Black hole scan

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Gravitation theories selected by requiring that they have a unique anti–de Sitter vacuum with a fixed cosmological constant are studied. For a given dimension d, the Lagrangians under consideration are labeled by an integer $k=1,2,\ldots,[(d-1)/2]$. Black holes for each d and k are found and are used to rank these theories. A minimum possible size for a localized electrically charged source is predicted in the whole set of theories, except general relativity. It is found that the thermodynamic behavior falls into two classes: If $d-2k=1$, these solutions resemble the three dimensional black hole; otherwise, their behavior is similar to the Schwarzschild-AdS_4 geometry.

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I. INTRODUCTION

Black holes are much more than a particular class of exact solutions of the Einstein equations; they are an essential feature of the spacetime dynamics in almost any sensible theory of gravity. Within the framework of general relativity, the singularity theorems of Hawking and Penrose [1] show that singular configurations—such as the Schwarzschild black hole—are inevitable under quite generic initial conditions. Furthermore, the Schwarzschild solution describes the leading asymptotic behavior of the geometry for any localized distribution of matter. The existence of this solution at spacelike infinity is a central ingredient to prove the positivity of energy in general relativity [2]. On the other hand, black holes are also fundamental objects where the thermodynamics of the gravitational field and its connection with information theory is expected to shed light on the quantization problem.

In this paper, we survey the black hole solutions in a class of gravitation theories, selected by requiring that they have a unique anti–de Sitter vacuum with a fixed cosmological constant. For a given dimension d, the Lagrangians under consideration are labeled by an integer $k=1,2,\ldots,[(d-1)/2]$, where the Einstein-Hilbert Lagrangian corresponds to $k=1$. For each of these theories we examine their static, spherically symmetric solutions. The existence of physical black holes is then used as a criterion to assess the validity of those theories, leading to a natural splitting between theories with even and odd k.

Coupling these gravity theories with the Maxwell action predicts the smallest size of a spherically symmetric electrically charged source, except for $k=1$.

An important aspect of the black holes under consideration is their thermodynamics, which is expected to be a reflection of the underlying quantum theory. The canonical ensemble for minisuperspaces containing the black holes found in these theories is well defined provided a negative cosmological constant exists. It is found that black holes are unstable against decay by Hawking radiation, unless their horizon radius is large, compared to the AdS radius.

Among all theories under consideration, there is only one representative in each odd dimension, given by a Chern-Simons action, having physical black holes whose spectrum has a mass gap separating them from AdS spacetime. These black holes always reach thermal equilibrium with a heat bath, and have positive specific heat, which guarantees their stability under thermal fluctuations.

A. Higher dimensional gravity revisited

The standard higher dimensional extension of the four-dimensional Einstein-Hilbert (EH) action reads [3]

$$I_{EH}=-\frac{1}{2(d-2)\Omega_{d-2}G}\int d^dx\sqrt{-g}(R-2\Lambda). \tag{1}$$

String and M-theory corrections to this action would bring in higher powers of curvature—see, e.g., Refs. [6,7]. This may be a source of inconsistencies because higher powers of curvature could give rise to fourth order differential equations for the metric. This not only complicates the causal evolution, but in general would introduce ghosts and violate unitarity. However, Zwiebach [8] and Zumino [9] observed that ghosts are avoided if stringy corrections would only consist of the dimensional continuations of the Euler densities, so that the resulting field equations remain second order.

These theories are far from exotic. Indeed, they are described by the most general Lagrangians constructed with the same principles as general relativity, that is, general covariance and second order field equations for the metric. These theories were first discussed by Lanczos for $d=5$ in 1938 [10] and more recently by Lovelock for $d\geqslant 3$ [11].

The Lanczos-Lovelock (LL) action is a polynomial of degree $[d/2]$ in curvature,[1] which can also be written in terms of the Riemann curvature $R^{ab}=d\omega^{ab}+\omega^{a}{}_{c}\omega^{cb}$ and the vielbein e^a as[2]

$$I_G=\kappa\int\sum_{p=0}^{[d/2]}\alpha_p L^{(p)}, \tag{2}$$

where α_p are arbitrary constants, and $L^{(p)}$ is given by

$$L^{(p)}=\epsilon_{a_1\cdots a_d}R^{a_1a_2}\cdots R^{a_{2p-1}a_{2p}}e^{a_{2p+1}}\cdots e^{a_d}. \tag{3}$$

In first order formalism the action (2) is regarded as a functional of the vielbein and the spin connection, and the corresponding field equations obtained varying with respect to e^a and ω^{ab} read

$$\sum_{p=0}^{[(d-1)/2]}\alpha_p(d-2p)\mathcal{E}_a^p=0, \tag{4}$$

$$\sum_{p=1}^{[(d-1)/2]}\alpha_p p(d-2p)\mathcal{E}_{ab}^p=0, \tag{5}$$

where we have defined

$$\mathcal{E}_a^p:=\epsilon_{ab_1\cdots b_{d-1}}R^{b_1b_2}\cdots R^{b_{2p-1}b_{2p}}e^{b_{2p+1}}\cdots e^{b_{d-1}},$$

$$\mathcal{E}_{ab}^p:=\epsilon_{aba_3\cdots a_d}R^{a_3a_4}\cdots R^{a_{2p-1}a_{2p}}T^{a_{2p+1}}e^{a_{2p+2}}\cdots e^{a_d}.$$

Here $T^a=de^a+\omega^a{}_b e^b$ is the torsion 2-form.

Note that in even dimensions, the term $L^{(d/2)}$ is the Euler density and therefore does not contribute to the field equations. However, the presence of this term in the action—with a fixed weight factor—guarantees the existence of a well-defined variational principle for asymptotically locally AdS spacetimes [12,13]. Moreover, the Euler density should assign different weights to non-homeomorphic geometries in the quantum theory.

The first two terms in the LL action (2) are the cosmological and kinetic terms of the EH action (1) respectively, and therefore general relativity is contained in the LL theory as a particular case.

The linearized approximation of the LL and EH actions around a flat, torsionless background are classically equivalent [9]. However, beyond perturbation theory the presence of higher powers of curvature in the Lagrangian makes both theories radically different. In particular, black holes and big-bang solutions of Eq. (2), have different asymptotic behaviors from their EH counterparts in general. Hence, a generic solution of the LL action cannot be approximated by a solution of Einstein's theory.

[1]Here $[x]$ is the integer part of x.

[2]A wedge product between forms is understood throughout.

B. Drawbacks

For a given dimension and an arbitrary choice of coefficients α_p's, higher dimensional LL theories have some drawbacks. One difficulty is the fact that the dynamical evolution can become unpredictable because the Hessian matrix cannot be inverted for a generic field configuration. Thus, the velocities are multivalued functions of the momenta and therefore the passage from the Lagrangian to the Hamiltonian is ill defined [14,15].

A reflection of this problem can be viewed in the static, spherically symmetric solutions of Eqs. (4) and (5). For arbitrary α_p's there are negative energy solutions with horizons and positive energy solutions with naked singularities [16].

These problems can be curbed if the coefficients α_p's are chosen in a suitable way. The aim of the next section is to show that requiring the theories to possess *a unique* cosmological constant strongly restricts the coefficients α_p's. As a consequence, one obtains a set of theories labeled by an integer k which lead to well defined black hole configurations.

II. SELECTING SENSIBLE THEORIES

The field equations of LL theory (4) can be rearranged as a polynomial of kth degree in the curvature

$$\epsilon_{ab_1\cdots b_{d-1}}\beta_0\bar{R}_{\beta_1}^{b_1b_2}\cdots\bar{R}_{\beta_k}^{b_{2k-1}b_{2k}}e^{b_{2k+1}}\cdots e^{b_{d-1}}=0 \tag{6}$$

where $\bar{R}_{\beta_i}^{ab}:=R^{ab}+\beta_i e^a e^b$, and the coefficients β_i's are related to the α_p's through

$$\sum_{p}^{[(d-1)/2]}(d-2p)\alpha_p x^p=\beta_0\prod_i^k(x-\beta_i). \tag{7}$$

Equation (6) can possess, in general, several constant curvature solutions with different radii $r_i=|\beta_i|^{-1/2}$, making the value of the cosmological constant ambiguous. In fact, the cosmological constant could change in different regions of a spatial section, or it could jump arbitrarily as the system evolves in time [14,15].

