

Naked Reissner-Nordström singularities and the anomalous magnetic moment of the electron field

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We study the problem of the quantization of the massive charged Dirac field on a naked Reissner-Nordström background. We show that the introduction of an anomalous magnetic moment for the electron field allows a well-defined quantum theory for the one-particle Dirac Hamiltonian, because no boundary condition on the singularity is required. This means that would-be higher order corrections can play an essential role in determining physics on the naked Reissner-Nordström background and that a nonperturbative approach is required. Moreover, we show that bound states for the Dirac equation are allowed. Various aspects of the physical picture emerging from our study are also discussed, such as the possibility to obtain exotic atomic systems, the formation of black holes by electronic capture, and some interesting consistency problems involving quantum gravity.

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I. INTRODUCTION

The behavior of a charged massive Dirac field on a naked Reissner-Nordström background is investigated. It is known that the Dirac Hamiltonian in the case of minimal coupling with the Coulomb classical field of the singularity, requires the choice of a boundary condition on the singularity [1,2], and so it is affected by the same problem as the free Dirac equation. This problem, which amounts, on a mathematical footing, to the fact that the Hamiltonian is not essentially self-adjoint, makes quantum physics not well-defined on the given background. Some qualitative similarities occurring with the case of the Dirac equation in flat space-time in the presence of a strongly charged pointlike nucleus are also underlined in [2].

Here we study the problem further on. In particular, in order to estimate the relevance of would-be higher order quantum electrodynamics corrections, in Sec. II we introduce an anomalous magnetic moment in the Dirac equation. Surprisingly, in the case of the electron field, the presence of an anomalous magnetic moment, e.g., of the order of the usual flat space-time one, is shown to be sufficient for ensuring the essential self-adjointness of the one-particle Hamiltonian, because no boundary condition on the singularity is required. A further analogy with the flat space-time case appears, because it is known that the introduction of an anomalous magnetic moment ensures the essential self-adjointness of the

Dirac Hamiltonian in the external field of a highly charged pointlike nucleus, for any value of the atomic number Z . There is still a difference, associated with the existence, in the naked Reissner-Nordström case, of a lower bound on the absolute value of the anomalous magnetic moment which is necessary for the essential self-adjointness of the Hamiltonian. In any case, this lower bound is fully satisfied by an anomalous magnetic moment order of the flat space-time one.

Some qualitative spectral properties are studied in Sec. III. We show that essential spectrum contributions¹ from near the singularity are excluded and that eigenvalues exist and have to belong to the mass gap. Moreover, the presence of an infinite number of eigenvalues is verified.

In Sec. IV the possibility to construct a "naked-Reissner-Nordström atom" is sketched and related consistency problems are discussed. In particular, the possibility to get Reissner-Nordström black holes by electronic capture is qualitatively analyzed. Some puzzling consistency problems are enhanced, particularly, situations are sketched in which one is forced to introduce a full quantum gravity formalism.

The final discussion, Sec. V, takes into account also the cosmic censorship conjecture (CCC).

In Appendixes A and B physical dimensions involved in the problem and a further comparison with the case of flat space-time are found. In Appendix C, some proofs of results

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[†]Email address: martellini@mi.infn.it[‡]Email address: baldicchi@mi.infn.it¹The definition of essential spectrum and the physical interest of this result are discussed in Sec. III.

presented in the main text are given; moreover, for the sake of completeness, the enunciations of some of the theorems used are found.

II. DIRAC HAMILTONIAN WITH ANOMALOUS MAGNETIC MOMENT

In this section we check if the one-particle Hamiltonian is well-defined in the sense that no boundary conditions are required in order to obtain a self-adjoint operator. In other terms, we check if the Hamiltonian is essentially self-adjoint, that is, if a unique self-adjoint extension and a uniquely determined physics occur. [In order to give a qualitative idea about the problem of defining a self-adjoint extension of an operator, we note that the one-particle Hamiltonian operator we are going to obtain by variable separation is a differential operator which represents a formal differential expression in a suitable Hilbert space; with this formal expression, according to a general theory (see, e.g., [3,4]), are associated the minimal operator and the maximal operator.² The minimal operator is to be suitably extended in order to get a self-adjoint operator. It is a basic tool for defining the self-adjoint operators which can be associated with the original formal differential expression (they are the so-called self-adjoint realizations of the formal expression and correspond to self-adjoint extensions of the minimal operator).³ Often, the Hamiltonian one writes is meant to be identified with the corresponding minimal operator. The essential self-adjointness of the minimal operator means that a unique self-adjoint operator can be associated with the original formal expression. From a physical point of view, a unique self-adjoint extension of the (minimal operator associated with the) Hamiltonian means that the physics is uniquely defined. In the following, for our aim, we can limit ourselves to consider the minimal operator associated with our reduced Hamiltonian and to study its self-adjointness properties.]

We first define the one-particle Hamiltonian for Dirac massive particles on the naked Reissner-Nordström geometry. We use natural units $\hbar = c = G = 1$ and unrationalized electric units. The metric of the naked Reissner-Nordström manifold ($t \in R; r \in (0, +\infty); \Omega \in S^2$) is

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2, \quad (1)$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2};$$

M is the mass and Q is the charge, and $Q^2 > M^2$. The vector potential associated with the Reissner-Nordström solution is

$A_\mu = (-Q/r, 0, 0, 0)$. We choose $Q > 0$. The anomalous magnetic moment contribution in the Dirac equation is proportional to $\sigma^{\mu\nu}F_{\mu\nu}$, and is the covariant generalization of the usual flat space-time term [5]. We will consider explicitly the case of the electron field (charge $-e$). One gets

$$(\gamma^\mu D_\mu + m_e + \frac{1}{2}\mu_a \sigma^{\mu\nu} F_{\mu\nu})\psi = 0, \quad (2)$$

where μ_a is the anomalous magnetic moment of the Dirac field (see Appendix A). The spherical symmetry of the problem allows us to separate the variables and to study a reduced problem on a fixed eigenvalue sector of the angular momentum operator. For a complete deduction of the variable separation see, e.g., Refs. [1,6]. We get the following reduced Hamiltonian:

$$H_{red} = \begin{bmatrix} \sqrt{f}m_e - \frac{eQ}{r} & -f\partial_r + k\frac{\sqrt{f}}{r} + \mu_a\sqrt{f}\frac{Q}{r^2} \\ f\partial_r + k\frac{\sqrt{f}}{r} + \mu_a\sqrt{f}\frac{Q}{r^2} & -\sqrt{f}m_e - \frac{eQ}{r} \end{bmatrix}$$

where $f(r)$ is the same as in Eq. (1), $k = \pm(j + \frac{1}{2}) \in \mathbb{Z} - \{0\}$ is the angular momentum eigenvalue. The Hilbert space in which H_{red} is formally defined is the Hilbert space $L^2[(0, +\infty), 1/f(r)dr]^2$ of the two-dimensional vector functions $\vec{g} \equiv \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ such that

