Solution to the graceful exit problem in pre-big-bang cosmology

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We examine the string cosmology equations with a dilaton potential in the context of the pre-big-bang scenario with the desired scale factor duality, and give a generic algorithm for obtaining solutions with appropriate evolutionary properties. This enables us to find pre-big-bang type solutions with suitable dilaton behavior that are regular at $t=0$, thereby solving the graceful exit problem. However, to avoid fine-tuning of initial data, an ''exotic'' equation of state is needed that relates the fluid properties to the dilaton field. We discuss why such an equation of state should be required for reliable dilaton behavior at late times.

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I. INTRODUCTION

In this paper, we investigate the equations of string cosmology [1,2] in the string frame, allowing for a dilaton potential $V(\phi)$. The pre-big-bang scenario is motivated by the search for cosmological solutions with an $a(t)\rightarrow 1/a(-t)$ symmetry in the scale factor $a(t)$, which implements an analogue of the *T*-duality symmetry of M theory. However, one must distinguish between symmetries of the equations and those of their solutions. We look at cases in which the *equations* have such a scale factor symmetry, when solutions may or may not exhibit the same symmetry, and at cases in which the *solutions* obey the scale factor symmetry, even if the equations do not. In the latter case we obtain some solutions that seem to have most of the properties desired in the pre-big-bang scenario, in that they have the desired scale factor symmetry, the desired evolution of the dilaton field, and continuity at $t=0$ of $a(t), \phi(t), \dot{\phi}(t)$ and the Hubble parameter $H(t)\equiv \dot{a}(t)/a(t)$ [but allowing a discontinuity in $\dot{H}(t)$ and $\ddot{\phi}(t)$ there, implying a corresponding discontinuity in $\partial V/\partial\phi$], thus providing a solution to the graceful exit problem [3,4]. However, to obtain the desired dilaton behavior at recent times, we need to employ an ''exotic'' equation of state as discussed below.

There are ''no-go'' theorems that exclude such regular transitions in the presence of a perfect fluid and Kalb-Ramond sources. A ''lowest order'' Einstein frame analysis by [4] discusses graceful exit in generalized phase space, and derives a set of necessary conditions for transition from a classical dilaton-driven inflationary pre-big-bang phase to a radiation-dominated era, joined at $t=0$ in a Planck epoch of maximal finite curvature $\dot{H}(t)$. They show that a successful exit requires violation of the null energy condition (NEC). Classical sources tend to obey the NEC, but various new kinds of effective sources generating non-singular evolution have been considered that do not. Thus, failing invocation of higher order curvature terms, some kind of exotic behavior of matter is necessary in order to obtain a graceful exit from the pre-big-bang phase.

In this paper we follow Gasperini [1] by working in the string frame. The relation to the Einstein frame is left for another paper. It should be made clear from the start that our solutions are rather special in the spectrum of pre-big-bang models; those we concentrate on in the main show an exact scale factor duality in the solutions, and thus we do not consider here the more exciting possibility of a phase of early kinetic-dilaton dominated inflation which leads to an early phase which is not radiation dual but is genuinely stringy inflationary vacuum. Nevertheless, the set of solutions investigated here helps to understand the spectrum of possibilities available within the broad pre-big-bang set of ideas.

II. STRING COSMOLOGY EQUATIONS

One can determine the general equations of string cosmology by extremizing the lowest order effective action of dilaton gravity:

$$S=-\frac{1}{2\lambda_s^{d-1}}\int d^{d+1}x\sqrt{|g|}e^{-\phi}\left[R+(\nabla\phi)^2-\frac{1}{12}H^2+V(\phi)\right]+\int d^{d+1}x\sqrt{|g|}L_m, \tag{1}$$

where ϕ is the scalar dilaton, $H=dB$ (antisymmetric tensor field strength), $V(\phi)$ is the dilaton potential, λ_s is the fundamental string length scale, and L_m is the Langrangian density of other matter sources. To derive string cosmology equations for the $d=3$, homogeneous, isotropic, conformally flat background we will follow Gasperini [1] in assuming $B=0$, a perfect fluid minimally coupled to the dilaton, and a Bianchi type I metric (see Appendix C of [1] for details). Unlike Gasperini we assume $V(\phi)\neq 0$, to obtain the string cosmology equations in the following canonical form:

$$H^2=\frac{e^{\phi}}{6}\rho+H\dot{\phi}+\frac{V}{6}-\frac{\dot{\phi}^2}{6}, \tag{2}$$

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$$\dot{H}+H^2=e^{\phi}\left(\frac{p}{2}-\frac{\rho}{3}\right)-H\dot{\phi}+\frac{\dot{\phi}^2}{3}-\frac{V'}{2}-\frac{V}{3}, \tag{3}$$

$$\ddot{\phi}=-3H\dot{\phi}+\dot{\phi}^2-V-V'+e^{\phi}\left(\frac{3p}{2}-\frac{\rho}{2}\right), \tag{4}$$

where $V'=\partial V/\partial\phi$. When combined, these imply the standard energy conservation equation

$$\dot{\rho}=-3H(\rho+p). \tag{5}$$

In a relationship analogous to that between the classical Friedmann equation and Raychaudhuri equation:

Equation (2) is the first integral of

Eq. (3) provided that Eq. (4) and Eq. (5) hold. (6)

These four equations will be the basis for the analysis in this paper.

One of the primary motivations for the pre-big-bang scenario [5] is that when $V(\phi)=0$, these equations are invariant under the following transformation:

$$a(t)\to\hat{a}(t)=a^{-1}(t) \tag{7}$$

provided that the dilaton transforms as $\phi\to\hat{\phi}=\phi-6\ln a$ and the energy density and pressure as $\rho\to\rho'=a^6\rho$, $p\to p'=-pa^{-6}$. Thus, if $a(t)$ is a solution, so is $\hat{a}(t)$ for suitable ϕ,ρ,p. Since the string cosmology equations are also invariant under time reversal symmetry,

$$a(t)\to\overline{a(t)}=a(-t), \tag{8}$$

the deceleration associated with standard post-big-bang cosmology can be associated with an accelerated evolution prior to the big bang by the generalized transformation

$$a(t)\to\tilde{a}(t)=a^{-1}(-t), \tag{9}$$

where $\tilde{a}(t)$ is a solution for suitable ϕ,ρ,p because $a(t)$ is. The solution has *T*-duality symmetry if, for each *t*,

$$a(t)=\tilde{a}(t)=a^{-1}(-t). \tag{10}$$

However, if we assume $V(\phi)\neq 0$ as in Eqs. (2)–(4), then in general the equations are not invariant under the symmetry, Eq. (10), even if the solutions are. We will look at both cases in what follows, but generically allowing a potential that does not preserve the symmetry. Note that if we assume matter with *the same* equation of state before and after $t=0$, then the matter equations also will not be invariant under the scale factor symmetry. One has to decide what is more physically meaningful: matter with a universal equation of state applicable at all times or that whch has a discontinuous equation of state that preserves this symmetry. In what follows, we adopt the first option. We return to discuss this choice in the conclusion.