On the other hand, solving Eq. (6) for a given global isometry leads in general to several solutions with different asymptotic behaviors. Some of these solutions are ''spurious'' in the sense that perturbations around them yield ghosts. For instance, if α_1 and α_2 were the only nonvanishing coefficients in the LL action (3), two different static, spherically symmetric solutions would be obtained, which are asymptotically (A)dS and flat respectively. The perturbations around the latter solution are gravitons, while those on the former are spurious in the sense described above [17].

These problems are overcome demanding the theory to have *a unique* cosmological constant.

Requiring the existence of *a unique* cosmological constant implies that locally maximally symmetric solutions possess only one fixed radius, that is $R^{ab}=-\beta e^a e^b$. This in turn means that the polynomial (7) must have only one real root. Hence, the coefficients α_p's are fixed through Eq. (7), so that the real β's in Eq. (6) are all equal, allowing—for

$d \geqslant 7$—an arbitrary number of distinct imaginary β's which must come in conjugate pairs. Under this assumption, solutions representing localized sources of matter approach a constant curvature spacetime with a fixed radius in the asymptotic region.

In what follows, we consider the simplest class of such theories, namely, we assume the field equations to be of the form (6) with only one real $\beta := 1/l^2$, and no complex roots.[3] These theories are described by the action

$$I_k = \kappa \int \sum_{p=0}^{k} c_p^k L^{(p)}, \tag{8}$$

which is obtained from Eq. (2) with the choice

$$\alpha_p := c_p^k = \begin{cases} \dfrac{l^{2(p-k)}}{(d-2p)} \dbinom{k}{p}, & p \leqslant k, \\ 0, & p > k, \end{cases} \tag{9}$$

where $1 \leqslant k \leqslant [(d-1)/2]$.

For a given dimension d, the coefficients c_p^k give rise to a family of inequivalent theories, labeled by the integer $k \in \{1, \ldots, [(d-1)/2]\}$ which represents the highest power of curvature in the Lagrangian. This set of theories possesses only two fundamental constants, κ and l, related to the gravitational constant G_k and the cosmological constant Λ through[4]

$$\kappa = \frac{1}{2(d-2)!\Omega_{d-2} G_k}, \tag{10}$$

$$\Lambda = -\frac{(d-1)(d-2)}{2l^2}. \tag{11}$$

The field equations for the action I_k in Eq. (8) read

$$\epsilon_{ba_1 \cdots a_{d-1}} \bar{R}^{a_1 a_2} \cdots \bar{R}^{a_{2k-1} a_{2k}} e^{a_{2k+1}} \cdots e^{a_{d-1}} = 0, \tag{12}$$

$$\epsilon_{aba_3 \cdots a_d} \bar{R}^{a_3 a_4} \cdots \bar{R}^{a_{2k-1} a_{2k}} T^{a_{2k+1}} e^{a_{2k+2}} \cdots e^{a_{d-1}} = 0, \tag{13}$$

with $\bar{R}^{ab} := R^{ab} + (1/l^2) e^a e^b$.

Examples

There are special cases of interest which are obtained for particular values of the integer k.

(i) The Einstein-Hilbert action in d dimensions, Eq. (1), is recovered setting $k=1$ in Eq. (8).

(ii) At the other end of the range, $k = [(d-1)/2]$, even and odd dimensions must be distinguished. These cases are exceptional in that they are the only ones which allow sectors with non-trivial torsion [18], as discussed in Appendix A. When $d = 2n-1$, the maximum value of k is $n-1$, and the corresponding Lagrangian is a Chern-Simons (CS) $(2n-1)$-form defined through Eq. (A3). For $d=2n$ and $k=n-1$, the action can be written as the Pfaffian of the 2-form $\bar{R}^{ab} = R^{ab} + (1/l^2) e^a e^b$ and, in this sense, it has a Born-Infeld-like (BI)-like form given by Eq. (A1).[5]

(iii) In three and four dimensions Eq. (9) defines only one possible theory which corresponds to EH. As is well known, the EH action is equivalent to CS theory in three dimensions [19], and for $d=4$ the EH action coincides with the BI action up to the Euler density.

(iv) In five and six dimensions, there are only two inequivalent theories which correspond to $k=1,2$. In five dimensions, $k=1$ represents EH and $k=2$ leads to CS. For $d=6$, one obtain EH and BI respectively.

(v) For $d \geqslant 7$ there exist other interesting possibilities which are neither EH, BI nor CS. For instance, consider the theory given by the action I_k in Eq. (8) with $k=2$, which exists only for dimensions greater than 4. In this case the Lagrangian reads

$$L = \kappa \left(\frac{l^{-4}}{d} L^{(0)} + \frac{2l^{-2}}{d-2} L^{(1)} + \frac{1}{d-4} L^{(2)} \right), \tag{14}$$

with

$$L^{(0)} = \epsilon_{a_1 \cdots a_d} e^{a_1} \cdots e^{a_d}, \tag{15}$$

$$L^{(1)} = \epsilon_{a_1 \cdots a_d} R^{a_1 a_2} e^{a_3} \cdots e^{a_d}, \tag{16}$$

$$L^{(2)} = \epsilon_{a_1 \cdots a_d} R^{a_1 a_2} R^{a_3 a_4} e^{a_5} \cdots e^{a_d}. \tag{17}$$

Here $L^{(0)}$ and $L^{(1)}$ are proportional to the standard cosmological and kinetic terms for the EH action, and $L^{(2)}$ is proportional to the four dimensional Gauss-Bonnet density [20],

$$\mathfrak{R}^2 := (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2), \tag{18}$$

where $R^{\mu\nu\alpha\beta}$, $R^{\mu\nu}$ and R are the Riemann, Ricci and scalar curvatures, respectively. The action in standard tensor components reads

$$I_2 = \frac{-2(d-3)!\kappa}{l^2} \int_M d^d x \sqrt{-g} \left[\frac{l^2 \mathfrak{R}^2}{2(d-3)(d-4)} + R - \Lambda \right], \tag{19}$$

with Λ given by Eq. (11). In sum, the theory with $k=2$ is described by a Lagrangian which is a linear combination of Gauss-Bonnet density, the EH Lagrangian and the volume term with fixed weights.

[3] A negative cosmological constant is assumed for later convenience, but this analysis does not depend on its sign.

[4] Here the gravitational constant has natural units given by $[G_k] = (\text{length})^{d-2k}$.

[5] Strictly speaking one must add the Euler density to the Lagrangian in Eq. (8) with the coefficient $\alpha_n = c_n^{n-1} := l^2/2n$, which does not modify the field equations. Therefore, the same BI Lagrangian Eq. (A1) is recovered from (9) but now the index p ranges from 0 to n.

Each of the theories described by I_k for all k possesses a unique cosmological constant. In fact, as is apparent from Eqs. (12) and (13), spacetimes satisfying $\bar{R}^{ab}=0$ are the only locally maximally symmetric solutions. This ensures that localized matter fields give rise to solutions which are asymptotically AdS spacetimes.

III. STATIC AND SPHERICALLY SYMMETRIC SOLUTIONS

In this section, we test the theories described by I_k analyzing their static, spherically symmetric solutions including their electrically charged extensions. It is shown that they possess well behaved black holes, resembling the Schwarzschild-AdS and Reissner-Nordström-AdS solutions. The subset of theories with k odd differ from their even counterparts, because in the first case there is a unique black hole solution, whereas in the latter, an additional solution with a naked singularity exists.

A. Pure gravity

Consider static and spherically symmetric solutions of Eqs. (12) and (13) for a fixed value of the label k. In Schwarzschild-like coordinates, the metric can be written as

$$ds^2=-N^2(r)f^2(r)dt^2+\frac{dr^2}{f^2(r)}+r^2d\Omega_{d-2}^2. \tag{20}$$

Replacing this *Ansatz* in the field equations (12) and (13) leads to the following equations for N and f^2 [21]:

$$\frac{dN}{dr}=0,$$
$$\frac{d}{dr}\left(r^{d-1}\left[F(r)+\frac{1}{l^2}\right]^k\right)=0, \tag{21}$$

where the function $F(r)$ is given by

$$F(r)=\frac{1-f^2(r)}{r^2}. \tag{22}$$

Integrating Eqs. (21) yields

$$N=N_\infty,$$
$$f^2(r)=1+\frac{r^2}{l^2}-\sigma\left(\frac{C_1}{r^{d-2k-1}}\right)^{1/k}, \tag{23}$$

where the integration constant N_∞ relates coordinate time to the proper time of an observer at spatial infinity and in what follows is chosen equal to 1. Here $\sigma=(\pm1)^{(k+1)}$, and the integration constant C_1 is identified as

$$C_1=2G_k(M+C_0),$$

where M stands for the mass, as is discussed in detail in Sec. III C.