$$\int_0^{+\infty} \frac{dr}{f(r)} (|g_1(r)|^2 + |g_2(r)|^2) < \infty.$$

As a domain for the minimal operator associated with H_{red} we can choose the following subset of $L^2[(0, +\infty), 1/f(r)dr]^2$: the set $C_0^\infty(0, +\infty)^2$ of the two-dimensional vector functions \vec{g} whose components are smooth and of compact support [3]. It is useful to define a new variable x as in [2]

$$\frac{dx}{dr} = \frac{1}{f(r)}$$

$$x = r + M \log\left(\frac{r^2 - 2Mr + Q^2}{Q^2}\right) + (2M^2 - Q^2) \frac{1}{\sqrt{Q^2 - M^2}} \arctan\left(\frac{r - M}{\sqrt{Q^2 - M^2}}\right) + C$$

and to choose the arbitrary integration constant C in such a way that $x \in (0, +\infty)$. The reduced Hamiltonian becomes

$$H_{red} = D_0 + V(x) \quad (3)$$

where

$$D_0 = \begin{bmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{bmatrix}$$

and

²The maximal operator is defined on the largest possible domain in the Hilbert space which is mapped into the Hilbert space itself. The minimal operator is defined as the restriction of the maximal one, such that the adjoint of the minimal operator is equal to the maximal operator [3]. See also Ref. [4].

³They are formally obtained as self-adjoint restrictions of the maximal operator.

$$V[r(x)] = \begin{bmatrix} \sqrt{f}m_e - \frac{eQ}{r} & +k\frac{\sqrt{f}}{r} + \mu_a\sqrt{f}\frac{Q}{r^2} \\ +k\frac{\sqrt{f}}{r} + \mu_a\sqrt{f}\frac{Q}{r^2} & -\sqrt{f}m_e - \frac{eQ}{r} \end{bmatrix}.$$

The Hilbert space of interest for the Hamiltonian (3) is $L^2[(0, +\infty), dx]^2$. We have to check if the reduced Hamiltonian is essentially self-adjoint; with this aim, we check if the solutions of the equation

$$H_{red}g = \lambda g \quad (4)$$

are square integrable in a right neighborhood of $x=0$ and in a left neighborhood of $x=+\infty$. The so-called Weyl alternative generalized to a system of first order ordinary differential equations (Ref. [3], theorem 5.6) states that, if the integrability condition in a right neighborhood of $x=0$ is verified for all the solutions corresponding to a fixed value of $\lambda \in \mathbb{C}$, then it is verified for every $\lambda \in \mathbb{C}$ and the so-called limit circle case (LCC) is said to occur. This occurrence of LCC implies the necessity to introduce boundary conditions in order to obtain a self-adjoint operator. If at least one solution not square integrable exists for every $\lambda \in \mathbb{C}$, then no boundary condition is required and the limit point case (LPC) is said to be verified. The same reasoning is to be applied for $x=+\infty$. The Hamiltonian operator is essentially self-adjoint if the LPC is verified both at $x=0$ and at infinity (cf., Ref. [3], theorem 5.7). It is known [1,2] that, if $\mu_a = 0$, the reduced Hamiltonian is not essentially self-adjoint on the set $C_0^\infty(0, +\infty)^2$. In fact, the limit circle case (LCC) at $x=0$ occurs, whereas the limit point case (LPC) is verified at infinity.

We show that the introduction of the anomalous magnetic moment allows us to get the LPC also at $x=0$ for suitable values of μ_a (the LPC is trivially verified at infinity). In our case one gets the following system of first order equations in the variable r :

$$\begin{aligned} \partial_r g_1 + \left(\frac{k}{\sqrt{f}r} + \mu_a \frac{1}{\sqrt{f}} \frac{Q}{r^2} \right) g_1 + \left[-\frac{m_e}{\sqrt{f}} + \frac{1}{f} \left(-\frac{eQ}{r} - \lambda \right) \right] g_2 \\ = 0 \\ -\partial_r g_2 + \left(\frac{k}{\sqrt{f}r} + \mu_a \frac{1}{\sqrt{f}} \frac{Q}{r^2} \right) g_2 + \left[\frac{m_e}{\sqrt{f}} + \frac{1}{f} \left(-\frac{eQ}{r} - \lambda \right) \right] g_1 \\ = 0. \end{aligned}$$

For $r \rightarrow 0 \Leftrightarrow x \rightarrow 0$ we get the following asymptotic expansion for the eigenvalue equation (4):

$$\begin{aligned} \partial_r g_1 + \frac{\mu_a}{r} g_1 + \left(\frac{k}{Q} + \mu_a \frac{M}{Q^2} \right) g_1 = O(r) \\ \partial_r g_2 - \frac{\mu_a}{r} g_2 - \left(\frac{k}{Q} + \mu_a \frac{M}{Q^2} \right) g_2 = O(r). \end{aligned}$$

The anomalous magnetic moment contribution is such that the coefficients of the asymptotic expansion are no more regular near $r=0$. The solutions in a right neighborhood of $r=0$ behave as

$$\begin{aligned} g_1(r) &\sim a_1 r^{-\mu_a}, \\ g_2(r) &\sim a_2 r^{+\mu_a}. \end{aligned} \quad (5)$$

Solutions of Eq. (4) belong to $L^2[(0, R), 1/f(r)dr]^2$ for $R > 0$ if

$$\int_0^R dr \frac{1}{f(r)} (|g_1(r)|^2 + |g_2(r)|^2) < \infty \quad (6)$$

which means in our case

$$\int_0^R dr r^{2 \pm 2\mu_a} < \infty. \quad (7)$$

The above condition implies $|\mu_a| < \frac{3}{2}$. For an explicit evaluation it is necessary to resort all the physical dimensions; see Appendix A herein. If the anomalous magnetic moment value is assumed to be the same as in flat space-time, for $|\mu_a|$ in Eq. (7) one gets $0.00058 \cdot e^*/m_e^* \sim 10^{18}$, where e^* and m_e^* are the lengths associated with the electron charge e and the electron mass m_e , respectively; then the LPC holds and the reduced Hamiltonian is essentially self-adjoint. It can be noted that the essential self-adjointness property of the reduced Hamiltonian does not depend on the charge Q and on the mass M of the singularity.

There is a preliminary problem. A perturbative evaluation of the anomalous magnetic moment is not available, the free and also the minimally coupled Dirac equation on the naked Reissner-Nordström background being not well-defined. Nevertheless, it is legitimate to consider the anomalous magnetic moment as a parameter modeled on the standard QED theory. If it is not at least eighteen magnitude orders smaller than the standard flat space-time anomalous magnetic moment, then the interacting theory becomes well-defined. The possibility to have a uniquely defined physics *only* for the interacting theory is nontrivial and, to some extent, unexpected. We interpret the fact that a well-defined physics for the free theory is not available, but the interacting theory can avoid this pathologic behavior, as the breakdown of the ‘‘perturbative approach’’ to the physics. In other words, would-be higher order dynamical effects in perturbation theory actually play a fundamental role in determining physics; they determine indeed a unique self-adjoint extension of the Hamiltonian.