Flat dilaton potential with exotic equation of state

To obtain equations of motion preserving the scale factor symmetry, Eq. (10), we assume the simplest potential, namely a flat potential

$$V(\phi)=\kappa, \tag{11}$$

where κ is a constant, and then investigate the behavior of the universe. In order to reliably obtain proper limiting behavior of the dilaton, we assume that the equation of state

$$p=\frac{\rho}{3}+\frac{2}{3}e^{-\phi}\kappa \tag{12}$$

holds at all times (this choice, which is not invariant under the duality symmetry, is discussed further in the following sections). One can immediately see that at late times if $\phi\to\text{const}$, as we will show follows from this choice, then this equation of state simply reduces to radiation plus a constant.

We are interested in getting satisfactory dynamics for $H(t)$ and $\phi(t)$, or, equivalently, for $\chi(t)\equiv\dot{\phi}$. To see when this occurs, we manipulate the string cosmology equations (2)–(4) subject to Eqs. (11),(12) to obtain the two-dimensional phase space with coordinates (χ,H) governed by the following equations:

$$\dot{H}=\frac{\chi^2}{6}-2H^2+\frac{\kappa}{6}, \tag{13}$$

$$\dot{\chi}=\chi(\chi-3H), \tag{14}$$

the latter following because of the choice (12). Having chosen the constant κ, we can set initial conditions (χ_0,H_0) at $t=0$, and then extend the solution to positive and negative values of t by use of these equations. For $\kappa<0$, there are no fixed points in the phase plane, and on every trajectory both H and χ diverge as $|t|\to\infty$. For $\kappa=0$, i.e. no dilaton potential, there is one fixed point at the origin, but for any initial condition (set at $t=0$), χ and H will diverge either as time is run forward or backward.

The interesting dynamics is obtained when $\kappa>0$. There are then fixed points at $A_+:\ (0,\sqrt{\kappa/12})$ (a source), $A_-:\ (0,\sqrt{\kappa/12})$ (a sink), $B_+:\ (\sqrt{3\kappa},\sqrt{\kappa/3})$ (a saddle point), and $B_-:(-\sqrt{3\kappa},-\sqrt{\kappa/3})$ (a saddle point). In the phase plane depicted in Fig. 1, we claim the initial conditions in the region I bounded by A_+, B_+ and A_- and the separatrixes joining them give satisfactory dynamics of both H and χ which include $\chi\to 0$ as $|t|\to\infty$, $\chi>0$ for all times, so $\phi(t)$ is monotonic, H remains finite, and a ''bounce'' occurs that avoids the initial singularity. Since region I is bounded by fixed points that have coordinates proportional to $\sqrt{\kappa}$, increasing κ will give one a larger region of initial conditions that lead to a nonsingular universe with proper dilaton dynamics. We can obtain a solution on the boundary of region I (evolving along the line joining A_- to A_+, which does not lie in I) that is invariant under symmetry (8) by setting $\chi_0=0$, $H_0=0$ at $t=0$, but this solution, given explicitly by $a(t)=\cosh^{1/2}(\sqrt{\kappa/3}t)$, is not invariant under the symmetry

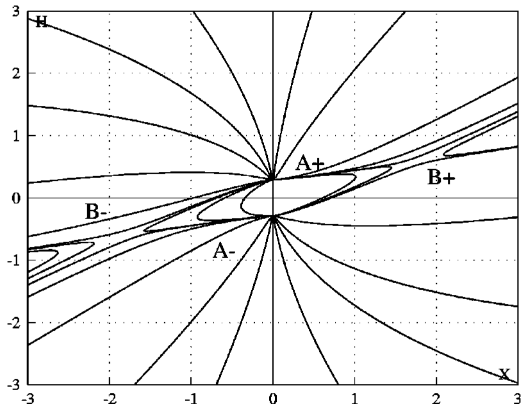

FIG. 1. Phase portrait representing the solution space of Eqs. (13),(14) with $\kappa>0$.

(10). A drawback of all these models is that inflation will not stop at $t>0$, but as discussed in the Conclusion, the string cosmology equations derived in Sec. II do not apply to the present cosmological regime without modification, so it is possible that a radiation dominated evolution started after the time when these equations no longer applied. In any case this gives a specific family of solutions where the equations display the desired symmetry (10) but the solutions do not—which is not very surprising, given the prevalence of broken symmetries in physics.

III. OBTAINING DESIRED DYNAMICS FROM A DILATON POTENTIAL

In this section, we generalize the method introduced by Ellis and Madsen [6] through which they obtain a classical scalar potential associated with a specified $a(t)$ in the standard gravitational equations. No field has been observed that coincides with a dilaton potential $V(\phi)$, so we assume that it is a freely disposable function. We show that by a suitable choice of $V(\phi)$ one can obtain almost any behavior for $a(t)$ or, alternatively, for $\phi(t)$. We first present an algorithm for determining $V(\phi)$ from a desired $a(t)$ or a desired $\phi(t)$, and then present an analytically smooth solvable example. This solution illustrates our main point, but has little physical relevance [although it does satisfy the symmetry (10)]. In the following section we use these methods to obtain two solutions that resemble the standard ''pre-big-bang scenario,'' but with continuity of $a(t)$ and $H(t)$ and with satisfactory dynamics of $\phi(t)$. The associated dilaton potentials are *ad hoc* because they are derived from the desired behavior of the universe, rather than from a field theory model; as discussed in many inflationary and quintessence studies, see, e.g., [7,8].

