Triad representation of the Chern-Simons state in quantum gravity

Robert Paternoga and Robert Graham
Fachbereich Physik, Universität-GH Essen, D-45117 Essen, Germany
(Received 30 March 2000; published 14 September 2000)

We investigate a triad representation of the Chern-Simons state of quantum gravity with a nonvanishing cosmological constant. It is shown that the Chern-Simons state, which is a well-known exact wave functional within the Ashtekar theory, can be transformed to the real triad representation by means of a suitably generalized Fourier transformation, yielding a complex integral representation for the corresponding state in the triad variables. It is found that topologically inequivalent choices for the complex integration contour give rise to linearly independent wave functionals in the triad representation, which all arise from the *one* Chern-Simons state in the Ashtekar variables. For a suitable choice of the normalization factor, these states turn out to be gauge invariant under arbitrary, even topologically nontrivial gauge-transformations. Explicit analytical expressions for the wave functionals in the triad representation can be obtained in several interesting asymptotic parameter regimes, and the associated semiclassical 4-geometries are discussed. In restriction to Bianchi-type homogeneous 3-metrics, we compare our results with earlier discussions of homogeneous cosmological models. Moreover, we define an inner product on the Hilbert space of quantum gravity, and choose a natural gauge condition fixing the time gauge. With respect to this particular inner product, the Chern-Simons state of quantum gravity turns out to be a *non-normalizable* wave functional.

PACS number(s): 04.60.Ds, 11.10.Jj, 11.15.Kc

I. INTRODUCTION

After four decades of vigorous research, a consistent quantization of general relativity remains as one of the most fundamental problems in theoretical physics. Aside from string theory [1,2], a promising approach to this problem is provided by a *canonical* quantization of gravity. Since early attempts in the 1960s [3,4], canonical quantum gravity enjoyed a renaissance after Ashtekar's discovery of complex spin-connection variables [5,6], which replaced [7] the metric variables used up till then. The new *Ashekar representation* of general relativity turned out to be closely related to a Yang-Mills theory of a local SO(3)-gauge group [5], and therefore many ideas and concepts known from Yang-Mills theory could be carried over to the theory of gravity. In particular, the *loop representation*, which had just been investigated within Yang-Mills theory [8], furnished yet another representation of general relativity [5,9,10], and, moreover, a remarkable connection between gravity and knot theory [9,11]. Later on, the loop representation of general relativity advanced to a mathematically rigorous theory within the framework of discretized models of gravity, the so-called quantum spin networks [12,13].

As one crucial advantage of the Ashtekar representation, the constraint operators of quantum gravity took a polynomial from in the new spin-connection variables, and explicit solutions were found. Among the different quantum states discussed up till then [14,15], the *Chern-Simons* state [16,17] played an outstanding role, since it was the only wave functional with a well-defined semiclassical limit.[1] A loop representation of the Chern-Simons state was investigated, and turned out to be closely related to the Kauffman brackets [20]. Moreover, this particular state was found to make an obvious connection between quantum gravity and topological field theory [20,21].

However, a physical interpretation of the Chern-Simons state within the Ashtekar representation implied several problems, which arose from the *reality conditions* underlying Ashtekar's complex theory of gravity [5]. Different *real* versions of Ashtekar's theory were suggested [22–24], but the corresponding quantum constraint equations turned out to be nonpolynomial, lacking the Chern-Simons state as a solution.

Amazingly, a rather natural way to circumvent the problems associated with Ashtekar's reality conditions has never been investigated: If we would be able to transform the Chern-Simons state from the Ashtekar to the metric representation, the geometrical meaning of the fundamental variables would be obvious, and no further reality conditions would be needed. In addition, questions concerning the normalizability of the Chern-Simons state are much easier to discuss in the real metric variables than in the complex Ashtekar spin-connection variables. It is therefore interesting to find an explicit transformation connecting these two representations, and to study the Chern-Simons state in the metric representation.

Recently, we examined this problem in the framework of the *homogeneous* Bianchi-type IX model [25–27]. As an intermediate step, we introduced the *triad representation* of general relativity, which is trivially connected to the metric representation we were interested in. Then it turned out that the Chern-Simons state in the Ashtekar representation can be transformed to the triad variables by a suitably generalized Fourier transformation. Topologically inequivalent choices for the *complex* integration contour in the Fourier integral gave rise to different, linearly independent quantum states in the triad representation, which all arose from the *one* Chern-Simons state in the Ashtekar variables. We found explicit

[1]Strictly speaking, this is only true for a nonvanishing cosmological constant, where de Sitter-like 4-geometries are described by the semiclassical Chern-Simons state [15,18]. The case of a vanishing cosmological constant has been investigated by Ezawa in [19], where it turned out that the semiclassical 4-geometries will in general suffer from different pathologies.

integral representations for the corresponding states in the triad variables, and gave semiclassical interpretations of the wave functions in different asymptotic parameter regimes.

In the present paper, we now want to push these results for the homogeneous model a big step further, and will ask for the corresponding form of the *inhomogeneous* Chern-Simons state in the triad representation. For technical reasons, we will restrict ourselves to model universes, where the spatial hypersurfaces of constant time are compact and without boundaries, but of arbitrary topology. In order to recover the Chern-Simons state as a quantum state of gravity, we should allow for a nonvanishing cosmological constant, which, by the way, is in complete agreement with current cosmological data [28,29].

The rest of this paper is organized as follows: In Sec. II we define our notation and start from the metric representation of classical general relativity. We introduce the triad and the Ashtekar variables, and give new representations of the constraint observables in terms of a single tensor density, which is closely related to the curvature of the Ashtekar spin connection. A canonical quanization of the theory is performed in Sec. III. Choosing a particular factor ordering for the constraint operators of quantum gravity, we discuss the corresponding operator algebra, and show on a formal level (i.e., without regularization and then performing the limit) that is closes without any quantum corrections. The transformation connecting the Ashtekar and the triad representation is explained in detail, and is then used to derive a formal functional integral representation for the Chern-Simons state in the triad representation. In Sec. IV we study several asymptotic expansions of this functional integral in some physically interesting parameter regimes. In particular, we are interested in the semiclassical form of the Chern-Simons state, which then will allow for a discussion of the semiclassical 4-geometries. A separate Sec. IV B1 is dedicated to the behavior of the Chern-Simons state under large, topologically nontrivial SO(3)-gauge transformations. The value of the Chern-Simons state on Bianchi-type homogeneous 3-manifolds is computed and compared with earlier results obtained within the framework of homogeneous models. In Sec. V we define a formal inner product on the Hilbert space of quantum gravity, which is gauge fixed with respect to the time-redefinition invariance, and examine the normalizability of the Chern-Simons state. Finally, we summarize our conclusions in Sec. VI. Three Appendixes deal with certain technical details. In Appendix A, we discuss the solvability of the saddle-point equations, which determine the semiclassical Chern-Simons state, and show how the solutions of these equations correspond to *divergence-free* triads in the limit of a vanishing cosmological constant. In Appendix B, then five divergence-free triads are calculated for homogeneous Bianchi-type IX metrics, and the corresponding values of the Chern-Simons state are given. In order to comment on possible boundary conditions satisfied by the Chern-Simons state, a further Appendix C deals with the asymptotic behavior of particular semiclassical 4-geometries, which arise for a special class of initial 3-metrics.

II. TRIAD REPRESENTATION AND ASHTEKAR VARIABLES

In order to set the stage and to define our notation let us briefly recall the ADM Hamiltonian formulation of general relativity [3,30,31] in terms of the densitized inverse triad $\tilde{e}^{i}{}_{a}$ and its canonically conjugate momentum p_{ia}. This will be called the triad representation for short [5,17,23,32,33].

The most commonly used form of the ADM formulation [3] employs as generalized coordinates the metric tensor h_{ij} on a family of spacelike 3-manifolds foliating space-time. Alternatively one may also employ the inverse metric tensor h^{ij} with $h^{ij}h_{jk}=\delta^{i}_{k}$, or, what will be done here, the densitized inverse metric

$$\tilde{\tilde{a}}^{ij}=hh^{ij} \tag{2.1}$$

with $h=\det(h_{ij})$.[2] Then the canonically conjugate momenta $\underset{\sim}{\pi}_{ij}$, which form a tensor density of weight -1, become

$$\underset{\sim}{\pi}_{ij}=\frac{\delta L}{\delta\dot{\tilde{\tilde{a}}}^{ij}}=\frac{1}{\gamma\sqrt{h}}K_{ij}, \tag{2.2}$$

where $\gamma=16\pi G$ is a convenient abbreviation containing Newton's constant G, and K_{ij} is the usual extrinsic curvature describing the embedding of the 3-manifold in space-time. The quantity L in Eq. (2.2) is the Lagrangian defined by the Einstein-Hilbert action [30,31], in which we include a cosmological term with a cosmological constant Λ. This choice of variables implies a symplectic structure on phase-space defined by the Poisson brackets

$$\{\tilde{\tilde{a}}^{ij}(x),\underset{\sim}{\pi}_{kl}(y)\}=\frac{1}{2}(\delta^{i}_{k}\delta^{j}_{l}+\delta^{i}_{l}\delta^{j}_{k})\delta^{3}(x-y),$$

$$\{\tilde{\tilde{a}}^{ij}(x),\tilde{\tilde{a}}^{kl}(y)\}=0=\{\underset{\sim}{\pi}_{ij}(x),\underset{\sim}{\pi}_{kl}(y)\}. \tag{2.3}$$

Indices i,j will be raised and lowered by h^{ij} and its inverse. In order to move on to the triad representation let us introduce the densitized inverse triad $\tilde{e}^{i}{}_{a}$ via

$$\tilde{e}^{i}{}_{a}\cdot\tilde{e}^{j}{}_{a}=\tilde{\tilde{a}}^{ij}, \tag{2.4}$$

and define an enlarged phase space by introducing canonically conjugate momenta p_{ia} of the $\tilde{e}^{i}{}_{a}$ with Poisson brackets

$$\{\tilde{e}^{i}{}_{a}(x),p_{jb}(y)\}=\delta^{i}_{j}\delta_{ab}\delta^{3}(x-y),$$

$$\{p_{ia}(x),p_{jb}(y)\}=0. \tag{2.5}$$

In the following we shall also make use of the triad 1-forms e_{ia} and the triad vectors $e^{i}{}_{a}=\tilde{e}^{i}{}_{a}/\sqrt{h}$. In order to guarantee that Eq. (2.3) is compatible with Eqs. (2.4), (2.5), we relate $\underset{\sim}{\pi}_{ij}$ to p_{ja} via

[2]Here and in the following densities of positive weight are denoted by an upper and densities of negative weight by a lower tilde.

$$\underset{\sim}{\pi}_{ij}=\frac{1}{2\sqrt{h}}\,e_{ia}p_{ja}\,, \tag{2.6}$$

which serves to satisfy the first of Eqs. (2.3). Furthermore we introduce the three additional constraints

$$\tilde{\mathcal{J}}:=\varepsilon_{abc}\tilde{e}^{i}{}_{b}p_{ic}\overset{!}{=}0. \tag{2.7}$$

Here the Levi-Cevitta tensor ε_{abc} is defined by

$$\varepsilon_{abc}:=\varepsilon(e_{ia})\cdot[abc], \tag{2.8}$$

where $\varepsilon(e_{ia})\in\{\pm 1\}$ measures the orientation of the triad e_{ia}, and $[abc]$ is the totally antisymmetric Levi-Cevitta symbol normalized such that $[123]=+1$. On the constraint hypersurface defined by Eq. (2.7) the quantity $\underset{\sim}{\pi}_{ij}$ defined by Eq. (2.6) is easily checked to be symmetric in i,j as required by Eq. (2.2) and to satisfy the last of Eqs. (2.3).

The ADM Hamiltonian [30,31]

$$H^{\mathrm{ADM}}=\int d^{3}x(N\tilde{\mathcal{H}}_{0}^{\mathrm{ADM}}+N^{i}\tilde{\mathcal{H}}_{i}) \tag{2.9}$$

with Lagrangian parameters N,N^{i} and constraints $\tilde{\mathcal{H}}_{0}^{\mathrm{ADM}},\tilde{\mathcal{H}}_{i}$ given in terms of $\tilde{\tilde{a}}^{ij},\underset{\sim}{\pi}_{ij}$ is easily rewritten in terms of the triad representation using Eqs. (2.4), (2.6). This yields (cf. [5,17,23,33])

$$\tilde{\mathcal{H}}_{0}^{\mathrm{ADM}}=-\frac{\gamma}{4}\,e_{ia}\tilde{\varepsilon}^{ijk}\varepsilon_{abc}p_{jb}p_{kc}+\frac{1}{\gamma}\,e_{ia}\tilde{\varepsilon}^{ijk}F_{jka}+\frac{2\Lambda}{\gamma}\sqrt{h},$$

$$\tilde{\mathcal{H}}_{i}=\partial_{j}(\tilde{e}^{j}{}_{a}p_{ia})-\tilde{e}^{j}{}_{a}\partial_{i}p_{ja}\,, \tag{2.10}$$

where $\tilde{\varepsilon}^{ijk}$ is the spatial Levi-Cevitta tensor density,[3] and $F_{jka}=\partial_{j}\omega_{ka}-\partial_{k}\omega_{ja}+\varepsilon_{abc}\omega_{jb}\omega_{kc}$ is the curvature of the Riemannian spin connection $\omega_{ia}=-\frac{1}{2}\varepsilon_{abc}e_{jb}\nabla_{i}e^{j}{}_{c}$. The additional constraints (2.7) must of course be added to the Hamiltonian (2.9) with new Lagrangian parameters Ω_{a}.

The introduction of the complex Ashtekar variables [5,6,34]

$$\mathcal{A}_{ia}=\omega_{ia}\pm\frac{i\gamma}{2}p_{ia} \tag{2.11}$$

instead of the canonical momenta p_{ia} is now convenient in order to simplify the constraints. In the framework of this paper we shall use the variables $\mathcal{A}_{ia}$ just as auxiliary quantities. In Eq. (2.11) either "+" or "−" may be chosen, but we will keep this option open by using both signs together. The two choices are classically equivalent, but lead to inequivalent quantizations in the quantum theory. The Poisson brackets in the new variables then take the form

$$\{\tilde{e}^{i}{}_{a}(x),\mathcal{A}_{jb}(y)\}=\pm\frac{i\gamma}{2}\,\delta^{i}_{j}\delta_{ab}\delta^{3}(x-y), \tag{2.12}$$

$$\{\mathcal{A}_{ia}(x),\mathcal{A}_{jb}(y)\}=0. \tag{2.13}$$

The second of these relations follows from the fact that the Riemannian spin connection ω_{ia} can be expressed as [7,17]

$$\omega_{ia}=\frac{\delta\phi}{\delta\tilde{e}^{i}{}_{a}}, \tag{2.14}$$

with

$$\phi:=-\frac{1}{2}\int d^{3}x\tilde{\varepsilon}^{ijk}e_{ia}\partial_{j}e_{ka}\,. \tag{2.15}$$

Employing $\mathcal{A}_{ia}$ as a new and complex spin connection it is convenient to use also its associated curvature

$$\mathcal{F}_{ija}=\partial_{i}\mathcal{A}_{ja}-\partial_{j}\mathcal{A}_{ia}+\varepsilon_{abc}\mathcal{A}_{ib}\mathcal{A}_{jc}\,. \tag{2.16}$$

Then the constraints take the more pleasing form (cf. [5,17,23])

$$\tilde{\mathcal{H}}_{0}^{\mathrm{ADM}}\equiv\tilde{\mathcal{H}}_{0}\mp i\,\partial_{i}(e^{i}{}_{a}\tilde{\mathcal{J}}_{0})\overset{!}{=}0, \tag{2.17}$$

with

$$\tilde{\mathcal{H}}_{0}=\frac{1}{\gamma}e_{ia}\left[\tilde{\varepsilon}^{ijk}\mathcal{F}_{jka}+\frac{2}{3}\Lambda\tilde{e}^{i}{}_{a}\right], \tag{2.18}$$

$$\tilde{\mathcal{H}}_{i}\equiv\mp\frac{2i}{\gamma}[\tilde{e}^{j}{}_{a}\partial_{j}\mathcal{A}_{ia}-\tilde{e}^{j}{}_{a}\partial_{i}\mathcal{A}_{ja}+\mathcal{A}_{ia}\partial_{j}\tilde{e}^{j}{}_{a}]\overset{!}{=}0, \tag{2.19}$$

$$\tilde{\mathcal{J}}_{a}\equiv\pm\frac{2i}{\gamma}[\partial_{i}\tilde{e}^{i}{}_{a}+\varepsilon_{abc}\tilde{e}^{i}{}_{c}\mathcal{A}_{ib}]\overset{!}{=}0, \tag{2.20}$$

and the Hamiltonian

$$H=\int d^{3}x(N\tilde{\mathcal{H}}_{0}+N^{i}\tilde{\mathcal{H}}_{i}+\Omega_{a}\tilde{\mathcal{J}}_{a}), \tag{2.21}$$

endowed with the symplectic structure (2.12), (2.13), is dynamically equivalent to the Arnowitt-Deser-Misner (ADM) Hamiltonian (2.9). In fact, as long as $\Lambda\neq 0$, the constraints (2.18)–(2.20) can all be expressed in terms of the single tensor density $\tilde{\mathcal{G}}^{i}_{\Lambda,a}$ defined by

$$\tilde{\mathcal{G}}^{i}_{\Lambda,a}=\frac{1}{2}\tilde{\varepsilon}^{ijk}\mathcal{F}_{jka}+\frac{1}{3}\Lambda\tilde{e}^{i}{}_{a}\,, \tag{2.22}$$

namely,

$$\tilde{\mathcal{H}}_{0}\equiv\frac{2}{\gamma}e_{ia}\tilde{\mathcal{G}}^{i}_{\Lambda,a}\overset{!}{=}0, \tag{2.23}$$

[3]With our definition of ε_{abc} in Eq. (2.8) the spatial Levi-Cevitta tensor density is naturally obtained as $\tilde{\varepsilon}^{ijk}=\sqrt{h}\varepsilon_{abc}e^{i}{}_{a}e^{j}{}_{b}e^{k}{}_{c}$.