According to our proposal, the problem of uniqueness of the quantum evolution on a naked Reissner-Nordström manifold becomes a nontrivial dynamical problem in a nonperturbative domain. A nonperturbative approach is still problematic, because one should also verify that no physically relevant term has been neglected in the truncation of the effective action which gives rise to Eq. (2). Nevertheless, no matter how limited our exploration of such a domain may be, our result opens up a new interesting level in the discussion

of physics on nonglobally hyperbolic manifolds. A further discussion is found in the conclusions.

It is remarkable that, to some extent, there can be found an analogy with the standard flat space-time Dirac equation in the Coulomb external field of a highly charged pointlike nucleus. In fact, it is known that in flat space-time the anomalous magnetic moment solves self-adjointness problems even in the case of a heavily charged pointlike nucleus [7]. In flat space-time the free Dirac Hamiltonian is, of course, essentially self-adjoint and the Dirac Hamiltonian in an external Coulomb field and is essentially self-adjoint too as far as $Z \leq 118$. For bigger Z , the essential self-adjointness can be restored by introducing the anomalous magnetic moment [7].⁴ In the naked Reissner-Nordström case, instead, neither the free Dirac Hamiltonian nor the one which is minimally coupled with the external Coulomb field of the singularity is essentially self-adjoint.

Moreover, in the case of the Dirac equation in the field of a charged pointlike nucleus it is evident that, as far as the effective coupling of a pointlike particle $Z\alpha_e = Z/137$ approaches one, the perturbative approach loses its validity and a nonperturbative approach is necessary.

In the following, some physical properties of our one-particle Hamiltonian are discussed.

III. SPECTRAL PROPERTIES

We now study some qualitative spectral properties of the reduced Hamiltonian (3). It will be found that the essential spectrum $\sigma_e(H_{red})$ (defined below) coincides with the complement of the interval $(-m_e, m_e)$, and that an infinite number of eigenvalues is confined in the mass gap.

A. Essential spectrum

The essential spectrum $\sigma_e(B)$ of a self-adjoint operator B consists of all points of the spectrum except for isolated eigenvalues of finite multiplicity. It corresponds to the union of the continuous spectrum, of the eigenvalues embedded in the continuous spectrum, or at the edges of the continuous spectrum, of the limit points for the eigenvalues, and of the eigenvalues having infinite multiplicity [8,9] (the latter case cannot occur for ordinary differential operators [3]). The physical interest is associated with the possibility to find, by means of qualitative spectral methods, a set which is the complement in the spectrum of the set composed by isolated eigenvalues (“bound states”). In fact, for any self-adjoint operator B the spectrum can be decomposed into the union of two disjoint sets: $\sigma(B) = \sigma_e(B) \cup \sigma_d(B)$, where $\sigma_d(B)$ is the discrete spectrum, i.e., the set containing all the isolated eigenvalues of finite multiplicity.

Let us consider the operators H_0 and H_∞ which are defined as the restrictions of H_{red} to the intervals $(0, c]$ and $[c, \infty)$, where $c > 0$ is arbitrary. By using the so-called de-

composition method (Ref. [3], p. 165), the essential spectrum of our Hamiltonian operator can be decomposed into the union of the essential spectra of the operators H_0 and H_∞ , in the sense that $\sigma_e(H_{red}) = \sigma_e(H_0) \cup \sigma_e(H_\infty)$ (see also Ref. [2]). The restriction H_∞ of H_{red} gives the same essential spectrum contribution as the one calculated in [2]:

$$\sigma_e(H_\infty) = (-\infty, -m_e] \cup [m_e, +\infty), \quad (8)$$

as it can be easily verified by using theorems 16.5 and 16.6 of Ref. [3] (see Appendix C both for the enunciations and for their application to our case and cf., Ref. [10] for an application to Kerr-Newman black holes). The case of H_0 , which is the restriction to the right neighborhood of the singularity, is a little more involved than in Ref. [2], because the LPC at the singularity is verified. Nevertheless, a careful application of theorem 2 appearing in Ref. [11] allows us to obtain the following result: The essential spectrum of the Dirac Hamiltonian restricted to a right neighborhood of the singularity $r=0$ is empty. We first discuss the physical meaning of this result; then we give some more detail. The absence of an essential spectrum contribution coming from near $r=0$ can be interpreted by means of an analogy with standard scattering centers. In fact, avoiding essential spectrum contribution from near the center amounts to verifying that the one-particle scattering problem is well-defined, in the sense that particles are not “captured” for long periods of time near the centers and the scattering matrix is unitary. In our case we can analogously say that Dirac particles do not spend an infinite amount of time near the singularity when scattering takes place. See also Ref. [12] for the case of other timelike singularities.

Giving all the details about the cited theorem would require a long digression. We limit ourselves to underline that, according to the aforementioned theorem, if the LPC is verified at $r=0$, in order that in $(0, R]$ there can be only a discrete spectrum contribution it is sufficient to verify that for an arbitrary $R > 0$ it holds

$$\int_0^R dr \frac{1}{f} \left| k \frac{\sqrt{f}}{r} + \mu_a \sqrt{f} \frac{Q}{r^2} \right| = \infty. \quad (9)$$

In our case the above integral diverges because the anomalous magnetic moment gives rise to a term which is not integrable in a right neighborhood of $r=0$. We refer the interested reader to Ref. [11] for more details. Actually, a more naive argument can also be used. In the case of a Schrödinger-like second order operator τ in $(0, R]$, if the LPC is verified in $r=0$, the absence of continuous spectrum for real $\lambda \in (\lambda_1, \lambda_2)$ is obtained if the asymptotic behavior of the solutions of the differential equation $(\tau - \lambda)f = 0$ near the origin is such that one is always square-integrable. In the case of the separated Dirac operator H_{red} the analogous argument is found in Ref. [3] (theorem 11.7), and in our case there is always a square-integrable solution of $(H_{red} - \lambda)g = 0$ for each $\lambda \in \mathbb{R}$, as Eq. (5) shows.

Finally, note that, as for the analogous equation in flat space-time, the interval $(-m_e, m_e)$ represents a gap in the Hamiltonian spectrum between the continuum positive en-

⁴For $119 \leq Z < 137$ a privileged self-adjoint extension can be selected on physical grounds, so that the nontrivial part of the problem from a physical point of view arises for $Z \geq 137$.

ergy states and the negative energy ones, and the discrete spectrum (isolated eigenvalues) can be located only in $(-m_e, m_e)$.