A. Algorithms

We proceed by providing the following *general algorithm* for determining a dilaton potential $V(\phi)$ that produces a desired $a(t)$:

(1) Specify a desired monotonic function for the scale parameter $a(t)$, consequently determining $H(t)$ and $\dot{H}(t)$,

(2) Choose an equation of state and solve for $\rho(a)$ from Eq. (5);[1] as $a(t)$ is known, this determines $\rho(t)$.

(3) Eliminate V and V' from Eq. (4) by use of Eqs. (2) and (3) to obtain a differential equation relating $H(t)$, $\phi(t)$, $\rho(t)$, and their time derivatives.

(4) Solve the equation obtained in step (3) for $\phi(t)$.

(5) Substitute the now known functions $\phi(t)$, $\rho(t)$, and $H(t)$ and $a(t)$ into the rearranged version of Eq. (2),

$$V(t)=6H^2-e^{\phi}\rho-6H\dot{\phi}+\dot{\phi}^2 \tag{15}$$

to obtain $V(t)$.

(6) Invert $\phi(t)$ to obtain $t(\phi)$.

(7) Transform $V(t)$ as follows: $V(t)=V(t(\phi))\Rightarrow V(\phi)$. This is possible for each range of t on which $\phi(t)$ is monotonic [if it is not monotonic on some range of t, in general $V(\phi)$ will not be well defined because it will not be single valued for the corresponding values of ϕ].

Thus, provided $\phi(t)$ determined from step (3) is monotonic, we find a $V(\phi)$ that corresponds to a given monotonic function $a(t)$. Because we have now satisfied Eqs. (5), (2) and the equation obtained in step (3), the latter depending essentially on Eq. (4), it follows from statement (6) that Eq. (3) will be true also, so we have satisfied all the equations of the theory for this matter description (cf. [6]); hence we have a solution of the desired form.

Alternatively, we can give an algorithm for determining a dilaton potential $V(\phi)$ that produces a desired dilaton evolution[2] $\phi(t)$ by proceeding in the same way as above, except for minor changes: replace step (1) by the following:

$(1')$ Specify the desired monotonic function for the dilaton, $\phi(t)$, in step (2), leave ρ in the form $\rho(a)$, and replace step (4) by

$(4')$ solve the equation obtained in step 3) for $a(t)$ [or for $H(t)$].

The rest of the algorithm is as before.

Finally, note that we can carry out these procedures piecewise: for example we can specify $a(t)$ for some range of t and $\phi(t)$ for some adjoining range of t, or different behaviors for $a(t)$ for different ranges of t, then join the solutions together, ensuring that $a(t)$, $H(t)$, $\phi(t)$ and $\chi(t)$ are continuous where these ranges meet.

B. Exponential scale factor behavior with no matter

To demonstrate the procedure, we give a simple analytically solvable example with a pure scalar field, i.e. $\rho=p=0$. Consider an exponential expansion as in classical inflation,

$$a(t)=e^{wt}\Rightarrow H=w,\ \dot{H}=0, \tag{16}$$

[1] If the equation of state is a function of V or V', these quantities then will have to be eliminated using Eqs. (2) and (3) before solving Eq. (5).

[2] It is important to note that one has freedom to choose only $a(t)$ or $\phi(t)$, not both.

where w is a positive constant. This solution has the desired symmetry (10).

In this case the differential equation for $\phi(t)$ takes the form

$$\ddot{\phi}=H\dot{\phi} \tag{17}$$

Using Eq. (17) and Eq. (15) we obtain

$$\phi(t)=\phi_0+\frac{\dot{\phi}_0}{w}(e^{wt}-1), \tag{18}$$

a monotonic function as required, and

$$V(t)=6w^2-6\dot{\phi}_0we^{wt}+\dot{\phi}_0^2e^{2wt}. \tag{19}$$

After inserting the inverted equation (18),

$$t(\phi)=\frac{1}{w}\log\left[\frac{w}{\dot{\phi}_0}\left(\phi-\phi_0+\frac{\dot{\phi}_0}{w}\right)\right], \tag{20}$$

into Eq. (19), one obtains

$$V(\phi)=w^2\left(\phi-3-\phi_0+\frac{\dot{\phi}_0}{w}\right)^2-3w^2 \tag{21}$$

which is simply a quadratic potential plus a constant. Clearly the behavior for $\phi(t)$ is unphysical since $\phi(t)\to\infty$ instead of asymptoting to a constant. However, this gives a transparent example where even though the scale factor symmetry (10) is broken in the equations [because $V(\phi)$ is not constant], the solution obeys that symmetry.

IV. ''PRE-BIG-BANG'' BEHAVIOR

In this section we try to use the methods just explained to obtain solutions that resemble the ''pre-big-bang scenario'' but with satisfactory dynamics of $\phi(t)$ and a continuous transition from the pre-big-bang to post-big-bang phases. In these examples, we seek solutions that evolve from a string perturbative vacuum, i.e. $H\to 0$ and $e^{\phi}\to 0$ (no interactions), to the present scenario where e^{ϕ}, which acts as the coupling constant, asymptotes to a constant. We will assume the following behavior of the universe:

$$a(t)=(t+1)^{1/2},\quad t\geqslant 0\Rightarrow H(t)=\frac{1}{2(t+1)} \tag{22}$$

determining $a(t)$ for $t\geqslant 0$, and by the symmetry (10),

$$a(t)=(-t+1)^{-1/2},\quad t\leqslant 0\Rightarrow H(t)=\frac{1}{2(-t+1)} \tag{23}$$

determining $a(t)$ for $t\leqslant 0$. Both $a(t)$ and $H(t)$ are continuous at $t=0$ with $a(0)=1$, $H(0)=1/2$, but $\dot{H}(t)$ is not continuous there.

This behavior, which is essentially radiation dominated evolution of the universe for positive times and power-law inflation for negative times, is motivated by the ''pre-big-bang'' scenario introduced in [5], and exactly obeys the scale factor symmetry (10). Note that we have shifted the origin of time in each branch from that customarily used, in order to get a smooth evolution through $t=0$; this of course makes no difference to the desired physical behavior, for we can choose the origin of time to be wherever we want (and the equations are invariant under time translation $t\to t'=t+c$). Although the power law inflation ends with the scale-factor value $a(0)=1$, required by continuity together with the symmetry (10), the solution has sufficient inflation for any purpose because it involves an infinite number of e-foldings (it starts with the asymptotical value $a=0$ as $t\to-\infty$).