$$\tilde{\mathcal{H}}_i = \pm\frac{2i}{\gamma}\varepsilon_{ijk}\tilde{e}^j{}_a\tilde{\mathcal{G}}^k_{\Lambda,a} - \mathcal{A}_{ia}\tilde{\mathcal{J}}_a \overset{!}{=} 0, \tag{2.24}$$

$$\tilde{\mathcal{J}}_a = \pm\frac{6i}{\gamma\Lambda}\mathcal{D}_i\tilde{\mathcal{G}}^i_{\Lambda,a} \overset{!}{=} 0, \tag{2.25}$$

where $\mathcal{D}_i$ is the covariant derivative with respect to the connection $\mathcal{A}_{ia}$. For $\Lambda=0$ the relation of the constraint $\mathcal{J}_a$ with $\tilde{\mathcal{G}}^i_{\Lambda,a}$ is lost. A simple way to satisfy all the constraints (2.23)–(2.25) for $\Lambda\neq 0$ is to restrict the phase space by the nine conditions

$$\tilde{\mathcal{G}}^i_{\Lambda,a} \overset{!}{=} 0. \tag{2.26}$$

Equations (2.26) are more restrictive than the seven Eqs. (2.23)–(2.25) which they imply, i.e., we can only hope to get special solutions in this manner. Remarkably, Eqs. (2.26), if imposed as initial conditions, remain satisfied for all times under the time evolution generated by the Hamiltonian (2.21). This follows from the Poisson brackets

$$\left\{\int d^3x N^i\tilde{\mathcal{H}}_i, \int d^3y \lambda_{ja}\tilde{\mathcal{G}}^j_{\Lambda,a}\right\}$$

$$= \int d^3z (N^i\partial_i\lambda_{ja} + \lambda_{ia}\partial_j N^i)\tilde{\mathcal{G}}^j_{\Lambda,a}, \tag{2.27}$$

$$\left\{\int d^3x \Omega_a\tilde{\mathcal{J}}_a, \int d^3y \lambda_{jb}\tilde{\mathcal{G}}^j_{\Lambda,b}\right\}$$

$$= \int d^3z \Omega_a\varepsilon_{abc}\lambda_{jb}\tilde{\mathcal{G}}^j_{\Lambda,c}, \tag{2.28}$$

$$\left\{\int d^3x N\tilde{\mathcal{H}}_0, \int d^3y \lambda_{ja}\tilde{\mathcal{G}}^j_{\Lambda,a}\right\}$$

$$= \pm\frac{i}{2}\int d^3z \frac{N}{\sqrt{h}}(e_{ia}e_{jb} - 2e_{ib}e_{ja})\tilde{\varepsilon}^{jkl}\mathcal{D}_k\lambda_{lb}\tilde{\mathcal{G}}^i_{\Lambda,a}, \tag{2.29}$$

which may be verified with some labor using Eqs. (2.22) and (2.23)–(2.25). They imply that on the subspace $\tilde{\mathcal{G}}^i_{\Lambda,a}=0$ of phase space

$$\{H, \tilde{\mathcal{G}}^i_{\Lambda,a}\} = 0; \tag{2.30}$$

i.e., this subspace is conserved.

Equations (2.26) bear a superficial formal similarity to Einstein's field equations

$$G^\mu_{\Lambda,\nu} := G^\mu{}_\nu + \Lambda\delta^\mu_\nu = 0 \tag{2.31}$$

in four space-time dimensions ($\mu,\nu=0,1,2,3$) with the 4-dimensional Einstein tensor $G^\mu{}_\nu$ satisfying the Bianchi identity

$$\nabla_\mu G^\mu{}_\nu \equiv 0 \tag{2.32}$$

and also $\nabla_\mu G^\mu_{\Lambda,\nu}=0$, because the affine connection satisfies the metric postulate. Since $\tilde{\mathcal{G}}^i_{\Lambda,a}$ similarly decomposes in a curvature part satisfying a Bianchi identity

$$\mathcal{D}_i(\tilde{\varepsilon}^{ijk}\mathcal{F}_{jka}) \equiv 0, \tag{2.33}$$

and a cosmological term proportional to Λ it is a three-dimensional analog of $G^\mu_{\Lambda,\nu}$. The analogy extends even to $\mathcal{D}_i\tilde{\mathcal{G}}^i_{\Lambda,a}=0$, which holds due to the Bianchi identity but requires in addition for the constraint (2.20). However, it has to be kept in mind that the spin connection $\mathcal{A}_{ia}$ and the densitized inverse triad $\tilde{e}^i{}_a$ in $\tilde{\mathcal{G}}^i_{\Lambda,a}$ are still *independent* variables. Equations (2.26) therefore are not a closed set of field equations on the spatial manifolds.

III. QUANTIZATION

Canonical quantization in the triad representation is achieved by imposing the commutation relations

$$[\tilde{e}^i{}_a(x), p_{jb}(y)] = i\hbar\,\delta^i_j\delta_{ab}\delta^3(x-y) \tag{3.1}$$

and representing $p_{ia}(x)$ as

$$p_{ia} = \frac{\hbar}{i}\frac{\delta}{\delta\tilde{e}^i{}_a(x)}. \tag{3.2}$$

This implies for the $\mathcal{A}_{ia}$ the representation

$$\mathcal{A}_{ia}(x) = \omega_{ia}(x) \pm \frac{\gamma\hbar}{2}\frac{\delta}{\delta\tilde{e}^i{}_a(x)}, \tag{3.3}$$

where $\omega_{ia}(x)$, given by Eqs. (2.14), (2.15), is a functional of $\tilde{e}^j_b(y)$ and a diagonal operator in this representation. We now have to choose a special factor ordering in the constraint operators $\tilde{\mathcal{J}}_a$, $\tilde{\mathcal{H}}_i$, and $\tilde{\mathcal{H}}_0$. It turns out that $\tilde{\mathcal{J}}_a$ does not suffer from an ordering ambiguity. We choose the factor ordering in $\tilde{\mathcal{H}}_0$ and $\tilde{\mathcal{H}}_i$ as given in Eqs. (2.18) and (2.19) in order to achieve closure of the algebra of the generators. Explicitly, the generators are then given by Eqs. (2.18)–(2.20) or Eqs. (2.23)–(2.25) with the ordering of $\tilde{e}^i{}_a, \mathcal{A}_{jb}$ given there. The algebra of the infinitesimal generators is obtained as[4]

$$\left[\int d^3x \xi^i\tilde{\mathcal{H}}_i, \int d^3y \varphi_a\tilde{\mathcal{J}}_a\right] = i\hbar\int d^3z (\xi^i\partial_i\varphi_a)\tilde{\mathcal{J}}_a, \tag{3.4}$$

$$\left[\int d^3x \xi^i\tilde{\mathcal{H}}_i, \int d^3y \eta^j\tilde{\mathcal{H}}_j\right] = i\hbar\int d^3z (\xi^i\partial_i\eta^j - \eta^i\partial_i\xi^j)\tilde{\mathcal{H}}_j, \tag{3.5}$$

[4]The algebra of the constraint operators has been discussed intensively in the literature; see, e.g., [5,17,35]. The factor ordering and the corresponding operator algebra considered here are in agreement with Ashtekar's results in [5].

$$\left[\int d^3x\,\varphi_a\tilde{\mathcal{J}}_a,\int d^3y\,\psi_b\tilde{\mathcal{J}}_b\right]=i\hbar\int d^3z\,\epsilon_{abc}\varphi_a\psi_b\tilde{\mathcal{J}}_c, \quad (3.6)$$

$$\left[\int d^3x\,\varphi_a\tilde{\mathcal{J}}_a,\int d^3y\,N\tilde{\mathcal{H}}_0\right]=0, \quad (3.7)$$

$$\left[\int d^3x\,\xi^i\tilde{\mathcal{H}}_i,\int d^3y\,N\tilde{\mathcal{H}}_0\right]=i\hbar\int d^3z(\xi^i\partial_iN)\tilde{\mathcal{H}}_0, \quad (3.8)$$

$$\left[\int d^3x\,N\tilde{\mathcal{H}}_0,\int d^3y\,M\tilde{\mathcal{H}}_0\right]=i\hbar\int d^3z(N\partial_iM-M\partial_iN)\times h^{ij}(\tilde{\mathcal{H}}_j+\mathcal{A}_{ja}\tilde{\mathcal{J}}_a). \quad (3.9)$$

On the right-hand side of these equations all generators appear on the right, which means that the algebra closes, at least formally (i.e., in the absence of any regularization procedure), without any quantum corrections.

Following Dirac [36], physical states $\Psi[\tilde{e}^i_a]$ must satisfy

$$\tilde{\mathcal{J}}_a\Psi[\tilde{e}^i_a]\overset{!}{=}0 \quad \text{Lorentz invariance}, \quad (3.10)$$

$$\tilde{\mathcal{H}}_i\Psi[\tilde{e}^i_a]\overset{!}{=}0 \quad \text{diffeomorphism invariance}, \quad (3.11)$$

$$\tilde{\mathcal{H}}_0\Psi[\tilde{e}^i_a]\overset{!}{=}0 \quad \text{time-redefinition invariance}. \quad (3.12)$$

Moreover, since the Lorentz constraint (3.10) guarantees only invariance under local SO(3)-gauge transformations of the triad $\tilde{e}^i_a$, while the full symmetry group is given by O(3), we further have to impose a discrete, global parity requirement

$$\mathcal{P}\Psi[\tilde{e}^i_a]:=\Psi[-\tilde{e}^i_a]\overset{!}{=}+\Psi[\tilde{e}^i_a], \quad (3.13)$$

where $\mathcal{P}$ denotes the parity operator acting on functionals of the triad.

As in the classical theory, the constraints (3.10)–(3.12) on physical states are all satisfied if the stronger conditions

$$\tilde{\mathcal{G}}^i_{\Lambda,a}\Psi[\tilde{e}^i_a]\overset{!}{=}0 \quad (3.14)$$

hold, where $\tilde{\mathcal{G}}^i_{\Lambda,a}$ is the tensor density defined by Eqs. (2.22), (2.16) in terms of the operators $\tilde{e}^i_a$ and $\mathcal{A}_{ia}$ given by Eqs. (2.4), (2.11). Remarkably, the quantum operators $\tilde{\mathcal{G}}^i_{\Lambda,a}$ turn out to commute among themselves. It can be seen from Eqs. (2.23)–(2.25), which must now be read as operator equations, that Eqs. (3.10)–(3.12) are implied by Eq. (3.14). The subspace of physical states satisfying Eq. (3.14) is the quantum version of the invariant subspace of classical phase space defined by Eqs. (2.26).

To find the solutions of Eqs. (3.14) it is useful to proceed in two steps. First, it is convenient to perform a similarity transformation (cf. [17])

$$\Psi=\exp\left[\mp\frac{2}{\gamma\hbar}\phi\right]\cdot\Psi', \quad (3.15)$$

where ϕ was defined in Eq. (2.15). Under this transformation, the operators $\mathcal{A}_{ia}$ according to Eq. (3.3) transform like

$$\exp\left[\pm\frac{2}{\gamma\hbar}\phi\right]\cdot\mathcal{A}_{ia}\cdot\exp\left[\mp\frac{2}{\gamma\hbar}\phi\right]=\pm\frac{\gamma\hbar}{2}\frac{\delta}{\delta\tilde{e}^i_a}, \quad (3.16)$$

and Eq. (3.14) becomes explicitly

$$\left[\tilde{\varepsilon}^{imn}\left(\pm\gamma\hbar\,\partial_m\frac{\delta}{\delta\tilde{e}^n{}_a}+\frac{\gamma^2\hbar}{4}\varepsilon_{abc}\frac{\delta^2}{\delta\tilde{e}^m{}_b\,\delta\tilde{e}^n{}_c}\right)+\frac{2\Lambda}{3}\tilde{e}^i{}_a\right]\Psi'=0. \quad (3.17)$$

As a second step, we now consider a representation of $\Psi'[\tilde{e}^i{}_a]$ by a generalized Fourier integral

$$\Psi'[\tilde{e}^i{}_a]=\int_\Gamma\mathcal{D}^9[\mathcal{A}_{ia}]\exp\left[\pm\frac{2}{\gamma\hbar}\int d^3x\,\tilde{e}^i{}_a\mathcal{A}_{ia}\right]\cdot\Psi[\mathcal{A}_{ia}], \quad (3.18)$$

where the complex integration manifold Γ is chosen in such a way that partial integrations with respect to $\mathcal{A}_{ia}$ are permitted without any boundary terms. Besides these restrictions, Γ may be chosen arbitrarily to guarantee the existence of the functional integral (3.18) (cf. discussions of the homogeneous Bianchi IX model [25,27]). Different choices of Γ within these restrictions, which cannot be deformed into each other continuously without crossing a singularity of the integrand, will, in general, correspond to different solutions. Under the transformation (3.18) the fundamental operators $\mathcal{A}_{ia},\tilde{e}^i{}_a$ transform like

$$\tilde{e}^i{}_a\cdot\Psi'\mapsto\mp\frac{\gamma\hbar}{2}\frac{\delta\tilde{\Psi}}{\delta\mathcal{A}_{ia}},\qquad\frac{\delta\Psi'}{\delta\tilde{e}^i{}_a}\mapsto\pm\frac{2}{\gamma\hbar}\mathcal{A}_{ia}\cdot\tilde{\Psi}, \quad (3.19)$$

and Eq. (3.17) becomes

$$\left[\tilde{\varepsilon}^{ijk}\mathcal{F}_{jka}\mp\frac{\gamma\hbar\Lambda}{3}\frac{\delta}{\delta\mathcal{A}_{ia}}\right]\tilde{\Psi}=0. \quad (3.20)$$

Up to a normalization factor $\mathcal{N}$, the unique solution of Eq. (3.20) is the Chern-Simons state (cf. [17])

$$\tilde{\Psi}_{\mathrm{CS}}[\mathcal{A}_{ia}]=\mathcal{N}\exp\left[\pm\frac{3}{\gamma\hbar\Lambda}\mathcal{S}_{\mathrm{CS}}[\mathcal{A}_{ia}]\right], \quad (3.21)$$

with the Chern-Simons functional

$$\mathcal{S}_{\mathrm{CS}}[\mathcal{A}_{ia}]=\int d^3x\,\tilde{\varepsilon}^{ijk}(\mathcal{A}_{ia}\partial_j\mathcal{A}_{ka}+\tfrac{1}{3}\varepsilon_{abc}\mathcal{A}_{ia}\mathcal{A}_{jb}\mathcal{A}_{kc}). \quad (3.22)$$

In the $\tilde{e}^i{}_a$ representation the corresponding wave functional is given by

$$\Psi_{\rm CS}[\tilde{e}^i{}_a]=\mathcal{N}\int_\Gamma \mathcal{D}^9[\mathcal{A}_{ia}]\exp\left[\pm\frac{1}{\gamma\hbar}\left(\int d^3x\,\tilde{\varepsilon}^{ijk}e_{ia}\mathcal{D}_j e_{ka}+\frac{3}{\Lambda}\mathcal{S}_{\rm CS}[\mathcal{A}_{ia}]\right)\right]. \tag{3.23}$$

We shall not attempt a mathematical existence proof of this functional integral and we treat it on a formal level in the following. The state (3.23) is obviously diffeomorphism invariant, and it is also gauge invariant under sufficiently small gauge transformations (i.e., those which are continuously connected to the identical transformation):[5] The contribution from the similarity transformation (3.16) and the Fourier term from Eq. (3.18) fit perfectly together to give the first gauge-invariant term in the exponent of Eq. (3.23), while the second term proportional to $\mathcal{S}_{\rm CS}$ is a well-known gauge-invariant functional. The wave functional $\Psi_{\rm CS}[\tilde{e}^i{}_a]$ given in Eq. (3.23) further turns out to be parity invariant, as it was required by the condition (3.13).

However, for a trivial choice of the prefactor $\mathcal{N}$ in Eq. (3.23) the state $\Psi_{\rm CS}[\tilde{e}^i{}_a]$ *fails* to be invariant under *large* gauge-transformations of the triad, since the Chern-Simons functional in Eq. (3.23) transforms non-trivially under such transformations (cf. [5,37]). At this point it is helpful to notice that the prefactor $\mathcal{N}$ in Eq. (3.23), underlying the only restriction

$$\frac{\delta\mathcal{N}}{\delta\tilde{e}^i{}_a}\overset{!}{=}0, \tag{3.24}$$

is just required to be constant under infinitesimal variations of $\tilde{e}^i{}_a$, while it may still depend on topological invariants of the triad. In Sec. IV B1 we will make use of this remarkable freedom, choosing the normalization factor $\mathcal{N}$ in such a way that the state $\Psi_{\rm CS}[\tilde{e}^i{}_a]$ becomes invariant even under large gauge-transformations of the triad with a non-trivial winding number.

Unfortunately, the integration manifold Γ in Eq. (3.23) cannot be given explicitly, but we will argue that several topologically inequivalent choices for Γ do exist, which give rise to linearly independent quantum states $\Psi_{\rm CS}[\tilde{e}^i{}_a]$. These different states in the $\tilde{e}^i{}_a$ representation all arise from the *one* Chern-Simons state in the $\mathcal{A}_{ia}$ representation, a phenomenon which is well known from discussions of the homogeneous Bianchi IX model in earlier papers [25,27]. Together these states span the subspace of physical states corresponding to the invariant subspace of phase space defined classically by $\tilde{\mathcal{G}}^i_{\Lambda,a}=0$.