B. Discrete spectrum

Here we are interested in the discrete spectrum of the one-particle Hamiltonian. In the gap $(-m_e, m_e)$ there is an infinite number of discrete eigenvalues. The interested reader is referred to Appendix C for a proof, which is based on theorems given in Ref. [13]. The presence of an infinite number of eigenvalues can be considered as a nontrivial result (note that the proof contained in Appendix C for the existence of an infinite number of eigenvalues holds also when there is no anomalous magnetic moment for the electron field). In fact, in the case of Reissner-Nordström black holes, no isolated eigenvalue is allowed, as it is shown in Refs. [1,2] and in Ref. [10] (in Ref. [1] a stronger result is given: no eigenvalue exists, no matter if isolated or embedded in the continuous spectrum). In fact, the presence of the black hole horizon does not allow a gap in the essential spectrum of the one-particle Dirac Hamiltonian operator [10]. Then, also from this point of view, naked singularities differ with respect to black holes.

C. Purely absolutely continuous spectrum

We are interested in determining if there are eigenvalues embedded into the continuous spectrum. Naively, it could be expected that eigenvalues are allowed to dive into the continuum as Z increases. A careful application of theorem 16.7 of Ref. [3] (theorem 16.7 of Ref. [3] and its application to our case are found herein in Appendix C) shows that the complement of the closed interval $(-m_e, m_e)$ belongs to the purely absolutely continuous spectrum. This means that the states with energy in $(-\infty, -m_e) \cup (m_e, +\infty)$ are scattering states with no eigenvalue embedded. The physical consequences of this result are very interesting. The eigenvalues have to be confined in the mass gap. So, contrary to the naive expectation, by increasing Z (Z finite), the bound-state energy cannot increase arbitrarily. The repulsive nature of the anomalous magnetic moment term should be the reason for such a behavior.

IV. A NAKED-REISSNER-NORDSTRÖM ATOM?

The existence of stationary states⁵ we have shown in the previous section allows us to speculate naively about the possibility to dress a naked Reissner-Nordström singularity by means of a cloud of electrons, and to obtain, as a consequence, a quantum-mechanical object (atomic system). In fact, one *a priori* can fill the bound state energy levels by means of electrons and, by pursuing this dressing process, the charged singularity can also be neutralized. Moreover, one can introduce a sort of “quantum radius” of the singu-

larity, a length scale which appears only at the quantum level and corresponds to the Bohr radius for standard atoms.⁶

In the following, we limit ourselves to a qualitative analysis of the “dressing” of a naked singularity. Quantitative evaluations imply very subtle numerical computations, because of the nontrivial form of Dirac equation (2) in our case.

A qualitative picture involves substantially two cases. In the case of a complete dressing of the singularity the space-time metric for a distant observer outside the outermost electronic shell (characterized by a radius we will call o-radius) is the Schwarzschild one, at least as far as multi-pole electromagnetic field contributions associated with the electronic shells can be neglected. The “dressed singularity” is characterized by a mass order of the original naked singularity one (if the total mass of the surrounding electrons is negligible; see the discussion below). Naively, to this neutral system an effective Schwarzschild radius (s-radius in the following) can also be assigned. If the dressing is only partial, then the external metric becomes a Reissner-Nordström one but with a reduced charge-to-mass ratio with respect to the original naked solution. For an exotic atomic system whose “nucleus” is represented by a naked Reissner-Nordström singularity and whose orbitals are filled with standard electrons, an electromagnetic spectrum associated with allowed transitions between atomic levels is also expected.

We will also verify that a too naive marriage between general relativity (Reissner-Nordström singularity playing the role of “nucleus”) and quantum mechanical orbits (electron states surrounding the singularity) is not free from ambiguities and possible inconsistencies.

We start by making some estimates. With this aim, we restore the physical dimensions and write the charge-to-mass ratio as

$$\frac{Q^*}{M^*} = \frac{\sqrt{\alpha_e} \frac{Q}{e}}{\frac{M}{m_{\text{pl}}}} = \sqrt{\alpha_e} Z \frac{m_{\text{pl}}}{M} \approx 9.35 \times 10^{-40} Z \frac{M_s}{M}, \quad (10)$$

where Q^* and M^* are the lengths associated with Q and M respectively (see also Appendix A); M_s is the Sun mass. A naked Reissner-Nordström singularity is characterized by $Q^*/M^* \equiv 1 + d^2 > 1$, that is

$$Z = 1.07 \times 10^{39} \frac{M}{M_s} (1 + d^2). \quad (11)$$

The parameter $d > 0$ points out “how naked” the singularity is, i.e., how much bigger than one the charge-to-mass ratio is. The mass of the singularity being equal, the amount of electrons neutralizing the naked singularity is lowest when $d^2 \ll 1$. Below we make some estimates for Z in the case of small d :

$$\begin{aligned} M = M_s &\Rightarrow Z \sim 10^{39}, \\ M = 10^{-16} M_s &\Rightarrow Z \sim 10^{23}, \\ M = m_{\text{pl}} &\Rightarrow Z \sim 12. \end{aligned}$$

⁵For the case of absence of anomalous magnetic moment, see Ref. [1].

⁶The authors are indebted to A. Treves for this suggestion.

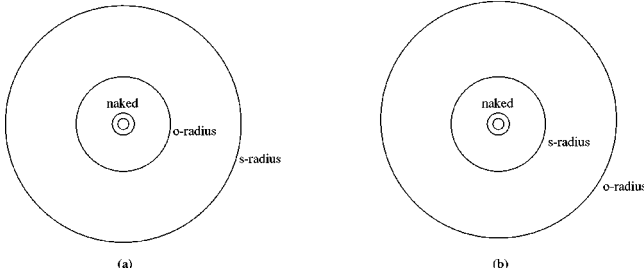


FIG. 1. Possible dressings of a naked singularity neutralized by electron capture (naive classical picture). (a) An effective black hole solution is generated, because the electronic shells are within the s -radius. (b) An electronic cloud available for external observers is displayed.

Then, in order to neutralize a naked Reissner-Nordström singularity with a mass order of the Sun mass and with a charge-to-mass ratio only slightly bigger than one, at least order of 10^{39} electrons would be required. We see also that a value of Z order of the standard atomic values is possible only if the mass of the singularity is order of the Planck mass. For small d , it is consistent to neglect the electron contribution to the total mass of the exotic atomic system. In fact, from Eq. (11) one deduces that there are about 21 orders of magnitude between the mass M and the total electron mass contribution ($M_s \sim 10^{60} m_e$), and this means that electron contribution to the mass starts being non-negligible only if $d^2 \sim 10^{20}$. It is then straightforward to estimate the s -radius of the neutralized system by means of the mass M :

$$\begin{aligned} M = M_s &\Rightarrow r_s \sim 3 \text{ km}, \\ M = 10^{-16} M_s &\Rightarrow r_s \sim 300 \text{ fm}, \\ M = m_{\text{pl}} &\Rightarrow r_s = 2l_{\text{pl}}. \end{aligned}$$

A. Dressing and black hole formation problem

Herein we check if a Reissner-Nordström naked singularity could become a Reissner-Nordström black hole by means of the capture of $N < Z$ electrons; particularly, the radius of the electronic shells is compared with the Reissner-Nordström black hole radius r_+ associated with the dressed solution. (See Fig. 1.) A discussion of related consistency

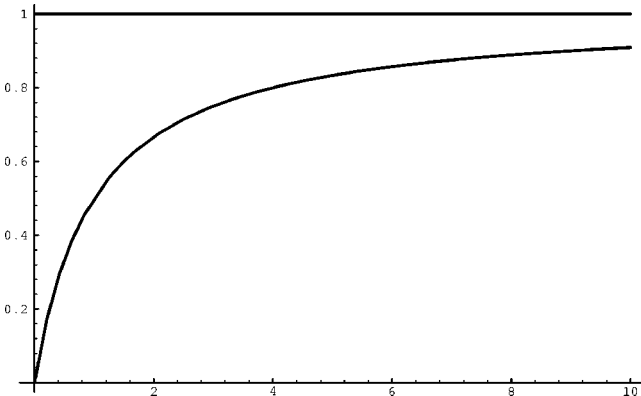


FIG. 2. A plot of the ratio N/Z as a function of d^2 is shown. For $d^2 = 9$ the ratio is already order of 0.9.

problems follows.

