;\nu} + u^*_{;\mu}u_{;\nu}) - \frac{1}{6}(u_{;\mu\nu}u^* + u^*_{;\mu\nu}u) \\ &\quad - \frac{1}{6}g_{\mu\nu}u_{;\tau}u^{*;\tau}. \end{aligned} \tag{3.20}$$

Kay and Wald [7] have shown that there does not exist a Hadamard state which respects the symmetries of the space-time and is regular everywhere in Kerr space-time. In the absence of such a ''true Hartle-Hawking'' vacuum we consider the following states, which are attempts in the literature to define a thermal state with most (but not all) of the properties of the Hartle-Hawking state.

The first state is that introduced by Candelas, Chrzanowski and Howard [8], which is constructed by thermalizing the ''in'' and ''up'' modes with respect to their natural energy, so

$$\begin{aligned} G_{CCH}(x,x') &= \langle CCH|\hat{\Phi}(x)\hat{\Phi}(x')|CCH\rangle \\ &= \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\coth\left(\frac{\pi\tilde{\omega}}{\kappa}\right) u^{up}_{\omega lm}(x)u^{up*}_{\omega lm}(x')\right. \\ &\quad \left. + \int_0^\infty d\omega\coth\left(\frac{\pi\omega}{\kappa}\right) u^{in}_{\omega lm}(x)u^{in*}_{\omega lm}(x')\right). \end{aligned} \tag{3.21}$$

and

$$\langle CCH|\hat{T}_{\mu\nu}|CCH\rangle = \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\coth\left(\frac{\pi\tilde{\omega}}{\kappa}\right)\times T_{\mu\nu}[u^{up}_{\omega lm},u^{up*}_{\omega lm}] + \int_0^\infty d\omega\coth\left(\frac{\pi\omega}{\kappa}\right)T_{\mu\nu}[u^{in}_{\omega lm},u^{in*}_{\omega lm}]\right). \tag{3.22}$$

As such, it might naturally be described as the ''past Hartle-Hawking'' vacuum, however, given the discussion above it is not surprising that as we shall show in detail below, this definition gives a state which does not respect the simultaneous t-ϕ reversal invariance of Kerr space-time.

The second state we shall consider is that introduced by Frolov and Thorne [5] who used the ''η formalism'' to treat the quantization of the superradiant modes. They derived the following expressions in the state, denoted here by $|FT\rangle$, which they claim defined the Hartle-Hawking vacuum (at least close to the horizon):

$$\begin{aligned}G_{FT}(x,x') &= \langle FT|\eta\hat{\Phi}(x)\eta\hat{\Phi}(x')\eta|FT\rangle \\ &= \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\coth\left(\frac{\pi\tilde{\omega}}{\kappa}\right)u^{up}_{\omega lm}(x)u^{up*}_{\omega lm}(x') + \int_0^\infty d\omega\coth\left(\pi\frac{\tilde{\omega}}{\kappa}\right)u^{in}_{\omega lm}(x)u^{in*}_{\omega lm}(x')\right)\end{aligned} \tag{3.23}$$

and

$$\langle FT|\hat{T}_{\mu\nu}|FT\rangle = \sum_{l,m}\left(\int_0^\infty d\tilde{\omega}\coth\left(\frac{\pi\tilde{\omega}}{\kappa}\right)T_{\mu\nu}[u^{up}_{\omega lm},u^{up*}_{\omega lm}] + \int_0^\infty d\omega\coth\left(\pi\frac{\tilde{\omega}}{\kappa}\right)T_{\mu\nu}[u^{in}_{\omega lm},u^{in*}_{\omega lm}]\right). \tag{3.24}$$

Thus, the Frolov-Thorne state differs in its choice of the appropriate ''energy'' for the thermal factor corresponding to the ''in'' modes. This state is formally invariant under simultaneous t-ϕ reversal. Frolov and Thorne claim that the state defined by Eq. (3.24) is regular out to the speed-of-light surface and is ill-defined outside. However, the Kay-Wald result is essentially local and the Frolov-Thorne state appears to violate the spirit if not the letter of the result proved by Kay and Wald.

Below and in subsequent papers in this series where we address the issues numerically, we shall show that the Frolov-Thorne state is fundamentally flawed while the Candelas-Chrzanowski-Howard state is workable but cannot claim to represent an equilibrium state.

IV. CONSTRAINTS ON THE STRESS TENSOR

We now investigate how much information can be gathered about the stress-energy tensor in Kerr space-time from general physical principles. We shall have in mind the physical vacua which have been defined in the previous section.

A. Solution of the conservation equations

In this section, we consider the solution of the conservation equations $\nabla_\nu T_\mu{}^\nu = 0$. To avoid the calculation of Christoffel symbols, since $T_{\mu\nu}$ is a symmetric tensor, the conservation equations may be written in the alternative form [9]

$$\partial_\nu(T_\mu{}^\nu\sqrt{-g}) = \frac{1}{2}\sqrt{-g}(\partial_\mu g_{\alpha\beta})T^{\alpha\beta} \tag{4.1}$$

where g is the determinant of the matrix of metric coefficients given by $g=-\rho^4\sin^2\theta$. Since we are interested in the renormalized stress tensor for states which respect the symmetries of the space-time, we assume that the stress-energy tensor, like the metric, is independent of t and ϕ. The $\mu=t$ and $\mu=\phi$ equations then become, respectively,

$$\begin{aligned}&\partial_r(\rho^2\sin\theta T_t{}^r) + \partial_\theta(\rho^2\sin\theta T_t{}^\theta) = 0 \\ &\partial_r(\rho^2\sin\theta T_\phi{}^r) + \partial_\theta(\rho^2\sin\theta T_\phi{}^\theta) = 0.\end{aligned} \tag{4.2}$$

These may be integrated immediately over r to yield [10]

$$\begin{aligned}T_{tr} &= \frac{K(\theta)}{\Delta} - \frac{1}{\Delta\sin\theta}\partial_\theta\left(\sin\theta\int_{r_+}^r T_{t\theta}dr'\right) \\ T_{\phi r} &= \frac{L(\theta)}{\Delta} - \frac{1}{\Delta\sin\theta}\partial_\theta\left(\sin\theta\int_{r_+}^r T_{\phi\theta}dr'\right)\end{aligned} \tag{4.3}$$

where $K(\theta)$ and $L(\theta)$ are arbitrary functions of θ alone.