For even k, the ambiguity of sign expressed through σ in Eqs. (23) implies that there are two possible solutions provided $C_1>0$. The solution with $\sigma=1$ describes a real black hole with *a unique* event horizon surrounding the singularity at the origin. The solution with $\sigma=-1$ has a naked singularity with positive mass.

If k is odd, there is no ambiguity of sign because σ cannot be different from unity; therefore in that case there exists a unique static, spherically symmetric solution, which corresponds to a black hole with positive mass.

The black hole mass for any value of k is a monotonically increasing function of the horizon radius r_+, which reads

$$M(r_+)=\frac{r_+^{d-2k-1}}{2G_k}\left(1+\frac{r_+^2}{l^2}\right)^k-C_0. \tag{24}$$

The additive constant C_0 is chosen so that the horizon shrinks to a point for $M\rightarrow0$; hence

$$C_0=\frac{1}{2G_k}\delta_{d-2k,1}, \tag{25}$$

which vanishes in all cases except for CS theory.

Summarizing, for a given dimension $d\geqslant3$ the full set of $[(d-1)/2]$ inequivalent theories given by the action I_k in Eq. (8) possess asymptotically AdS black hole solutions whose line elements read

$$ds^2=-\left[1+\frac{r^2}{l^2}-\left(\frac{2G_kM+\delta_{d-2k,1}}{r^{d-2k-1}}\right)^{1/k}\right]dt^2 +\frac{dr^2}{1+\frac{r^2}{l^2}-\left(\frac{2G_kM+\delta_{d-2k,1}}{r^{d-2k-1}}\right)^{1/k}}+r^2d\Omega_{d-2}^2. \tag{26}$$

One can see from Eq. (26) that for $k=1$, the three dimensional black hole [22] and Schwarzschild-AdS solutions of the d-dimensional Einstein-Hilbert action with negative cosmological constant are recovered. The black hole solutions corresponding to BI and CS theories [23] are obtained also from Eq. (26) setting $k=[(d-1)/2]$.

The whole set of black hole metrics given by Eq. (26) share a common causal structure when $M>0$, which coincides with the familiar one described by the Penrose diagram of the four dimensional Schwarzschild-AdS solution. Nevertheless, the presence of the Kronecker delta within the metrics (26) signals the existence of two possible black hole vacua ($M=0$) with different causal structures. The generic case holds for the whole set of theories except CS, whose line elements are described by Eq. (26) with $d-2k\neq1$, that is

$$ds^2=-\left[1+\frac{r^2}{l^2}-\left(\frac{2G_kM}{r^{d-2k-1}}\right)^{1/k}\right]dt^2+\frac{dr^2}{1+\frac{r^2}{l^2}-\left(\frac{2G_kM}{r^{d-2k-1}}\right)^{1/k}}+r^2d\Omega_{d-2}^2. \quad (27)$$

Analogously with the Schwarzschild-AdS metric, this set possesses a continuous mass spectrum, whose vacuum state is the AdS spacetime. The other case is obtained only for $d=2n-1$ dimensions, and it is a peculiarity of CS theories, whose black hole solutions are recovered from Eq. (26) with $k=n-1$, which read

$$ds^2=-\left(1+\frac{r^2}{l^2}-(2G_{n-1}M+1)^{1/(n-1)}\right)dt^2+\frac{dr^2}{1+\frac{r^2}{l^2}-(2G_{n-1}M+1)^{1/(n-1)}}+r^2d\Omega_{d-2}^2. \quad (28)$$

In that case, the black hole vacuum $(M=0)$ differs from AdS spacetime. Although this configuration has no constant curvature for $d>3$, it possesses the same causal structure as the three-dimensional zero mass black hole. Another common feature with $2+1$ dimensions is the existence of a mass gap between the zero mass black hole and AdS spacetime, where the later is obtained for $M=-1/2G_{n-1}$.

B. Coupling to the electromagnetic field

The standard coupling with the electromagnetic field is obtained adding to the gravitational action I_k in Eq. (8) the Maxwell term[6]

$$I_M=-\frac{1}{4\epsilon\Omega_{d-2}}\int\sqrt{-g}F^{\mu\nu}F_{\mu\nu}d^dx. \quad (29)$$

Electrically charged solutions which are static and spherically symmetric can be found through the *Ansatz* (20), and requiring that and the only non-vanishing component of the electromagnetic field strength be

$$F_{0r}=-\partial_rA_0(r). \quad (30)$$

The field equations for N, f^2 and A_0 read

$$\frac{dN}{dr}=0,$$

$$\frac{d}{dr}(r^{d-2}p)=0,$$

$$\frac{dA_0}{dr}+Np=0,$$

$$\frac{d}{dr}\left(r^{d-1}\left[F(r)+\frac{1}{l^2}\right]^k\right)=\frac{G_k}{\epsilon}r^{d-2}p^2, \quad (31)$$

where $F(r)$ is defined in Eq. (22), and $p(r)$ is a redefinition of the electric field:

$$p=\frac{1}{N}F_{0r}. \quad (32)$$

Integrating these equations yields

$$N=N_\infty=1,$$

$$p(r)=\epsilon\frac{Q}{r^{d-2}},$$

$$A_0(r)=\phi_\infty+\frac{\epsilon}{(d-3)}\frac{Q}{r^{d-3}},$$

$$f^2(r)=1+\frac{r^2}{l^2}-\sigma g_k(r), \quad (33)$$

with $\sigma=(\pm1)^{(k+1)}$ and

$$g_k(r)=\left(\frac{2G_kM+\delta_{d-2k,1}}{r^{d-2k-1}}-\frac{\epsilon G_k}{(d-3)}\frac{Q^2}{r^{2(d-k-2)}}\right)^{1/k}. \quad (34)$$

The integration constants M and Q in Eq. (34) are the mass and the electric charge of the black hole respectively, as is shown in the next subsection.

Equations (33) provide the electrically charged extension of the vacuum solution (23).[7] The presence of σ in Eq. (33) leads to a similar picture as in the uncharged case. When k is odd, there is a unique electrically charged black hole solution because σ is always equal to 1, but when k is even, the solution with $\sigma=1$ represents a black hole, and the solution with $\sigma=-1$ possess a naked singularity.

Therefore, electrically charged asymptotically AdS black hole solutions are obtained from Eq. (33) with $\sigma=1$, whose line element reads—for $d>3$—as

$$ds^2=-\left(1+\frac{r^2}{l^2}-g_k(r)\right)dt^2+\frac{dr^2}{1+\frac{r^2}{l^2}-g_k(r)}+r^2d\Omega_{d-2}^2, \quad (35)$$

[6] The constant ϵ is related with the ''vacuum permeability'' through $\epsilon=1/\Omega_{d-2}\epsilon_0$. Its natural units are $[\epsilon]=(\text{length})^{d-4}$.

[7] The expression (34) is valid for $d>3$. The three dimensional case is discussed in Refs. [22,24].

where $g_k(r)$ is given by Eq. (34).

As is naturally expected, the set of black holes described by Eq. (35), reduce to the d-dimensional Reissner-Nordström-AdS solution for $k=1$. The electrically charged black hole solutions corresponding to BI and CS theories [23] are also recovered for $d=2n$ and $d=2n-1$ respectively, as it can be seen replacing $k=n-1$ in Eq. (35).

For a generic value of the label k, in analogy with standard Reissner-Nordström-AdS geometry, the black hole solutions given by Eq. (35) possess in general two horizons located at the roots of $f^2(r)$. They satisfy $0<r_-<r_+$ provided the mass is bounded from below as $M\geqslant h_k(Q)$, where h_k is a monotonically increasing function of the electric charge. Both horizons merge when the bound is saturated, corresponding to the extreme case, that is $r_+=r_-$ for $M=h_k(Q)$. Solutions with $M<h_k(Q)$ possess naked singularities which should be considered unphysical. Thus, for a given electric charge, the existence of a lower bound on M is in agreement with the cosmic censorship principle.