In general, we write $M \equiv y m_{\text{pl}}$, where $y \in (0, +\infty)$ is a real positive number. Then

$$\frac{Q^*}{M^*} = \frac{Z e^*}{y l_{\text{pl}}} = 1 + d^2 > 1; \quad (12)$$

the second equality above fixes the value of y as follows:

$$y = \frac{Z e^*}{(1 + d^2) l_{\text{pl}}}. \quad (13)$$

When N electrons are captured, from the point of view of an observer which is far from the outermost electronic shell, the effective charge is $Q_{\text{eff}} = (Z - N)e$, and the effective mass is $M_{\text{eff}} = y m_{\text{pl}} + N m_e$, so that

$$R \equiv \frac{Q_{\text{eff}}^*}{M_{\text{eff}}^*} = \frac{(Z - N)(1 + d^2)}{Z + N(1 + d^2)} \frac{m_e^*}{e^*}. \quad (14)$$

A necessary condition in order to get a horizon is $R \leq 1$, which can be obtained for

$$N \geq Z \frac{d^2}{(1 + d^2) \left(1 + \frac{m_e^*}{e^*} \right)}. \quad (15)$$

(see Fig. 2 for a plot of N/Z). Correspondingly, the black hole radius would be

$$r_+ = M_{\text{eff}}^* (1 + \sqrt{1 - R^2}). \quad (16)$$

We choose again to work in the limit of $d \ll 1$, and, in particular, as a sample estimate, we impose the condition $Z d^2 = 1$. Then one finds that $N = 1$ is enough to obtain $R < 1$; moreover, one finds $y \sim 8.54 \times 10^{-2} Z$, $r_+ \sim y l_{\text{pl}}$.

Then we consider two cases (a) and (b) which appear meaningful.

Case (a). For $Z = 100$ the radius is $r_+ \sim 8.5 l_{\text{pl}}$ (the mass is $M = 8.5 m_{\text{pl}}$) and it is plausible that r_+ is smaller than the Bohr radius.

Case (b). For $Z = 10^{24}$, one gets $M \sim 10^{15}$ Kg and $r_+ \sim 10^3$ fm, and an almost “atomic” scale appears to be available, to be compared with a huge value of the atomic number which would make plausible that the Bohr radius is smaller than the estimated r_+ .⁷

At first sight, the second example can allow a picture of transformation of the naked singularity into a black hole by means of the capture of a single electron, but this conclusion

⁷It is also plausible that it is not necessary to approach $Z = 10^{24}$ in order to get $r_+ > r_{\text{Bohr}}$.

is puzzling. A single electron in case (b) could be enough to induce the appearance of a black hole horizon, in spite of the fact that its backreaction is negligible (one has $m_e \ll M$, $e \ll Q$, which should allow for a safe external field approximation in the Dirac equation). Moreover, in case (a), where the backreaction effect of one electron is more significant, the electronic capture is not able to transform the naked singularity into a black hole. A qualitative reason for this paradoxical behavior could be that in case (b) the naked solution is much closer to the extremal limit $Q^*/M^* = 1$ than in case (a) ($d^2 = 10^{-24}$ against $d^2 = 0.01$), so that it should be affected by a much bigger instability with respect to electronic capture. Even assuming the plausibility of such a picture, the mechanism of the generation of the black hole remains unclear. However, note that, at the classical level, the transformation of naked Reissner-Nordström singularities into Reissner-Nordström black holes by means of bombardment with charged test particles is allowed in Ref. [14]. A further remark is that, after the generation of a black hole, the dressing mechanism by means of electronic orbits would stop, because no discrete eigenvalue is allowed for a Reissner-Nordström black hole (cf., Ref. [10]; an anomalous magnetic moment contribution does not affect the absence of discrete spectrum for the electron field on a Reissner-Nordström black hole manifold).

For a quantum object like ours, the notion of “orbit” is probabilistic and a comparison of the expectation value of r (Bohr radius) with the classical black hole radius runs the risk of being too naive. In fact, qualitatively, the electron field is distributed with radial probability density $P_e(r)$ around the “naked nucleus.” As a consequence, even in the case that $r_+ < r_{\text{Bohr}}$ there can be a significant nonzero probability that the electron is within the black hole radius r_+ . This implies that there can be a significant nonzero probability \mathcal{P} that the solution is a black hole:

$$\mathcal{P}(\text{black hole}) = \mathcal{P}(\text{electron between } 0 \text{ and } r_+) \in (0,1). \quad (17)$$

In other words, the metric seems to be necessarily associated with a probability \mathcal{P} to be a black hole and $1 - \mathcal{P}$ to be a naked singularity surrounded by an electron. Then, serious self-consistency problems can arise if the parameters of the effective dressed solution correspond to a black hole solution. When $\mathcal{P}(\text{black hole})$ is significantly different from 0 (or 1), the above picture turns out to associate with the metric a probabilistic interpretation, and a consistent treatment of the problem requires a quantum gravity approach.

B. Further consistency considerations

Concerning the radius of the innermost electronic orbits, we make some qualitative considerations which involve the actual availability of the external field approximation for the gravitational background. For high Z the Coulomb field interaction could give rise to extremely small innermost orbits, and for, say, $Z \geq Z_0$ one could find an orbit radius smaller than the Planck length, in evident conflict with the bound on the minimal length l_{pl} imposed by quantum gravity. In order to be more explicit, let us assume, on a purely heuristic foot-

ing, that the innermost electronic radius scales as $1/Z$ and satisfies the same law as the Bohr radius of a hydrogenlike atom:⁸ $r_{\text{Bohr}} = (0.529/Z) \times 10^{-10}$ m. Then, for $Z > 10^{25}$ the problem we are discussing surely takes place, because $r_{\text{Bohr}} < l_{\text{pl}}$ (for $Z \sim 10^{39}$ one finds, e.g., $r_{\text{Bohr}} \sim 10^{-50}$ m). If the full problem (i.e., naked singularity geometry and anomalous magnetic moment contribution) displays an analogous behavior at least for $Z \geq Z_0$, then consistency problems of the semiclassical approach for $Z \geq Z_0$ arise. As a consequence, overcoming the problem for boundary conditions on the singularity could be insufficient to ensure a full self-consistency of physics at least under suitable conditions (e.g., for $Z \geq Z_0$), due to a possible breakdown of the external field approximation for the gravitational part of the path integral. See on this topic also the discussion in Ref. [15].