IV. ASYMPTOTIC EXPANSIONS OF THE CHERN-SIMONS STATE

Since the functional integral occurring in the $\tilde{e}^i{}_a$ representation of the Chern-Simons state (3.23) is too complicated to be performed analytically, we will restrict ourselves to an asymptotic evaluation of the wave functional (3.23) in several interesting parameter regimes. The possible different asymptotic regimes can be displayed by rewriting the Chern-Simons state (3.23) in dimensionless quantities. Therefore, we introduce the three fundamental length scales of the theory, namely, the Planck scale

$$a_{\rm Pl}:=\sqrt{\gamma\hbar}, \tag{4.1}$$

the cosmological scale parameter,

$$a_{\cos}:=\left(\int d^3x\sqrt{h}\right)^{1/3}, \tag{4.2}$$

and a third length scale, which is associated with the cosmological constant Λ,

$$a_\Lambda:=\sqrt{\frac{3}{\Lambda}}. \tag{4.3}$$

These three length scales give rise to the definition of two dimensionless parameters, for example,

$$\kappa:=\left(\frac{a_{\cos}}{a_\Lambda}\right)^2=\frac{\Lambda}{3}a_{\cos}^2,\quad \mu:=\left(\frac{a_{\cos}}{a_{\rm Pl}}\right)^2=\frac{a_{\cos}^2}{\gamma\hbar}. \tag{4.4}$$

Moreover, we may rescale the triad fields with the help of the cosmological scale parameter $a_{\cos}$ to arrive at dimensionless field variables denoted by a prime:[6]

$$e'_{ia}=a_{\cos}^{-1}e_{ia},\quad \tilde{e}'^i{}_a=a_{\cos}^{-2}\tilde{e}^i{}_a,\quad \sqrt{h'}=a_{\cos}^{-3}\sqrt{h}. \tag{4.5}$$

Making use of Eqs. (4.1)–(4.5) the Chern-Simons state (3.23) reduces to the form

$$\Psi_{\rm CS}[\tilde{e}^i{}_a]=\mathcal{N}\int_\Gamma \mathcal{D}^9[\mathcal{A}_{ia}]\exp[\pm\mu F], \tag{4.6}$$

where the exponent F is defined by

$$F:=\int d^3x\,\tilde{\varepsilon}^{ijk}e'_{ia}\mathcal{D}_j e'_{ka}+\frac{1}{\kappa}\mathcal{S}_{\rm CS}[\mathcal{A}_{ia}]. \tag{4.7}$$

A. The semiclassical limit $\mu\to\infty$

Because of the Gaussian saddle-point form of Eq. (4.6) with respect to the parameter μ it is natural to study the limit $\mu\to\infty$ first. This limit corresponds to the regime $a_{\cos}\gg a_{\rm Pl}$ and also to the formal limit $\hbar\to 0$ [cf. Eq. (4.4)], so we shall

[5] Here and in the following, we shall refer to the SO(3)-gauge invariance just as "gauge invariance" for short. The diffeomorphism and the time-redefinition invariance, which are of course inherent gauge symmetries of the theory as well, will allways be mentioned separately.

[6] By definition, the Ashtekar variables $\mathcal{A}_{ia}$ carry no dimension and need not be rescaled.

refer to it as the *semiclassical* limit for short. In the limit $\mu\to\infty$ the asymptotic form of the integral (4.6) becomes in leading order of μ

$$\Psi_{\rm CS}[\tilde{e}^i{}_a] \overset{\mu\to\infty}{\propto} \mathcal{N}\left|\frac{\mu\,\delta^2 F}{\delta\mathcal{A}_{ia}(x)\,\delta\mathcal{A}_{jb}(y)}\right|^{-1/2}\cdot\exp[\pm\mu F], \tag{4.8}$$

where an infinite prefactor in Eq. (4.8) has been omitted. The asymptotic expression (4.8) has to be evaluated at a saddle point of the exponent F with respect to $\mathcal{A}_{ia}$, which is obtained by solving the saddle-point equations

$$\frac{\delta F}{\delta\mathcal{A}_{ia}} = 2\tilde{e}'^{\,i}{}_a + \frac{1}{\kappa}\tilde{\varepsilon}^{ijk}\mathcal{F}_{jka} \overset{!}{=} 0. \tag{4.9}$$

Equations (4.9) more explicitly take the form

$$\tilde{\varepsilon}^{ijk}\left(\partial_j\mathcal{A}_{ka} + \frac{1}{2}\varepsilon_{abc}\mathcal{A}_{jb}\mathcal{A}_{kc}\right) = -\frac{\Lambda}{3}\tilde{e}^i{}_a, \tag{4.10}$$

and coincide with the classical equations $\tilde{\mathcal{G}}^i_{\Lambda,a}=0$ as they should, since the latter constitute the classical limit of the gravitational Chern-Simons state. The saddle-point equations (4.10) must be read as determining implicitly the complex spin connection $\mathcal{A}_{ia}$ for any given real triad $\tilde{e}^i{}_a$, for which we wish to evaluate $\Psi_{\rm CS}[\tilde{e}^i{}_a]$. Since $\tilde{e}^i{}_a$ carries information about the coordinate system and the local SO(3)-gauge degrees of freedom, the solutions $\mathcal{A}_{ia}$ of Eq. (4.10) for a *given* triad $\tilde{e}^i{}_a$ have *no* further gauge freedom. This is why we expect a discrete, finite set of solutions $\mathcal{A}_{ia}$ of Eq. (4.10) for a fixed triad $\tilde{e}^i{}_a$. A detailed mathematical discussion of the solvability properties of the semiclassical saddle-point equation (4.10) will be given in Appendix A1.

For a fixed triad $\tilde{e}^i{}_a$ the number of the different gauge fields $\mathcal{A}_{ia}$ solving Eq. (4.10) will depend on the topology of the spatial manifold $\mathcal{M}_3$: For example, if $\mathcal{M}_3$ has the topology of the 3-sphere S^3, five distinct solutions $\mathcal{A}_{ia}$ of the corresponding saddle-point equations are found for spatially *homogeneous* 3-manifolds, which are described by the Bianchi IX model (cf. [25,27]). It follows from the arguments given in Appendix A1 that this number of saddle points is preserved under sufficiently small inhomogeneous perturbations of the triad $\tilde{e}^i{}_a$. We therefore find *five* physically inequivalent solutions $\mathcal{A}_{ia}$ in this case. If we consider manifolds $\mathcal{M}_3$ with the topology of the 3-torus T^3, the subset of homogeneous manifolds is described by the Bianchi I model, restricting the number of independent solutions $\mathcal{A}_{ia}$ of Eq. (4.10) to two, as in this homogeneous model. Considering other topologies of $\mathcal{M}_3$, the number of inequivalent saddle points will differ further. However, we will see in Sec. IV B that, for *any* given topology of the spatial 3-manifold $\mathcal{M}_3$, the number of distinct saddle points $\mathcal{A}_{ia}$ of Eq. (4.10) should *at least* be two.

Given a topology of $\mathcal{M}_3$ and a saddle-point solution $\mathcal{A}_{ia}$ of Eq. (4.10), the evaluation of Eq. (4.8) at this saddle point gives a possible semiclassical contribution to the Chern-Simons state $\Psi_{\rm CS}[\tilde{e}^i{}_a]$ in the limit $\mu\to\infty$. It will depend on the choice of the integration contour Γ in Eq. (4.6) whether this particular saddle point contributes to the functional integral or not. Under gauge or coordinate transformations of the triad $\tilde{e}^i{}_a$ the fixed solution $\mathcal{A}_{ia}$ of Eq. (4.10) transforms like a spin connection, since Eq. (4.10) is a coordinate- and gauge-covariant equation. Consequently, the semiclassical expression (4.8) remains unchanged under (sufficiently small) gauge transformations, as indeed must be the case, since $\Psi_{\rm CS}$, also for $\mu\to\infty$, was constructed as as a coordinate- and gauge-invariant state. Therefore, we may solve Eq. (4.10) in any desired gauge for $\tilde{e}^i{}_a$, fixing automatically a gauge for the solutions $\mathcal{A}_{ia}$.

Any *possible* saddle-point contribution (4.8) for a given saddle point $\mathcal{A}_{ia}$ can be chosen to become the *dominant* contribution to the functional integral in Eq. (4.6) in the limit $\mu\to\infty$ by choosing the complex integration manifold Γ suitably. So the number of linearly independent semiclassical wave functionals $\Psi_{\rm CS}[\tilde{e}^i{}_a]$ equals the number of inequivalent saddle points $\mathcal{A}_{ia}$ of Eq. (4.10). This is also the number of linearly independent *exact* wave functionals $\Psi_{\rm CS}[\tilde{e}^i{}_a]$, because the complex integration manifold Γ, constructed as a contour of steepest descent to a given saddle point $\mathcal{A}_{ia}$, satisfies the requirements for Γ in Eq. (3.18) and may therefore be used to define an exact wave functional (4.6). We conclude that the *one* Chern-Simons state (3.21) in the complex Ashtekar representation generates a discrete, finite set of linearly independent gravitational states in the real triad representation, which differ by the topology of the integration manifolds Γ connecting the two representations via Eq. (3.18). The number of the different Chern-Simons states in the $\tilde{e}^i{}_a$ representation depends on the topology of the spatial manifold $\mathcal{M}_3$ and should at least be *two*.

We will now try to construct explicit solutions $\mathcal{A}_{ia}$ of the nonlinear, partial differential equations (4.10). In general, analytical solutions of this complicated set of equations are not available, so we will restrict ourselves to asymptotic solutions in the two different limits $\kappa\to\infty$ and $\kappa\to 0$, which will be treated in Secs. IV B and IV C, respectively.

B. The limit of large scale parameter $\mu\to\infty$, $\kappa\to\infty$

According to our definition of the parameters μ and κ in Eq. (4.4), the limit $\kappa\to\infty$ within the semiclassical limit $\mu\to\infty$ can be realized by taking the scale parameter $a_{\rm cos}$ of the spatial manifold sufficiently large, $a_{\rm cos}\gg a_{\rm Pl}a_\Lambda$. In this special asymptotic regime, solutions of Eq. (4.10) can be found by inserting the ansatz

$$\mathcal{A}_{ia} \overset{\kappa\to\infty}{\sim} \sqrt{\kappa}\,c^{(0)}_{ia} + \mathcal{O}(\kappa^0) \tag{4.11}$$

into the saddle-point equations

$$\tilde{\varepsilon}^{ijk}\left(\partial_j\mathcal{A}_{ka} + \frac{1}{2}\varepsilon_{abc}\mathcal{A}_{jb}\mathcal{A}_{kc}\right) = -\kappa\tilde{e}'^{\,i}{}_a. \tag{4.12}$$

Then we find the two solutions

$$c^{(0)}_{ia} = \pm i e'_{ia}, \tag{4.13}$$

or, equivalently,

$$\mathcal{A}_{ia} \overset{\kappa\to\infty}{\sim} \pm i\sqrt{\frac{\Lambda}{3}}e_{ia}+\mathcal{O}(\kappa^0). \tag{4.14}$$

We should stress that the two signs occurring in Eqs. (4.13), (4.14) are *independent* of the double sign in Eq. (2.11), i.e., for both possible definitions (2.11) of the Ashtekar variables we find two independent, complex conjugate solutions $\mathcal{A}_{ia}$ of the saddle-point equations (4.12) in the limit $\kappa\to\infty$, corresponding to two semiclassical wave functions via Eq. (4.8). To avoid confusion, we will discuss only one of these solutions in the following, which is obtained by choosing the upper sign in Eqs. (4.13), (4.14). The corresponding results for the second solution may then be obtained at any time by a complex conjugation.

The result (4.14) can be improved by calculating the coefficients $c_{ia}^{(n)}$ of the asymptotic series

$$\mathcal{A}_{ia} \overset{\kappa\to\infty}{\sim} \sum_{n=0}^{\infty} c_{ia}^{(n)}\kappa^{(1-n)/2}. \tag{4.15}$$

All coefficients in Eq. (4.15) can be calculated analytically, since, in any order of κ, the non-Abelian term in Eq. (4.12) contains the unknown coefficient $c_{ia}^{(n)}$, while the nonlocal term in Eq. (4.12) is known from the previous orders. Consequently, the recursion equations determining $c_{ia}^{(n)}$ are just algebraic equations at each space point, which, moreover, are linear and analytically solvable for $n>0$. The first three terms of the series (4.15) turn out to be

$$\mathcal{A}_{ia} \overset{\kappa\to\infty}{\sim} \underbrace{i\sqrt{\frac{\pi}{3}}e_{ia}}_{\mathcal{O}(\kappa^{1/2})}+\underbrace{\omega_{ia}}_{\mathcal{O}(\kappa^0)}+\underbrace{i\sqrt{\frac{3}{\Lambda}}\left(\frac{R}{4}e_{ia}-e_{ja}R^j{}_i\right)}_{\mathcal{O}(\kappa^{-1/2})}+\mathcal{O}(\kappa^{-1}). \tag{4.16}$$

To calculate the corresponding saddle-point contribution to the semiclassical Chern-Simons state via Eq. (4.8) we need the Gaussian prefactor and the exponent F defined in Eq. (4.7), evaluated at the saddle point $\mathcal{A}_{ia}$. The asymptotic form of the Gaussian prefactor becomes in the limit $\kappa\to\infty$

$$\left|\frac{\mu\,\delta^2 F}{\delta\mathcal{A}_{ia}(x)\,\delta\mathcal{A}_{jb}(y)}\right|^{-1/2} \overset{\kappa\to\infty}{\propto} \mathbf{h}^{-3/4}, \tag{4.17}$$

with the abbreviation

$$\mathbf{h} := \prod_{x\in\mathcal{M}_3} h(x). \tag{4.18}$$

The exponent in Eq. (4.8) for $\kappa\to\infty$ can be expanded as follows:

$$F \overset{\kappa\to\infty}{\sim} \frac{1}{\gamma\hbar\mu}\left[i\sqrt{\frac{3}{\Lambda}}\int d^3x\sqrt{h}\left(\frac{4\Lambda}{3}-R\right)+\frac{3}{\Lambda}\mathcal{S}_{\rm CS}(\omega_{ia})\right] +\mathcal{O}(\kappa^{-3/2}). \tag{4.19}$$

Here the contribution ϕ from the similarity transformation (3.15) has disappeared, because it precisely cancels with the contribution ω_{ia} in the asymptotic series (4.16) of $\mathcal{A}_{ia}$. The first term in Eq. (4.19) derives from the contributions of order $\kappa^{1/2}$ and $\kappa^{-1/2}$ to the asymptotic series of $\mathcal{A}_{ia}$ given in Eq. (4.16). It defines a real action

$$S=\pm\frac{1}{\gamma}\sqrt{\frac{3}{\Lambda}}\int d^3x\sqrt{h}\left(\frac{4\Lambda}{3}-R\right), \tag{4.20}$$

giving rise to a well-defined, semiclassical time evolution. The term of order κ^{-1} in the expansion (4.16), which was not given explicitly there, because it is rather lengthy, determines the asymptotic form of the second term in Eq. (4.19), which is real valued and therefore governs the asymptotic behavior of $|\Psi_{\rm CS}|^2$. Surprisingly, this contribution again turns out to be a Chern-Simons functional, but with $\mathcal{A}_{ia}$ replaced by the real Riemannian spin connection ω_{ia}. As one can check quite easily, this functional $\mathcal{S}_{\rm CS}[\omega_{ia}]$ has the interesting property that it is also invariant under *local* scale transformations of the triad $e_{ia}\mapsto\exp[\zeta(x)]e_{ia}$.

Inserting the results (4.17) and (4.19) into Eq. (4.8), we find for the semiclassical Chern-Simons state in the $\tilde{e}^i{}_a$ representation

$$\Psi_{\rm CS} \underset{\mu\to\infty}{\overset{\kappa\to\infty}{\propto}} \mathcal{N}\cdot\mathbf{h}^{-3/4}\cdot\exp\left[\pm\frac{1}{\gamma\hbar}\left(i\sqrt{\frac{3}{\Lambda}}\int d^3x\sqrt{h}\left(\frac{4\Lambda}{3}-R\right)+\frac{3}{\Lambda}\mathcal{S}_{\rm CS}[\omega_{ia}]\right)\right], \tag{4.21}$$

where the complex conjugate solution $\Psi_{\rm CS}^*$ is equally possible, if we choose the second saddle-point solution in Eqs. (4.13), (4.14). It is remarkable that this result is universal in the sense that it does not depend on the topology of the spatial 3-manifold $\mathcal{M}_3$.

1. Large gauge transformations

An unsatisfactory feature of the asymptotic state (4.21) is the fact that its exponent is *not* invariant under large gauge transformations with a nonvanishing winding number: As is well known [5,37], in general the Chern-Simons functional $\mathcal{S}_{\rm CS}[\omega_{ia}]$ transforms inhomogeneously under local gauge transformations of the triad,

$$e_{ia}\mapsto\Omega_{ab}e_{ib}\Rightarrow\mathcal{S}_{\rm CS}[\omega_{ia}]\mapsto\mathcal{S}_{\rm CS}[\omega_{ia}]+\tfrac{1}{6}I(\mathbf{\Omega}), \tag{4.22}$$

with $(\Omega_{ab})=\mathbf{\Omega}\in O(3)$ being an arbitrary rotation matrix. The quantity $I(\mathbf{\Omega})$ occurring in Eq. (4.22) is defined by

$$I(\mathbf{\Omega}):=\int d^3x\,\tilde{\varepsilon}^{ijk}\,{\rm Tr}[\mathbf{\Omega}^T\partial_i\mathbf{\Omega}\cdot\mathbf{\Omega}^T\partial_j\mathbf{\Omega}\cdot\mathbf{\Omega}^T\partial_k\mathbf{\Omega}] \tag{4.23}$$

and is known as the Cartan-Maurer invariant [37]. Its value is restricted to be of the form

$$I(\mathbf{\Omega})=I_0\cdot w(\mathbf{\Omega}), \tag{4.24}$$

where the winding number $w(\mathbf{\Omega})$ is an integer, and I_0 is a constant depending only on the topology of the 3-manifold $\mathcal{M}_3$.