The $\mu=r$ and $\mu=\theta$ equations are, respectively,

$$\begin{aligned}F(r,\theta) &= \partial_r(\rho^2 T_r{}^r) + \Delta^{-1}\csc\theta\,\partial_\theta(\rho^2\sin\theta T_\theta{}^r) - rT_\theta{}^\theta - \Delta^{-1}(ra^2\sin\theta - \Lambda)T_r{}^r \\ G(r,\theta) &= \partial_r(\rho^2 T_\theta{}^r) + \csc\theta\,\partial_\theta(\rho^2\sin\theta T_\theta{}^\theta) + a^2\sin\theta\cos\theta T_r{}^r + a^2\sin\theta\cos\theta T_\theta{}^\theta\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}F(r,\theta) &= \rho^{-2}[-\Lambda T^{tt} + 2a\Lambda\sin^2\theta T^{t\phi} + \sin^2\theta(-\Lambda a^2\sin^2\theta + r\rho^4)T^{\phi\phi}] \\ G(r,\theta) &= \frac{a^2(r^2+a^2-\Delta)}{\rho^2\Delta(r^2+a^2)}\sin\theta\cos\theta[(r^2+a^2)^2T_{tt} + 2a(r^2+a^2)T_{t\phi} + a^2T_{\phi\phi}] + \frac{\rho^2\cos\theta}{(r^2+a^2)\sin^3\theta}T_{\phi\phi}\end{aligned} \tag{4.5}$$

with $\Lambda = M(r^2 - a^2\cos^2\theta)$. Here we have two equations in six unknowns each of which is a function of two variables r and θ.

One other symmetry immediately apparent from the form of the metric is invariance under the transformation

$$\theta \rightarrow \tilde{\theta} = \pi - \theta. \tag{4.6}$$

The components of the stress-energy tensor will also possess this symmetry, so in particular

$$\partial_\theta(T_{\mu\nu}) = 0 \quad \text{when} \quad \theta = \pi/2. \tag{4.7}$$

This does not imply that any components of $T_{\mu\nu}$ vanish, so $T_{r\theta}$ is non-zero in general. However, from the conservation equations (4.4), it follows that

$$T_{r\theta} = 0 \quad \text{when} \quad \theta = \pi/2. \tag{4.8}$$

The other symmetry of the geometry which should be mentioned here is invariance under simultaneous t-ϕ reversal, that is, $t \rightarrow -t$ and $\phi \rightarrow -\phi$. The stress tensor for a state satisfying this invariance must have $T_{tr} = T_{t\theta} = T_{\phi r} = T_{\phi\theta} = 0$ and correspondingly $K(\theta) = L(\theta) = 0$. It might be thought that this simple symmetry of the space-time should be mirrored by the stress tensor for the physical vacua in which we are interested. However, as discussed above this is not the case, because of the superradiant modes. Neither the Boulware vacuum $|B^-\rangle$ nor the Unruh vacuum $|U^-\rangle$ defined in Sec. III B is invariant under simultaneous t-ϕ reversal. This in contrast to the situation for Schwarzschild black holes, where the Boulware vacuum is time-reversal invariant, although the Unruh vacuum is not, due to the Hawking flux. In Schwarzschild space-time, the Hartle-Hawking state is also time-reversal invariant. Of the two Hartle-Hawking-like states, $|CCH\rangle$ is not invariant under simultaneous t-ϕ reversal but $|FT\rangle$ is. In Sec. V we shall consider further the symmetry and other properties of these states.

B. The trace anomaly

As is well known, conformally invariant field theories on a curved background $g_{\mu\nu}$ possess a conformal anomaly which means that the renormalized stress tensor has a trace even though the classical stress tensor must be trace-free. As it arises from the renormalization procedure, the trace anomaly is a geometrical scalar, depending only on the geometry and the nature of the quantum field under consideration, not on the actual quantum state. All methods of regularization agree that it has the form

$$\langle \hat{T}^\alpha_\alpha \rangle_{ren} = k_1 C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + k_2\left(R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2\right) + k_3 \nabla_\alpha \nabla^\alpha R \tag{4.9}$$

in four dimensions. Here k_1, k_2, k_3 are constants which are independent of the space-time geometry and depend only on the quantum field. For example, for a massless scalar field, $k_1 = k_2 = k_3 = (2880\pi^2)^{-1}$. Although all methods of regularization agree on the values of k_1, k_2, k_3 for scalar and neutrino fields, and on k_1 and k_2 for the electromagnetic field, there is disagreement on the value of k_3. Dimensional regularization gives $k_3 = 0$ whilst both point separation and ζ-function renormalization give $k_3 = -(96\pi^2)^{-1}$. This discrepancy is unimportant for us as $R = 0$ for a Kerr black hole. For a Kerr black hole of mass M and angular momentum Ma,

$$C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = 48\rho^{-12}\{M^2r^8 - 15M^2r^4a^2\cos^2\theta + 15M^2r^2a^4\cos^4\theta - M^2a^6\cos^6\theta\} \tag{4.10}$$

where, as before, $\rho^2 = r^2 + a^2\cos^2\theta$. The trace anomaly is, of course, finite except at a curvature singularity of the space-time.