An important difference with the Reissner-Nordström-AdS case $(k=1)$ is shared by all electrically charged black hole solutions with $k\neq 1$, as can be inferred evaluating the scalar curvature for the metrics (35), given by

$$R=\frac{1}{r^{d-2}}\frac{d^2}{dr^2}\left[r^{d-2}\left(g_k(r)-\frac{r^2}{l^2}\right)\right]. \tag{36}$$

For any $k\neq 1$, Eq. (36) has a branch point unbounded singularity at the zero of the function $g_k(r)$. This is a real timelike singularity located at

$$r_e=\left(\frac{\epsilon}{2(d-3)}\frac{Q^2}{\left(M+\frac{1}{2G_k}\delta_{d-2k,1}\right)}\right)^{1/(d-3)}, \tag{37}$$

which can be reached in a finite proper time. However, an external observer is protected from it because it is surrounded by both horizons, i.e., $0<r_e<r_-<r_+$.

When k is even, spacetime cannot be extended to $r<r_e$, because in that case the metrics (35) would become complex. This means that the manifold possesses a real boundary at $r=r_e$, and therefore, r_e *is the smallest possible size* of a spherical body endowed of electric charge Q and mass M.

For odd values of $k\neq 1$ there is no obstruction to define spacetime within the region $r<r_e$. However, as it can be seen from Eq. (36), there is an additional timelike singularity located at $r=0$. In that case, a spherical source with electric charge Q and mass M, whose radius is smaller than r_e possesses an exterior geometry described by Eq. (35) which cannot be empty, since it has a singularity at $r=r_e$. This means that the original source generates ''a shield,'' which acts as the effective source of the external geometry. Hence, again, r_e is the *smallest size* for the source.

This means that the presence of electric charge brings in a new length scale into the system, except when one deals with the EH action. For CS theory $(d=2k+1)$, the radius r_e depends on the gravitational constant. However, in the generic case, which is given by the set of theories which are neither EH or CS, the radius r_e depends only on intrinsic features of the source and it is completely independent from gravity. That is, r_e is independent of the label k, the gravitational constant G_k and the cosmological constant—or equivalently the AdS radius l—that is

$$r_e=\left(\frac{\epsilon}{2(d-3)}\frac{Q^2}{M}\right)^{1/(d-3)}, \tag{38}$$

which has the same expression as the classical radius of the electron in d dimensions. It is noteworthy that r_e is encoded in the geometry.

Remarkably, the only theory within the family discussed here, which is unable to predict a minimum size for the source is general relativity.

C. Mass and electric charge from boundary terms

In order to identify the integration constants appearing in the black hole solutions (26) and (35) with the mass and electric charge, it is convenient to carry out the canonical analysis [25]. The total action can be written in Hamiltonian form as

$$I_T=I_G+I_M+B, \tag{39}$$

where I_G and I_M are the canonical actions for gravity and electromagnetism, respectively,

$$I_G=\int d^dx(\pi^{ij}\dot{g}_{ij}-N^{\perp}H_{G\perp}-N^iH_{Gi}), \tag{40}$$

$$I_M=\int d^dx(p^i\dot{A}_i-N^{\perp}H_{M\perp}-N^iH_{Mi}-A_0\partial_ip^i), \tag{41}$$

and B stands for a boundary term which is needed so that the action attains an extremum on the classical solution. Here $H_{G\mu}$ and $H_{M\mu}$ are the Hamiltonian generators of diffeomorphisms on the gravitational and electromagnetic phase spaces, respectively (see Ref. [14]).

In case of static, spherically symmetric spacetimes, a general theorem [26] implies that the extremum of the action can be found through a minisuperspace model, which is obtained replacing the *Ansätze* (20) and (30) into the action, as well. Hence, one deals with a simple one-dimensional model which allows fixing the boundary term B as a function of the integration constants requiring the total action (39) to have an extremum on the classical solutions. The minisuperspace action takes the form

$$I_T=\Delta t\int\frac{N}{2}\left[\frac{d}{dr}\left\{\frac{r^{d-1}}{G_k}\left[F(r)+\frac{1}{l^2}\right]^k\right\}-\frac{1}{\epsilon}r^{d-2}p^2\right]dr$$
$$+\frac{1}{\epsilon}\Delta t\int A_0\frac{d}{dr}(r^{d-2}p)dr+B, \tag{42}$$

where $N:=N^{\perp}(r)f^{-2}(r)$, and p is a redefinition of the canonical momentum p^r, conjugate to A_r,

$$p=\frac{1}{N}F_{0r}=\frac{\epsilon\Omega_{d-2}}{r^{d-2}\sqrt{\gamma}}p^{r}, \tag{43}$$

and γ is the determinant of the angular metric.

The action (42) is a functional of the fields N, f^2, A_0 and p, whose variation leads to a bulk term which vanishes on the field equations (31). Thus, the variation of the action (42) on shell is a boundary term given by

$$\delta I_T=\Delta t\int \frac{d}{dr}\left(N\frac{r^{d-1}}{2G_k}\delta\left[F(r)+\frac{1}{l^2}\right]^k\right)dr + \frac{1}{\epsilon}\Delta t\int \frac{d}{dr}(A_0 r^{d-2}\delta p)dr+\delta B, \tag{44}$$

which means that the action is stationary on the black hole solution provided

$$\delta B=-\Delta t(N_\infty \delta M+\phi_\infty \delta Q). \tag{45}$$

Since δM is multiplied by the proper time separation at infinity, one identifies M and Q as the mass and the electric charge up to additive constants. The additive constant related with the mass is called C_0 and it is fixed in Eq. (25), requiring that the horizon shrink to a point for $M\to 0$. The additive constant related to the electric charge vanishes, demanding that the electrically charged solution (35) reduce to the uncharged one (26) for $Q=0$. Therefore, the boundary term that must be added to the action is

$$B=-\Delta t(M+\phi_\infty Q)+B_0, \tag{46}$$

where N_∞ has been chosen equal to 1, and B_0 is an arbitrary constant without variation. This proves that the integration constants M and Q appearing in the black hole metrics (35) and (26) are the mass and the electric charge respectively.

These results are confirmed also through an alternative method which holds for even dimensions, as is discussed in Appendix B.

D. Asymptotically flat limit ($l\to\infty$)

The black hole metrics (26) and (35) tend asymptotically to an AdS spacetime with radius l, whose curvature satisfies $R^{ab}\to -l^{-2}e^a e^b$ at the boundary. Then, their asymptotically flat limit is obtained by taking $l\to\infty$. Thus, instead of taking the vanishing limit of the volume term ($\alpha_0\to 0$), the vanishing cosmological constant limit of the action I_k is obtained setting $l\to\infty$ in Eq. (9). This procedure is consistent with taking the same limit in the field equations (12) and (13).

When $l\to\infty$ the only non-vanishing term in Eq. (9) is the kth one; consequently the action is obtained from Eq. (2) with the following choice of coefficients:

$$\alpha_p := \tilde{c}_p^k=\frac{1}{(d-2k)}\delta_p^k . \tag{47}$$

Therefore, replacing Eq. (47) in Eq. (2), a new family of Lagrangians labeled by the integer $k\in\{1,2,\dots,[(d-1)/2]\}$, is obtained. For a fixed value of k, the Lagrangian is given just by $L^{(k)}$ defined in Eq. (3), so that the action reads

$$\tilde{I}_k=\frac{\kappa}{(d-2k)}\int \epsilon_{a_1\cdots a_d}R^{a_1a_2}\cdots R^{a_{2k-1}a_{2k}}e^{a_{2k+1}}\cdots e^{a_d}, \tag{48}$$

where κ is defined in Eq. (10). The field equations coincide with the $l\to\infty$ limit of Eqs. (12), (13), which merely amounts to replacing $\overline{R}^{ab}$ by R^{ab}.

Note that for $k=1$, the standard EH action without cosmological constant is recovered, while for $k=2$ the Lagrangian is the Gauss-Bonnet density (18).

Static and spherically symmetric solutions of Eq. (48) lead to a similar picture as in the electrically (un)charged asymptotically AdS case: when k is odd, one obtains only one solution describing a black hole, but for even values of k, two different solutions exist; one of them describes a black hole, while the other possesses naked singularities even when the mass bound holds.

It is simple to verify that black hole solutions of the action (48) correspond to the vanishing cosmological constant limit of the solutions for pure gravity (26). This also holds for the electrically charged solutions (35).