A consequence of Eq. (4.22) is that the asymptotic Chern-Simons state (4.21) will not be invariant under general gauge transformations of the triad, at least as long as we make a trivial choice for the normalization factor $\mathcal{N}$ in Eq. (4.21). However, as we pointed out in Sec. III, the factor $\mathcal{N}$ does not need to be *completely* independent of the triad—it is still allowed to depend on topological invariants, such as the Cartan-Maurer invariant. This is why we are free to choose the normalization factor $\mathcal{N}$ according to

$$\mathcal{N} \propto \exp\left[\mp \frac{I(\hat{\mathbf{\Omega}})}{2\gamma\hbar\Lambda} \right], \tag{4.25}$$

where $\hat{\mathbf{\Omega}}$ is a special gauge transformation rotating the triad e_{ia} into a *gauge-fixed* triad g_{ia} of the 3-metric $h_{ij}=e_{ia}e_{ja}$. Then the requirement (3.24) remains to be satisfied, and, in addition, the Chern-Simons state (4.21) becomes invariant under arbitrary gauge transformations of the triad e_{ia}, since the inhomogeneous term in Eq. (4.22) is canceled precisely by a suitable contribution from the prefactor $\mathcal{N}$ according to Eq. (4.25). With our special choice (4.25) of the normalization factor $\mathcal{N}$ we circumvent the definition of the so-called ''Θ angle,'' which can be introduced alternatively to solve the problem associated with large gauge transformations [5,37]. As a special, gauge-fixed triad g_{ia} in the definition of $\hat{\mathbf{\Omega}}$ may serve the ''Einstein triad'' that can be constructed by solving the eigenvalue problem of the three-dimensional Einstein tensor $G^i{}_j$:[7]

$$G^i{}_j g^j{}_a = \lambda_{\bar{a}} g^i{}_{\bar{a}}, \quad g^i{}_a g_{ib} = \delta_{ab}. \tag{4.26}$$

2. Restriction to Bianchi-type homogeneous 3-manifolds

It is very instructive to specialize the asymptotic state (4.21) to the case of spatially homogeneous 3-manifolds. For homogeneous manifolds of one of the nine Bianchi types, the 3-metric can be expressed in terms of *invariant* triad 1-forms $\mathbf{e}_a = \boldsymbol{\iota}_a = \iota_{ia} dx^i$ as (cf. [38])

$$\boldsymbol{h} = \boldsymbol{\iota}_a \otimes \boldsymbol{\iota}_a, \quad d\boldsymbol{\iota}_a = -\tfrac{1}{2} m_{ba} \varepsilon_{bcd} \boldsymbol{\iota}_c \wedge \boldsymbol{\iota}_d, \tag{4.27}$$

with a *spatially constant* structure matrix $\boldsymbol{m}=(m_{ab})$. We should restrict ourselves to compactified, homogeneous 3-manifolds, such that the volume

$$V = \tfrac{1}{6} \varepsilon_{abc} \int \boldsymbol{\iota}_a \wedge \boldsymbol{\iota}_b \wedge \boldsymbol{\iota}_c \tag{4.28}$$

is finite. If we introduce the scale-invariant structure matrix $\boldsymbol{M}$ as

$$\boldsymbol{M} = a_{\cos} \cdot \boldsymbol{m}, \tag{4.29}$$

the asymptotic Chern-Simons state (4.21) takes the following value for Bianchi-type homogeneous 3-manifolds:

$$\Psi_{\rm CS} \overset{\kappa\to\infty}{\underset{\mu\to\infty}{\propto}} \mathcal{N} \cdot \mathbf{h}^{-3/4} \cdot \exp\left[\pm \mu \left(4i\sqrt{\kappa} - \frac{i}{\sqrt{\kappa}} \left[\mathrm{Tr}\, \boldsymbol{M}^2 - 2\,\mathrm{Tr}\, \boldsymbol{M}^T\boldsymbol{M} + \frac{1}{2}\mathrm{Tr}^2\, \boldsymbol{M} \right] - \frac{1}{\kappa}\left[\mathrm{Tr}\, \boldsymbol{M}^2\boldsymbol{M}^T - \frac{1}{6}\mathrm{Tr}\, \boldsymbol{M}(\mathrm{Tr}\, \boldsymbol{M}^2 + 2\,\mathrm{Tr}\, \boldsymbol{M}^T\boldsymbol{M}) + 2\det \boldsymbol{M} \right] \right) \right]. \tag{4.30}$$

For homogeneous manifolds of Bianchi-type IX, the determinant of the matrix $\boldsymbol{M}$ is given by $\det \boldsymbol{M} = 8\mathcal{V}$, where $\mathcal{V}$ is the dimensionless, invariant volume of the unit 3-sphere, so the matrix $\boldsymbol{M}$ may be parameterized by a diagonal, traceless matrix $\boldsymbol{\beta}$ via

$$\boldsymbol{M} = 2\sqrt[3]{\mathcal{V}} e^{2\boldsymbol{\beta}}. \tag{4.31}$$

Using the identity

$$\mathrm{Tr} e^{2\boldsymbol{\beta}} \cdot \mathrm{Tr} e^{4\boldsymbol{\beta}} = \mathrm{Tr} e^{6\boldsymbol{\beta}} + \mathrm{Tr} e^{2\boldsymbol{\beta}} \cdot \mathrm{Tr} e^{-2\boldsymbol{\beta}} - 3, \tag{4.32}$$

and introducing the rescaled parameter $\kappa' := \mathcal{V}^{-2/3}\kappa/4$, we find for Bianchi-type IX homogeneous 3-manifolds:

$$\Psi_{\rm CS} \overset{\kappa\to\infty}{\underset{\mu\to\infty}{\propto}} \mathcal{N} \cdot \mathbf{h}^{-3/4} \cdot \exp\left[\pm \frac{24\mathcal{V}}{\gamma\hbar\Lambda} \left(4i\sqrt{\kappa'}^3 - i\sqrt{\kappa'} \left[\mathrm{Tr} e^{-2\boldsymbol{\beta}} - \frac{1}{2}\mathrm{Tr} e^{4\boldsymbol{\beta}} \right] - \frac{1}{2}\left[\mathrm{Tr} e^{6\boldsymbol{\beta}} - \mathrm{Tr} e^{2\boldsymbol{\beta}} \cdot \mathrm{Tr} e^{-2\boldsymbol{\beta}} + 7 \right] \right) \right]. \tag{4.33}$$

Thus, up to a quantum correction in the Gaussian prefactor, we reproduce exactly the result obtained earlier within the framework of the homogeneous Bianchi IX model in [25] [cf. Eq. (5.18) there]. To compare the results explicitly, we have to identify κ' with the parameter κ in [25], and to set $\gamma = 16\pi$, $\mathcal{V} = 4\pi^2$.

In the case of flat 3-metrics, which are of Bianchi type I, the structure matrix $\boldsymbol{M}$ turns out to vanish, and Eq. (4.30) reduces to

$$\Psi_{\rm CS} \overset{\kappa\to\infty}{\underset{\mu\to\infty}{\propto}} \mathcal{N} \cdot \mathbf{h}^{-3/4} \cdot \exp[\pm 4i\mu\sqrt{\kappa}] = \mathcal{N} \cdot \mathbf{h}^{-3/4} \cdot \exp\left[\pm \frac{4i}{\gamma\hbar}\sqrt{\frac{\Lambda}{3}} \int d^3x \sqrt{h} \right], \tag{4.34}$$

a result which also follows directly from Eq. (4.21) by setting $R=0$, $\mathcal{S}_{\rm CS}[\omega_{ia}]=0$.

3. Semiclassical 4-geometries

Let us now ask for the semiclassical trajectories and the corresponding semiclassical 4-geometries, which are described by the state (4.21) in the limit $\mu \to \infty$, $\kappa \to \infty$, i.e., in the limit of large scale parameters $a_{\cos} \gg a_{\rm Pl}, a_\Lambda$. Choosing the Lagrangian multipliers trivially as $N=1$, $N^i=0$, $\Omega_a=0$ in Eq. (2.21), we find

[7] Here a bar over an index indicates that *no* summation with respect to this index should be performed.

$$\dot{\tilde{e}}^i{}_a = -\{H, \tilde{e}^i{}_a\} = \pm i \tilde{\varepsilon}^{ijk} \mathcal{D}_j e_{ka} = -\frac{\gamma}{2} \tilde{\varepsilon}^{ijk} \varepsilon_{abc} p_{jb} e_{kx}, \tag{4.35}$$

where the dot denotes a derivative with respect to the classical ADM time variable t introduced in Sec. II. The semiclassical momentum p_{ia} is given in terms of the action (4.20) of the wave function (4.21) by

$$p_{ia} = \frac{\delta S}{\delta \tilde{e}^i{}_a}, \tag{4.36}$$

or can equivalently be extracted from the asymptotic saddle point $\mathcal{A}_{ia}$ according to Eq. (4.16) in connection with Eq. (2.11):

$$p_{ia} \overset{\kappa\to\infty}{\sim} \pm \frac{2}{\gamma}\left[\sqrt{\frac{\Lambda}{3}} e_{ia} + \sqrt{\frac{3}{\Lambda}}\left(\frac{R}{4} e_{ia} - e_{ja} R^j{}_i\right)\right]. \tag{4.37}$$

Thus, for large scale parameters $a_{\cos}$ the classical evolution of the triad $\tilde{e}^i{}_a$ is determined by the equation

$$\mp \dot{\tilde{e}}^i{}_a \overset{a_{\cos}\to\infty}{\sim} 2\sqrt{\frac{\Lambda}{3}} \tilde{e}^i{}_a + \sqrt{\frac{3}{\Lambda}} \tilde{e}^j{}_a G^i{}_j, \tag{4.38}$$

which describes a de Sitter–like time evolution in leading order $a_{\cos}$,

$$\tilde{e}^i{}_a(x,t) \overset{a_{\cos}\to\infty}{\sim} \tilde{e}^i{}_{a,\infty}(x) \cdot \exp\left[\mp 2\sqrt{\frac{\Lambda}{3}} \cdot t\right], \tag{4.39}$$

with corrections described by the second term of Eq. (4.38) containing the three-dimensional Einstein tensor $G^i{}_j$.

Figure 1 shows an embedding of the asymptotic 4-geometry (4.39) into a flat Minkowski space, where the time direction has been chosen according to the lower sign in Eq. (4.39). As is well known for inflationary models such as the one discussed within this paper, the spatial, Riemannian 3-manifolds $(\mathcal{M}_3, \boldsymbol{h})(t)$ tend to homogenize in the course of time t.

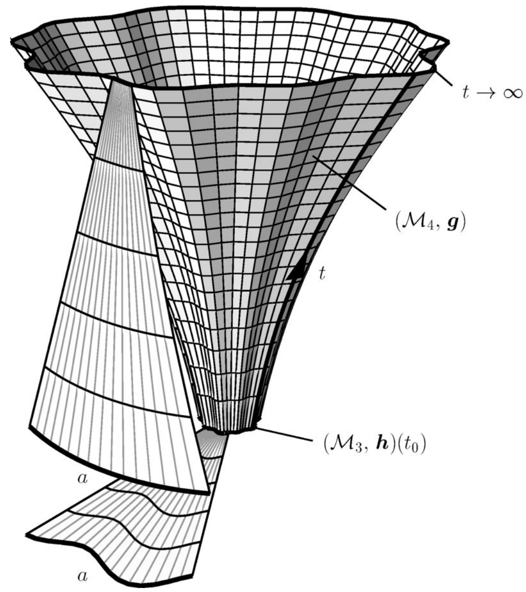

FIG. 1. Geometrical illustration of the generalized de Sitter–4-geometry (4.39). The spatial 3-manifolds $(\mathcal{M}_3, \boldsymbol{h})(t)$ are represented by one-dimensional curves; possible inhomogeneities are indicated by small deformations of these curves. The resulting space-time 4-manifold $(\mathcal{M}_4, \boldsymbol{g})$ according to Eq. (4.39) then corresponds to a two-dimensional, Lorentzian manifold, which has been embedded into a flat, three-dimensional Minkowski space. Portions of the marginal spatial 3-manifolds, which are of the *same* length scale a, have been magnified to illustrate the increase in homogeneity in the course of evolution.

C. The semiclassical vacuum limit $\mu\to\infty$, $\kappa\to 0$

Apart from the limit $\kappa\to\infty$, there exists another asymptotic regime, where an analytical treatment of the semiclassical saddle-point equations (4.12) is tractable, namely, the limit $\kappa\to 0$. By virtue of the relationships (4.4), a discussion of the Chern-Simons state (4.6) in the limit $\mu\to\infty$, $\kappa\to 0$ corresponds to an investigation of the asymptotic regime $a_\Lambda \gg a_{\cos} \gg a_{\rm Pl}$. This limit may be realized by considering the special case of a vanishing cosmological constant $\Lambda\to 0$ within the semiclassical limit, which will be called the semiclassical vacuum limit for short.

To find solutions of Eqs. (4.12) in the limit $\kappa\to 0$ we proceed analogously to Sec. IV B, and try a power series ansatz of the form

$$\mathcal{A}_{ia} \overset{\kappa\to 0}{\sim} \sum_{n=0}^{\infty} C_{ia}^{(n)} \kappa^n. \tag{4.40}$$

Then we find in the lowest order of κ

$$\tilde{\varepsilon}^{ijk}\left(\partial_j C_{ka}^{(0)} + \frac{1}{2}\varepsilon_{abc} C_{jb}^{(0)} C_{kc}^{(0)}\right) = 0, \tag{4.41}$$

i.e., $C_{ia}^{(0)}$ has to be a *flat* gauge field, which is of the general form

$$C_{ia}^{(0)} = -\tfrac{1}{2}\varepsilon_{abc}\Omega_{db}\partial_i\Omega_{dc} \quad \text{with} \quad \boldsymbol{\Omega}\in O(3). \tag{4.42}$$

The matrix $\boldsymbol{\Omega}(x)$ is a free integration field, as long as we restrict ourselves to the leading order $\mathcal{O}(\kappa^0)$ of the saddle-point equations (4.12). However, in the next-to-leading order $\mathcal{O}(\kappa^1)$, we find the equations

$$\tilde{\varepsilon}^{ijk}\mathcal{D}_j^{(0)} C_{ka}^{(1)} := \tilde{\varepsilon}^{ijk}(\partial_j C_{ka}^{(1)} + \varepsilon_{abc} C_{jb}^{(0)} C_{kc}^{(1)}) \overset{!}{=} -\tilde{e}'^i{}_a, \tag{4.43}$$

which imply additional restrictions for the coefficients $C_{ia}^{(0)}$, and thus for the matrix $\boldsymbol{\Omega}$ in Eq. (4.42). These *integrability conditions* for the equations (4.43) can be obtained by operating on Eq. (4.43) with $\mathcal{D}_i^{(0)}$ from the left: Then the left-hand side becomes proportional to the curvature of $C_{ia}^{(0)}$, which vanishes by virtue of Eq. (4.41), and a multiplication of the resulting equations with $a_{\cos}^2$ yields

$$\mathcal{D}_i^{(0)}\tilde{e}^i{}_a \equiv \partial_i \tilde{e}^i{}_a + \varepsilon_{abc} C_{ib}^{(0)} \tilde{e}^i{}_c \stackrel{!}{=} 0. \tag{4.44}$$

If we insert the general solution (4.42) into Eq. (4.44), we arrive at the three integrability conditions

$$\partial_i(\Omega_{ab}\tilde{e}^i{}_b) \stackrel{!}{=} 0, \tag{4.45}$$

which fix the integration field $\mathbf{\Omega}(x)$ in Eq. (4.42). Moreover, the special triad fields

$$\tilde{d}^i{}_a := \Omega_{ab}\tilde{e}^i{}_b, \tag{4.46}$$

with $\mathbf{\Omega}$ chosen according to Eq. (4.45), turn out to have the geometrically interesting property of being *divergence free*. Therefore, we may use the different possible divergence-free triads $\tilde{d}^i{}_a$ of a given Riemannian manifold $(\mathcal{M}_3, \boldsymbol{h})$ to parametrize the saddle points $\mathcal{A}_{ia}$ in the limit $\kappa \to 0$ via Eqs. (4.46) and (4.42).

For a given divergence-free triad $\tilde{d}^i{}_a$, which characterizes uniquely one saddle-point solution $\mathcal{A}_{ia}$ in the limit $\kappa \to 0$, we now wish to calculate the corresponding saddle-point contribution (4.8) to the Chern-Simons state (4.6) in the limit $\mu \to \infty$, $\kappa \to 0$. We first expand the exponent F defined in Eq. (4.7) for $\kappa \to 0$, and find, in particular, that the Chern-Simons functional $\mathcal{S}_{\mathrm{CS}}[\mathcal{A}_{ia}]$ is given by

$$\mathcal{S}_{\mathrm{CS}}[\mathcal{A}_{ia}] \stackrel{\kappa\to 0}{\sim} \tfrac{1}{6} I(\mathbf{\Omega}) + \mathcal{O}(\kappa^2). \tag{4.47}$$

Here $\mathbf{\Omega}$ is the special rotation matrix defined in Eq. (4.45), connecting the given divergence-free triad $\tilde{d}^i{}_a$ with an arbitrary triad $\tilde{e}^i{}_a$, for which we want to evaluate $\Psi_{\mathrm{CS}}[\tilde{e}^i{}_a]$. In Eq. (4.47) a contribution of order $\mathcal{O}(\kappa^1)$ is missing, since this term becomes proportional to the curvature of the *flat* gauge field $C_{ia}^{(0)}$. Using Eq. (4.47), the exponent F of the semiclassical Chern-Simons state takes the following form in the limit $\kappa \to 0$:

$$F \stackrel{\kappa\to 0}{\sim} \frac{I(\mathbf{\Omega})}{6\kappa} + \int d^3x \,\tilde{\varepsilon}^{ijk} d'_{ia}\partial_j d'_{ka} + \mathcal{O}(\kappa). \tag{4.48}$$

The Cartan-Maurer invariant $I(\mathbf{\Omega})$ in Eq. (4.48) can be contracted with the Cartan-Maurer invariant $I(\hat{\mathbf{\Omega}})$ in the definition (4.25) of the normalization factor $\mathcal{N}$ to give

$$I(\mathbf{\Omega}) - I(\hat{\mathbf{\Omega}}) \equiv I(\mathbf{\Omega}\cdot\hat{\mathbf{\Omega}}^T) =: I_0 \cdot \hat{w}[d_{ia}]. \tag{4.49}$$

Here $\hat{w}[d_{ia}]$ denotes the winding number of the divergence-free triad d_{ia} with respect to the Einstein triad g_{ia} defined in Eq. (4.26), which is a functional of d_{ia} only: For a given divergence-free triad d_{ia} we know the 3-metric $h_{ij} = d_{ia}d_{ja}$, and therefore the Einstein triad g_{ia}.