We may now replace one of the stress tensor components by the trace. Hence we may substitute

$$T_\theta{}^\theta = T_\alpha{}^\alpha - T_t{}^t - T_r{}^r - T_\phi{}^\phi \tag{4.11}$$

to yield

$$\begin{aligned}\tilde{F}(r,\theta) &= \partial_r(\rho^2 T_r{}^r) + \Delta^{-1}\csc\theta\,\partial_\theta(\rho^2\sin\theta\, T_\theta{}^r) + T_r{}^r \\ &\quad - \Delta^{-1}(ra^2\sin^2\theta - \Lambda)T_r{}^r \\ \tilde{G}(r,\theta) &= \partial_r(\rho^2 T_\theta{}^r) - \csc\theta\,\partial_\theta(\rho^2\sin\theta\, T_r{}^r)\end{aligned} \tag{4.12}$$

where

$$\tilde{F}(r,\theta) = F(r,\theta) + rT_\alpha{}^\alpha - rT_t{}^t - rT_\phi{}^\phi$$

$$= rT_\alpha{}^\alpha + \frac{(M-r)}{\Delta^2}[(r^2+a^2)^2T_{tt} + 2a(r^2+a^2)T_{t\phi} + a^2T_{\phi\phi}] + \frac{2r}{\Delta}[(r^2+a^2)T_{tt} + aT_{t\phi}] \tag{4.13a}$$

$$\tilde{G}(r,\theta) = G(r,\theta) - a^2\sin\theta\cos\theta(T_\alpha{}^\alpha - T_t{}^t - T_\phi{}^\phi) - \csc\theta\,\partial_\theta(\rho^2\sin\theta[T_\alpha{}^\alpha - T_t{}^t - T_\phi{}^\phi])$$

$$= -\frac{1}{\Delta\sin\theta}\partial_\theta(\sin\theta[(r^2+a^2)^2T_{tt} + 2a(r^2+a^2)T_{t\phi} + a^2T_{\phi\phi}]) + 2a\cot\theta(a\sin^2\theta T_{tt} + T_{t\phi})$$

$$+ a^2\sin\theta\,\partial_\theta T_{tt} + 2a\,\partial_\theta T_{t\phi} + \csc^2\theta\,\partial_\theta T_{\phi\phi} - \rho^2\partial_\theta T_\alpha{}^\alpha + \cos\theta(a^2\sin\theta - \rho)^2 T_\alpha{}^\alpha. \tag{4.13b}$$

Equations (4.12) can be written in the alternative form:

$$\partial_r(\Delta^{1/2}\rho^2\sin\theta T_r{}^r)+\partial_\theta(\Delta^{-1/2}\rho^2\sin\theta T_\theta{}^r)=\Delta^{1/2}\tilde{F}(r,\theta)\sin\theta$$

$$\partial_r(\rho^2\sin\theta T_\theta{}^r)-\partial_\theta(\rho^2\sin\theta T_r{}^r)=\tilde{G}(r,\theta)\sin\theta. \tag{4.14}$$

These equations can now be integrated over r to give

$$T_r{}^r=\frac{R(\theta)}{\Delta^{1/2}\rho^2}+\frac{1}{\Delta^{1/2}\rho^2\sin\theta}\int_{r_+}^r(\Delta^{1/2}\tilde{F}(r',\theta)\sin\theta-\Delta^{-1/2}\partial_\theta(\rho^2\sin\theta T_\theta{}^r))dr'$$

$$T_\theta{}^r=\frac{S(\theta)}{\rho^2}+\frac{1}{\rho^2\sin\theta}\int_{r_+}^r(\tilde{G}(r',\theta)\sin\theta+\partial_\theta(\rho^2\sin\theta T_r{}^r))dr' \tag{4.15}$$

where $R(\theta)$, $S(\theta)$ are arbitrary functions of θ alone. Choice of a particular vacuum state will place restrictions on the four arbitrary functions $K(\theta)$, $L(\theta)$, $R(\theta)$ and $S(\theta)$ and also on $\tilde{F}(r,\theta)$ and $\tilde{G}(r,\theta)$, which depend on three unknown stress tensor components, T_{tt}, $T_{t\phi}$ and $T_{\phi\phi}$. The solutions (4.15) are particularly useful for finding the behavior of the stress tensor close to the event horizon, but we still have the coupling between $T_r{}^r$ and $T_\theta{}^r$.

Uncoupled equations for $T_r{}^r$ and $T_\theta{}^r$ can be obtained from Eq. (4.14) in the form

$$\Delta^{1/2}\partial_r[\Delta^{1/2}\partial_r\mathcal{T}_1]+\partial_\theta^2\mathcal{T}_1=\Delta^{1/2}\partial_r(\mathcal{F})-\partial_\theta(\mathcal{G})$$

$$\Delta^{1/2}\partial_r[\Delta^{1/2}\partial_r(\mathcal{T}_2)]+\partial_\theta^2(\mathcal{T}_2)=\Delta^{1/2}\partial_r(\mathcal{G})+\partial_\theta(\mathcal{F}) \tag{4.16}$$

where

$$\mathcal{T}_1=T_r{}^r\Delta^{1/2}\rho^2\sin\theta \quad \mathcal{T}_2=T_\theta{}^r\rho^2\sin\theta$$

$$\mathcal{F}=\Delta\tilde{F}\sin\theta \quad \mathcal{G}=\Delta^{1/2}\tilde{G}\sin\theta. \tag{4.17}$$

We now define a new variable x by

$$x=2\Delta^{1/2}+2r-2M, \tag{4.18}$$

in terms of which the equations (4.16) now have the usual polar form of the Laplacian:

$$x\partial_x[x\partial_x\mathcal{T}_1]+\partial_\theta^2\mathcal{T}_1=x\partial_x\mathcal{F}-\partial_\theta\mathcal{G} \tag{4.19a}$$

$$x\partial_x[x\partial_x\mathcal{T}_2]+\partial_\theta^2\mathcal{T}_2=x\partial_x\mathcal{G}+\partial_\theta\mathcal{F}. \tag{4.19b}$$

The domain of these equations is $x\in((r^2-a^2)^{1/2},\infty)$, $\theta\in(0,\pi)$, that is the punctured half-plane. By constructing a Green's function for this domain, a unique solution for $\mathcal{T}_1$ and $\mathcal{T}_2$ can be found if they are specified on the boundary, provided we know $\mathcal{F}$ and $\mathcal{G}$ throughout the region. Therefore, we need to know $T_r{}^r$ and $T_\theta{}^r$ on the event horizon [where $x=(r^2-a^2)^{1/2}$], and the three components of the stress tensor, T_{tt}, $T_{t\phi}$ and $T_{\phi\phi}$ everywhere outside the event horizon. From Eq. (4.17), it can be seen that $\mathcal{T}_1$ and $\mathcal{T}_2$ must vanish on the axis $\theta=0,\pi$ provided that $T_r{}^r$ and $T_\theta{}^r$ are well-defined there. Therefore this reduces the number of boundary functions which are unknown.