1. Q=0

The asymptotically flat solutions without electric charge are given by

$$ds^2=-\left[1-\left(\frac{2G_kM}{r^{d-2k-1}}\right)^{1/k}\right]dt^2+\frac{dr^2}{1-\left(\frac{2G_kM}{r^{d-2k-1}}\right)^{1/k}} + r^2 d\Omega_{d-2}^2 . \tag{49}$$

The generic cases correspond to $d-2k-1\neq 0$, for which the metrics (49) represent black hole solutions with an event horizon located at $r_+=(2G_kM)^{1/(d-2k-1)}$. As usual, their common vacuum geometry is the flat Minkowski spacetime, and their causal structure is described through the standard Penrose diagram of the Schwarzschild solution. In case of $k=1$ (EH), the Schwarzschild solution is recovered from Eq. (49) for $d>3$. Exceptional cases occur when $d=2k+1$, for which the action (48) corresponds to a CS theory for the Poincaré group $ISO(d-1,1)$. Their static, spherically symmetric solutions (49) do not describe black holes because they have a naked singularity at the origin. This can be inferred from Eq. (28) because when $l\to\infty$ the horizon recedes to infinity. For instance, in three dimensions, the solution (49) represent a conical spacetime [27].

2. Q≠0

The electrically charged asymptotically flat black hole solutions can be obtained for $d>3$ from Eq. (35) in the limit $l\to\infty$. As for the uncharged solutions, the generic case holds for $d-2k-1\neq 0$, whose line elements read

$$ds^2=-[1-g_k(r)]dt^2+\frac{dr^2}{1-g_k(r)}+r^2d\Omega_{d-2}^2, \quad (50)$$

with $g_k(r)$ given by

$$g_k(r)=\left(\frac{2G_kM}{r^{d-2k-1}}-\frac{\epsilon G_k}{(d-3)}\frac{Q^2}{r^{2(d-k-2)}}\right)^{1/k}. \quad (51)$$

For different generic values of the label k, the black hole solutions given by Eq. (50) resemble the Reissner-Nordström one, possessing two horizons which are found solving $g_k(r)=1$. As usual, these horizons satisfy $0<r_-<r_+$ provided the mass is bounded from below by

$$Q^2\leqslant\frac{(d-2k-1)}{\epsilon G_k}\left(\frac{(d-3)G_kM}{d-k-2}\right)^{(2d-2k-4)/(d-2k-1)}. \quad (52)$$

The extreme case occurs when both horizons coalesce, that is

$$r_+=r_-=\left(\frac{(d-3)G_kM}{d-k-2}\right)^{1/(d-2k-1)}, \quad (53)$$

so that the bound (52) is saturated.

The d-dimensional Reissner-Nordström solution is obtained from Eq. (50) setting $k=1$. Equation (52) reproduces the well-known four-dimensional bound given by

$$Q_{EH}^2\leqslant\frac{GM^2}{\epsilon}, \quad (54)$$

which is saturated when $r_+=r_-=G_kM$, as can be seen from Eq. (53) for $d=4$ and $k=1$.

A further example corresponds to the electrically charged black hole in the vanishing cosmological constant limit of the BI action. The bound and the extreme radius are obtained in that case from Eqs. (52) and (53) for $d=2n$ and $k=n-1$:

$$Q_{BI}^2\leqslant\frac{1}{\epsilon G_{n-1}}\left[\frac{(2n-3)G_{n-1}M}{n-1}\right]^{2(n-1)}$$

$$r_+=r_-=\frac{(2n-3)G_{n-1}M}{n-1}. \quad (55)$$

The full set of asymptotically flat electrically charged black hole solutions (50) shares a common feature with its asymptotically AdS counterparts given by Eq. (35) in the generic case $(d-2k-1\neq0)$. That is the existence of a timelike singularity for $k\neq1$ located at the zero of $g_k(r)$ in Eq. (51) given by

$$r_e=\left(\frac{\epsilon}{2(d-3)}\frac{Q^2}{M}\right)^{1/(d-3)}, \quad (56)$$

which satisfies $0<r_e<r_-<r_+$ and is again interpreted as *the smallest possible size* of a spherical body with electric charge Q and mass M. Then one concludes that this feature is absent only when one deals with the EH action with or without cosmological constant.

IV. THERMODYNAMICS

A. Temperature

As usual, we define the black hole temperature by the condition that in the Euclidean sector, the solution be well defined (smooth) at the horizon. This means that the Euclidean time is a periodic coordinate with period

$$\tau=4\pi\left(\left.\frac{df^2}{dr}\right|_{r_+}\right)^{-1}, \quad (57)$$

which is identified with $\beta=1/\kappa_BT$, where κ_B is the Boltzmann constant. Thus, the Hawking temperature is given by

$$T=\frac{1}{4\pi\kappa_B}\left.\frac{df^2}{dr}\right|_{r_+}. \quad (58)$$

For the electrically uncharged cases, the black hole temperature for the set of metrics (26) is

$$T=\frac{1}{4\pi\kappa_Bk}\left((d-1)\frac{r_+}{l^2}+\frac{(d-2k-1)}{r_+}\right). \quad (59)$$

For all k such that $d-2k-1\neq0$, the function $T(r_+)$ exhibits the same behavior as the standard Schwarzschild-AdS black hole (which is obtained for $k=1$); that is, the temperature diverges at $r_+=0$. It has a minimum at r_c given by

$$r_c=l\sqrt{\frac{d-2k-1}{d-1}}, \quad (60)$$

and grows linearly for large r_+. Considering $k=n-1$, formula (59) reproduces the known results for BI $(d=2n)$ and CS $(d=2n-1)$ black holes [23]. The temperature (59) reaches an absolute minimum at r_c equal to

$$T_c=\frac{\sqrt{(d-2k-1)(d-1)}}{2\pi\kappa_Bkl}, \quad (61)$$

provided the existence of a nonvanishing cosmological constant $(l\neq\infty)$.

In case of CS theory, that is when $d-2k-1=0$, $T(r_+)$ is not divergent at all, its absolute minimum is at $r_c=0$ and $T_c=0$. Thus, CS black holes are the only exceptional cases among all the possibilities considered here. Both, CS and generic cases are depicted in Fig. 1.

B. Specific heat and thermal equilibrium

As seen in Sec. III A, the black hole mass is a monotonically increasing function of r_+; therefore the behavior of $T(M)$ is qualitatively similar to that of $T(r_+)$.

Using Eqs. (59) and (24), the specific heat $C_k=\partial M/\partial T$ can be expressed as a function of r_+:

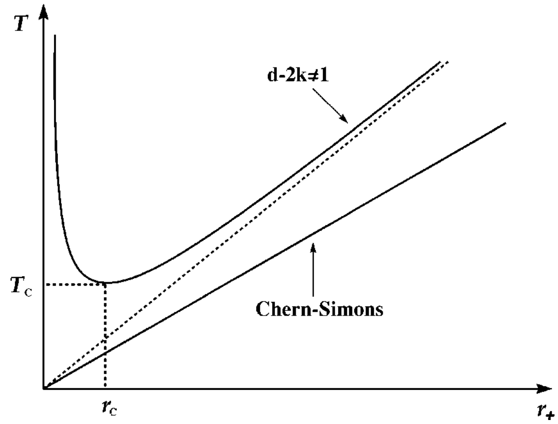

FIG. 1. The black hole temperature is plotted as a function of the horizon radius r_+. For $d-2k\neq 1$ the temperature reaches an absolute minimum T_c at $r_+=r_c$.

$$C_k=k\frac{2\pi\kappa_B}{G_k}r_+^{d-2k}\left(\frac{r_+^2+r_c^2}{r_+^2-r_c^2}\right)\left(1+\frac{r_+^2}{l^2}\right)^{k-1}. \tag{62}$$

In case of $d-2k-1\neq 0$, the specific heat (62) possesses an unbounded discontinuity at $r_+=r_c$ (see Fig. 2), signaling a phase transition. The specific heat C is positive for $r_+>r_c$, and has the opposite sign for $r_+<r_c$.

Again, the CS case is exceptional. Setting $d=2n-1$ and $k=n-1$ in Eq. (62), the specific heat is found as

$$C_{CS}=(n-1)\frac{2\pi\kappa_B}{G_{n-1}}r_+\left(1+\frac{r_+^2}{l^2}\right)^{n-2}, \tag{63}$$

which is a continuous monotonically increasing positive function of r_+ and does not diverge for any finite value of r_+ [28].