Inserting the results (4.49), (4.48) into (4.8), we find the following saddle-point contribution to the Chern-Simons state (4.6) in the limit $\mu \to \infty, \kappa \to 0$:

$$\lim_{\kappa\to 0} \Psi_{\mathrm{CS}} =: \Psi_{\mathrm{vac}}$$

$$\stackrel{\mu\to\infty}{\propto} \exp\left[\pm\frac{1}{\gamma\hbar}\left(\frac{I_0\hat{w}[d_{ia}]}{2\Lambda} + \int d^3x\,\tilde{\varepsilon}^{ijk} d_{ia}\partial_j d_{ka}\right)\right], \tag{4.50}$$

where the Gaussian prefactor, which contains a complicated, nonlocal functional determinant, has been hidden in the proportionality sign.

From the result (4.50) we can see the gauge invariance of the semiclassical vacuum state Ψ_{vac}, since this state does not depend explicitly on the triad $\tilde{e}^i{}_a$, but only on the 3-metric $h_{ij} = e_{ia}e_{ja}$, to which we have chosen a *fixed* divergence-free triad $\tilde{d}^i{}_a$. It is remarkable that for the one unique choice (4.25) of the prefactor $\mathcal{N}$ gauge invariance, even under large gauge-transformations, can be achieved in both of the two quite different limits $\kappa \to \infty$ and $\kappa \to 0$.

The *existence* of divergence-free triads to a given 3-metric h_{ij} is discussed in Appendix A2. There, we also argue that in general there will even exist different, topologically inequivalent divergence-free triads, giving rise to linearly independent semiclassical vacuum states via Eq. (4.50).

1. Restriction to Bianchi-type A homogeneous 3-manifolds

We now wish to evaluate the semiclassical vacuum state (4.50) for the special case of Bianchi-type homogeneous 3-manifolds. For such manifolds, it follows directly from Eq. (4.27) that the divergence of the *invariant* triad $\vec{\imath}_a = \iota^i{}_a\partial_i$ can be expressed in terms of the structure matrix $\boldsymbol{m}$ as

$$\vec{\nabla}\cdot\vec{\imath}_a = \frac{1}{\sqrt{h}}\,\partial_i \tilde{\imath}^i{}_a = \varepsilon_{abc} m_{bc}. \tag{4.51}$$

Consequently, the invariant triad $\vec{\imath}_a$ of Bianchi-type homogeneous 3-manifolds is divergence free, if, and only if the structure matrix $\boldsymbol{m}$ is *symmetric*, i.e., if the 3-manifold is of Bianchi type A. If we restrict ourselves to this special class of manifolds in the following, at least *one* divergence-free triad $\vec{d}_a^{(0)} = \vec{\imath}_a$ is known, and we can calculate the corresponding value of the semiclassical vacuum state (4.50):

$$\Psi_{\mathrm{vac}}^{(0)} \stackrel{\mu\to\infty}{\propto} \exp\left[\mp\frac{V}{\gamma\hbar}\mathrm{Tr}\boldsymbol{m}\right]. \tag{4.52}$$

Here we made use of the fact that for 3-manifolds of Bianchi type A, the invariant triad $\vec{\imath}_a$ and the Einstein triad $\vec{g}_a$ differ only by a *spatially constant* rotation $\hat{\mathbf{\Omega}}$, implying a vanishing winding number $\hat{w}[\iota_{ia}] = 0$ in Eq. (4.50). A further specialization of the result (4.52) to Bianchi-type IX homogeneous manifolds gives

$$\Psi_{\mathrm{vac}}^{(0)} \stackrel{\mu\to\infty}{\propto} \exp\left[\mp\frac{2\mathcal{V}}{\gamma\hbar}(a_1^2 + a_2^2 + a_3^2)\right], \tag{4.53}$$

where we have introduced the three scale parameters a_b via

$$\boldsymbol{m} =: 2\,\mathrm{diag}\left[\frac{a_1}{a_2 a_3}, \frac{a_2}{a_3 a_1}, \frac{a_3}{a_1 a_2}\right] \;\Rightarrow\; V = \mathcal{V} a_1 a_2 a_3, \tag{4.54}$$

with the same, dimensionless volume $\mathcal{V}$ of the unit 3-sphere that already occurred in Sec. IV B2. The saddle-point value (4.53) corresponds to the ''wormhole state'' of the Bianchi type-IX model [25,27]. Within the framework of the homogeneous Bianchi type-IX model, four further semiclassical vacuum states are known, which, in the inhomogeneous approach of the present paper, correspond to nontrivial divergence-free triads of Bianchi type-IX manifolds via Eq. (4.50). These topologically nontrivial divergence-free triads of Bianchi type-IX metrics and the resulting values of the semiclassical vacuum state (4.50) will be discussed separately in Appendix B.

As a further restriction of the state (4.52) one may consider again the case of flat Bianchi-type-I manifolds, where the structure matrix $\boldsymbol{m}$, and therefore the exponent of Eq. (4.52), vanishes. Thus, for flat 3-manifolds the behavior of the semiclassical vacuum state is governed by the Gaussian prefactor, which we do not know explicitly.

2. Semiclassical 4-geometries

The semiclassical trajectories and the associated 4-geometries, which are generated by the state (4.50) in the limit $\kappa \to 0$, $\mu \to \infty$, can be calculated by solving the evolution equations (4.35) with the flat, semiclassical spin connection $\mathcal{A}_{ia}$ derived in Sec. IV C. However, in contrast to the limit $\kappa \to \infty$ discussed in Sec. IV B3, we here arrive at *imaginary* evolution equations, since the semiclassical action of the wave functional Ψ_{vac} according to Eq. (4.50) is purely imaginary. Following Hawking [39], a geometrical interpretation may still be given in terms of an imaginary time variable $\tau := it$, converting the Lorentzian signature of the four-dimensional space-time into a positive, Euclidian signature. Then the semiclassical evolution equations can conveniently be expressed in terms of the divergence-free triad d_{ia}, which characterizes the flat Ashtekar spin connection $\mathcal{A}_{ia}$ in the limit $\kappa \to 0$:

$$\frac{d}{d\tau}\tilde{d}^i{}_a = \pm \tilde{\varepsilon}^{ijk}\partial_j d_{ka} \;\Leftrightarrow\; \frac{d}{d\tau} d_{ia} = \mp \omega_{ia}. \tag{4.55}$$

Here ω_{ia} in the second equation is the Riemannian spin connection of the divergence-free triad d_{ia}. Obviously, the gauge condition $\partial_i \tilde{d}^i{}_a = 0$ remains preserved in the course of evolution, as must be the case.

Stationary solutions of Eqs. (4.55) are given by $\omega_{ia} = 0$, i.e., *flat* 3-manifolds $(\mathcal{M}_3, \boldsymbol{h})$. With our trivial choice of the Lagrangian multipliers $N = 1$, $N^i = 0$, these correspond to locally flat, positive definite semiclassical space-time manifolds $(\mathcal{M}_4, \boldsymbol{g})$. Further solutions of Eq. (4.55) can be constructed with help of the scaling ansatz

$$d_{ia}(x, \tau) = \mp \tau \cdot d'_{ia}(x), \tag{4.56}$$

which implies $d'_{ia}(x) = \omega_{ia}(x)$, and therefore a simple form for the Ricci tensor of the spatial 3-manifold:

$$R^i{}_j = \frac{2}{\tau^2}\delta^i_j. \tag{4.57}$$

Consequently, the spatial manifold has to be a 3-sphere with radius τ, and the four-dimensional line element becomes

$$ds^2 = d\tau^2 + \tau^2 d\Omega_3^2, \tag{4.58}$$

with $d\Omega_3^2$ being the line element of the unit 3-sphere. As for the stationary solutions mentioned above, the line element (4.58) describes a locally flat, positive definite 4-manifold.

Because of the nonlinearity of the evolution equations (4.55), the general behavior of the solution is quite complicated and cannot be discussed here. However, a complete discussion of the possible semiclassical trajectories can be given within the narrow class of Bianchi-type-IX homogeneous 3-manifolds; cf. [25]. There it turns out that the semiclassical evolution governed by the invariant, divergence-free triad $\vec{d}_a^{(0)} = \vec{\imath}_a$, which corresponds to the wormhole state (4.53) via Eq. (4.50), gives rise to asymptotically *flat* 4-geometries in the limit of large scale parameters $a_{\cos}$. Moreover, a second divergence-free triad of these Bianchi-type-IX homogeneous 3-manifolds, which is given in Appendix B, is known to evolve in such a way that *compact, regular* 4-manifolds are approached in the limit of vanishing scale parameter $a_{\cos}$.[8]

One may now ask if such a universal behavior of the semiclassical trajectories, which can be found within the Bianchi IX model, carries over to the inhomogeneous case. Unfortunately, this does not seem to be the case: In Appendix C we explicitly solve the evolution equations (4.55) for a particular class of initial 3-manifolds, and find that these solutions satisfy neither the condition of asymptotical flatness in the limit $a_{\cos} \to \infty$, nor the ''no-boundary'' proposal suggested by Hartle and Hawking [39–41]. Thus we conclude that, in the inhomogeneous case, the semiclassical vacuum state given in Eq. (4.50) will in general *not* be subject to any specific boundary condition, such as the ''no-boundary'' proposal or the condition of asymptotical flatness.

V. NON-NORMALIZABILITY OF THE CHERN-SIMONS STATE IN A PHYSICAL INNER PRODUCT

We now want to argue that the gravitational Chern-Simons state $\Psi_{\mathrm{CS}}[\tilde{e}^i{}_a]$ according to Eq. (3.23) does *not* constitute a normalizable physical state on the Hilbert space of quantum gravity. Therefore, we will derive a physical inner product on the configuration space of real triads, which we want to be gauge fixed with respect to the time reparametrization invariance of general relativity. In this particular inner product, we then will try to calculate the corresponding norm of the Chern-Simons state $\Psi_{\mathrm{CS}}[\tilde{e}^i{}_a]$.

To derive a physical inner product within the framework of the Faddeev-Popov calculus [42,43], we first have to find

[8]The semiclassical vacuum state, corresponding to this second divergence-free triad via Eq. (4.50), is the ''no-boundary state'' of the Bianchi IX model.

a kinematical inner product, denoted by $\langle\cdot|\cdot\rangle$ in the following, with respect to which the quantum constraint operators $\tilde{\mathcal{H}}_0$, $\tilde{\mathcal{H}}_i$, and $\tilde{\mathcal{I}}_a$ are formally Hermitian. Since the complex Hamiltonian constraint operator $\tilde{\mathcal{H}}_0$ defined in Eq. (2.18) cannot be Hermitian with respect to *any* inner product on the configuration space, we replace $\tilde{\mathcal{H}}_0$ by its real version $\tilde{\mathcal{H}}_0^{\mathrm{ADM}}$ given in Eq. (2.17), with the factor ordering suggested there. With the help of the commutators (3.4)–(3.9) one can check quite easily that the algebra of $\tilde{\mathcal{H}}_0^{\mathrm{ADM}}$, $\tilde{\mathcal{H}}_i$ and $\tilde{\mathcal{J}}_a$ still closes without any quantum corrections. However, the explicit commutators turn out to be much more complicated than the corresponding commutators of $\tilde{\mathcal{H}}_0$, $\tilde{\mathcal{H}}_i$, $\tilde{\mathcal{J}}_a$ given in Eqs. (3.4)–(3.9), and will not be given here.

Since the quantum state Ψ_{CS} given in Eq. (3.23) is also annihilated by the operator $\tilde{\mathcal{H}}_0^{\mathrm{ADM}}$, the substitution $\tilde{\mathcal{H}}_0 \mapsto \tilde{\mathcal{H}}_0^{\mathrm{ADM}}$ has no negative consequences for the theory, but the positive effect is that we can now define a kinematical inner product, with respect to which the operators $\tilde{\mathcal{H}}_0^{\mathrm{ADM}}$, $\tilde{\mathcal{H}}_i$, and $\tilde{\mathcal{J}}_a$ are Hermitian. This product turns out to be

$$\langle\Psi|\Phi\rangle = \int \mathcal{D}^9[e_{ia}]\Psi^*[e_{ia}]\cdot\Phi[e_{ia}], \tag{5.1}$$

where the functional integral has to be performed over all real triads $e_{ia}(x)$. While $\tilde{\mathcal{H}}_0^{\mathrm{ADM}}$ and $\tilde{\mathcal{J}}_a$ are formally Hermitian in the product (5.1), $\tilde{\mathcal{H}}_i$ is Hermitian only if we take a regularization of the theory, where terms containing the singular object $(\partial_i\delta)(0)$ vanish.[9] If we can achieve this, we have found a kinematical inner product on the configuration space of all real triads $e_{ia}(x)$, and can continue with the Faddeev-Popov calculus by choosing a gauge condition $\tilde{\chi}[e_{ia}]=0$ fixing the time gauge. The corresponding physical inner product is then obtained as

$$\langle\langle\Psi\|\Phi\rangle\rangle_{\mathrm{phys}} = \langle\Psi|\delta[\tilde{\chi}]\cdot|J_H||\Phi\rangle, \tag{5.2}$$

with the Faddeev-Popov functional determinant

$$J_H := \det\left(\frac{i}{\hbar}[\tilde{\mathcal{H}}_0^{\mathrm{ADM}}(x),\tilde{\chi}(y)]\right). \tag{5.3}$$

A rather natural way to fix the time gauge is to consider 3-geometries with a given volume form $\sqrt{h(x)}$, for which there remain only two local degrees of freedom. Therefore we assume $\tilde{v}(x)$ to be a fixed, positive scalar density of weight $+1$ on the spatial manifold $\mathcal{M}_3$, normalized such that[10]

$$\int d^3x\,\tilde{v}(x) \overset{!}{=} 1. \tag{5.4}$$

Furthermore, let a_χ be an arbitrary, positive scale parameter. Then the gauge condition

$$\tilde{\chi} := \sqrt{h(x)} - a_\chi^3\tilde{v}(x) \overset{!}{=} 0 \tag{5.5}$$

is a diffeomorphism- and SO(3)-gauge invariant equation fixing the volume form of the 3-metric. In particular, it follows from Eq. (5.5) that the length scale a_χ and the cosmological scale $a_{\cos}$ introduced in Eq. (4.2) must be equal. In the gauge (5.5), the physical norm associated with the inner product (5.2) obviously depends on the scale parameter a_χ and the choice of $\tilde{v}(x)$, but we can consider the limit $a_\chi \to \infty$,

$$\|\Psi\|_\infty^2 := \lim_{a_\chi\to\infty}\langle\langle\Psi\|\Psi\rangle\rangle_{\mathrm{phys}}, \tag{5.6}$$

which, in the case of the Chern-Simons state $\Psi=\Psi_{\mathrm{CS}}$, will turn out to be independent of $\tilde{v}(x)$. For an explicit calculation of Eq. (5.6), we need the Faddeev-Popov commutator occurring in Eq. (5.3), which turns out to be

$$\frac{i}{\hbar}[\tilde{\mathcal{H}}_0^{\mathrm{ADM}}(x),\tilde{\chi}(y)] = \frac{\gamma}{4}\delta^3(x-y)\tilde{J}(x), \tag{5.7}$$

with

$$\tilde{J}(x) := \frac{i\hbar}{2}\left[e_{ia}(x)\frac{\delta}{\delta e_{ia}(x)} + \frac{\delta}{\delta e_{ia}(x)}e_{ia}(x)\right]. \tag{5.8}$$

The Faddeev-Popov functional determinant J_H according to Eq. (5.3) follows as

$$J_H = \prod_{x\in\mathcal{M}_3}\frac{\gamma}{4}\tilde{J}(x), \tag{5.9}$$

which, acting on the wave functional Ψ_{CS}, measures the space product of the current $\tilde{J}(x)$ of Ψ_{CS} in the $h(x)$ direction of superspace. Since we are dealing with the limit $a_\chi = a_{\cos}\to\infty$, the exact quantum state Ψ_{CS} given in Eq. (3.23) may be substituted by the asymptotic state (4.21) for explicit calculations. Then the current of Ψ_{CS} in the $h(x)$ direction turns out to have the same sign at each space point for large scale parameters $a_\chi = a_{\cos}$,[11] so we do *not* need to take the modulus of the Faddeev-Popov determinant in Eq. (5.2), as the general calculus in [42] would prescribe. More explicitly, we find the result

$$J_H\cdot\Psi_{\mathrm{CS}}|_{\tilde{\chi}=0} \overset{a_\chi\to\infty}{\propto} \mathbf{h}^{1/2}\cdot\Psi_{\mathrm{CS}}|_{\tilde{\chi}=0}, \tag{5.10}$$

where $\mathbf{h}$ was defined in Eq. (4.18), so the physical norm (5.6) becomes in the limit $a_\chi\to\infty$:

[9] Some authors argue that this should be possible; cf. Matschull [33].

[10] For example, the quantity $\tilde{v}$ may be chosen as the rescaled volume element of a maximally symmetric 3-metric on $\mathcal{M}_3$.

[11] This property of Ψ_{CS} in the limit $a_{\cos}\to\infty$ reminds one of the Vilenkin proposal for the wave function of the Universe discussed in [44,45].

$$\|\Psi_{\rm CS}\|_\infty^2 \propto \int \mathcal{D}^9[e_{ia}]\mathbf{h}^{1/2}|\Psi_{\rm CS}|^2\delta[\tilde{\chi}]. \qquad (5.11)$$

If we now introduce the new integration variables $\sqrt{h}$, and eight locally scale-invariant fields β_κ, the functional integral in Eq. (5.11) becomes

$$\|\Psi_{\rm CS}\|_\infty^2 \propto \int \mathcal{D}[\sqrt{h}]\mathcal{D}^8[\beta_\kappa]w[\beta_\kappa]\mathbf{h}^{3/2}|\Psi_{\rm CS}|^2\delta[\sqrt{h}-a_{\cos}^3\tilde{v}]$$
$$= \int \mathcal{D}^8[\beta_\kappa]w[\beta_\kappa]\exp\left[\pm\frac{6}{\gamma\hbar\Lambda}\hat{\mathcal{S}}_{\rm CS}[\beta_\kappa]\right], \qquad (5.12)$$

where

$$\hat{\mathcal{S}}_{\rm CS}[\beta_\kappa] := \mathcal{S}_{\rm CS}[\omega_{ia}] - \tfrac{1}{6}I(\hat{\mathbf{\Omega}}) \qquad (5.13)$$

is a locally scale-invariant functional describing the exponent of $|\Psi_{\rm CS}|^2$ according to Eqs. (4.21) and (4.25). The weight function $w[\beta_\kappa]$ occurring in Eq. (5.12) depends on the choice of the new integration variables β_κ. Since the integrand of Eq. (5.12) is locally scale invariant, the integral is independent of the choice of $\tilde{v}(x)$ in Eq. (5.5), as announced above, so the gauge condition $\tilde{\chi}=0$ can be omitted in the second line of Eq. (5.12).