Although it looks like $\mathcal{T}_1$ vanishes on the event horizon, the analysis of Sec. IV D will show that even for a quantum state which is regular on the event horizon, $T_r{}^r$ diverges as Δ^{-1} as $r\to r_+$, giving a divergent value for $\mathcal{T}_1$ on the horizon. This means that the Green's function method is not directly applicable to Eq. (4.19a). However, the second equation can be solved uniquely using a Green's function, and the solution then fed into Eq. (4.15) to give the behavior of $T_r{}^r$. Note that our calculations in Sec. V confirm that, for the Unruh and (past and future) Boulware vacua, the function $\mathcal{T}_2$ vanishes sufficiently quickly at infinity that the Green's function method gives a unique solution.

C. The Killing-Yano tensor

So far in our analysis we have exploited the Killing vector symmetries of the Kerr geometry to assume that the stress tensor is a function only of r and θ. The Kerr geometry also possesses a Killing-Yano tensor [11], which is a skew-symmetric tensor $f_{\mu\nu}$ satisfying

$$\nabla^{(\mu}f^{\nu)\lambda}=0. \tag{4.20}$$

We shall now show that the consequence of the existence of the Killing-Yano tensor is that $T_{x\theta}=0$, when $x=t$ or $x=\phi$, for the quantum states we are interested in.

For any quantum state, the renormalized expectation value of the quantum stress tensor can be calculated using the technique of point splitting:

$$\langle T_{\mu\nu}\rangle_{\rm ren}=\lim_{x\to x'}[T_{\mu\nu}(x,x')-T^{\rm div}_{\mu\nu}(x,x')] \tag{4.21}$$

where $T_{\mu\nu}(x,x')$ is the point-separated stress tensor for our particular quantum state and $T^{\rm div}_{\mu\nu}(x,x')$ are the divergent subtraction terms. The unrenormalized stress tensor components for the quantum states in which we are interested are given as mode sums (3.19a)–(3.24), the mode sum contribution to $T_{A\theta}$ for $A=t$ or $A=\phi$ being

$$T_{A\theta}[u,u^*]=\Re e\left[\frac{1}{3}(u_{;A}u^*_{;\theta}+u^*_{;A}u_{;\theta})-\frac{1}{6}(u_{;A\theta}u^*+u^*_{;A\theta}u)\right]. \tag{4.22}$$

The existence of the Killing-Yano tensor has the result that the wave equation for a massless scalar field on the Kerr geometry is separable [2], with the mode solutions given by Eq. (2.2). In addition, we have

$$u_{;A\theta}=u_{,A\theta}-\Gamma^t_{A\theta}u_{,t}-\Gamma^\phi_{A\theta}u_{,\phi}. \tag{4.23}$$

From the mode functions, $u_{,A}=iku$ where $k=-\omega$ if $A=t$ and $k=m$ if $A=\phi$; also

$$u_{,\theta}\propto-(r^2+a^2)^{-1/2}e^{-i\omega t+im\phi}R_{\omega lm}(r)S'_{\omega lm}(\cos\theta)\sin\theta; \tag{4.24}$$

and $u_{,x\theta}=iku_{,\theta}$. Since the spheroidal harmonics $S_{\omega lm}$ are real, the quantities appearing in Eq. (4.22) are all purely imaginary and hence $T_{x\theta}[u,u^*]=0$. Therefore the point-separated stress tensor components $T_{A\theta}(x,x')$ vanish for all of the states under consideration. In addition, it is shown in [5] that the subtraction terms $T^{\rm div}_{A\theta}(x,x')$ are also zero, so that $\langle T_{x\theta}\rangle_{\rm ren}$ vanishes for all the states we are considering here. This property was proved by Frolov and Thorne [5] for $|FT\rangle$ but we have shown here that this is a quite general result.

D. Behavior on the event and Cauchy horizons

Next we shall investigate the behavior of the stress tensor at the future and past event and Cauchy horizons. It is convenient to introduce co-ordinate systems that are regular at the horizons. We first introduce two double-null co-ordinate systems u, v, θ, $\phi_\pm$ by

$$u=t-r_*,\quad v=t+r_*,\quad \phi_\pm=\phi-\frac{a}{r_\pm^2+a^2}t=\phi-\Omega_\pm t, \tag{4.25}$$

where the last equation defines Ω_+ and Ω_- which are the angular velocity of the event and Cauchy horizons respectively, and r_* is the ''tortoise'' co-ordinate given by Eq. (2.8). The two sets of Kruskal co-ordinates $U_\pm$, $V_\pm$ are then defined by

$$U_\pm=-e^{-\kappa_\pm u},\quad V_\pm=e^{\kappa_\pm v}. \tag{4.26}$$

From the definition of $U_\pm$, $V_\pm$ and r_*,

$$U_\pm V_\pm=-e^{2\kappa_\pm r_*}=-e^{2\kappa_\pm r}(r-r_\pm)|r-r_\mp|^{\kappa_\pm/\kappa_\mp}. \tag{4.27}$$

The exterior region corresponds to $U_+<0$, $V_+>0$ with the past event horizon at $V_+=0$ and the future event horizon at $U_+=0$. These coordinates may be extended to cover the event horizons in a regular fashion but are singular at the Cauchy horizons. Correspondingly, the coordinates U_- and V_- may be extended to cover the Cauchy horizons ($U_-=0$ and $V_-=0$) in a regular fashion but are singular at the event horizons.