On the other hand, if one introduces *ab initio* a box with radius $\sim l_{\text{pl}}$ around the singularity [15], the problem of imposing a boundary condition near the origin becomes again unavoidable even in presence of an anomalous magnetic moment. A more naive approach consists in assuming that the problem is well-posed only when the would-be Bohr radius starts being bigger than the Planck length, that is only for $Z \leq Z_0$.

Solving the problem of constructing explicitly the exotic naked-Reissner-Nordström atom is beyond the aim of our work. We limit ourselves to note that our naive picture of “dressing” looks like the one in Ref. [15] but there are fundamental differences. The charged particles which dress the singularity are not related to the Klein paradox and are not in principle due to vacuum instability, whose presence on the given background cannot be revealed by means of a static approach (see also Ref. [2]). In our picture the electrons are captured from the space region around the singularity. Moreover, in our work, no boundary condition on the singularity is required for the quantized field and a discussion about a possible black hole formation appears.

V. CONCLUSIONS

We have shown that at least in the case of the Dirac field, a uniquely defined physics can be retrieved on a naked Reissner-Nordström background in four dimensions, by means of the introduction of an anomalous magnetic moment which can also be much smaller than in flat space-time. A substantial breakdown of the perturbative approach to physics is the suggestion we propose for interpreting our result. It is remarkable, as a consequence, that the problem of a well-posed physics on a naked Reissner-Nordström background can involve nontrivially would-be higher order terms. This is verified for the charged massive Dirac field, and it would be interesting to investigate if any higher order corrections could restore the essential self-adjointness also in the case of other fundamental fields (e.g., the electromagnetic field or

⁸For high Z , of course, a relativistic approach is necessary for a hydrogenlike atom and the nonrelativistic formula loses its meaning. Herein, the formula is used well beyond its validity range, but in the frame of a purely heuristic reasoning.

the uncharged Dirac particles like the neutrino).

We can also discuss the relation of our result with the CCC. The CCC was formulated with the aim to avoid the indefiniteness of physics on nonglobally hyperbolic manifolds associated with naked curvature singularities. Studies involving quantum fields, on the other hand, have shown that a well-behaved physics can be recovered for free quantum fields on the manifold of a class of naked singularities [12]. This allows us to relax the need for the CCC for the aforementioned class. We have shown that there is a possibility to relax this need even in the case of a Dirac field on a naked Reissner-Nordström manifold. We have yet to underline that our test in a nonperturbative domain is interesting but not definitive, just because of the substantial lack of a criterion allowing us to justify approximations for the effective action calculation in a nonperturbative domain, and because of the lack of tools allowing us to treat a full quantum calculation for all the fields (which would avoid problems with the external field approximation). Nevertheless, our analysis shows that a further level of discussion has to be introduced.

In the second part of our work, we have analyzed some aspects of the physics associated with our Hamiltonian. A spectral analysis of the reduced Hamiltonian has been performed and it has been verified that an infinite set of eigenvalues is present, contrary to what happens in the case of black hole Reissner-Nordström solutions. Then the dressing of the singularity by means of the formation of an ‘‘exotic’’ atomic system and related problems have been discussed.

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APPENDIX A: DIMENSIONS

We here resort all physical dimensions. The function $f(r) = 1 - 2M^*/r + (Q^*)^2/r^2$ is characterized by the lengths which are associated with the mass M and the charge Q of the Reissner-Nordström solution, respectively:

$$M^* \equiv \frac{G}{c^2} M = l_{\text{pl}} \frac{M}{m_{\text{pl}}},$$

$$Q^* \equiv \sqrt{\frac{G}{c^4}} Q = \sqrt{\frac{l_{\text{pl}}}{m_{\text{pl}} c^2}} Q = l_{\text{pl}} \sqrt{\alpha_e} \frac{Q}{e}.$$

By posing $Q = Z \cdot e$ one gets $Q^* = l_{\text{pl}} \sqrt{\alpha_e} Z$ and

$$\frac{Q^*}{M^*} = \frac{\sqrt{\alpha_e} \frac{Q}{e}}{\frac{M}{m_{\text{pl}}}} = \sqrt{\alpha_e} Z \frac{m_{\text{pl}}}{M}. \quad (\text{A1})$$

It is useful to recall that in the case of the electron one has

$$e^* = 8.54 \times 10^{-2} l_{\text{pl}},$$

$$m_e^* = 4.18 \times 10^{-23} l_{\text{pl}},$$

$$\alpha_e = \left(\frac{e^*}{l_{\text{pl}}} \right)^2.$$

The anomalous magnetic moment of the electron is given by

$$\mu_a \equiv -a \mu_{\text{Bohr}} \quad (\text{A2})$$

where a is a dimensionless constant⁹ and μ_{Bohr} is the standard Bohr magneton:

$$\mu_{\text{Bohr}} = \frac{e \hbar}{2 m_e c}. \quad (\text{A3})$$

We also write the reduced Hamiltonian as follows

$$H_{\text{red}} = \begin{bmatrix} A & C_- \\ C_+ & B \end{bmatrix},$$

where the physical dimensions in each entry are resorted:

$$A = + \sqrt{f} (m_e c^2) - Z (\alpha_e \hbar c) \frac{1}{r},$$

$$B = - \sqrt{f} (m_e c^2) - Z (\alpha_e \hbar c) \frac{1}{r},$$

$$C_+ = + (\hbar c) f \partial_r + k (\hbar c) \sqrt{f} \frac{1}{r} - a \frac{(\hbar c)^2}{2 m_e c^2} (Z \alpha_e) \sqrt{f} \frac{1}{r^2},$$

$$C_- = - (\hbar c) f \partial_r + k (\hbar c) \sqrt{f} \frac{1}{r} - a \frac{(\hbar c)^2}{2 m_e c^2} (Z \alpha_e) \sqrt{f} \frac{1}{r^2}.$$

The asymptotic expansion of the eigenvalue equation for $r \rightarrow 0$ is [each term is divided by $(\hbar c)$ so that it has dimensions of the inverse of a length]

$$\partial_r g_1 + \frac{\mu_a Q}{Q^* \hbar c} \frac{1}{r} g_1 = O(1),$$

$$\partial_r g_2 - \frac{\mu_a Q}{Q^* \hbar c} \frac{1}{r} g_2 = O(1).$$

We are interested in the dimensionless ratio $|\mu_a| Q / Q^* \hbar c$ which corresponds to the absolute value of the μ_a appearing in Eq. (7)

$$\frac{|\mu_a| Q}{Q^* \hbar c} = \frac{a \mu_{\text{Bohr}} e}{e^* \hbar c}, \quad (\text{A4})$$

⁹The first perturbative order in QED in flat space-time gives $a = \alpha_e / (2\pi)$.

then

$$\begin{aligned} \frac{ae^2}{e^*2m_e c^2} &= \frac{a\hbar\alpha_e}{e^*2m_e c} = \frac{al_{\text{pl}}m_{\text{pl}}\alpha_e}{e^*2m_e} = \frac{am_{\text{pl}}e^{*2}}{e^*2m_e l_{\text{pl}}} \\ &= \frac{a}{2} \frac{e^*}{m_e^*} = 1.18 \times 10^{18}. \end{aligned}$$

So one gets that $ae^*/m_e^* \gg 1$ if the value of a is not much smaller than the flat space-time one.