As a result, we find that the diffeomorphism-, gauge-, and locally scale-invariant functional $\hat{\mathcal{S}}_{\rm CS}[\beta_\kappa]$, which is closely related to the Chern-Simons functional of the Riemannian spin connection ω_{ia}, governs the ''probability'' distribution associated with the Chern-Simons state (4.21) in the limit $a_{\cos}\to\infty$. Since the functional $\mathcal{S}_{\rm CS}[\omega_{ia}]$ is obviously unbounded from above and below, we conclude that the norm (5.12) cannot be finite, even if we fix the remaining gauge freedoms concerning the diffeomorphism and the local SO(3)-gauge transformations.

However, we should keep in mind that the result (5.12) has been derived for a very special choice of the gauge condition $\tilde{\chi}$ according to Eq. (5.5). Since different gauge fixings of the Hamiltonian constraint give rise to *inequivalent* physical inner products on the Hilbert space of quantum gravity,[12] there may still exist other choices of $\tilde{\chi}$, for which the Chern-Simons state $\Psi_{\rm CS}[\tilde{e}^i{}_a]$ turns out to be normalizable.

VI. DISCUSSION AND CONCLUSION

The main purpose of this paper was to derive and discuss a triad representation of the Chern-Simons state, which is a well-known exact wave functional of quantum gravity within Ashtekar's theory of general relativity. In particular, we were interested in an explicit transformation connecting the real triad representation with the complex Ashtekar representation. Therefore, we first investigated this transformation on the classical level in Sec. II. Here we also derived new representations for the constraint observables $\tilde{\mathcal{H}}_0$, $\tilde{\mathcal{H}}_i$, and $\tilde{\mathcal{J}}_a$ in terms of a single tensor density $\tilde{\mathcal{G}}^i_{\Lambda,a}$ defined in Eq. (2.22), which is closely related to the curvature $\mathcal{F}_{ija}$ of the Ashtekar spin connection $\mathcal{A}_{ia}$.

Then, in Sec. III, we performed a canonical quantization of the theory in the triad representation. In the particular factor ordering for the quantum constraint operators $\tilde{\mathcal{H}}_0$, $\tilde{\mathcal{H}}_i$, and $\tilde{\mathcal{J}}_a$ suggested by Eqs. (2.23)–(2.25), we found that the constraint algebra closes formally without any quantum corrections.

On the quantum-mechanical level, the transformation from the Ashtekar to the triad representation turned out to be given by a generalized Fourier transformation (3.18) and a subsequent similarity transformation (3.15). Here it was essential to allow for an arbitrary *complex* integration manifold Γ in the Fourier integral (3.18), restricted only by the condition that partial integrations should be permitted without getting any boundary terms.

Making use of the transformations (3.15) and (3.18), we then recovered the Chern-Simons state of quantum gravity by searching for a wave functional which is annihilated by $\tilde{\mathcal{G}}^i_{\Lambda,a}$. The Chern-Simons state in the triad representation turned out to be given by the formal complex functional integral (3.23). In our approach the Ashtekar variables played only the role of convenient auxiliary quantities. The *reality conditions* originally introduced by Ashtekar in [5] enter nowhere explicitly, but lie hidden in the choice of the integration contour Γ for the functional integrals in Eqs. (3.18) and (3.23).

We did not try to perform the complex functional integral occurring in Eq. (3.23) analytically, but restricted ourselves to semiclassical expansions of the Chern-Simons state, which were treated in Sec. IV. Rewriting the state $\Psi_{\rm CS}[\tilde{e}^i{}_a]$ in suitable dimensionless field parameters, the functional integral turned out to be of a Gaussian saddle-point form in the semiclassical limit $\mu\to\infty$, and the semiclassical Chern-Simons state was determined by solutions of the saddle-point equations (4.10). Here it depended on the choice of the integration contour Γ, which particular saddle-points contributed to the functional integral (3.23) via Eq. (4.8). In order to prove the consistency of the semiclassical expansions, we argued for the solvability of the saddle-point equations (4.10) in a separate Appendix A1 from a mathematical point of view, where it turned out that saddle-point solutions will exist at least under the restriction $R(x)\neq 2\Lambda$.

We were able to find explicit analytical results for the semiclassical Chern-Simons state in the two asymptotic regimes $\kappa=\Lambda a_{\cos}^2/3\to\infty$ and $\kappa\to 0$, which were discussed in Secs. IV B and IV C, respectively.

In the limit $\kappa\to\infty$, two different solutions of the saddle-point equations (4.10) could be found, giving rise to the linearly independent asymptotic states $\Psi_{\rm CS}$ and $\Psi^*_{\rm CS}$ given in Eq. (4.21). For a suitable choice of the normalization factor $\mathcal{N}$ according to Eq. (4.25), these asymptotic states turned out to be invariant under arbitrary, even topologically nontrivial SO(3)-gauge transformations of the triad. In the special case of Bianchi-type homogeneous 3-metrics, we obtained the ex-

[12]This is a peculiarity of the Hamiltonian constraint, and in contrast to gauge-fixing procedures associated with $\tilde{\mathcal{H}}_i$ or $\tilde{\mathcal{J}}_a$, for which the Faddeev-Popov calculus guarantees a *unique* physical inner product [42,43].

plicit result (4.30) for the value of the asymptotic Chern-Simons state (4.21), which, by a further restriction to Bianchi-type-IX metrics, coincided with the corresponding result known from discussions of the homogeneous Bianchi-type-IX model.

The asymptotic Chern-Simons state (4.21) in the limit $\kappa\to\infty$ gives rise to a well-defined semiclassical time evolution, which we discussed in Sec. IV B3. There it turned out that for large scale parameters $a_{\cos}$ the semiclassical 4-geometries associated with the Chern-Simons state are given by inhomogeneously generalized de Sitter space-times.

In the limit $\kappa\to 0$, the semiclassical saddle-point contributions to the Chern-Simons state can be characterized by divergence-free triads $\vec{d}_a$ of the Riemannian 3-manifold $(\mathcal{M}_3,\boldsymbol{h})$ via Eq. (4.50). Thus we had to answer the nontrivial question of whether divergence-free triads to a given 3-metric will in general exist, which was done in Appendix A2.

In restriction to homogeneous manifolds of Bianchi type A, *one* divergence-free triad was explicitly known, giving rise to the result (4.52). In particular, we were able to recover the ''wormhole-state'' (4.53), which is a well-known vacuum state within the homogeneous Bianchi IX model. For Bianchi-type-IX manifolds, four further divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha\in\{1,2,3,4\}$, were constructed in Appendix B. They gave rise to four additional saddle-point contributions $\Psi_{\text{vac}}^{(\alpha)}$, $\alpha\in\{1,2,3,4\}$, to the vacuum Chern-Simons state, which, however, were restricted to occur simultaneously. We concluded that, together with the wormhole state, only *two* linearly independent values of the vacuum Chern-Simons state are realized for Bianchi-type-IX manifolds.

Since these two values should continue to exist under sufficiently small, inhomogeneous perturbations of the 3-metric, and since also in the limit $\kappa\to\infty$ exactly two different values of the semiclassical Chern-Simons state were found, one may assume that the one Chern-Simons state in the Ashtekar representation corresponds to two linearly independent states in the triad representation.

Within the narrow class of Bianchi-type-IX metrics, the semiclassical 4-geometries associated with the vacuum Chern-Simons state (4.50) are satisfying physically interesting boundary conditions, namely, either the ''no-boundary'' condition proposed by Hartle and Hawking [39–41], or the condition of asymptotical flatness at large scale parameters $a_{\cos}$. However, this does *not* remain true for general 3-metrics, as we have shown by exhibiting a counter example in Appendix C. We conclude that, in general, the Chern-Simons state will not satisfy the ''no-boundary'' condition or the condition of asymptotical flatness. Nevertheless, as we have remarked in Sec. V, the asymptotic state (4.21) in the limit $\kappa\to\infty$ reminds one of the Vilenkin proposal for the wave function of the Universe [44,45].

In Sec. V, we investigated the normalizability of the Chern-Simons state (3.23) in the triad representation. We defined a kinematical inner product on the Hilbert space of quantum gravity, and by performing a special gauge fixing for the time gauge we arrived at the physical inner product (5.2). Unfortunately, the Chern-Simons state turned out to be *non-normalizable* with respect to this particular inner product. However, as we have pointed out, there may still exist other gauge-fixing procedures (e.g., the one suggested by Smolin and Soo in [18]), which render the Chern-Simons state to be normalizable.

ACKNOWLEDGMENTS

Support of this work by the Deutsche Forschungsgemeinschaft through the Sonder-forschungsbereich ''Unordnung und große Fluktuationen'' is gratefully acknowledged. We further wish to thank Professor Abresch from the Ruhr-Universität Bochum for many fruitful discussions and important ideas concerning the mathematical problems discussed in Appendix A.

APPENDIX A: ON THE SOLVABILITY OF THE SADDLE-POINT EQUATIONS

The solvability of the semiclassical saddle-point equations (4.10) is essential in order to justify the consistency of the asymptotical expansions of the Chern-Simons state discussed in Sec. IV. Therefore, it is worth studying the solvability properties of the nonlinear, partial differential equations (4.10) from a mathematical point of view, which will be done in Sec. A1. Applying the results of Sec. A1 to the special case of a vanishing cosmological constant Λ, we will then, in Sec. A2, be able to prove the existence of divergence-free triads of Riemannian 3-manifolds, which determine the semiclassical vacuum state (4.50).

1. The general case $\Lambda\neq 0$

If we want to discuss the solvability of the saddle-point equations (4.10) within the theory of partial differential equations (cf. [46]), it is *not* advisable to study this problem in the particular form (4.10), since the spatial derivative operator, which is given by the curl of the gauge field $\mathcal{A}_{ia}$, is known to be *nonelliptic*. However, we will show that it is possible to consider a set of second order partial differential equations instead, which will turn out to be elliptic in leading derivative order, thus allowing for solvability statements concerning the solutions $\mathcal{A}_{ia}$.

Let us first introduce new variables

$$\mathcal{K}_{ij}:=(\omega_{ia}-\mathcal{A}_{ia})e_{ja}\equiv\mp iK_{ji} \tag{A1}$$

instead of the gauge fields $\mathcal{A}_{ia}$, where e_{ia} denotes a fixed triad for which we want to solve the set of equations (4.10). Up to a Wick rotation, the tensor $\mathcal{K}_{ij}$ plays the role of the semiclassical extrinsic curvature tensor K_{ij} [cf. Eqs. (2.2), (2.6), and (2.11)]. If we rewrite the saddle-point equations (4.10) in terms of the new variables $\mathcal{K}_{ij}$, they become

$$\begin{aligned}\mathcal{G}^i_{\Lambda,j}&:=\frac{1}{\sqrt{h}}\tilde{\mathcal{G}}^i_{\Lambda,a}e_{ja}\\&=G^i_{\Lambda,j}+{}^*\mathcal{K}^i{}_j-\frac{1}{\sqrt{h}}\tilde{\varepsilon}^{ikl}\nabla_k\mathcal{K}_{lj}\overset{!}{=}0,\end{aligned} \tag{A2}$$

where

$$*\mathcal{K}^i{}_j := \tfrac{1}{2}\,\tilde{\varepsilon}^{ikl}\underset{\sim}{\varepsilon}_{jmn}\mathcal{K}_k{}^m\mathcal{K}_l{}^n \tag{A3}$$

are the cofactors of the matrix elements $\mathcal{K}_i{}^j$, and $G^i_{\Lambda,j}$ is the usual, three-dimensional Einstein tensor with a cosmological term. In analogy to Eq. (2.25), the set of equations (A2) implies the three Gauss constraints

$$\tilde{\mathcal{J}}_a = \pm\frac{6i}{\gamma\Lambda}[\nabla_j\mathcal{G}^j_{\Lambda,i} - \sqrt{h}\underset{\sim}{\varepsilon}_{ijk}\mathcal{K}_l{}^j\mathcal{G}^{lk}_\Lambda]\tilde{e}^i{}_a \equiv \pm\frac{2i}{\gamma}e_{ia}\tilde{\varepsilon}^{ijk}\mathcal{K}_{jk} \overset{!}{=} 0, \tag{A4}$$

which require the tensor $\mathcal{K}_{ij}$ to be symmetric in i and j. Therefore, if we take $\mathcal{K}_{ij}$ to be symmetric in the following, the Gauss-constraints (A4) are satisfied identically, and the first line of Eq. (A4) takes the form of three generalized Bianchi identities. We thus conclude that the set of equations (A2) constitutes only six independent equations for the six fields $\mathcal{K}_{ij} = \mathcal{K}_{ji}$ we are searching for.

Beside the Gauss-constraints (A4), four further equations are implied by Eq. (A2) via Eqs. (2.23) and (2.24), namely, the Hamiltonian constraint

$$\tilde{\mathcal{H}}_0^{\mathrm{ADM}} = \frac{2\sqrt{h}}{\gamma}\mathcal{G}^i_{\Lambda,i} \equiv \frac{\sqrt{h}}{\gamma}(\mathcal{K}^2 - \mathcal{K}^i{}_j\mathcal{K}^j{}_i + 2\Lambda - R) \overset{!}{=} 0, \tag{A5}$$

and the three diffeomorphism constraints

$$\tilde{\mathcal{H}}_i = \mp\frac{2ih}{\gamma}\underset{\sim}{\varepsilon}_{ijk}\mathcal{G}^{jk}_\Lambda \equiv \pm\frac{2i\sqrt{h}}{\gamma}(\nabla_j\mathcal{K}^j{}_i - \nabla_i\mathcal{K}) \overset{!}{=} 0, \tag{A6}$$

respectively. Here $\mathcal{K}$ in Eqs. (A5) and (A6) denotes the trace of $(\mathcal{K}^i{}_j)$. Remarkably, the Hamiltonian constraint (A5) is a purely algebraic equation for $\mathcal{K}_{ij}$, which will be solved explicitly later on, while the diffeomorphism constraints (A6) are linear equations and contain information about the divergence of the fields $\mathcal{K}_{ij}$.

Moreover, since Eqs. (A2) contain the curl of the fields $\mathcal{K}_{ij}$, Eqs. (A2) and (A6) together may be used to construct a second-order derivative operator similar to the Laplace-Beltrami operator of $\mathcal{K}_{ij}$. Let us therefore consider the following second-order differential equations:

$$\Delta_{ij} := \sqrt{h}[\underset{\sim}{\varepsilon}_{jmn}\nabla_i\mathcal{G}^{mn}_\Lambda - \underset{\sim}{\varepsilon}_{imn}h^{mk}\nabla_k\mathcal{G}^n_{\Lambda,j} + \tfrac{1}{2}\,\underset{\sim}{\varepsilon}_{ijk}\nabla_n\mathcal{G}^{nk}_\Lambda] \overset{!}{=} 0, \tag{A7}$$

which must be satisfied for solutions $\mathcal{K}_{ij}$ of Eq. (A2). The first term in Eq. (A7) can be simplified with the help of Eq. (A6), and gives in the leading derivative order the gradient of the divergence of $\mathcal{K}_{ij}$ and, in addition, the Hessian of $\mathcal{K}$. Making use of Eqs. (A2), the second term in Eq. (A7) contributes the curl of $\mathcal{K}_{ij}$, i.e., taking the first two terms in Eq. (A7) together, we arrive at

$$\Delta_{ij} = \nabla_i\nabla_j\mathcal{K} - \Delta\mathcal{K}_{ij} + \mathcal{O}(\nabla_i\mathcal{K}_{jk}) \tag{A8}$$

in leading derivative order. By virtue of Eqs. (A4), the third term in Eq. (A7) contains only first-order derivatives of $\mathcal{K}_{ij}$. It has been added to obtain simple expressions for the trace and the antisymmetric part of Δ_{ij}, which are given by

$$h^{ij}\Delta_{ij} \equiv 0, \quad \tilde{\varepsilon}^{ijk}\Delta_{jk} \equiv \frac{\gamma}{2}h^{ij}\nabla_j\tilde{\mathcal{H}}_0^{\mathrm{ADM}}. \tag{A9}$$

Instead of solving the nine equations (A7), we may therefore consider the six equations

$$\Delta_{(ij)} := \tfrac{1}{2}(\Delta_{ij} + \Delta_{ji}) \overset{!}{=} 0, \quad \tilde{\mathcal{H}}_0^{\mathrm{ADM}} \overset{!}{=} 0 \tag{A10}$$

to determine the six fields $\mathcal{K}_{ij}$.

In a next step, we will now solve the Hamiltonian constraint (A5) explicitly. At any space point $x \in \mathcal{M}_3$, Eq. (A5) describes a five-dimensional hyperboloid in the six-dimensional space spanned by $\mathcal{K}_{ij}$, as long as

$$\forall x \in \mathcal{M}_3: \quad R(x) \neq 2\Lambda, \tag{A11}$$

which will be assumed in the following. This five-dimensional hyperboloid may be parametrized with the help of a stereographic projection; hence the general solution of the Hamiltonian constraint can be written in the form

$$\mathcal{K}^i{}_j = \frac{\sqrt{R-2\Lambda}}{1-\mathrm{Tr}\boldsymbol{\mathcal{Q}}^2}\left[\frac{1+\mathrm{Tr}\boldsymbol{\mathcal{Q}}^2}{\sqrt{6}}\delta^i_j + 2\,\boldsymbol{\mathcal{Q}}^i{}_j\right], \quad \mathrm{Tr}\boldsymbol{\mathcal{Q}}^2 \neq 1, \tag{A12}$$

where $\boldsymbol{\mathcal{Q}}$ is a symmetric, *traceless* matrix. Matrices $\boldsymbol{\mathcal{Q}}$ with $\mathrm{Tr}\boldsymbol{\mathcal{Q}}^2 = 1$ correspond to coordinate singularities of the stereographic projection, and thus have to be excluded in Eq. (A12). Inserting the general solution (A12) of $\tilde{\mathcal{H}}_0^{\mathrm{ADM}} = 0$ into the first of Eqs. (A10), we arrive at five equations for the five fields $\mathcal{Q}^i{}_j$, which remain to be determined.