The stress tensor components in these Kruskal co-ordinate systems are

$$\begin{aligned}T_{U_\pm U_\pm}=\kappa_\pm^{-2}U_\pm^{-2}\bigg[&\frac{1}{4}T_{tt}+\frac{1}{2}\Omega_\pm T_{t\phi}+\frac{1}{4}\Omega_\pm^2T_{\phi\phi}\\&-\frac{\Delta}{2(r^2+a^2)}T_{rt}+\frac{\Delta^2}{4(r^2+a^2)^2}T_{rr}\\&-\frac{\Delta\Omega_\pm}{2(r^2+a^2)}T_{r\phi}\bigg]\end{aligned} \tag{4.28a}$$

$$\begin{aligned}T_{U_\pm V_\pm}=-\kappa_\pm^{-2}U_\pm^{-1}V_\pm^{-1}\bigg[&\frac{1}{4}T_{tt}+\frac{1}{2}\Omega_\pm T_{t\phi}+\frac{1}{4}\Omega_\pm^2T_{\phi\phi}\\&-\frac{\Delta^2}{4(r^2+a^2)^2}T_{rr}\bigg]\end{aligned} \tag{4.28b}$$

$$\begin{aligned}T_{V_\pm V_\pm}=\kappa_\pm^{-2}V_\pm^{-2}\bigg[&\frac{1}{4}T_{tt}+\frac{1}{2}\Omega_\pm T_{t\phi}+\frac{1}{4}\Omega_\pm^2T_{\phi\phi}\\&+\frac{\Delta}{2(r^2+a^2)}T_{rt}+\frac{\Delta^2}{4(r^2+a^2)^2}T_{rr}\\&+\frac{\Delta\Omega_\pm}{2(r^2+a^2)}T_{r\phi}\bigg]\end{aligned} \tag{4.28c}$$

$$T_{U_\pm\theta}=\kappa_\pm^{-1}U_\pm^{-1}\frac{\Delta}{2(r^2+a^2)}T_{r\theta} \tag{4.28d}$$

$$T_{V_\pm\theta}=\kappa_\pm^{-1}V_\pm^{-1}\frac{\Delta}{2(r^2+a^2)}T_{r\theta} \tag{4.28e}$$

$$\begin{aligned}T_{U_\pm\phi_\pm}=-\kappa_\pm^{-1}U_\pm^{-1}\bigg[&\frac{1}{2}T_{t\phi}+\frac{1}{2}\Omega_\pm T_{\phi\phi}\\&-\frac{\Delta}{2(r^2+a^2)}T_{r\phi}\bigg]\end{aligned} \tag{4.28f}$$

$$\begin{aligned}T_{V_\pm\phi_\pm}=\kappa_\pm^{-1}V_\pm^{-1}\bigg[&\frac{1}{2}T_{t\phi}+\frac{1}{2}\Omega_\pm T_{\phi\phi}\\&+\frac{\Delta}{2(r^2+a^2)}T_{r\phi}\bigg]\end{aligned} \tag{4.28g}$$

with $T_{\theta\theta}=T_{\theta\theta}$, $T_{\theta\phi_\pm}=0$ and $T_{\phi_\pm\phi_\pm}=T_{\phi\phi}$, where we have set $T_{t\theta}=T_{\theta\phi}=0$. It follows immediately that regularity of the stress tensor on any horizon requires that $T_{\theta\theta}$, $T_{\phi\phi}$ and $T_{r\theta}$ be finite as the horizon is approached.

For a general stress tensor with $T_{t\theta}=T_{\theta\phi}=0$, we have by Eqs. (4.3)

$$T_{tr}=\frac{K(\theta)}{\Delta},\quad T_{\phi r}=\frac{L(\theta)}{\Delta}. \tag{4.29}$$

In this case, consideration of the $T_{U_\pm\phi_\pm}$ and $T_{V_\pm\phi_\pm}$ components shows that regularity requires

$$T_{t\phi}(r,\theta)=\pm\frac{L(\theta)}{r_\pm^2+a^2}-\Omega_\pm T_{\phi\phi}(r_\pm,\theta)+O(r-r_\pm)\quad \text{as}\quad r\to r_\pm, \tag{4.30}$$

where the positive sign is taken for regularity on the future horizon ($U_\pm=0$) and the negative sign on the past horizon ($V_\pm=0$). Note that if $L(\theta)$ is non-zero, only one of these conditions can be met on either the future or past event horizon. Regularity of the $T_{U_\pm U_\pm}$, $T_{V_\pm V_\pm}$ and $T_{U_\pm V_\pm}$ components implies that

$$T_{tt}=\pm\frac{K(\theta)-\Omega_\pm L(\theta)}{r^2+a^2}+\Omega_\pm^2T_{\phi\phi}+O(r-r_\pm) \tag{4.31a}$$

$$T_{rr} = \pm [K(\theta) + \Omega_{\pm} L(\theta)] \frac{r^2 + a^2}{\Delta^2} + O(r - r_{\pm})^{-1} \tag{4.31b}$$

as $r \to r_{\pm}$, with the positive sign for regularity on the future horizon, and the negative sign for the past horizon, as before. Finiteness of $T_{r\theta}$ as the horizon is approached implies that the function $S(\theta)$ in Eq. (4.15) vanishes, whilst the form (4.31b) of T_{rr} near the horizon tells us that $R(\theta)$ in Eq. (4.15) is also identically zero. It should be stressed that the forms (4.30)–(4.31b) are compatible with the solution of the conservation equations (4.15) with R and S identically zero.

We note that our analysis is in agreement with that of [10], in that unless both $K(\theta)$ and $L(\theta)$ vanish identically, the stress tensor must diverge at one of the event horizons, and at least one of the Cauchy horizons. The past and future Boulware vacua are not expected to be regular on either event horizon. For the Unruh vacuum state, it is expected that the divergences occur on the past event horizon and future Cauchy horizon [10]. For $|FT\rangle$ simultaneous t-ϕ invariance required that $K(\theta)$ and $L(\theta)$ vanish consistent with regularity. On the other hand, for $|CCH\rangle$ there was no requirement that $K(\theta)$ and $L(\theta)$ vanish and so one expects that there will be divergences on the past event horizon in line with the Unruh vacuum. We shall return to this issue in Sec. V.