A. Pre-big-bang behavior with radiation equation of state

First we assume the radiation equation of state holds at all times, that is,

$$p=\frac{\rho}{3}, \tag{24}$$

which, using Eqs. (5) and (22),(23), implies

$$\rho(\pm t)=\rho_0(\pm t+1)^{\mp 2} \tag{25}$$

where ρ_0 is a positive constant and ''$+t$'' refers to the post-big-bang era, ''$-t$'' to the pre-big-bang era. Notice that both ρ and $\dot{\rho}$ are continuous at $t=0$.

The equation for ϕ now takes the form

$$\ddot{\phi}=\frac{2}{3}e^{\phi}\rho+H\dot{\phi}+2\dot{H}. \tag{26}$$

Substituting in Eqs. (22) and (25), we could not find an analytical solution to Eq. (26), so we investigate the three dimensional phase space with coordinates (t,ϕ,χ), given from Eqs. (26),(22),(23),(25) by

$$\dot{\phi}=\chi,\quad \dot{\chi}=\frac{2}{3}e^{\phi}\rho_0(\pm t+1)^{\mp 2}+\frac{\chi}{2(\pm t+1)}\mp\frac{1}{(\pm t+1)^2}, \tag{27}$$

where the top sign holds for $t>0$ and the bottom sign for $t<0$. We can set initial data at $t=0$, and then investigate the phase plane orbits as we run the trajectory forward and backward in time in such a way that χ and ϕ are continuous through $t=0$. Then $\dot{\chi}$ is discontinuous there, but we have no problem in joining the solutions for $t>0$ and $t<0$.

For $t>0$, there is an exceptional integral curve $\gamma(t)$ given by $(t,\tilde{\phi}_0,0)$, where $\tilde{\phi}_0\equiv\ln(3/2\rho_0)$; this is the only integral curve with a fixed value of ϕ and χ. Note that setting ϕ_0 and χ_0 determines the initial point in the phase space, and specifying ρ_0 determines the location of this exceptional curve. In the 2-dimensional sub-spaces $t=$const with coordinates (ϕ,χ), the curve $\gamma(t)$ has coordinates $(\tilde{\phi}_0,0)$ for all t, and represents a set of saddle points parametrised by t. To get exactly the desired dilaton dynamics in the future ($\chi>0$, $e^{\phi}\to$const$\Rightarrow\chi\to 0$ as $t\to\infty$), one must restrict the initial conditions (ϕ_0,χ_0) to start precisely on the stable branch of these saddle points, which intersects the surface $t=0$ in a single curve $(0,\gamma_+(\chi),\chi)$ passing through the exceptional point $\gamma_0=(0,\tilde{\phi}_0,0)$ (for more details, see Appendix A). However, there is actually slightly more freedom than this in finding physically relevant initial conditions because if a trajectory starts close enough to the stable branch (but not exactly on it), then the trajectory will stay close to the fixed point for an arbitrarily long period of time before ϕ and χ diverge, and this may suffice for physical purposes even if the solution eventually diverges (cf. the discussion of intermediate isotropisation in [9]). Nevertheless, the physically relevant set of solutions is very unstable and requires very precise fine-tuning, in order to obtain the desired dilaton dynamics, lying in a small open neighborhood $\mathcal{D}_+$ of the curve $\phi_0=\gamma_+(\chi_0)$ in the initial data set at $t=0$. Indeed we have found it very difficult to obtain numerical solutions with the desired behavior because of this instability.

For $t<0$ there are no points with a fixed value of ϕ and χ (because we assume $\rho_0>0$). To get the desired dilaton dynamics in the past ($\chi>0$, $e^{\phi}\to 0$ as $t\to-\infty$) one must further restrict the initial conditions, the problem being that Eq. (27) is an inhomogeneous equation for χ with a time-varying source function (albeit a source function that decays away as $t\to\pm\infty$). We can obtain the desired behavior if $y_0=\frac{2}{3}e^{\phi_0}\rho_0\ll 1$, i.e. $\phi_0\ll\ln(3/2\rho_0)$ (details are given in Appendix A). This is a sufficient condition; there will be a wider domain $\mathcal{D}_-$ of initial data at $t=0$, containing this set, which will ensure that at early enough times the desired behavior is attained.

To get a satisfactory solution for all time, for a given choice of ρ_0, one must set the initial conditions to lie in both $\mathcal{D}_+$ and $\mathcal{D}_-$, so the crucial issue is whether they intersect or not. We have not attained finality on this point. It may be that the ''no-go'' theorems with a potential [3] imply that they do not intersect, but this implication is not entirely clear, as the conditions of those theorems may not correspond precisely to the conditions we contemplate here. If they do intersect, we can attain the desired behavior $\chi\to 0$ and $e^{\phi}\to 0$ when time runs backward as well as $e^{\phi}\to$const as time runs forward and, in principle, one can obtain a continuous $V(\phi)$ associated with the unstable solution described above because every function is continuous on the right hand side of Eq. (15). Furthermore, $\phi(t)$ is monotonic and continuous, and therefore invertible, so one can complete step (6) of the algorithm set out in Sec. III A. However, attaining such solutions will require extreme fine-tuning of the initial data, and this is very difficult to do because one does not know where the stable branch of the saddle point intersects $t=0$. Thus, if such solutions do exist, the extreme fine-tuning required for their initial data makes them seem impracticable as cosmologies despite their other desirable properties.

B. ''Pre-big-bang'' behavior with exotic equation of state

Finally, we assume the identical ''pre-big-bang'' behavior of the last example [Eqs.(22),(23)], but we obtain a stable solution with a different equation of state. The instability in the last example arises because of our choice of equation of state, as can be seen by inspection of Eq. (4), which we write now as

$$\dot{\chi}=-3H\chi+\chi^2+\beta, \tag{28}$$

where

$$\beta\equiv -V-V'+e^{\phi}\left(\frac{3p}{2}-\frac{\rho}{2}\right). \tag{29}$$

As mentioned before, we want to obtain $e^{\phi}\to$ constant, i.e. $\chi\to 0$, at late times, which implies $\beta\to 0$ in Eq. (28). If we choose the radiation equation of state as in the last example [Eq. (24)], then $\beta=-V-V'$. Therefore, requiring $\beta\to 0$ as $t\to\infty$ puts a heavy restriction on the dilaton potential, namely $V\to e^{-\phi}$ at late times. Consequently, there is a fine-tuning problem if you use the radiation equation of state.