We now want to argue that the effective set of partial differential equations obtained this way is soluble with respect to $\mathcal{Q}^i{}_j$. Let us therefore consider a background solution $\overline{\mathcal{Q}^i}{}_j$ of these equations, which we assume to be known for sufficiently simple parameter fields $\tilde{e}^i{}_a$ and Λ.[13] Under infinitesimal perturbations of the parameter fields $\tilde{e}^i{}_a$ and Λ, the new solution $\mathcal{Q}^i{}_j$ will differ from the background solution $\overline{\mathcal{Q}^i}{}_j$ by an infinitesimal amount

$$\mathcal{Q}^i{}_j = \tilde{\mathcal{Q}}^i{}_j + \epsilon\cdot\mathcal{Q}'^j{}_j + \mathcal{O}(\epsilon^2), \tag{A13}$$

and in the following it will be sufficient to show that the fields $\mathcal{Q}'^i{}_j$ exist to any given background solution $\overline{\mathcal{Q}^i}{}_j$. Inserting the perturbation ansatz (A13) into $\Delta_{(ij)} = 0$, we arrive at five linear partial differential equations $\Delta'_{(ij)} = 0$ in $\mathcal{O}(\epsilon)$ determining the fields $\mathcal{Q}'^i{}_j$. To show that these equations are soluble with respect to $\mathcal{Q}'^i{}_j$, we will restrict ourselves to a discussion of the symbol of $\Delta'_{(ij)} = 0$, which we will show to

[13]Explicit solutions $\mathcal{A}_{ia}$ of the saddle-point equations (4.10), which correspond to the fields $\mathcal{Q}^i{}_j$ via Eqs. (A1) and (A12), are in fact known for various homogeneous 3-manifolds, such as Bianchi-type-IX manifolds; cf. [25].

be elliptic (cf. [46]). The symbol $\sigma(\boldsymbol{k})$ of a linear differential operator is obtained by computing the action on a Fourier mode

$$\mathcal{Q}'^{I}{}_{j}(x)=\hat{\mathcal{Q}}^{i}{}_{j}(\boldsymbol{k})\cdot e^{ik_l x^l}, \tag{A14}$$

in leading order of the wave vector $\boldsymbol{k}$. For the operator $\Delta'_{(ij)}$ under study, we obtain

$$\begin{aligned}\sigma_{ij}(\Delta'_{(mn)};\boldsymbol{k})=-2\,\frac{\sqrt{R-2\Lambda}}{(1-\mathrm{Tr}\bar{\boldsymbol{\mathcal{Q}}}^2)^2}\Bigg[&\sqrt{6}k_ik_j\bar{\mathcal{Q}}^{mn}\hat{\mathcal{Q}}_{mn}\\&-|k|^2\Bigg((1-\mathrm{Tr}\bar{\boldsymbol{\mathcal{Q}}}^2)\hat{\mathcal{Q}}_{ij}+\sqrt{\frac{2}{3}}\,\bar{\mathcal{Q}}^{mn}\\&\times(h_{ij}+\sqrt{6}\,\bar{\mathcal{Q}}_{ij})\hat{\mathcal{Q}}_{mn}\Bigg)\Bigg].\end{aligned} \tag{A15}$$

The symbol $\sigma(\boldsymbol{k})$ is called elliptic, if it has a trivial kernel for $\mathbf{k}\neq\mathbf{0}$. Then the linear differential operator is invertible in the leading derivative order, and solutions of the linear differential equations will exist. To prove the ellipticity of the symbol (A15), it remains to be shown that the linear equations

$$\sqrt{6}qn_in_j=\sqrt{\frac{2}{3}}\,q(h_{ij}+\sqrt{6}\,\bar{\mathcal{Q}}_{ij})+(1-\mathrm{Tr}\bar{\boldsymbol{\mathcal{Q}}}^2)\hat{\mathcal{Q}}_{ij} \tag{A16}$$

have only the trivial solution $\hat{\mathcal{Q}}_{ij}=0$ for $\boldsymbol{n}\neq\mathbf{0}$, where we have introduced the abbreviations

$$q:=\bar{\mathcal{Q}}^{ij}\hat{\mathcal{Q}}_{ij}, \quad \boldsymbol{n}:=\frac{k}{|\boldsymbol{k}|}\Rightarrow|\boldsymbol{n}|=1. \tag{A17}$$

Contracting Eqs. (A16) with $\bar{\mathcal{Q}}^{ij}$, we obtain the necessary implication

$$q(1+\mathrm{Tr}\bar{\boldsymbol{\mathcal{Q}}}^2-\sqrt{6}\,\bar{\mathcal{Q}}^{ij}nm_in_j)\overset{!}{=}0, \tag{A18}$$

i.e., if we can show that the bracket in Eq. (A18) is different from zero, Eq. (A18) implies $q=0$, and therefore $\hat{\mathcal{Q}}_{ij}=0$ via Eq. (A16), so the ellipticity of $\sigma(\boldsymbol{k})$ according to Eq. (A15) would have been proven.

It now follows from a simple estimate for symmetric matrices $\bar{\boldsymbol{\mathcal{Q}}}$ that the vanishing of the bracket in Eq. (A18) implies[14]

$$1+\sum_{i=1}^{3}\bar{\mathcal{Q}}_i^2\leq\sqrt{6}\max_{i=1}^{3}\{\bar{\mathcal{Q}}_i\}, \tag{A19}$$

where the $\bar{\mathcal{Q}}_i$ denote the three eigenvalues of the matrix $\bar{\boldsymbol{\mathcal{Q}}}$. Since $\bar{\boldsymbol{\mathcal{Q}}}$ is traceless, these three eigenvalues may be parametrized by

$$\bar{\mathcal{Q}}_j=\sqrt{\frac{2}{3}}\,\varrho\cos\left(\theta+\frac{2\pi j}{3}\right),\quad j\in\{1,2,3\}$$

with

$$\varrho\geq0,\quad 0\leq\theta<2\pi. \tag{A20}$$

Then the relation (A19) takes the form

$$1+\varrho^2\leq2\rho\Leftrightarrow(1-\rho)^2\leq0, \tag{A21}$$

and is obviously only satisfied for $\varrho=1$. Moreover, because of the identity $\mathrm{Tr}\bar{\boldsymbol{\mathcal{Q}}}^2=\varrho^2$, the particular value $\varrho=1$ corresponds to the coordinate singularity of the stereographic projection used in Eq. (A12), and is hence not permitted by construction. Thus the relation (A19) has been brought to a contradiction, and we conclude that the bracket in Eq. (A18) cannot vanish, which finishes our proof of the ellipticity of the symbol $\sigma(\boldsymbol{k})$ given in Eq. (A15).

Summarizing our results, we have shown that the set of linear partial differential equations $\Delta'_{(ij)}=0$, which determines the fields $\mathcal{Q}'^{i}{}_{j}$, is elliptic, and therefore soluble in leading derivative order. It follows that the solutions $\mathcal{Q}^{i}{}_{j}$ of the nonlinear set of equations $\Delta_{(ij)}=0$ continue to exist under infinitesimal perturbations of the parameter fields $\tilde{e}^{i}{}_{a}$ and Λ. Therefore, solutions $\mathcal{K}_{ij}$ of Eq. (A7), and also solutions $\mathcal{A}_{ia}$ of the saddle-point equations (4.10), can be obtained via Eqs. (A12) and (A1) for a wide range of parameter fields $\tilde{e}^{i}{}_{a}$ and Λ, as long as the only restriction $R\neq2\Lambda$ met in Eq. (A11) is satisfied.

2. Divergence-free triads in the limit $\Lambda\to0$

In this section we want to discuss how suitable *flat* gauge field $\mathcal{A}_{ia}$ may be used to construct divergence-free triads $\vec{d}_a$ of a given Riemannian 3-manifold $(\mathcal{M}_3,\boldsymbol{h})$. Such a flat gauge field on $\mathcal{M}_3$ can be obtained by pursuing any *fixed* solution $\mathcal{A}_{ia}[\tilde{e}^{i}{}_{a},\Lambda]$ of the saddle-point equations (4.10) in the limit $\Lambda\to0$. Using the arguments of Sec. A1, this will be possible for 3-manifolds with $R(x)\neq0$. By virtue of Eq. (2.25), the corresponding gauge field $\mathcal{A}_{ia}$ will not only be flat, but it will in addition satisfy the three Gauss constraints

$$\mathcal{D}_i\tilde{e}^{i}{}_{a}\equiv\partial_i\tilde{e}^{i}{}_{a}+\varepsilon_{abc}\mathcal{A}_{ib}\tilde{e}^{i}{}_{c}=0, \tag{A22}$$

where $\tilde{e}^{i}{}_{a}$ is a fixed but arbitrary triad of the 3-metric $\boldsymbol{h}$.

Let us now consider the parallel transport associated with the gauge field $\mathcal{A}_{ia}$: Given a vector $\vec{v}(0)=v_{a,0}\vec{e}_a$ at a point P_0 of $\mathcal{M}_3$, and a curve $\mathcal{C}$: $x^i=f^i(u)$, $0\leq u\leq1$, connecting P_0 with a second point P_1, we define a vector field $\vec{v}(u)$ along $\mathcal{C}$ by solving the equations of parallel transport,

$$\frac{\mathcal{D}v_a}{\mathcal{D}u}:=\frac{\partial v_a}{\partial u}+\varepsilon_{abc}\frac{\partial f^i}{\partial u}\mathcal{A}_{ib}v_c\overset{!}{=}0,\quad v_a(0)\overset{!}{=}v_{a,0}. \tag{A23}$$

[14]Here and in the following, we have to restrict ourselves to *real-valued* matrices $\bar{\boldsymbol{\mathcal{Q}}}$, which correspond to real or complex solutions $\mathcal{A}_{ia}$ of the saddle-point equations (4.10) via Eqs. (A12) and (A1) in the two different cases $R>2\Lambda$ or $R<2\Lambda$, respectively.

Since the gauge field $\mathcal{A}_{ia}$ is flat, the resulting vector $\vec{v}(1)$ at point P_1 does not depend on the particular choice of $\mathcal{C}$ (cf. [47]), i.e., if we restrict ourselves to the case of simply connected manifolds $\mathcal{M}_3$ in the following, the parallel transport of $\vec{v}(0)$ along arbitrary curves $\mathcal{C}\subset\mathcal{M}_3$ will define a well-defined vector field $\vec{v}(x)$ on $\mathcal{M}_3$. By construction, this vector field $\vec{v}(x)$ turns out to be covariantly constant with respect to $\mathcal{A}_{ia}$,

$$\mathcal{D}_i v_a \equiv \partial_i v_a + \varepsilon_{abc}\mathcal{A}_{ib} v_c \equiv 0, \tag{A24}$$

and, as a consequence of Eq. (A22), the vector field $\vec{v}(x)$ is in addition divergence free,

$$\vec{\nabla}\cdot\vec{v} \equiv \frac{1}{\sqrt{h}}\mathcal{D}_i(v_a \vec{e}^{\,i}{}_a) = \frac{1}{\sqrt{h}}(\underbrace{\mathcal{D}_i v_a}_{0} \vec{e}^{\,i}{}_a + v_a \underbrace{\mathcal{D}_i \vec{e}^{\,i}{}_a}_{0}) = 0. \tag{A25}$$

Moreover, it follows from Eq. (A24) that the parallel transport according to Eq. (A23) conserves the scalar product of two vectors $\vec{v}$ and $\vec{w}$:

$$\partial_i(\vec{v}\cdot\vec{w}) \equiv \mathcal{D}_i(v_a w_a) = \underbrace{\mathcal{D}_i v_a}_{0} w_a + v_a \underbrace{\mathcal{D}_i w_a}_{0} = 0. \tag{A26}$$

From Eqs. (A25) and (A26) it is then obvious that a divergence-free triad $\vec{d}_a(x)$ of the Riemannian 3-manifold $(\mathcal{M}_3, \boldsymbol{h})$ can be constructed by choosing three orthonormal vectors $\vec{d}_a$ at a point P_0, and parallel propagating these vectors along arbitrary curves $\mathcal{C}\subset\mathcal{M}_3$. Since the only freedom in this construction arises from the choice of $\vec{d}_a$ at a single point P_0, this divergence-free triad $\vec{d}_a(x)$ associated with the flat gauge field $\mathcal{A}_{ia}$ turns out to be unique up to global rotations.

APPENDIX B: THE VACUUM STATE ON BIANCHI-TYPE-IX HOMOGENEOUS MANIFOLDS

In this appendix we want to discuss the semiclassical vacuum state (4.50) in the special case of Bianchi-type IX homogeneous 3-manifolds. While one saddle-point contribution, the so-called ''wormhole-state,'' is given by the result (4.53), four further semiclassical vacuum states are known within the framework of the homogeneous Bianchi IX model [25,27]. In the inhomogeneous approach of the present paper, these additional states should correspond to topologically nontrivial divergence-free triads of Bianchi-type-IX manifolds via Eq. (4.50). Such special triads can indeed be constructed from the divergence-free triads of the unit 3-sphere, which will be discussed first in Sec. B1. The divergence-free triads of Bianchi-type-IX manifolds and the corresponding saddle-point contributions to the vacuum Chern-Simons state will then be given in Sec. B2.

1. Divergence-free triads of the unit 3-sphere

The 3-sphere is a maximally symmetric 3-manifold with six Killing vectors $\vec{\xi}_a^{\pm}$, representing the commutator algebra

$$[\vec{\xi}_a^{\pm}, \vec{\xi}_b^{\pm}] = \pm 2[abc]\vec{\xi}_c^{\pm}, \quad [\vec{\xi}_a^{+}, \vec{\xi}_b^{-}] = \vec{0} \tag{B1}$$

of the symmetry group SO(4)≅SO(3)×SO(3). From the second of these commutation relations it follows that the three vector fields $\vec{\xi}_a^{-}$ are the left-invariant vector fields to the Killing vectors $\vec{\xi}_a^{+}$, and vice versa; i.e., the metric tensor of the unit 3-sphere can be expanded in *both* of the two sets $\vec{\xi}_a^{\pm}$ with *spatially constant* coefficients. In particular, if we choose the normalization of $\vec{\xi}_a^{\pm}$ as in the first of Eqs. (B1), the invariant vector fields $\vec{\xi}_a^{\pm}$ form automatically two different sets of invariant triads $\vec{\imath}_a^{\,\pm} := \vec{\xi}_a^{\pm}$ to the metric $\boldsymbol{h}$ of the unit 3-sphere:

$$\vec{\imath}_a^{\,+} \otimes \vec{\imath}_a^{\,+} = \boldsymbol{h} = \vec{\imath}_a^{\,-} \otimes \vec{\imath}_a^{\,-}. \tag{B2}$$

According to Eqs. (B1) and (4.27), both invariant triads $\vec{\imath}_a^{\,\pm}$ have a symmetric structure matrix $\boldsymbol{m}$, and are thus divergence free by virtue of Eq. (4.51). Since they are triads to the same metric $\boldsymbol{h}$, they must be connected by a gauge transformation $\boldsymbol{E}\in O(3)$:

$$\vec{\imath}_a^{\,+} = E_{ab}\overline{\vec{\imath}b.} \tag{B3}$$

The matrix $\boldsymbol{E}$ has a spatially nontrivial dependence, and may of course be calculated explicitly in any given coordinate system on S^3.[15] However, in the following the explicit form of the rotation matrix $\boldsymbol{E}$ will not be needed.

2. Divergence-free triads of Bianchi-type-IX homogeneous manifolds

Anisotropic manifolds of Bianchi-type IX can be described by choosing an invariant triad of the unit 3-sphere, for example, $\vec{\imath}_a^{\,+}$, and rescaling this triad with three scale parameters $a_b>0$:

$$\vec{\imath}_a := D_{ab}\vec{\imath} b^{+} \quad \text{with} \quad \boldsymbol{D}^{-1} := \mathrm{diag}(a_1, a_2, a_3). \tag{B4}$$

Then $\vec{\imath}_a$ is the invariant triad of a Bianchi-type-IX manifold, and the metric tensor is given by $\boldsymbol{h} = \vec{\imath}_a \otimes \vec{\imath}_a$. In the general, anisotropic case, only three of the six vector fields $\vec{\xi}_a^{\pm}$ discussed in Sec. B1 remain as Killing vectors of the 3-metric $\boldsymbol{h}$, namely, the fields $\vec{\xi}_a^{-}$. We will assume that the invariant triad $\vec{\imath}_a$ given in Eq. (B4) is positive oriented. As pointed out in Sec. IV C1, this triad $\vec{d}_a^{(0)} := \vec{\imath}_a$ is automatically divergence free, and gives rise to the ''wormhole'' saddle-point contribution (4.53) to the semiclassical vacuum state.