At this stage, we need to step back and see how much information about the stress tensor we have managed to obtain from our approach. We began with ten stress tensor components, each a function of the two variables r and θ. The Killing-Yano symmetry revealed that two of these components $T_{t\theta}$ and $T_{\theta\phi}$ vanished identically, whilst another component could be eliminated by using the known trace anomaly, leaving seven unknown functions of r and θ. Using the conservation equations, we need to know three functions of r and θ (corresponding to T_{tt}, $T_{t\phi}$ and $T_{\phi\phi}$), and four functions of θ (K, L, T_{rr} and $T_{r\theta}$ on the event horizon). Finally, for a state which is regular on one of the event horizons, this reduces to three functions of θ since the behavior of T_{rr} is given in terms of K and L. In addition, we know the behavior of the three unknown components, T_{tt}, $T_{t\phi}$ and $T_{\phi\phi}$ on the event horizon, in terms of K, L and $T_{\phi\phi}$. Thus our analysis has significantly reduced the number of degrees of freedom of the stress tensor in Kerr space-time. Of course, this reduction is rather less significant than the corresponding analysis for Schwarzschild black holes [12], but this was to be expected due to the fact that Kerr has fewer symmetries than Schwarzschild.

V. ASYMPTOTIC BEHAVIOR OF THE PHYSICAL VACUA

In this section we shall consider the asymptotic behavior of the physical states of interest near the event horizon and at infinity. This will provide a consistency check on the analysis of the previous section. We shall also use the properties of the Unruh and Boulware vacua (whose asymptotic behaviors are well understood) to reveal information about the states $|CCH\rangle$ and $|FT\rangle$. It is known that the divergent terms which have to be subtracted from the unrenormalized expectation value of the stress tensor are independent of the quantum state under consideration. Therefore we shall consider the differences in expectation values of the stress tensor in two different states, since these can be calculated without renormalization. Such differences in expectation values will be traceless tensors since the trace anomaly is the same for all quantum states.

We shall begin by concentrating on the Unruh vacuum, since its stress tensor has been calculated in the asymptotic regimes by Punsley [13] using an equivalence principle approach. This will provide a useful check of our calculations. First, we consider the behavior at infinity, and calculate

$$\begin{aligned}
&\langle U^{-}|\hat{T}_{\mu\nu}|U^{-}\rangle_{ren} - \langle B^{-}|\hat{T}_{\mu\nu}|B^{-}\rangle_{ren} \\
&\quad = \langle U^{-}|\hat{T}_{\mu\nu}|U^{-}\rangle - \langle B^{-}|\hat{T}_{\mu\nu}|B^{-}\rangle \\
&\quad = \sum_{l,m} \int_0^{\infty} \frac{2d\tilde{\omega}}{e^{2\pi\tilde{\omega}/\kappa} - 1} T_{\mu\nu}[u^{up}_{\omega lm}, u^{up*}_{\omega lm}].
\end{aligned} \tag{5.1}$$

Using the asymptotic form of the mode functions (2.11), we have, as $r \to \infty$,

$$\begin{aligned}
&\langle U^{-}|\hat{T}^{\nu}_{\mu}|U^{-}\rangle_{ren} - \langle B^{-}|\hat{T}^{\nu}_{\mu}|B^{-}\rangle_{ren} \\
&\quad \sim \frac{1}{4\pi^2 r^2} \sum_{l,m} \int_0^{\infty} \frac{\omega\, \mathrm{d}\tilde{\omega}}{\tilde{\omega}(e^{2\pi\tilde{\omega}/\kappa} - 1)} |B^{-}_{\omega lm}|^2 |S_{\omega lm}(\cos\theta)|^2 \\
&\quad \times \begin{pmatrix} -\omega & \omega & 0 & m \\ -\omega & \omega & 0 & m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{5.2}$$

In order to obtain the behavior of the Unruh vacuum at future null infinity, we need to consider the ‘‘past’’ Boulware vacuum at infinity. The ‘‘past’’ Boulware vacuum contains at future null infinity an outward flux of particles due to the Unruh-Starobinskii effect [6], so that, as we approach $\mathfrak{I}^{+}$,

$$\begin{aligned}
&\langle B^{-}|\hat{T}^{\nu}_{\mu}|B^{-}\rangle_{ren} \sim \langle B^{-}|\hat{T}^{\nu}_{\mu}|B^{-}\rangle - \langle B^{+}|\hat{T}^{\nu}_{\mu}|B^{+}\rangle \\
&\quad \sim \frac{1}{4\pi^2 r^2} \sum_{l,m} \int_0^{\omega_{min}} \frac{\omega d\omega}{\tilde{\omega}(e^{2\pi\tilde{\omega}/\kappa} - 1)} |B^{-}_{\omega lm}|^2 \\
&\quad \times |S_{\omega lm}(\cos\theta)|^2 \begin{pmatrix} \omega & -\omega & 0 & -m \\ \omega & -\omega & 0 & -m \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{5.3}$$

Adding these two tensors gives the asymptotic behavior of the Unruh vacuum at future null infinity as

$$\langle U|\hat{T}_\mu^\nu|U\rangle_{ren}\sim\frac{1}{4\pi^2 r^2}\sum_{l,m}\int_0^\infty\frac{\omega d\omega}{\tilde{\omega}(e^{2\pi\tilde{\omega}/\kappa}-1)}|B^-_{\omega lm}|^2 \times|S_{\omega lm}(\cos\theta)|^2\begin{pmatrix}-\omega & \omega & 0 & m\\ -\omega & \omega & 0 & m\\ 0&0&0&0\\ 0&0&0&0\end{pmatrix}. \tag{5.4}$$

This is in agreement with the form obtained in [13], and represents the expected thermal flux at infinity. It should be noted that, despite initial appearances, the integrands are regular when $\tilde{\omega}=0$ due to the Wronskian relations (2.12d) which ensure that $|B^-_{\omega lm}|^2=O(\tilde{\omega}^2)$ as $\tilde{\omega}\to 0$. From Eq. (5.4) we can read off the forms of the functions K and L (4.29) for the Unruh vacuum:

$$K_{U^-}(\theta)=\frac{1}{4\pi^2}\sum_{l,m}\int_0^\infty\frac{-\omega^2 d\omega}{\tilde{\omega}(e^{2\pi\tilde{\omega}/\kappa}-1)}|B^-_{\omega lm}|^2|S_{\omega lm}(\cos\theta)|^2$$

$$L_{U^-}(\theta)=\frac{1}{4\pi^2}\sum_{l,m}\int_0^\infty\frac{-m\omega d\omega}{\tilde{\omega}(e^{2\pi\tilde{\omega}/\kappa}-1)} \times|B^-_{\omega lm}|^2|S_{\omega lm}(\cos\theta)|^2. \tag{5.5}$$