The presence of a negative cosmological constant makes it possible for the family of black hole solutions (26) to reach thermal equilibrium, as is possible for the Schwarzschild-AdS$_4$ spacetime [29] and for the three-dimensional black hole. Let us assume that any black hole described by Eq. (26) is immersed in a thermal bath of temperature $T_0>T_c$. If $d-2k-1\neq 0$, the thermal behavior splits in two branches: for $r_+<r_c$, the specific heat is negative and therefore black hole state is driven away from that with temperature T_0; for $r_+>r_c$, the black hole state is attracted towards the equilibrium configuration at temperature T_0 (see Fig. 3). Thus, the temperature T_0 corresponds to two equilibrium states of radii r_u (unstable) and r_s (locally stable), with $r_u<r_c<r_s$. Neglecting quantum tunneling processes, there are two possible scenarios: if the initial black hole state has $r_+<r_u$, the black hole cannot reach equilibrium because it evaporates until its final stage. Otherwise, for $r_+>r_u$, the black hole evolves towards an equilibrium configuration at $r_+=r_s$.

If the heat bath has temperature below T_c, the black hole cannot reach a stable equilibrium state and must evaporate, as depicted in Fig. 4.

None of the above arguments hold for the Chern-Simons case. When $d-2k=1$, the specific heat (63) is always positive; therefore the equilibrium configuration is always reached, independently from the initial black hole state and for any finite temperature of the heat bath.

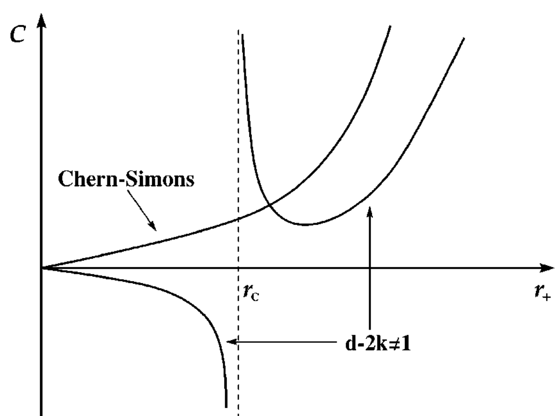

FIG. 2. The specific heat C_k is plotted as a function of the horizon radius. For a generic theory, $d-2k\neq 1$, C_k has a simple pole at $r_+=r_c$. For the exceptional case, $d=2k+1$ (CS), the specific heat is a continuous, monotonically increasing, positive function of r_+.

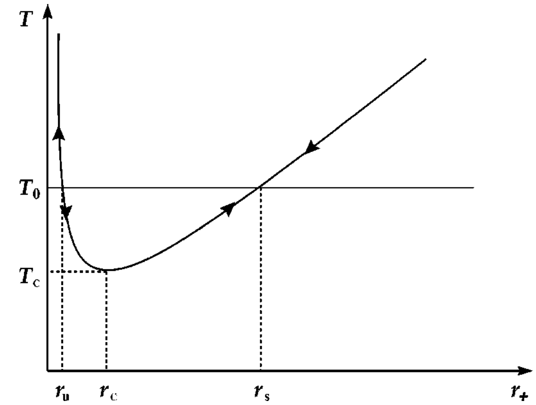

FIG. 3. In the generic case, $d-2k\neq 1$, the black hole can reach thermal equilibrium with a bath of temperature higher than T_c, provided the horizon radius satisfies $r_+>r_u$.

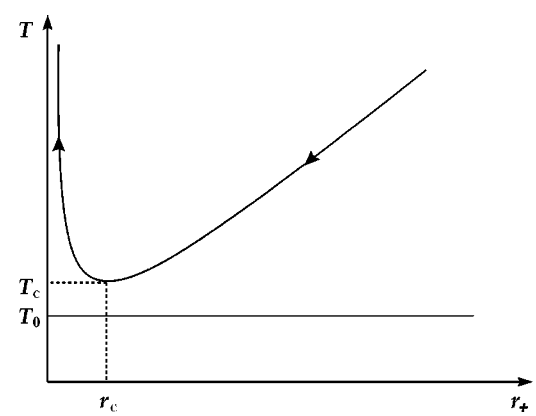

FIG. 4. In the generic case, $d-2k\neq 1$, the black hole cannot reach thermal equilibrium with a bath of temperature lower than T_c.

C. Entropy

It is well known that the partition function which describes the black hole thermodynamics is obtained through the Euclidean path integral in the saddle point approximation around the black hole solution [30]. That is,

$$Z \approx e^{-I_E},$$

which means that the Euclidean action evaluated on the black hole configuration is identified with β times the free energy of the system

$$I_E = \beta M - \frac{S}{\kappa_B} + \beta \sum_i \mu_i Q_i, \tag{64}$$

where the μ_i's are the chemical potentials corresponding to the charges Q_i. The Euclidean minisuperspace action is given by the Wick-rotated form of Eq. (42), that is

$$I_E = -\beta \int_{r_+}^{\infty} \frac{N}{2} \left[\frac{d}{dr} \left\{ \frac{r^{d-1}}{G_k} \left[F(r) + \frac{1}{l^2} \right]^k \right\} - \frac{1}{\epsilon} r^{d-2} p^2 \right] dr - \frac{1}{\epsilon} \beta \int_{r_+}^{\infty} A_0 \frac{d}{dr}(r^{d-2}p) dr + B_E. \tag{65}$$

In what follows we shall consider the electrically uncharged cases only. The bulk part of the Euclidean action is a linear combination of the constrains and, therefore, its on-shell value is given by the boundary term B_E. This boundary piece is determined by the requirement that I_E be stationary on the black hole geometry. Varying Eq. (65) leads to

$$\delta I_E = -\frac{\beta N_\infty}{2G_k} \int_{r_+}^{\infty} \frac{d}{dr} \left\{ r^{d-1} \delta \left[F(r) + \frac{1}{l^2} \right]^k \right\} dr + \delta B_E, \tag{66}$$

on shell. From this expression, one finds

$$\delta B_E = \beta \delta M - \frac{2\pi k}{G_k} r_+^{d-2k-1} \left(1 + \frac{r_+^2}{l^2} \right)^{k-1} \delta r_+,$$

where N_∞ has been set equal to one and we have used $df^2/dr|_{r_+} = 4\pi\beta^{-1}$. From Eq. (64) one identifies

$$\delta S = k \frac{2\pi\kappa_B}{G_k} r_+^{d-2k-1} \left(1 + \frac{r_+^2}{l^2} \right)^{k-1} \delta r_+, \tag{67}$$

which is integrated into

$$S_k = k \frac{2\pi\kappa_B}{G_k} \int_0^{r_+} r^{(d-2k-1)} \left(1 + \frac{r^2}{l^2} \right)^{k-1} dr. \tag{68}$$

This is a monotonically increasing function of r_+, in agreement with the second law of thermodynamics. In Eq. (68) the lower limit in the integral has been fixed by the condition $S_k(r_+ = 0) = 0$ for the whole set of black holes given by Eq. (26).

For the EH action (that is for $k=1$), expression (68) readily reproduces, for the Schwarzschild-AdS solution,

$$S_{EH} = \frac{2\pi\kappa_B}{(d-2)G} r_+^{d-2},$$

which in standard units [3] is the celebrated ''area law''

$$S_{EH} = \frac{\kappa_B}{\tilde{G}} \frac{A}{4}.$$

For $k = [(d-1)/2]$ (BI and CS), formula (68) reduces to the known results [23]. The theory described by I_2 in Eq. (19) is an intrinsically higher dimensional one, and the corresponding black hole entropy is given by

$$S_2 = \frac{4\pi\kappa_B}{G_2} r_+^{d-4} \left[\frac{1}{(d-4)} + \frac{r_+^2}{(d-2)l^2} \right]. \tag{69}$$

Hence, the area law is a peculiarity of the Einstein-Hilbert theory ($k=1$), while for $k \neq 1$ the entropy (68) becomes proportional to the area in the large r_+ limit, that is

$$S_k \approx k \frac{2\pi\kappa_B}{(d-2)G_k l^{2(k-1)}} r_+^{d-2} = k \frac{G}{G_k l^{2(k-1)}} S_{EH}, \tag{70}$$

with $r_+ \gg l$.