In the present example, we assume $\beta=0$ for all times, which from Eq. (28) demands the exotic equation of state

$$p=\frac{\rho}{3}+\frac{2}{3}e^{-\phi}(V+V') \tag{30}$$

at all times [note that Eq. (12) is the special case resulting when $V'=0$]. Using this equation of state implies

$$\rho(t)=\int 2He^{-\phi}(12H\dot{\phi}+6\dot{H}-3\dot{\phi}^2)dt \tag{31}$$

which allows the density to go through zero and become negative. We discuss this equation further in the Conclusion.

The differential equation that relates $H(t)$ to $\phi(t)$ is simply Eq. (28) with $\beta=0$:

$$\ddot{\phi}=-3H\dot{\phi}+\dot{\phi}^2. \tag{32}$$

For arbitrary $a(t)$, this can be solved (with $a_0=1$ and $\chi_0\equiv\dot{\varphi}_0$) by

$$\exp[\phi_0-\phi(t)]=1-\chi_0\int_0^t a^{-3}(t)dt. \tag{33}$$

For the specific case given by Eqs. (22),(23) we obtain from this the analytical solution

$$\phi(t)=+\phi_0-\ln|1-2\chi_0[1-(1+t)^{-1/2}]| \tag{34}$$

for $t>0$ and

$$\phi(t)=+\phi_0-\ln\left|1-\frac{2\chi_0}{5}[1-(1-t)^{5/2}]\right| \tag{35}$$

for $t<0$. Inverting Eq. (34) we obtain

$$t(\phi)=\frac{4\chi_0^2e^{2(\phi-\phi_0)}}{[(1-2\chi_0)e^{\phi-\phi_0}-1]^2}-1 \tag{36}$$

and inverting Eq. (35) we obtain

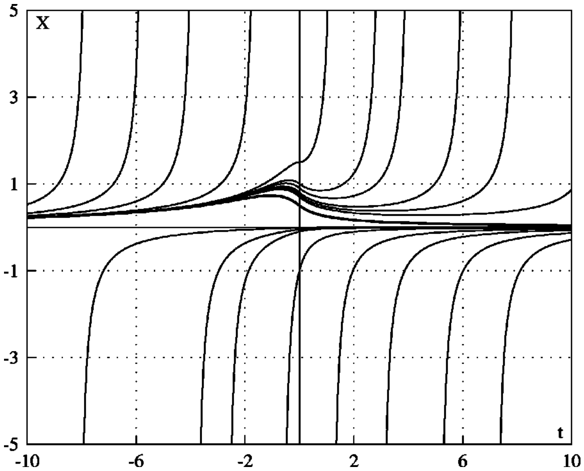

FIG. 2. Phase portrait representing the solution space of Eq. (38).

$$t(\phi)=1-\left[\frac{5e^{\phi-\phi_0}-5+2\chi_0}{2\chi_0}\right]^{2/5}. \tag{37}$$

Now we can solve Eq. (31) to obtain $\rho(a)$ and so $\rho(t)$ (see Appendix B for one particular case), and substitute our results into Eq. (15) to obtain the dilaton potential $V(\phi)$ that is associated with our specified ''pre-big-bang'' behavior. This is straightforward but tedious, and results in very complex analytic expressions [the real complexity coming through the expressions for $\rho(t)$ that occur as a consequence of the choice of the exotic equation of state]. Rather than giving these analytic expressions, we give a graph of the potential for one particular case in Fig. 2.

To discuss the relevant initial conditions, it is instructive to look at the phase plane (Fig. 2) with coordinates (t,χ), where $\chi=\dot{\phi}$ is governed by the equation

$$\dot{\chi}=-\frac{3}{2(\pm t+1)}\chi+\chi^2 \tag{38}$$

where we again use $+$ to represent $t>0$ and $-$ to represent $t<0$. One can easily see that $\chi=0$ ($\Rightarrow\dot{\chi}=0$) is an attractor, and represents a physically uninteresting solution with $\phi=\mathrm{const}$. Also $\chi=3/2(\pm t+1)$ is a nullcline, characterizing the other points where $\dot{\chi}=0$. This curve starts at $(0,\frac{3}{2})$ and drops symmetrically away to zero as $t\rightarrow\pm\infty$. Now we can solve Eq. (38) analytically for $t>0$, finding

$$\chi=\frac{1}{2(t+1)(1+C_+\sqrt{t+1})} \tag{39}$$

where $C_+=(1/2\chi_0-1)$ is positive iff $\chi_0<1/2$. The separatrix between the solutions that diverge and those that go asymptotically to zero as $t\rightarrow\infty$ is the special solution with $C_+=0$ which goes through $(0,\frac{1}{2})$, that is,

$$\chi=\frac{1}{2(t+1)} \tag{40}$$

which itself goes to zero as $t\rightarrow\infty$. If we specify the initial conditions at $t=0$ such that ϕ_0 is free and $0<\chi_0<\frac{1}{2}\Leftrightarrow C_+>0$, then as we run the trajectories forward in time $\chi\rightarrow 0$. In this case, for large positive values of t, Eq. (39) will be approximately

$$\chi=\frac{1}{2C_+\, t\sqrt{2t}}>0 \tag{41}$$

[note that $\phi(t)$ is monotonic for $t>0$ because $\chi>0$ on these trajectories]. Let T_+ be such that Eq. (41) is valid for all $t>T_+>0$. Then, for $t>T_+$,

$$\phi(t)\simeq\int_{T_+}^{t}\frac{1}{2C_+t\sqrt{2t}}dt+\phi_{T_+}=\frac{1}{C_+\sqrt{2}}[T_+^{-1/2}-t^{-1/2}]+\phi_{T_+}. \tag{42}$$

Thus as $t\rightarrow\infty$, for all χ_0, $\phi(t)\rightarrow$ a constant value, say ϕ_∞, and $\exp\phi(t)\rightarrow\exp(\phi_\infty)$. (Note that it is essential to check this result even though $\chi\rightarrow 0$; cf. the discussion below of what happens as $t\rightarrow-\infty$). If we specify the initial conditions at $t=0$ such that ϕ_0 is free and $\frac{1}{2}<\chi_0\Leftrightarrow C_+<0$, as we run the trajectories forward in time, then $\chi\rightarrow\infty$ as $t\rightarrow t_0$ given by $1+C_+\sqrt{t_0+1}=0$, that is $t_0=[(2\chi_0)^2-1]/(2\chi_0-1)^2$. In this case for large values of χ, Eq. (38) can be approximated as follows:

$$\chi\gg\frac{3}{2(t+1)}\Rightarrow\dot{\chi}\simeq\chi^2\Rightarrow\chi\simeq 1/(t-t_0). \tag{43}$$

The solution diverges as $t\rightarrow t_0$ and the approximation, Eq. (41), never applies. This behavior conforms to that implied by Eq. (34), and may be seen clearly on the phase plane.