We now turn to the behavior of the Unruh vacuum at the event horizon. In Schwarzschild, the Hartle-Hawking state is regular on both event horizons, and so the behavior of the Unruh vacuum as $r\to r_+$ is found from

$$\langle U^-|\hat{T}_{\mu\nu}|U^-\rangle_{ren}\sim\langle U^-|\hat{T}_{\mu\nu}|U^-\rangle_{ren}-\langle H|\hat{T}_{\mu\nu}|H\rangle_{ren} =\langle U^-|\hat{T}_{\mu\nu}|U^-\rangle-\langle H|\hat{T}_{\mu\nu}|H\rangle. \tag{5.6}$$

In the absence of a Hartle-Hawking state for Kerr, we shall instead consider the differences of the stress tensors in the Unruh vacuum and the states $|FT\rangle$ and $|CCH\rangle$. These are given by

$$\langle U^-|\hat{T}_{\mu\nu}|U^-\rangle-\langle FT|\hat{T}_{\mu\nu}|FT\rangle =\sum_{l,m}\int_0^\infty\frac{-2d\omega}{e^{2\pi\tilde{\omega}/\kappa}-1}T_{\mu\nu}[u^{in}_{\omega lm},u^{in*}_{\omega lm}], \tag{5.7a}$$

$$\langle U^-|\hat{T}_{\mu\nu}|U^-\rangle-\langle CCH|\hat{T}_{\mu\nu}|CCH\rangle =\sum_{l,m}\int_0^\infty\frac{-2d\omega}{e^{2\pi\omega/\kappa}-1}T_{\mu\nu}[u^{in}_{\omega lm},u^{in*}_{\omega lm}]. \tag{5.7b}$$

As $r\to r_+$, one finds

$$\langle U^-|\hat{T}_\mu^\nu|U^-\rangle-\langle FT|\hat{T}_\mu^\nu|FT\rangle\sim\frac{1}{4\pi^2\rho^2}\sum_{l,m}\int_0^\infty\frac{d\omega}{\omega(e^{2\pi\tilde{\omega}/\kappa}-1)}|B^+_{\omega lm}|^2|S_{\omega lm}(\cos\theta)|^2 \times\begin{pmatrix}\Delta^{-1}(r_+^2+a^2)\omega\tilde{\omega} & -\omega\tilde{\omega} & 0 & \Delta^{-1}a\omega\tilde{\omega}\\ \Delta^{-2}(r_+^2+a^2)^2\tilde{\omega}^2 & -\Delta^{-1}(r_+^2+a^2)^2\tilde{\omega}^2 & O(1) & -\Delta^{-2}a(r_+^2+a^2)\tilde{\omega}^2\\ 0 & O(\Delta) & O(1) & 0\\ \Delta^{-1}(r_+^2+a^2)m\tilde{\omega} & m\tilde{\omega} & 0 & -\Delta^{-1}am\tilde{\omega}\end{pmatrix}. \tag{5.8}$$

The expression for $\langle U^-|\hat{T}_\mu^\nu|U^-\rangle-\langle CCH|\hat{T}_\mu^\nu|CCH\rangle$ is identical to Eq. (5.8), with the denominator $e^{2\pi\tilde{\omega}/\kappa}-1$ replaced by $e^{2\pi\omega/\kappa}-1$. In both cases the integrand is regular for all values of ω, by virtue of the Wronskian relations (2.12d). The difference in expectation values of the stress tensor in the Unruh and Frolov-Thorne states (5.8) agrees with the stress tensor for the Unruh vacuum found in [13], whereas when we have the state $|CCH\rangle$ instead of $|FT\rangle$ the thermal terms in the denominator do not agree. Furthermore, the tensor (5.8) is regular on the future event horizon but not on the past event horizon, the same behavior that we would expect for the Unruh vacuum. Therefore we can compare the tensor (5.8) with the behavior near the event horizon derived in Sec. IV D. There is exact agreement, using the functions $K_{U^-}(\theta)$ and $L_{U^-}(\theta)$ found from the expectation value of the stress tensor at infinity in the Unruh vacuum (5.5), and the Wronskian relations.

From the regularity of the tensor (5.8) on the future event horizon, we can conclude that the expectation value of the stress tensor in the state $|FT\rangle$ is regular on at least one event horizon (and, since it is invariant under simultaneous t, ϕ reversal, it will be regular on both event horizons). Thus, it may appear that the state $|FT\rangle$ in fact has the properties that we require of the Hartle-Hawking state. However, whilst the expectation value of the stress tensor in the state $|FT\rangle$ is regular on the event horizon, the expectation value of $\hat{\Phi}^2$ is not. We calculate, as $r\to r_+$,

$$\langle U^-|\hat{\Phi}^2|U^-\rangle - \langle FT|\hat{\Phi}^2|FT\rangle \sim \frac{1}{4\pi^2(r_+^2+a^2)}\sum_{l,m}\int_0^\infty \frac{-2d\omega}{\omega(e^{2\pi\tilde{\omega}/\kappa}-1)} \times |B^+_{\omega lm}|^2 |S_{\omega lm}(\cos\theta)|^2. \tag{5.9}$$

The integrand in the above expression is regular at $\omega=0$ because of the Wronskian relations (2.12d), but has a pole at $\tilde{\omega}=0$, giving a divergent integral. If we attempt to calculate the difference in expectation values (5.7a) anywhere outside the event horizon, then the integral over ω also has a pole at $\tilde{\omega}=0$, leading to a divergent result. Therefore it seems that the regularity of the difference in expectation values of the stress tensor (5.8) at the event horizon does not reflect the true nature of *the state* $|FT\rangle$, and that this state *in fact fails to be regular almost everywhere*, both on or outside the event horizon, although it formally has attractive symmetry properties.