D. Asymptotically flat limit

In the limit $l \to \infty$, the geometry of the uncharged black hole is given by Eq. (49) whose corresponding temperature is

$$T^0 = \frac{1}{4\pi\kappa_B k} \frac{(d-2k-1)}{r_+}. \tag{71}$$

This gives a vanishing value for CS theory ($d-2k-1=0$), which is consistent with the fact that in that case, the geometry possesses a singularity which is not surrounded by a horizon in the limit $l \to \infty$, so that no temperature can be associated with it. For all the other cases ($d-2k-1 \neq 0$), the horizon is located at $r_+ = (2G_k M)^{1/(d-2k-1)}$, so that the black hole temperature (71) is a monotonically decreasing function of the mass. Therefore, thermal equilibrium can never be reached, consistently with the fact that the specific heat is always negative:

$$C^0 = -k \frac{2\pi\kappa_B}{G_k} r_+^{d-2k}. \tag{72}$$

The entropy is also an increasing function of r_+,

$$S_k^0 = k \frac{2\pi\kappa_B}{G_k} \frac{r_+^{(d-2k)}}{(d-2k)}, \tag{73}$$

which is consistent with the second law of thermodynamics. Note that formula (73) is proportional to the area of the horizon only for $k=1$ (EH). Thus, in the $l \rightarrow \infty$ limit, the area law cannot be recovered even as an approximation in the cases with $k \neq 1$.

E. Canonical ensemble

In four dimensions, Hawking and Page have shown that in the presence of a negative cosmological constant, the partition function in the canonical ensemble is well defined, unlike in case of a vanishing Λ [29]. The same argument can be extended for higher dimensions for the whole set of theories (8) labeled by k.

The partition function in the canonical ensemble reads

$$Z(\beta)=\int_0^{\infty} e^{-\beta M} \rho(M) dM, \tag{74}$$

where $\rho(M)=\exp(S_k/\kappa_B)$ is the density of states as a function of the energy. The convergence of this integral depends on the asymptotic behavior of S_k for large M,

$$S_k \approx a_{d,k} M^{(d-2)/(d-1)},$$

where $a_{d,k}$ is a positive constant. Thus, the integrand of Eq. (74) goes as $\exp(-\beta M+\kappa_B^{-1} a_{d,k} M^{(d-2)/(d-1)})$ and therefore the partition function converges.

This argument breaks down in the $l \rightarrow \infty$ limit: in that case, the entropy is

$$S_k^0 = a_{d,k}^0 M^{(d-2k)/(d-2k-1)},$$

with $a_{d,k}^0$ a different positive constant, which yields a divergent partition function.

The lesson one can draw from this exercise is that the presence of a negative cosmological constant is sufficient to render the canonical ensemble well defined for all the theories described here.

V. SUMMARY AND DISCUSSION

A. Theories described by the action I_k

We have examined a family of gravitation theories in dimension d, whose common feature is to possess vacuum solutions with maximal symmetry. This means that the theories—described by the action I_k—have a unique cosmological constant. For a given d there exist $[(d-1)/2]$ different theories labeled by the integer k, which is the highest power of curvature in the Lagrangian. For $k=1$, the EH action is recovered, while for the largest value of k, that is $k=[(d-1)/2]$, BI and CS theories are obtained. These three cases exhaust the different possibilities up to six dimensions, and new interesting cases arise for $d \geqslant 7$. For instance, the case with $k=2$, which is described by the action (19), exists only for $d>4$: In five dimensions this theory is equivalent to CS; for $d=6$ it is equivalent to BI, and for $d=7$ and up, it defines a new class of theories.

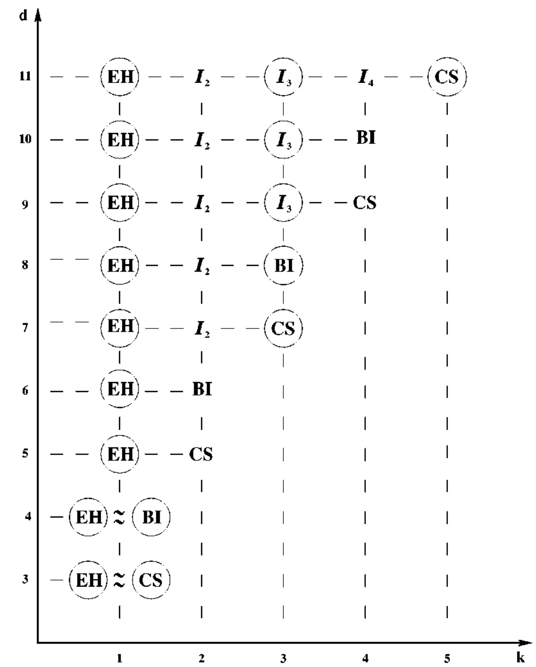

FIG. 5. Black hole scan: summary of all theories described by I_k up to 11 dimensions. The integer $k=1,\ldots,[(d-1)/2]$ represents the highest power of curvature in the action. The columns with odd k are singled out by cosmic censorship. The supersymmetric extensions of EH and CS theories are known. The supergravities for the remaining I_k's are unknown.

B. Special cases selected from cosmic censorship

A first distinction between the different theories mentioned above comes from the study of their spherically symmetric, static solutions. It is found that for odd k, physical black holes satisfying the cosmic censorship criterion exist. For even k, however, both physical black holes and solutions with naked singularities with positive mass exist. This already casts doubt on the soundness of this subset of theories. Moreover, the absence of a cosmic censorship principle would be in conflict with the existence of a positive energy theorem obtained from supersymmetry. This means that the supersymmetric extensions of the theories considered here can be expected to be very different for odd and even k. In fact, as it has been shown in [31], CS theories with even k—defined for $d=5,9,\ldots$—have a supersymmetric extension based on superunitary groups, whereas for odd k ($d=3,7,11,\ldots$) the corresponding supergravities are based on the orthosymplectic groups.

The different theories considered here are summarized in the scheme shown in Fig. 5.

Here we have highlighted the odd k columns as they would represent better candidates for physical theories based on the criterion of cosmic censorship versus supersymmetry.

Note that CS theories are the representatives of the lowest possible dimension for a given k. Moreover, CS gravity theories exhibit local AdS symmetry whereas all other gravitation theories of the same dimension only have local Lorentz invariance (see Appendix A).

Over the years, 11 dimensional spacetime has been believed to be the arena for the ultimate unified theory. From the present analysis, it follows that in $d=11$, the cases $k=1,3,5$ are of special interest. The supersymmetric extension for $k=1$ is the famous Cremmer-Julia-Scherk supergravity [32], which only exists if the cosmological constant vanishes [33]. The supersymmetric extension for $k=5$ with a finite Λ is also known [34,31], whose vanishing cosmological constant version is described in [35]. The corresponding supersymmetric extension of the gravity theory with $k=3$ is an open problem.

C. Black holes

For all dimensions and for any k, there exist well-behaved black hole solutions, in the sense that the singularities are hidden by an event horizon. For $d-2k\neq 1$, the causal structure of these black holes is the same as that of Schwarzschild-AdS and Reissner-Nordström-AdS spacetimes. However, this set of black holes differs from standard d-dimensional Schwarzschild and Reissner-Nordström solutions in that their asymptotic behavior, with respect to the vacuum, is given by $g_{00}-\overline{g}_{00}\approx r^{-[(d-2k-1)/k]}$. Again, the CS case stands separate from the rest, in that the causal structure of the vacuum is the same as that of $2+1$ dimensions, and analogously, there is a mass gap between the $M=0$ black hole and AdS spacetime ($M=-1/2G_{n-1}$). Furthermore, in the vanishing cosmological constant limit, the CS theory supports no static, spherically symmetric black holes.

In the electrically charged case, the black holes for $k\neq 1$ predict a minimum size for a physical source. It is noteworthy that the geometry encodes this restriction for all cases, except for the EH action.

D. Thermodynamics

The presence of a negative cosmological constant for the entire set of theories described by the action I_k makes it possible for black holes to reach thermal equilibrium with a heat bath. The AdS radius l acts as a regulator allowing the canonical ensemble to be well defined, unlike the case of zero cosmological constant. The black hole entropy obeys the area law only in the case $k=1$. For other values of k, the entropy respects the second law of thermodynamics, because $dS/dr_+>0$, but the area law is recovered only in the limit $r_+/l\rightarrow\infty$.

In the limit $\Lambda\rightarrow 0$, the area law never holds, except for $k=1$. In that limit, the temperature has no minimum and consequently the thermodynamic equilibrium cannot be reached.

The thermodynamic behavior is qualitatively the same as the Schwarzschild-AdS$_4$ black hole in the generic cases $d-2k\neq 1$. On the other hand, Chern-Simons black holes for odd dimensions behave like the $d=3$ case.