If we run the trajectories backward in time, starting from initial data with $\chi_0>0$, they will cross the nullcline and then drop to zero, never becoming negative because $\chi=0$ is an exceptional solution of the equations. Then $\phi(t)$ is monotonic for $t<0$ also because $\chi>0$ on these trajectories. Solving Eq. (38) analytically for $t<0$ gives

$$\chi=-\frac{5}{2}\frac{(t-1)\sqrt{-t+1}}{(t^2-2t+1)\sqrt{-t+1}+C_-} \tag{44}$$

where $C_-=(5/2\chi_0-1)$. This expression goes to zero for all $C_->-1$, corresponding to $\chi_0>0$ (note that it does not matter if C_- is positive or negative). For large negative t its value, for all C_-, will be approximately

$$\chi=d\phi/dt\simeq-\frac{5}{2t}. \tag{45}$$

Let T_- be such that Eq. (45) is valid for $t<T_-<0$. Then, for $t<T_-$,

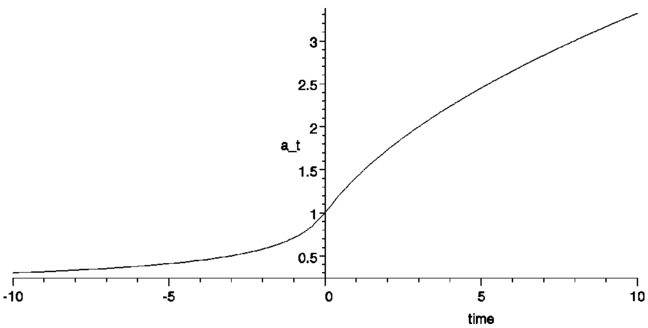

FIG. 3. The evolution of the scalefactor $a(t)$ as a function of time t, with $a(0)=1$, over the time interval $[-10,10]$. For negative times $t\leqslant 0$, there is power-law inflation, $t\geqslant 0$, followed by a radiation dominated phase of expansion for positive time $t\geqslant 0$.

$$\phi(t)\simeq\int_{T_-}^{t}\left(-\frac{5}{2t}\right)dt+\phi_{T_-}=\frac{5}{2}\ln\left(\frac{T_-}{t}\right)+\phi_{T_-}-\Rightarrow\exp\phi(t)\propto\left(\frac{T_-}{t}\right)^{5/2}. \tag{46}$$

Thus as $t\rightarrow-\infty$, for all χ_0, $\phi(t)\rightarrow-\infty$ even though $\chi\rightarrow 0$, and $\exp\phi(t)\rightarrow 0$, which is the dynamics we desire [1], and indeed is indicated already by Eq. (35). The value of C_- corresponding to the separatrix, Eq. (40), is $C_-=4$, which does not give any special behavior for $t<0$.

Typical results of the integrations for this case are given in Figs. 3–5.

In summary, one gets a stable solution for $0<\chi_0<1/2$, as one can see from the phase plane, with good ''pre-big-bang'' behavior and the desired dynamics for $\phi(t)$ for both large and small t. The shape of the potential is a bit unusual, but results directly from the specific requested ''pre-big-bang'' evolution, Eqs. (22),(23), and the chosen initial conditions. Smoothing out that behavior at $t=0$, so that the solution departs from the ''radiation'' form Eq. (22) at very early times while preserving the symmetry (10), will result in a smoothed out potential $V(\phi)$; we can choose $a(t)$ in this way so that $H(t)$ and hence $V(\phi)$ are continuous at $t=0$. Initial conditions can be set so that the matter has the desired late time behavior: $p/\rho\rightarrow 1/3$, $\rho\rightarrow 0$. However, it then has unusual behavior at early times in that both ρ and $h\equiv\rho+p$ go negative for some values of $t<0$. It is unclear if this should be regarded as a serious defect of the model or not, remembering that with the unusual equation of state adopted, the properties of matter are different than usual, and in particular the speed of sound will no longer be given by the usual expression. This needs further investigation. What is clear is that these solutions are not physically reliable as $t\rightarrow+\infty$ (see below), and they will have to be joined on to some other solution to give an adequate model of the universe with ordinary matter behavior at late times. However, as discussed below, that problem occurs in the entire family of pre-big-bang models, and so is not restricted to the models considered here.

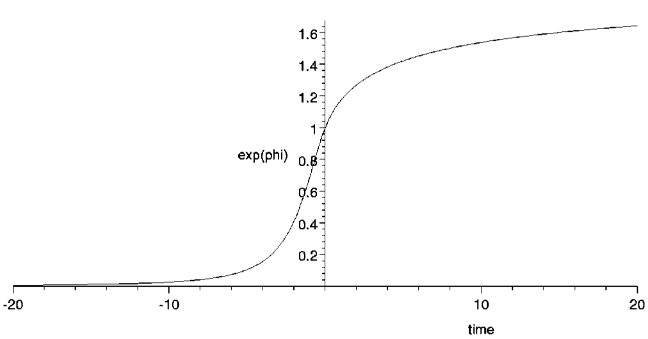

FIG. 4. The function $\exp[\phi(t)]$ as a function of time t, with $a(0)=1$, $\phi(0)=0$ and $\chi(0)=0.25$. Here $\exp[\phi(t)]$ increases monotonically from 0 at time $t=-\infty$ to 2 at $t=+\infty$.