To find further, topologically nontrivial divergence-free triads $\vec{d}_a$ of Bianchi-type-IX metrics, let us try an ansatz of the form

$$\vec{d}_a = E_{ba} O_{bc} \vec{\imath}_c, \tag{B5}$$

[15] For example, if we employ the Euler angles ψ, ϑ, φ as coordinates on the unit 3-sphere, the matrix $\boldsymbol{E}$ turns out to be precisely the well-known Euler-matrix $\boldsymbol{E}(\psi,\vartheta,\varphi)$ (for a definition of the Euler matrix, see, e.g., [48]).

where $\boldsymbol{O}=(O_{ab})\in \mathrm{SO}(3)$ is assumed to be spatially constant.[16] If we require the triad $\vec{d}_a$ according to Eqs. (B5), (B4) to be divergence free, we arrive at three equations for the matrix $\boldsymbol{O}$,

$$\vec{\nabla}\cdot\vec{d}_a=O_{bc}D_{cd}[\vec{\iota}_d^{\,+},E_{ba}]\overset{!}{=}0. \tag{B6}$$

The spatial derivatives of the matrix $\boldsymbol{E}$ with respect to the vector fields $\vec{\iota}_a^{\,+}$ can be calculated by inserting Eqs. (B3) into Eqs. (B1), and are given by

$$[\vec{\iota}_a^{\,+},E_{bc}]=2\varepsilon_{abd}E_{dc}. \tag{B7}$$

Therefore, the requirements (B6) can be simplified to the form

$$\varepsilon_{abc}O_{bd}D_{dc}\overset{!}{=}0, \tag{B8}$$

i.e., the matrix $\boldsymbol{O}$ has to be chosen in such a way that for any given diagonal matrix $\boldsymbol{D}$ the matrix $\boldsymbol{O}\cdot\boldsymbol{D}$ is *symmetric*. The only four solutions $\boldsymbol{O}\in\mathrm{SO}(3)$ of this problem turn out to be

$$\boldsymbol{O}^{(1)}=\mathrm{diag}(+1,-1,-1),\quad \boldsymbol{O}^{(2)}=\mathrm{diag}(-1,+1,-1),$$
$$\boldsymbol{O}^{(3)}=\mathrm{diag}(-1,-1,+1),\quad \boldsymbol{O}^{(4)}=\mathrm{diag}(+1,+1,+1). \tag{B9}$$

Hence the ansatz (B5) gives exactly four further divergence-free triads of Bianchi-type-IX homogeneous manifolds,

$$\vec{d}_a^{(\alpha)}=E_{ba}O_{bc}^{(\alpha)}\cdot\vec{\iota}_c,\quad \alpha\in\{1,2,3,4\}. \tag{B10}$$

We now wish to compute the semiclassical saddle-point contributions to the vacuum state (4.50), which correspond to the divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha\in\{1,2,3,4\}$. Therefore we first need the winding numbers $\hat{w}$ of these triads with respect to the Einstein triad $\vec{g}_a$ of Bianchi-type-IX metrics. Since the Einstein triad turns out to be given exactly by the invariant triad of the homogeneous 3-metric, $\vec{g}_a\equiv\vec{\iota}_a$, we have to calculate the Cartan-Maurer invariants (4.23) of the four rotation matrices

$$\boldsymbol{\Omega}^{(\alpha)}:=\boldsymbol{E}^T\cdot\boldsymbol{O}^{(\alpha)},\quad \alpha\in\{1,2,3,4\}. \tag{B11}$$

This can be done without knowing the matrix $\boldsymbol{E}$ in Eq. (B11) explicitly, because the spatial derivatives in Eq. (4.23) may be substituted by $\partial_j=\boldsymbol{\iota}_{ja}^{+}\cdot\vec{\iota}_a^{\,+}$, and then be eliminated with the help of Eq. (B7), yielding

$$I(\boldsymbol{\Omega}^{(\alpha)})=-8\int d^3x\,\varepsilon_{abc}\boldsymbol{\iota}_a^{+}\wedge\boldsymbol{\iota}_b^{+}\wedge\boldsymbol{\iota}_c^{+}=-48\mathcal{V}, \tag{B12}$$

where $\mathcal{V}=2\pi^2$ is the dimensionless volume of the unit 3-sphere. Since the constant I_0 in the definition (4.24) of the winding number has the numerical value $I_0=96\pi^2$ for manifolds with S^3 topology (cf. [37]), it follows that the ''absolute'' winding numbers of the triads $\vec{d}_a^{(\alpha)}$, $\alpha\in\{1,2,3,4\}$, are simply given by $\hat{w}=-1$.

To proceed in the computation of the semiclassical saddle-point contributions (4.50), we further have to evaluate the functional ϕ defined in Eq. (2.15) for the four divergence-free triads $\vec{e}_a=\vec{d}_a^{(\alpha)}$, $\alpha\in\{1,2,3,4\}$. Inserting the triads (B10) into ϕ according to Eq. (2.15), we first recover the wormhole exponent of Eq. (4.53), if the spatial derivative ∂_j acts on the invariant triad $\vec{\iota}_a$. In addition, we obtain a second term, which stems from the action of the derivative operator ∂_j on the spatially nontrivial matrix $\boldsymbol{E}$. This contribution can again be calculated by reexpressing the spatial derivative in terms of the vector fields $\vec{\iota}_a^{\,+}$, and making use of Eqs. (B7). In case of the divergence-free triad $\vec{d}_a^{(4)}$, we obtain the explicit result

$$\Psi_{\mathrm{vac}}^{(4)}\overset{\mu\to\infty}{\propto}\Psi_{\mathrm{vac}}^{(0)}\cdot\exp\left[\pm\frac{4\mathcal{V}}{\gamma\hbar}\left(-\frac{6}{\Lambda}+a_1a_2+a_2a_3+a_3a_1\right)\right], \tag{B13}$$

with $\Psi_{\mathrm{vac}}^{(0)}$ given in Eq. (4.53). The saddle-point value (B13) is known as the ''no-boundary'' state from the homogeneous Bianchi IX model. Three further semiclassical saddle-point contributions to the vacuum state (4.50), which correspond to the remaining divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha\in\{1,2,3\}$, are of the same form as $\Psi_{\mathrm{vac}}^{(4)}$ given in Eq. (B13), but with two of the three scale parameters a_b replaced by their negatives. In the framework of the Bianchi IX model, the corresponding states were referred to as ''asymmetric'' states. We conclude that all five saddle-point values $\Psi_{\mathrm{vac}}^{(\alpha)}$, $\alpha\in\{0,...,4\}$, known for the homogeneous Bianchi IX model can be recovered within the inhomogeneous approach of the present paper by evaluating the state (4.50) for the five topologically inequivalent divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha\in\{0,...,4\}$, of Bianchi-type-IX manifolds. Up to a Gaussian prefactor, which always lies hidden in the proportionality signs of Eqs. (4.53), (B13), the results are of the same form as in [25,27].

However, as we have shown in [27,49], the four semiclassical saddle-point contributions $\Psi_{\mathrm{vac}}^{(\alpha)}$, $\alpha\in\{1,2,3,4\}$, are restricted to occur *simultaneously* for symmetry reasons. This can also be seen within the present, inhomogeneous approach, since the four divergence-free triads $\vec{d}_a^{(\alpha)}$, $\alpha\in\{1,2,3,4\}$, all have the same winding number, and thus should enter into the value of the Chern-Simons state with the same topological right. We conclude that, in agreement with discussions of the nondiagonal Bianchi IX model, only two independent values of the vacuum Chern-Simons state are found for Bianchi-type-IX manifolds.

APPENDIX C: A NONFLAT 4-METRIC GENERATED BY THE VACUUM STATE

We now want to give special solutions of the vacuum evolution equations (4.55), such that the associated semiclassical 4-geometries satisfy neither the ''no-boundary'' condi-

[16] At least in the isotropic case $a_1=a_2=a_3$, this ansatz gives the second divergence-free triad $\vec{\iota}_a^{\,-}$ of the 3-sphere by virtue of Eq. (B3), if we simply choose $\boldsymbol{O}=\mathbf{1}$.

tion proposed by Hartle and Hawking [39–41], nor the condition of asymptotical flatness in the limit of large scale parameters $a_{\cos}$.[17] Let us therefore consider the class of 3-metrics

$$\boldsymbol{h}=\vec{\iota}_a\otimes\vec{\iota}_a, \tag{C1}$$

where the triad vector fields $\vec{\iota}_a=\iota^i_a\partial_i$ are given by

$$\vec{\iota}_1=\frac{1}{a_1}\partial_1,\quad \vec{\iota}_2=\frac{1}{a_2}\partial_2,$$

$$\vec{\iota}_3=\frac{1}{a_3}(\partial_3+x^2\partial_1+x^1\partial_2). \tag{C2}$$

The scale parameters a_b in Eq. (C2) are assumed to be spatially constant, and the triad $\vec{\iota}_a$ is taken to be positive oriented. Then the structure matrix $\boldsymbol{m}$ introduced in Eq. (4.27) takes the spatially constant form

$$\boldsymbol{m}=\mathrm{diag}\left[\frac{a_1}{a_2a_3},-\frac{a_2}{a_3a_1},0\right], \tag{C3}$$

i.e., the triad $\vec{\iota}_a$ is the invariant triad of a spatially homogeneous 3-manifold, which can be classified to be of Bianchi type VI_{-1}. Since the structure matrix $\boldsymbol{m}$ according to Eq. (C3) is symmetric, it follows directly from Eq. (4.51) that the invariant triad $\vec{\iota}_a$ is divergence free. The Killing vectors of the 3-metric (C1) must commute with the $\vec{\iota}_a$ and are given by

$$\vec{\xi}_1=\cosh x^3\partial_1+\sinh x^3\partial_2,\quad \vec{\xi}_2=\sinh x^3\partial_1+\cosh x^3\partial_2,$$

$$\vec{\xi}_3=\partial_3. \tag{C4}$$

They may be used to compactify the 3-manifold $\mathcal{M}_3$ with the metric (C1) in the three $\vec{\xi}_a-$ directions, giving rise to a manifold with the nontrivial topology $S^1\times T^2$. The compactified 3-manifold will then have a finite volume $V=\mathcal{V}a_1a_2a_3$, where the value of $\mathcal{V}>0$ depends on the particular choice of the compactification.

We are now interested in the semiclassical 4-geometries being generated by the evolution equations (4.55) in case of the divergence-free triad $\vec{d}_a=\vec{\iota}_a$. If we allow for an arbitrary lapse function N, they read

$$\frac{d}{d\tau}\tilde{\iota}^i{}_a=\pm N\tilde{\varepsilon}^{ijk}\partial_j\iota_{ka}. \tag{C5}$$

For the three-metric (C1) under study, Eqs. (C5) take the form

$$\frac{d}{d\tau}\sigma_1=\mp N\sqrt{\frac{\sigma_2\sigma_3}{\sigma_1}},\quad \frac{d}{d\tau}\sigma_2=\pm N\sqrt{\frac{\sigma_3\sigma_1}{\sigma_2}},\quad \frac{d}{d\tau}\sigma_3=0, \tag{C6}$$

where we have introduced the new variables

$$\sigma_1:=a_2a_3,\quad \sigma_2:=a_3a_1,\quad \sigma_3:=a_1a_2. \tag{C7}$$

Choosing the lapse function N as

$$N=\mp\frac{1}{2}(\sigma_1\sigma_2\sigma_3)^{-1/2}, \tag{C8}$$

the set of Eqs. (C6) is easily integrated and has the general solution

$$\sigma_1(\tau)=\sqrt{\tau_0+\tau},\quad \sigma_2(\tau)=\sqrt{\tau_0-\tau},$$

$$\sigma_3(\tau)\equiv\sigma_3=\mathrm{const};\quad |\tau|<\tau_0. \tag{C9}$$

Here we have chosen $\tau=0$ such that $\sigma_1(0)=\sigma_2(0)$, so only two integration constants $\tau_0>0$ and $\sigma_3>0$ remain in Eq. (C9).

In order to prove that the 4-geometry according to Eq. (C9) is nonflat, it is not sensible to compute the 4-dimensional Ricci or Einstein tensor, since these quantities vanish identically by construction, so we will consider the nontrivial components ${}^4R^{0i}{}_{0j}$ of the four-dimensional Riemann tensor instead. For a vanishing shift vector $N^i=0$, they are given by

$${}^4R^{0i}{}_{0j}=-\frac{1}{N}\frac{d}{dt}K^i{}_j+K^i{}_kK^k{}_j, \tag{C10}$$

with $K^i{}_j$ being the usual extrinsic curvature tensor. With help of the triad (C2), we may convert the spatial indices of ${}^4R^{0i}{}_{0j}$ into internal indices a, b, to obtain

$$\mathcal{R}_{ab}:=\iota_{ia}\iota^j{}_b\ {}^4R^{0i}{}_{0j}. \tag{C11}$$

For the metric (C1), $(\mathcal{R}_{ab})$ is a diagonal matrix with

$$\mathcal{R}_{33}=-\frac{1}{Na_3}\frac{d}{d\tau}\left(\frac{1}{N}\frac{da_3}{d\tau}\right), \tag{C12}$$

and analogous expressions for $\mathcal{R}_{11}$, $\mathcal{R}_{22}$. Making use of the evolution Eqs. (C6), we can eliminate the τ derivatives in Eq. (C12) to arrive at

$$\mathcal{R}_{33}=\frac{\sigma_1\sigma_2\sigma_3}{2}\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)^2, \tag{C13}$$

and inserting the general solution (C9) into Eq. (C13), we find

$$\mathcal{R}_{33}=\sigma_3\tau_0^2(\tau_0^2-\tau^2)^{-3/2}>0. \tag{C14}$$

Thus we have found a component of the Riemann tensor, which is nonzero for all times τ, $|\tau|<\tau_0$, so the semiclassical 4-geometries obtained by evolving the initial

[17]Here we assume the vacuum limit $\kappa\to0$ to be realized by considering a sufficiently small value for the cosmological constant Λ. Then it will be possible to take the cosmological scale parameter $a_{\cos}$ arbitrarily large at the same time; cf. Eq. (4.4).

3-geometries (C1) are *nowhere* flat. Moreover, the semiclassical 4-geometries do *not* satisfy the ''no-boundary'' condition: While the cosmological scale parameter

$$a_{\cos}=\mathcal{V}^{1/3}(\sigma_1\sigma_2\sigma_3)^{1/6}=\mathcal{V}^{1/3}\sigma_3^{1/6}(\tau_0^2-\tau^2)^{1/12} \quad \text{(C15)}$$

vanishes only at the timelike borders $|\tau|\rightarrow\tau_0$ of the semiclassical space-time manifolds, the corresponding curvature components $\mathcal{R}_{33}$ at the same time are tending to $+\infty$. Consequently, the semiclassical 4-manifolds are *not* regular or compact for vanishing scale parameters $a_{\cos}$.

[1] L. Brink and M. Henneaux, *Principles of String Theory* (Plenum, New York, 1988).
[2] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, (Cambridge University Press, Cambridge, England, 1987), Vols. I and II.
[3] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation, An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[4] J. A. Wheeler, in *Relativity and Topology*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1964).
[5] A. Ashtekar, *Lectures on Nonperturbative Canonical Gravity* (World Scientific, Singapore, 1991).
[6] A. Ashtekar, Phys. Rev. D **36**, 2955 (1987); Phys. Rev. Lett. **57**, 2244 (1986).
[7] M. Henneaux, C. Schomblond, and J. E. Nelson, Phys. Rev. D **39**, 434 (1989).
[8] R. Gambini and A. Trias, Phys. Rev. D **23**, 553 (1981); Nucl. Phys. **B278**, 436 (1986).
[9] R. Rovelli and L. Smolin, Nucl. Phys. **B331**, 80 (1990); Phys. Rev. Lett. **61**, 1155 (1988).
[10] T. Thiemann, J. Math. Phys. **39**, 1236 (1998).
[11] B. Brügmann and J. Pullin, Nucl. Phys. **B390**, 399 (1993).
[12] S. Major and S. Smolin, Nucl. Phys. **B473**, 267 (1996).
[13] C. Rovelli, J. Math. Phys. **36**, 6529 (1995).
[14] J. Griego, Nucl. Phys. **B467**, 332 (1996); Phys. Rev. D **53**, 6966 (1996).
[15] B. Brügmann, R. Gambini, and J. Pullin, Nucl. Phys. **B385**, 587 (1992); Gen. Relativ. Gravit. **25**, 1 (1993).
[16] M. P. Blencowe, Nucl. Phys. **B341**, 213 (1990).
[17] H. Kodama, Phys. Rev. D **42**, 2548 (1990).
[18] L. Smolin and C. Soo, Nucl. Phys. **B449**, 289 (1995).
[19] K. Ezawa, Phys. Rev. D **53**, 5651 (1996).
[20] L. Kauffman, *On Knots*, *Annals of Mathematics Studies* (Princeton University Press, Princeton, NJ, 1987).
[21] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[22] R. Loll, Phys. Rev. D **54**, 5381 (1996).
[23] J. F. Barbero, Phys. Rev. D **51**, 5507 (1995).
[24] G. Immirzi, Class. Quantum Grav. **14**, L177 (1997).
[25] R. Graham and R. Paternoga, Phys. Rev. D **54**, 2589 (1996).
[26] R. Paternoga and R. Graham, Phys. Rev. D **54**, 4805 (1996).
[27] R. Paternoga and R. Graham, Phys. Rev. D **58**, 083501 (1998).
[28] S. Perlmutter, M. S. Turner, and M. White, Phys. Rev. Lett. **83**, 670 (1999).
[29] S. Perlmutter *et al.*, Nature (London) **391**, 51 (1998).
[30] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[31] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, New York, 1973).
[32] A. Ashtekar, A. P. Balachandran, and S. G. Jo, Int. J. Mod. Phys. A **4**, 1493 (1989).
[33] H. J. Matschull, Class. Quantum Grav. **12**, 651 (1995).
[34] A. Ashtekar, Int. J. Mod. Phys. D **5**, 629 (1996).
[35] T. Jacobson and L. Smolin, Nucl. Phys. **B299**, 29 (1988).
[36] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).
[37] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, New York, 1997), Vol. II.
[38] M. P. Ryan, Jr. and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, NJ, 1975).
[39] J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).
[40] S. W. Hawking, Nucl. Phys. **B239**, 257 (1984).
[41] S. W. Hawking and J. C. Luttrell, Phys. Lett. **143B**, 83 (1984).
[42] L. D. Faddeev and V. N. Popov, Phys. Lett. **25B**, 29 (1967).
[43] R. P. Woodard, Class. Quantum Grav. **10**, 483 (1993).
[44] A. Vilenkin, Phys. Rev. D **33**, 3560 (1986); **37**, 888 (1988).
[45] S. Del Campo and A. Vilenkin, Phys. Lett. B **224**, 45 (1989).
[46] J. Rauch, *Partial Differential Equations* (Springer-Verlag, New York, 1991).
[47] J. L. Synge and A. Schild, *Tensor Calculus* (Dover, New York, 1978).
[48] H. Goldstein, *Classical Mechanics* (Addison-Wesley, New York, 1980).
[49] R. Paternoga, in *Current Topics in Mathematical Cosmology*, edited by M. Rainer and H. J. Schmidt (World Scientific, Singapore, 1998).