We list below the values of some factors appearing in our equation (for the anomalous magnetic moment the flat space-time value is assumed):

$$\begin{aligned} m_e c^2 &= 0.510999 \text{ MeV}, \\ \alpha_e &= 1/137.035989, \\ \hbar c &= 197.327053 \text{ MeV fm}, \\ \frac{(\hbar c)^2}{2m_e c^2} \alpha_e &= 278.02803 \text{ MeV(fm)}^2, \\ Q^* &= 1.38050219 \times 10^{-21} \text{ Z fm}, \\ a &= 0.001159. \end{aligned}$$

APPENDIX B: COMPARISON WITH FLAT SPACE-TIME

The asymptotic expansions of the potential $V[r(x)]$ as $x \rightarrow 0$ and $x \rightarrow +\infty$ are useful for a comparison with the Dirac equation in flat space-time $f=1$. We note that

$$x = \frac{r^3}{3Q^2} + O(r^4) \quad \text{for } r \rightarrow 0$$

and

$$x = r + 2M \log(r) + O(1) \quad \text{for } r \rightarrow +\infty$$

in such a way that $r \sim (3Q^2)^{1/3} x^{1/3}$ and $r \sim x$ respectively. Near the singularity one gets (only the leading order of each entry is displayed)

$$V[r(x)] \sim \begin{bmatrix} (m_e - e) \left(\frac{Q}{3}\right)^{1/3} x^{-1/3} & \frac{\mu_a}{3} x^{-1} \\ \frac{\mu_a}{3} x^{-1} & (-m_e - e) \left(\frac{Q}{3}\right)^{1/3} x^{-1/3} \end{bmatrix},$$

and near infinity

$$V[r(x)] \sim \begin{bmatrix} m_e + (-m_e M - eQ)x^{-1} & kx^{-1} \\ kx^{-1} & -m_e + (m_e M - eQ)x^{-1} \end{bmatrix}.$$

The difference with respect to the flat space-time Hamiltonian is mostly evident near the origin, but also near infinity there are subleading corrections to the behavior of the Dirac Hamiltonian in flat space-time.

APPENDIX C: THEOREMS ON DIRAC SYSTEMS AND SOME PROOFS

We list below the enunciated of some theorems we refer to in our paper. A Dirac operator of the form

$$H = \begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix} + P(x)$$

defined on $I=(a,b)$ will be considered; the potential

$$P(x) \equiv \begin{bmatrix} p_1(x) & p_{12}(x) \\ p_{12}(x) & p_2(x) \end{bmatrix}$$

is real symmetric, $|P(x)|$ is locally integrable, $p_1(x)$, $p_2(x)$, and $p_{12}(x)$ are real functions locally integrable [16]. $|\cdot|$ stays for a norm in $\mathbb{C}^{2 \times 2}$ (e.g., the Euclidean norm for matrices; see below).

In order to work with an operator having the form required by theorems appearing in Ref. [3], we introduce the unitary matrix

$$T \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and an operator $H_* \equiv TH_{red}T^\dagger$ which is unitarily equivalent to H_{red} (so it has the same spectrum and the same spectral properties as H_{red}) and matches the required form. In particular, we have

$$H_* = \begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix} + P(x)$$

where

$$P(x) \equiv TV[r(x)]T^\dagger$$

$$= \begin{bmatrix} -\sqrt{f}m_e - \frac{eQ}{r} & +k\frac{\sqrt{f}}{r} + \mu_a\sqrt{f}\frac{Q}{r^2} \\ +k\frac{\sqrt{f}}{r} + \mu_a\sqrt{f}\frac{Q}{r^2} & \sqrt{f}m_e - \frac{eQ}{r} \end{bmatrix}.$$

Theorem 16.5 of Ref. [3]: Assume that H is regular at a^{10} and that $b = +\infty$. If $P(x) \rightarrow P_0$ for $x \rightarrow +\infty$ and $\mu_- \leq \mu_+$ are the eigenvalues of P_0 , then for every self-adjoint extension H_1 of H it holds $\sigma_e(H_1) \cap (\mu_-, \mu_+) = \emptyset$.

This theorem is applied to $TH_\infty T^\dagger$. In our case, $a=c$ and the above operator is regular at c , moreover, it is easy to

¹⁰“Regular at a ” (where a is finite) means that the assumptions on the coefficients of the differential expression H are satisfied in $[a,b)$ instead of in (a,b) [3].

show, by taking the limit $\lim_{x \rightarrow +\infty} P(x)$, that $\mu_- = -m_e$ and $\mu_+ = m_e$. As a consequence of the above theorem, $\sigma_e(H_\infty) \cap (-m_e, m_e) = \emptyset$.

Theorem 16.6 of Ref. [3]: Assume that $b = +\infty$ and let $\mu_- \leq \mu_+$ be the eigenvalues of P_0 defined as above. If for some $d \in (a, +\infty)$

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \int_d^x dt |P(t) - P_0| = 0$$

then for every self-adjoint extension H_1 of H it holds $\sigma_e(H_1) \supset$ complement of (μ_-, μ_+) .

This theorem is again applied to $TH_\infty T^\dagger$. We can choose $d = c$. The Euclidean norm for $P(x)$ is defined as

$$|P(x)| = \sqrt{|p_1(x)|^2 + |p_2(x)|^2 + 2|p_{12}(x)|^2}.$$

In our case, $|P(x) - P_0|$ is order of $(1/x)$ as $x \rightarrow +\infty$; then $\int_d^x dt |P(t) - P_0|$ diverges and, by applying L'Hospital's rule to $1/x \int_d^x dt |P(t) - P_0|$ one finds that the above limit is zero. Theorem 16.6 allows us to conclude that $\sigma_e(H_\infty) \supset (-\infty, -m_e) \cup [m_e, +\infty)$. This result and the above one imply that $\sigma_e(H_\infty) = (-\infty, -m_e) \cup [m_e, +\infty)$.

