Gravity waves from instantons

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We perform a first principles computation of the spectrum of gravity waves produced in open inflationary universes. The background spacetime is taken to be the continuation of an instanton saddle point of the Euclidean no boundary path integral. The two-point tensor correlator is computed directly from the path integral and is shown to be unique and well behaved in the infrared. We discuss the tensor contribution to the cosmic microwave background anisotropy and show how it may provide an observational discriminant between different types of primordial instantons.

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I. INTRODUCTION

The inflationary universe scenario provides an appealing explanation for the smoothness and flatness of the present universe, as well as a mechanism for the origin of density fluctuations. Until recently it was believed that inflation inevitably predicted a flat $\Omega_0=1$ universe. However, in [1] it was shown that with mild fine-tuning an open universe is also possible. The potential must have a sharp false vacuum in which the field is assumed to have become trapped. The field is then assumed to tunnel out via an instanton known as the Coleman–De Luccia instanton [2], producing a bubble within which slow roll inflation occurs. The interior of the bubble produced via the Coleman–De Luccia instanton is an infinite inflating open universe.

Such models provide important counter-examples to the standard folklore but require quite contrived scalar field potentials. Recently, however, Hawking and one of us showed that open inflation can occur much more generally. We found a new class of instantons [3] that exist for essentially any inflationary potential and provide saddle points of the Euclidean path integral. The continuation of these instantons is similar to that of the Coleman–De Luccia instantons, and they define initial conditions for open inflationary universes. Although the Hawking-Turok instantons are singular, the singularity is mild enough for the quantization of perturbations to be well posed [3–5]. In this paper we compute the spectrum of gravity waves for both Coleman–De Luccia and Hawking-Turok instantons.

This paper is a companion to Ref. [5], where scalar fluctuations about open inflationary instantons were calculated. Here we perform an analogous calculation for the tensor fluctuations and discuss possible observable signals in the cosmic microwave background (CMB) anisotropy power spectrum. The calculation is performed in the framework of the Euclidean no boundary proposal due to Hartle and Hawking [6], as discussed in [5]. The correlator is computed in the Euclidean region where it is uniquely determined by a Gaussian integral, and then analytically continued in the coordinates of the classical background solution into the Lorentzian region of interest. Our main result for the tensor correlator (32) is given in a form which is straightforward to compute numerically. We defer detailed numerical calculations of the CMB anisotropies to a future paper [7] in which both scalar and tensor contributions for a variety of scalar potentials will be discussed.

There have over the last few years been many papers exploring similar calculations, mostly making one approximation or another [9–11]. Very recently, Garriga *et al.* have independently obtained formulas for the scalar and tensor correlators similar to ours [12,13]. These formulas have been numerically implemented in Ref. [14] which gives results for some examples of Coleman–De Luccia instantons calculated without approximation.

We feel that the derivation given here is significantly clearer than in those papers, and that our method has several important conceptual advantages. First, all earlier authors performed a mode by mode analysis. In this framework, one requires a prescription for the vacuum state for each perturbation mode and this is taken to be the state in which the positive frequency modes are regular on the lower half of the instanton. This prescription is rather *ad hoc*. In contrast, our method is to simply perform the Euclidean no boundary path integral. This automatically gives a unique Green function. There is no need for an additional prescription; indeed imposing one is contrary to the spirit of the no boundary proposal (see the discussion in [5]). The whole idea of the Euclidean no boundary proposal is that an essentially *topological* prescription should define the initial state of the universe. Analyticity arises because the background solution is a solution of a differential equation. Divergent fluctuation modes have infinite Euclidean action and are therefore suppressed in the path integral. Second, in the matching method of Garriga *et al.*, they devote a great deal of effort to determining the action for perturbations in region II, the part of the Lorentzian spacetime exterior to the open universe region. This introduces considerable technical complexity since the spatial hypersurfaces used in their canonical quantization approach are inhomogeneous in region II. Our approach is to analytically continue directly from the Euclidean region into the open universe. Region II is just a part of the continuation route with no special significance. Third, as emphasized in [5], we deal throughout directly with the real space correlator. In this approach ''super-curvature'' modes

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are automatically included and their relation to the "sub-curvature" modes is thereby made clear. A related fact is that we find the real space correlator to be infrared finite even in perfect de Sitter space, as mentioned below. Finally, Garriga *et al.* only give formulas for equal-time correlators. To compute the microwave anisotropies one requires the unequal-time correlator, which we give here. We are also careful to define the continuation of the conformal time coordinate into the Euclidean region, which is not explained in Ref. [14].