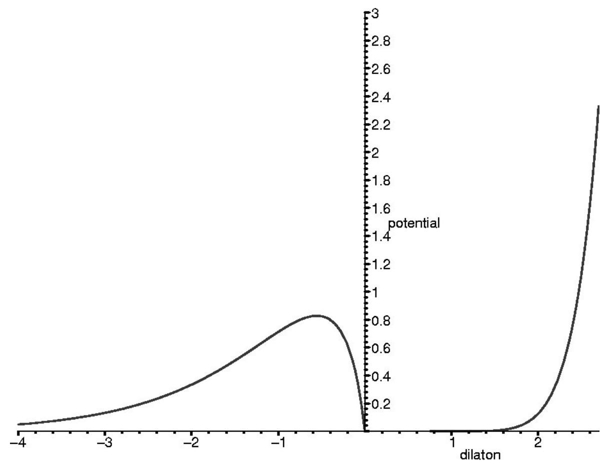

FIG. 5. The dilaton potential $V(\phi)$ as a function of time ϕ. We assume that $a(0)=1$, $\phi(0)=0$ and $\chi(0)=0.25$ and take the density $\rho(t)$ to have value $\rho(\infty)=0$ at time $t=\infty$. The potential $V(\phi)$ is continuous at all times, but non-differentiable at $\phi=0$. For $\phi\rightarrow-\infty$, $V(\phi)$ is asymptotically zero. To the right of $\phi=0$, the potential starts at $V\approx-0.005$ and goes to zero from below as ϕ goes to $\ln 2$, then increases to $+\infty$ as $\phi\rightarrow\infty$. Around $\phi=\ln 2$, both $V(\phi)$ and its gradient $V'(\phi)$ are zero. As time $t\rightarrow+\infty$, the dilaton field asymptotes to a constant value of $\ln 2$ in our model. The dilaton potential $V(\phi)$ approximates a fixed value of 0 as $\phi\rightarrow\ln 2$ asymptotically for large positive times.

V. DISCUSSION

We have given examples making very clear the distinction between the equations and the solution having the desired ''pre-big-bang'' symmetry. We have given a broad method of attaining the desired string cosmology solutions when there is a dilaton potential V not equal to zero, and used it to obtain ''pre-big-bang'' solutions that seem to have close to the desired properties. In the first case considered, the choice of the exact radiation equation of state (24) at all times leads to a very unstable situation where extreme fine-tuning of initial conditions is required to attain the desired results, and indeed there may be no initial data leading to the desired behavior in both the forward and backward directions of time. In the second case we impose an ''exotic'' equation of state (30) that links the fluid behavior to the potential in a way that generalizes the perfect fluid equation of state, and we obtain solutions of the desired type without the need for fine-tuning the initial data set at $t=0$.

This equation of state looks strange, and the resulting matter behavior is certainly unusual, but we have no solid handle to use in restricting equations of state in this early era, and we suggest that *it is essential to choose such an equation if one wants the solution to reliably tend to the ''classical'' form at late times*. This is because of the form of the equation for $\ddot{\phi}$; if we do not set $\beta=0$, where β is defined by Eq. (29), then almost always that desired classical state will not be attained, because of Eq. (28), but setting $\beta=0$, which leads to the desired behavior, leads immediately to our ''exotic'' equation of state. Insofar as that equation of state and resulting behavior are unsatisfactory, this indicates that *there is a problem with the form of the equation for* $\ddot{\phi}$, which comes directly from the standard variational principle employed in the context of the pre-big-bang scenario. The remedy probably lies in finding other scenarios with alternative forms of the variational principle, leading to other equations for $\ddot{\phi}$.

This is also indicated because the present form of the equations does not accommodate ordinary matter, the point being that the above analysis applies even if there is no dilaton potential. Suppose $V=0$; then Eq. (28) remains true, but now

$$\beta=e^{\phi}\left(\frac{3p}{2}-\frac{\rho}{2}\right), \tag{47}$$

so a reliable approach of the dilaton to a classical solution at late times, requiring $\beta=0$, demands the radiation equation of state (24); a baryon dominated epoch is not allowed.[3] This is usually dealt with by stating that Eqs. (2)–(5) do not apply at late times in the history of the universe — a different set of equations is to be used then, and the solutions for early times obtained from Eqs. (2)–(5) must be suitably joined to that late time evolution. However, given the vision of M theory as representing the fundamental theory of gravity, it should be able to describe that epoch too; this apparently requires some modified scenario and associated variational principle (note that although we have discussed the issue in the string frame, it also arises in essentially the same form in the Einstein frame). In any case, whether one accepts this argument or not, given the standard variational principle and equations, we argue that the ''exotic'' equation of state implied by setting $\beta=0$ is *necessary* to give the desired behavior; when adopted, it enables obtaining that behavior reliably (i.e. it eliminates the need for extreme fine-tuning of data set at $t=0$).

However, one should note here that we have perhaps been somewhat extreme in imposing this equation of state at all times. It is only really needed, in our approach, near the time of the turnaround, and one could obtain far more general behaviors by modifying what we have here in that light; what is required is that the quantity β must go to zero in the period when the dilaton is stabilized. It has also been pointed out to us that it is not clear why the deviation from its vanishing point should be absorbed completely in the pressure, and then promoted into the conservation equation; other models of the transition [10–12] successfully stabilize the dilaton at late times without this requirement, with suggestions for classical and quantum corrections in the effective action taking the place of the exotic fluid. Hence our proposal must just be seen as one of a range of possibilities in this regard.

Because we have not made the usual separation of our solution into a ''+'' and a ''−'' branch, it is not immediately clear why these solutions are not ruled out by the ''no-go'' theorems involving a dilaton potential [3]; this is presumably because those theorems exclude fluids with the equation of state we have assumed. We also have not examined the relation of these string-frame solutions to the corresponding Einstein-frame versions. These issues await investigation.

[3]Although of course by the algorithm given above we can simulate a matter dominated phase by a suitable choice of the potential V.