In the generic cases, black holes have a minimum temperature T_c at $r_+=r_c=l\sqrt{(d-2k-1)/(d-1)}$, so that—as is depicted in Fig. 3—those whose horizon radius exceed the unstable equilibrium position r_u can reach equilibrium with a heat bath of temperature higher than T_c. If the heat bath has a temperature below T_c, or $r_+<r_u$, the black holes evaporate.

In the CS case, the temperature grows linearly with r_+; hence there is no critical temperature and the thermal equilibrium is always attained.

In an equilibrium configuration, the free energy $F=M-TS$ can be expressed as a function of r_+. For fixed k the behavior of F can be found from Eqs. (24), (59) and (68) as

$$F(r_+\rightarrow 0)\sim\frac{r_+^{d-2k-1}}{2(d-2k)G_k}, \tag{75a}$$

$$F(r_+\rightarrow\infty)\sim-\frac{r_+^{d-1}}{2(d-2)G_k l^{2k}}. \tag{75b}$$

This change in sign has been interpreted as an indication that, for small r_+ the black hole would be unstable for decay into AdS spacetime, while for large r_+ the black hole would be stable [5]. This suggests that a phase transition would occur at $F(r_+)=0$. This conclusion, however, contradicts the fact that the phase transition actually occurs at the critical value r_c, where the specific heat C changes sign, and which does not coincide with the zero of $F(r_+)$. In particular, considering the EH action ($k=1$), the change of sign in F occurs at $r_+=l$ while $r_c=l\sqrt{(d-3)/(d-1)}<l$. Moreover, for the CS case, $d-2k=1$, there is no phase transition at all, although F still has a change in sign. The source of the disagreement lies in that the canonical ensemble is defined keeping T fixed, while the limits in Eqs. (75a) and (75b) do not respect this condition.

From all the evidence presented here, it is apparent that CS theories form an exceptional class: They are genuine gauge theories whose supersymmetric extension is known; their black hole spectrum has a mass gap separating it from AdS spacetime, and these black holes possess remarkable thermodynamical properties. CS black holes can reach thermal equilibrium with a heat bath at any temperature, and the positivity of the specific heat guarantees their stability under thermal fluctuations.

In contrast with the generic case, a small CS black hole is stable against decay by Hawking radiation. This suggests that, as in the three dimensional case, CS (super)gravities could have a well-defined quantum theory.

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APPENDIX A: CS AND BI THEORIES

Requiring that the integrability conditions of equation (4) do not impose further algebraic constraints on the curvature or the torsion beyond Eq. (5) implies that the coefficients α_p's in Eq. (2) satisfy a recursive equation, whose solution fixes them in terms of the gravitational and cosmological constants [18]. An equivalent way to express this is that the α_p's become fixed as in Eq. (9) with $k=[(d-1)/2]$, just requiring the existence of a sector in the theory with propagating torsion. Thus, in $d=2n$ dimensions, the Lagrangian reads

$$L=\frac{\kappa l^2}{2n}\epsilon_{a_1\cdots a_d}\bar{R}^{a_1a_2}\cdots\bar{R}^{a_{d-1}a_d}, \tag{A1}$$

where[8] $\bar{R}^{ab}:=R^{ab}+(1/l^2)e^ae^b$.

The expression (A1) is proportional to the Pfaffian of the 2-form $\bar{R}^{ab}$ and, in this sense, it has a Born-Infeld-like form [36]:

$$L=2^{n-1}(n-1)!\,\kappa l^2\sqrt{\det\left(R^{ab}+\frac{1}{l^2}e^ae^b\right)}. \tag{A2}$$

For $d=2n-1$ dimensions, the Lagrangian is given by the Euler-Chern-Simons form for the AdS group, whose exterior derivative is proportional to the Euler density in $2n$ dimensions,

$$dL^{AdS}_{G\,2n-1}=\frac{\kappa l}{2n}\epsilon_{A_1\cdots A_{2n}}\bar{R}^{A_1A_2}\cdots\bar{R}^{A_{2n-1}A_{2n}}=\bar{\kappa}\mathcal{E}_{2n}, \tag{A3}$$

where $\bar{R}^{AB}$ stands for the AdS curvature. This Lagrangian was discussed in [37] and also in [23] for torsion-free manifolds.

Additional terms which depend explicitly on the torsion are required by local supersymmetry [31,34] and they can be consistently added to the Lagrangian only for $d=4m-1$ [18].

These torsional Lagrangians are odd under parity and are obtained from the Chern characters associated with the AdS curvature in $4m$ dimensions. Furthermore, the coefficients in front of the different terms in these torsional Lagrangians are necessarily quantized. The odd dimensional action, with or without torsional terms, has a larger local symmetry given by $SO(d-1,2)$, so that beyond standard local Lorentz symmetry ($\delta e^a=\lambda^a{}_be^b$ and $\delta\omega^{ab}=-D\lambda^{ab}$), these theories are invariant also under local ''AdS translations:''

$$\delta e^a=-D\lambda^a$$
$$\delta\omega^{ab}=\frac{1}{l^2}(\lambda^ae^b-\lambda^be^a). \tag{A4}$$

[8]A positive cosmological constant is obtained making $l^2\to-l^2$.

APPENDIX B: CONSERVED CHARGES FROM A BACKGROUND-INDEPENDENT SURFACE INTEGRAL

If one deals with more general solutions possessing different isometries, the identification of the integration constants with the conserved charges through the minisuperspace trick does not work, because in general the reduced action does not lead to the true extremum of the original action. The Hamiltonian method provides a way to express the mass as a surface integral [25]. However, this procedure requires the invertibility of the symplectic matrix associated with the action I_k. This is impossible to perform globally in phase space, because there are field configurations for which the symplectic form degenerates. Therefore, no general formula could be found for an arbitrary field configuration.

A way to circumvent this problem is carried out in $d=2n$ following a recently proposed method [12,13] which is appropriate to deal with asymptotically AdS spacetimes.

Consider the action I_k defined in Eq. (8). In first order formalism, the existence of an extremum of I_k for asymptotically locally AdS spacetimes fixes the boundary term that must be added to the action as being proportional to the Euler density multiplied by a fixed weight factor. Hence, in order to cancel the boundary term coming from the variation of I_k, the total action including the boundary term—up to a constant—is given by

$$I_T=I_k+\kappa\alpha_n\int\mathcal{E}_{2n}, \tag{B1}$$

with

$$\alpha_n=c_n^k:=\frac{(-1)^{n+k+1}l^{2(n-k)}}{2n\binom{n-1}{k}}. \tag{B2}$$

The total action I_T is invariant under diffeomorphisms by construction, because I_k is written in terms of differential forms. Thus, Noether's theorem provides a conserved current ($d*J=0$) associated with this invariance, which can be locally written as $*J=dQ$. Assuming the topology of the manifold to be of the form $\mathcal{M}=R\times\Sigma$, this procedure yields a regularized and background-independent expression for the conserved charges associated with a Killing vector ξ, which is globally defined on the boundary of the spatial section $\partial\Sigma$. The surface integral reads

$$Q(\xi)=\int_{\partial\Sigma}\xi^\mu\omega^{ab}_\mu\mathcal{T}_{ab}, \tag{B3}$$

where, $\mathcal{T}_{ab}$ is the variation of the total Lagrangian with respect to the curvature

$$\mathcal{T}_{ab}:=\frac{\delta L_T}{\delta R^{ab}}=\sum_{p=1}^{n}c^k_pp\mathcal{T}^p_{ab}, \tag{B4}$$

with

$$\mathcal{T}^{p}_{ab}=\kappa\epsilon_{aba_3\cdots a_d}R^{a_3a_4}\cdots R^{a_{2p-1}a_{2p}}e^{a_{2p+1}}\cdots e^{a_d}, \tag{B5}$$

and where the coefficients c_p^k are defined through Eqs. (9) and (B2).

The mass is obtained from Eq. (B3) when $\xi=\partial_t$, without making further assumptions about the matching with a background geometry nor with its topology.

One way to check this result is evaluating the mass for the black hole metrics (26), which leads to the expected result

$$Q(\partial_t)=M. \tag{B6}$$

It is a simple exercise to check that formula (B3) vanishes when evaluated on any constant curvature spacetime—satisfying $\bar{R}^{ab}=R^{ab}+l^{-2}e^ae^b=0$—which admits at least one Killing vector. This means that spaces which are locally AdS have vanishing Noether charges for the whole set of theories defined by I_k in even dimensions. These spaces in general possess non-trivial topologies and could be regarded as different possible vacua. Hence one can find massive solutions which correspond to excitation of the corresponding vacuum in the same topological sector.

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