Theorem 16.7 of [3] (see also Ref. [17]): Consider H satisfying the LPC at $b = +\infty$ (LPC or LCC at a). Assume that $P(x)$ can be decomposed for some $c \in (a, +\infty)$ as follows:

$$P(x) = P_1(x) + P_2(x),$$

$$|P_1(x)| \in L_1(c, +\infty),$$

$$P_2(x) \in BV([c, +\infty)),$$

$$\lim_{x \rightarrow +\infty} P_2(x) = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}$$

with $\mu_1 < \mu_2$. Then, each self-adjoint extension of H has purely absolutely continuous spectrum in the complement of $[\mu_1, \mu_2]$. Cf., also Ref. [5], theorem 4.18.

In the theorem above, $BV([c, +\infty))$ represents the space of the functions of bounded variation on the interval $[c, +\infty)$. We recall that $f \in BV([c, +\infty))$ means that, for any partition $\Pi: c = x_0 < x_1 < \dots < x_n = b$ of the interval $[c, b]$, where $c < b < +\infty$, the variation

$$V_c^b(f) \equiv \sup_{\Pi} \sum_{k=0}^{n-1} |f(x_k) - f(x_{k+1})|$$

is finite, and, moreover,

$$\lim_{b \rightarrow +\infty} V_c^b(f) \equiv V_c^{+\infty}(f)$$

exists and is finite.

We apply this theorem to $TH_{red} T^\dagger$. This Dirac operator satisfies the hypotheses of the cited theorem. In fact, in the interval $[c, +\infty)$, where $c > 0$, each component of $P(x)$ is smooth and has derivative belonging to $L_1([c, +\infty))$, so that

$P(x) \in BV([c, +\infty))$ [note that the anomalous term could as well belong to $P_1(x)$]. This follows from the fact that, in general, if a function f is, e.g., continuously differentiable and its derivative f' belongs to $L_1([c, +\infty))$, then

$$\begin{aligned} \sum_{k=0}^n |f(x_k) - f(x_{k-1})| &= \sum_{k=0}^n \left| \int_{x_{k-1}}^{x_k} dt f'(t) \right| \\ &\leq \sum_{k=0}^n \int_{x_{k-1}}^{x_k} dt |f'(t)| \\ &= \int_c^b dt |f'(t)|, \end{aligned}$$

and the condition $f' \in L_1([c, +\infty))$ allows us to get the desired result. Moreover,

$$\lim_{x \rightarrow +\infty} P(x) = \begin{bmatrix} -m_e & 0 \\ 0 & m_e \end{bmatrix}.$$

Then our operator H_{red} has a purely absolutely continuous spectrum in the complement of the closed interval $[-m_e, m_e]$.

Note also that this holds also for the flat space-time case [where the anomalous contribution is monotone and bounded in $[c, +\infty)$, $c > 0$ (then it is of bounded variation) and is also a term which can belong to $P_1(x)$].

In order to verify that an infinite number of eigenvalues is contained in the mass gap of our one-particle Hamiltonian, we use theorem 2.3 of Ref. [13]. Some preliminary definitions are given below.

One considers for $x \in (0, +\infty)$ an operator L of the form

$$Ly \equiv J(y' - Sy) \quad (C1)$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$S = \begin{pmatrix} p(x) & c_1 + V_1(x) \\ c_2 - V_2(x) & -p(x) \end{pmatrix},$$

where c_1 and c_2 are positive numbers and $p(x)$, $V_1(x)$, and $V_2(x)$ are real, locally integrable functions [13].

We introduce also a nontrivial linear functional $G[\cdot]$ defined on real 2×2 matrices B by $G[B] = \langle Bu, u \rangle$, where u is a non-null two-vector and $\langle \cdot, \cdot \rangle$ is the inner product in \mathbb{R}^2 . $G[B]$ so defined is a positive functional according to the definition of Ref. [13]. Let I be the identity matrix and let P be the matrix

$$P \equiv JS = \begin{pmatrix} -c_2 + V_2(x) & p(x) \\ p(x) & c_1 + V_1(x) \end{pmatrix}.$$

Theorem 2.3 of Ref. [13] is:

Let $h > 0$, G be a nontrivial positive linear functional and assume P locally absolutely continuous. Then, for any self-adjoint extension L_1 of L the set $\sigma(L_1) \cap (-h, h)$ is infinite if the scalar differential equation,

$$-G[I]z'' + G[P^2 - h^2I + (P'J - JP')/2]z = 0$$

is oscillatory¹¹ either at 0 or at $+\infty$.

We verify that our Hamiltonian implements the conditions given in theorem 2.3 of Ref. [13]. In our case we have

$$c_1 = c_2 = m_e,$$

$$V_1(x) = (\sqrt{f} - 1)m_e + \frac{eQ}{r},$$

$$V_2(x) = -(\sqrt{f} - 1)m_e + \frac{eQ}{r},$$

$$p(x) = -\left(\frac{k\sqrt{f}}{r} + \mu_a\sqrt{f}\frac{Q}{r^2}\right).$$

As a consequence, in order to verify if the spectrum of the self-adjoint extension L_1 of L has an infinite number of ei-

¹¹“Oscillatory at infinity” means that in a left neighborhood $(b, +\infty)$, with $b > 0$, of $+\infty$ all the solutions of the above second order scalar differential equation admit infinitely many zeroes in $(b, +\infty)$ [13]. An analogous definition holds for “oscillatory at 0.”

genvalues in $(-m_e, m_e)$ it is sufficient to verify that the following scalar differential equation

$$-G[I]z'' + G[P^2 - m_e^2I + (P'J - JP')/2]z = 0 \quad (C2)$$

(where $P \equiv JS$) has an oscillatory behavior either at 0 or at $+\infty$. Note that in our case the self-adjoint extension of the reduced Hamiltonian is unique. We choose $u_+ = (1, 0)^T$ and also $u_- = (0, 1)^T$. Then one obtains a scalar equation in the form

$$-z'' + \Gamma_{\pm}(x)z = 0 \quad (C3)$$

where Γ_{\pm} is relative to the choice of the vector u_{\pm} . One has

$$\Gamma_+(x) = V_2^2(x) - 2m_eV_2(x) + p^2(x) + p'(x),$$

$$\Gamma_-(x) = V_1^2(x) + 2m_eV_1(x) + p^2(x) - p'(x),$$

and asymptotically for $x \rightarrow +\infty$ it holds

$$\Gamma_{\pm}(x) \sim \frac{-2m_e(m_eM \pm eQ)}{x}. \quad (C4)$$

One can use corollary 37, p. 1463, of Ref. [18] for the scalar equation (C3). If the limit $\lim_{x \rightarrow +\infty} x^2\Gamma_{\pm}(x) = \lim_{r \rightarrow +\infty} r^2\Gamma_{\pm}(r) < -\frac{1}{4}$, then the equation is oscillatory near $+\infty$. In our case $Q > 0$ and $x^2\Gamma_+(x) \rightarrow -\infty$ as $x \rightarrow +\infty$ [if $Q < 0$ then $x^2\Gamma_-(x) \rightarrow -\infty$], that is, the behavior is oscillatory. Cf., also the examples in Ref. [13]. Then

$$\sigma(H_{red}) \cap (-m_e, m_e) = \text{infinite set.} \quad (C5)$$

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