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APPENDIX PRE-BIG-BANG EVOLUTION FOR RADIATION

For given ρ_0, it is convenient to define $y=\frac{2}{3}e^{\phi}\rho_0$ and change variables to (t,y,χ). Equations (27) for $t>0$ become

$$\dot{y}=\chi y,\ \dot{\chi}=\frac{y-1}{(t+1)^2}+\frac{\chi}{2(t+1)}, \tag{A1}$$

In the 2-dimensional sub-spaces $t=\text{const}$ with coordinates (y,χ); the curve $\gamma(t)$ has coordinates (1,0) for all t, and represents a set of saddle points parametrized by t. To get exactly the desired dilaton dynamics in the future ($\chi>0$, $e^{\phi}\to\text{const}\Rightarrow\chi\to 0$ as $t\to\infty$), one must restrict the initial conditions (y_0,χ_0) to start precisely on the stable branch of these saddle points, which intersects the surface $t=0$ in a curve $(0,\gamma_+(\chi),\chi)$ passing through the exceptional point $\gamma_0=(0,1,0)$. One can obtain approximate solutions by rewriting the second equation of Eqs. (A1) in the form

$$\left(\frac{\chi}{(1+t)^{1/2}}\right)^{\cdot}=\frac{y-1}{(t+1)^{5/2}}.$$

Suppose y is almost constant for $t>T_+$, implying that χ is close to zero then. Then we can integrate to get

$$t>T_+\Rightarrow\chi=-\frac{2}{3}\frac{y-1}{1+t}+C_+\sqrt{1+t}$$

where C_+ determines the magnitude of χ at time T_+. The first part decays away as desired, but the second part grows with time unless $C_+=0$; this is the fine-tuning required to attain the desired behavior of χ.

To investigate $t<0$, it is again convenient to define $y=\frac{2}{3}e^{\phi}\rho_0$ and change variables to (t,y,χ). The equations for $t<0$ become

$$\dot{y}=\chi y,\quad \dot{\chi}=y(-t+1)^2+\frac{\chi}{2(-t+1)}+\frac{1}{(-t+1)^2}, \tag{A2}$$

implying that $\dot{\chi}>0$ for all $t<0$; hence χ necessarily decreases at all times in the past. The problem is that it can become negative, because $\chi=0$ is not an invariant set of the equation. We want a solution where χ remains positive for all time so that ϕ decreases for all time; this means we need χ to go to a positive value or zero, but not to become negative, and y to go to zero. As in the previous case one can obtain approximate solutions by rewriting the second equation of Eqs. (A2) in the form

$$(\chi(1-t)^{1/2})^{\cdot}=\frac{1}{(-t+1)^{3/2}}[1+y(-t+1)^4].$$

Suppose

$$y(-t+1)^4\ll 1 \quad \text{for} \quad t<T_-. \tag{A3}$$

Then we can ignore the second term on the right and integrate to get

$$t<T_-\Rightarrow\chi=\frac{2}{1-t}+\frac{C_-}{\sqrt{1-t}},$$

$$y=y_0\frac{1}{(1-t)^2}\exp(-2C_-\sqrt{1-t})$$

where C_-, y_0 represent the magnitude of χ,y at time T_-. This decays away as desired, and consistently preserves the inequality (A3) for all earlier times because the exponential always dominates the power law terms. The question then is whether for suitable initial conditions we can attain this inequality at some time T_-, requiring $y(T_-)\ll(1-T_-)^{-4}$. We can satisfy this with $T_-=0$ if $y_0=\frac{2}{3}e^{\phi_0}\rho_0\ll 1$, i.e. $\phi_0\ll\ln(3/2\rho_0)$.

APPENDIX B: DENSITY EVOLUTION WITH EXOTIC EQUATION OF STATE

The "pre-big-bang" evolution (22),(23) implies H and $\dot{H}$ in terms of a:

$$t\geqslant 0:\quad H(a)=\frac{1}{2a^2},\quad \dot{H}(a)=\frac{-1}{2a^4}, \tag{B1}$$

$$t\leqslant 0:\quad H(a)=\frac{a^2}{2},\quad \dot{H}(a)=\frac{a^4}{2}. \tag{B2}$$

Assuming the exotic equation of state (30) implied by setting $\beta=0$ at all times, from (33) we find φ in terms of a

$$t\geqslant 0:\quad \exp[\phi(a)]=\exp(\phi_0)\frac{a}{a(1-2\chi_0)+2\chi_0}, \tag{B3}$$

$$t\leqslant 0:\quad \exp[\phi(a)]=\exp(\phi_0)\frac{a^{5/2}}{a^{5/2}\left(1-\frac{2}{5}\chi_0\right)+\frac{2}{5}\chi_0}, \tag{B4}$$

and from Eqs. (39),(44) we find χ in terms of a,

$$t\geqslant 0:\quad \chi(a)=\frac{\chi_0}{a^2(2\chi_0+(1-2\chi_0)a)}, \tag{B5}$$

$$t\leqslant 0:\quad \chi(a)=-\frac{5a\chi_0}{2\chi_0+(5-2\chi_0)a^{5/2}}. \tag{B6}$$

A particularly simple case occurs when $\chi_0=\frac{1}{4}$. Then

$$t\geqslant 0:\quad \exp[\phi(a)]=\exp(\phi_0)\frac{2a}{a+1}, \tag{B7}$$

$$t\leqslant 0:\quad \exp[\phi(a)]=\exp(\phi_0)\frac{10a^{5/2}}{9a^{5/2}+1} \tag{B8}$$

and

$$t\geqslant 0:\quad \chi(a)=\frac{1}{2a^2(1+a)}, \tag{B9}$$

$$t\leqslant 0:\quad \chi(a)=-\frac{5a}{2(1+9a^{5/2})}. \tag{B10}$$

Now $\rho(t)$ is determined by Eq. (31); using the above expressions, for $t>0$ and $\chi_0=\frac{1}{4}$ this becomes

$$\frac{d\rho}{da}=-\frac{3}{a^5}-\frac{3}{4a^6(1+a)}$$

which can be solved to give

$$\rho(a)=C+\frac{3}{20a^5}+\frac{9}{16a^4}+\frac{1}{4a^3}-\frac{3}{8a^2}+\frac{3}{4a}+\frac{3}{4}\ln\left(\frac{a}{1+a}\right).$$

This implies $\rho(t)\to C+\frac{107}{80}-\frac{3}{4}\ln 2=C+0.81764\ldots$ as $t\to 0_+$ and $\rho(t)\to C$ as $t\to\infty$; hence choosing $C=0$, $\rho(t)\to 0.81764\ldots$ as $t\to 0_+$ and $\rho(t)\to 0$ as $t\to\infty$. Also $p/\rho\to 1/3$ as $t\to\infty$. The expression for $V(\varphi)$ in this case follows on putting this into Eq. (15) and using Eqs. (36), (22), and the various expressions above. Similar (more complicated) expressions can be obtained for $t<0$.

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