There is one exception to the regularity of the state $|FT\rangle$ which is that on the axis the terms with $m\neq 0$ (and, in particular, all superradiant modes) do not contribute. Thus, if one point is on the axis the $|FT\rangle$ and $|CCH\rangle$ two-point functions agree:

$$G^{FT/CCH}(t,r,\theta,\phi;t',r',0,\phi') = \sum_l \int_0^\infty \frac{d\omega\,\coth(\pi\omega/\kappa)}{\omega\sqrt{(r^2+a^2)(r'^2+a^2)}}[R^+_{\omega l0}(r)R^{+*}_{\omega l0}(r') + R^-_{\omega l0}(r)R^{-*}_{\omega l0}(r')]S_{\omega l0}(\cos\theta)S_{\omega l0}(1). \tag{5.10}$$

In the asymptotic regions, the integrals are dominated by the contribution from near $\omega=0$. In this limit the spheroidal functions reduce to Legendre polynomials

$$S_{0lm} = \frac{1}{\sqrt{4\pi}}P_l(\cos\theta), \quad \lambda(0)=l(l+1). \tag{5.11}$$

In addition, $T_{lm}(r)=R_{0lm}(r)/\sqrt{r^2+a^2}$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}\eta}(\eta^2-1)\frac{\mathrm{d}T_{lm}}{\mathrm{d}\eta} - \left[l(l+1) + \frac{m^2a^2}{(M^2-a^2)(\eta^2-1)}\right]T_{lm}=0 \tag{5.12}$$

where

$$\eta = \frac{2r-(r_++r_-)}{(r_+-r_-)} = \frac{r-M}{\sqrt{M^2-a^2}}, \tag{5.13}$$

with solutions $P_l^{ma/\sqrt{M^2-a^2}}(\eta)$ and $Q_l^{ma/\sqrt{M^2-a^2}}(\eta)$. In particular, a steepest descent analysis of Eq. (5.10) as $r'\to r_+$ yields

$$\begin{aligned} G^{FT/CCH}(t,r,\theta,\phi;t',r_+,0,\phi') &= \frac{\kappa_+}{16\pi^2\sqrt{M^2-a^2}}\sum_l (2l+1)Q_l\left(\frac{r-M}{\sqrt{M^2-a^2}}\right)P_l(\cos\theta) \\ &= \frac{\kappa_+}{8\pi^2}\frac{1}{r-M-\sqrt{M^2-a^2}\cos\theta}, \end{aligned} \tag{5.14}$$

where the second line follows from Heine's formula. This result was first given by Frolov [14] and enabled him to calculate the renormalized value of the expectation value of $\hat{\Phi}^2$ on the pole of the event horizon. Later with Zel'nikov [15] he extended this calculation to calculate the renormalized value of the expectation value of $\hat{T}_{\mu\nu}$ on the pole of the event horizon. Our point is that, unfortunately, these calculations were only possible because the troublesome superradiant modes do not contribute on the axis and have actually led to a false confidence concerning the Hartle-Hawking vacuum.

Finally we return the properties of the state $|CCH\rangle$. This has a different thermal factor from $|FT\rangle$ (3.22) which means that the difference in expectation values of the stress tensor in $|U^-\rangle$ and $|CCH\rangle$ at the event horizon is rather different from simply the stress tensor in the state $|U^-\rangle$. The difference in thermal factors also means that the state $|CCH\rangle$ is *not* invariant under simultaneous t-ϕ reversal.

However, the quantity $\langle U^-|\hat{T}^\nu_\mu|U^-\rangle - \langle CCH|\hat{T}^\nu_\mu|CCH\rangle$ is regular on the future event horizon (but not on the past), so, using the expected regularity of the Unruh vacuum, we can conclude that $\langle CCH|\hat{T}^\nu_\mu|CCH\rangle$ is also regular on the future event horizon (but not on the past). If we consider the difference in expectation values of $\hat{\Phi}^2$ at the event horizon, the answer is the same as Eq. (5.9), but with $e^{2\pi\tilde{\omega}/\kappa}$ replaced by $e^{2\pi\omega/\kappa}$. Using the Wronskian relations (2.12d), this gives a finite answer, further strengthening our argument that $|CCH\rangle$ is a regular state on the future event horizon.

VI. CONCLUSIONS

In this paper we have considered the renormalized stress energy tensor on Kerr space-time, and used the anticipated physical properties of this tensor (symmetry, conservation equations, and regularity conditions) in order to derive as much information as possible. As expected, the analysis is considerably more complex than the corresponding problem in Schwarzschild [12], and the solution gives us less information, although we are able to reduce the number of unknowns to three functions of r and θ and three functions of θ.

Our results are in agreement with the known form of the Unruh vacuum at the event horizon and at infinity. We also considered two candidates for the state analogous to the Hartle-Hawking state in Schwarzschild. From the Kay-Wald theorem [7], we know that there is no state in Kerr which is regular at the event horizon and everywhere outside, invariant under simultaneous t, ϕ reversal and thermal in nature. Of our two candidate states, one is invariant under t, ϕ reversal, but fails to be regular on the event horizon, whilst the other is regular on the event horizon but not invariant under simultaneous t, ϕ reversal. We should add that our conclusions are based on a *mode by mode* analysis and it is possible, though in our opinion unlikely, that subtle cancellations could rescue the Frolov-Thorne state.

A detailed numerical investigation would be necessary to elucidate further details of the properties of these states outside the event horizon. This paper has laid the foundation for such an investigation which we will present in following papers in this series.

It is possible to draw some conclusions on the basis of our analysis without resorting to a numerical investigation. For example, one can show that any state which is isotropic in a tetrad which co-rotates with the event horizon must become divergent on the velocity of light surface [16]. This implies that even if we could construct a state which is regular on the event horizon and has the desired thermal properties, then that state may well turn out not to be regular on the velocity of light surface, in agreement with the Kay-Wald theorem that the state must fail to be regular somewhere.

This paper has shown that while quantum field theory in Kerr space-time is more complex than in Schwarzschild, application of the same physical principles which have proved to be so valuable in Schwarzschild also makes the picture much clearer and more simple in Kerr.

ACKNOWLEDGMENTS

E.W. thanks Oriel College, Oxford, for financial support for this research, and the Department of Physics, University of Newcastle, Newcastle-upon-Tyne, for hospitality during the completion of this work.

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