The paper is organized as follows. In Sec. II we describe the relevant path integral and the model-dependent Schrödinger operator which occurs in the Euclidean action. We show that for singular instantons the singularity acts as a reflecting boundary, fixing Dirichlet boundary conditions for the perturbation modes [3–5]. The Euclidean tensor correlator is computed from the path integral in Sec. III. In this calculation we need several properties of maximally symmetric bitensors on S^3, which are described in the Appendix. Section IV describes the analytic continuation to the open universe. Finally, Sec. V is devoted to the Sachs-Wolfe integral to determine the contribution of gravitational waves to the CMB anisotropy. Here we comment on possible observational distinctions between Coleman–De Luccia and Hawking-Turok instantons.

We conclude this Introduction with two technical remarks. First, the question of discrete "super-curvature" modes arises in the tensor calculation just as in the scalar case [5]. Here, however, we find that although the relevant Schrödinger operator possesses a bound state just as in the case of scalar perturbations [5], here it does not generate a "super-curvature mode." Instead the relevant mode is a time-independent shift in the metric perturbation which may be gauged away. This is in agreement with Refs. [11,12]. Second, it has been claimed in much of the previous literature that the spectrum of gravity waves in pure de Sitter space is infrared divergent [15,9] but that the divergence disappears once the existence of the bubble wall is taken in account [10]. In our approach we find a different result. Neglecting the gauge mode previously mentioned, the two point correlator has a well-defined long-wavelength limit even in perfect de Sitter space. We shall investigate this issue further in future work.

II. PATH INTEGRAL FOR TENSOR FLUCTUATIONS

In quantum cosmology the basic object is the wave functional $\Psi[h_{ij},\phi]$, the amplitude for a three-geometry with metric h_{ij} and field configuration ϕ. It is formally given by a path integral

$$\Psi[h_{ij},\phi]\sim\int^{h_{ij},\phi}[\mathcal{D}g][\mathcal{D}\phi]e^{iS[g,\phi]}. \tag{1}$$

Following Hartle and Hawking [6] the lower limit of the path integral is defined by continuing to Euclidean time and integrating over all compact Riemannian metrics g and field configurations ϕ. If one can find a saddle point of Eq. (1), namely a classical solution satisfying the Euclidean no boundary condition, one can in principle at least compute the path integral as a perturbative expansion to any desired power in $\hbar$.

In this paper, we shall compute the two-point tensor fluctuation correlator about classical solutions describing the beginning of open inflationary universes, to first order in $\hbar$. The principles are described in [5], namely that we compute the correlator in the Euclidean region where the exponent iS in the path integral becomes $-S_E=-(S_0+S_2)$, where S_E is the Euclidean action, S_0 is the instanton action and S_2 the action for fluctuations. We shall keep the latter only to second order, this being all that is needed to compute the quantum fluctuations to leading order in $\hbar$. The correlator is then given by a Gaussian path integral

$$\langle t_{ij}(x)t_{i'j'}(x')\rangle=\frac{\int[\mathcal{D}\delta g][\mathcal{D}\delta\phi]e^{-S_2}t_{ij}(x)t_{i'j'}(x')}{\int[\mathcal{D}\delta g][\mathcal{D}\delta\phi]e^{-S_2}}. \tag{2}$$

The Lorentzian correlator is then obtained by analytically continuing in the coordinates of the background classical solution, into the open inflating region.

The $O(4)$ symmmetric instantons of interest possess a line element of the form $d\sigma^2+b^2(\sigma)d\Omega_3^2$ where $d\Omega_3^2$ is the line element on S^3. Both Hawking-Turok and Coleman–De Luccia instantons possess a regular pole which we take to be at $\sigma=0$. As σ approaches zero, we have $b(\sigma)\rightarrow\sigma$. The Coleman–De Luccia instantons have a second regular pole where $b\rightarrow\sigma_m-\sigma$ where σ_m is the maximum value of σ. In contrast Hawking-Turok instantons have $b\rightarrow(\sigma_m-\sigma)^{1/3}$ as $\sigma\rightarrow\sigma_m$. It is useful in both cases to introduce a conformal spatial coordinate satisfying $dX=d\sigma/b(\sigma)$, so that the line element takes the form

$$ds^2=b^2(X)(dX^2+d\Omega_3^2). \tag{3}$$

For Hawking-Turok instantons we define

$$X\equiv\int_{\sigma}^{\sigma_m}\frac{d\sigma'}{b(\sigma')}. \tag{4}$$

so $X=0$ corresponds to the singular pole and $X\rightarrow\infty$ to the regular pole. For Coleman–De Luccia instantons X may be conveniently defined by $\int_{\sigma}^{\sigma_t}d\sigma'/b(\sigma')$, where σ_t is the value of sigma for which b is a maximum, and then X ranges from $-\infty$ to $+\infty$. We write the perturbed line element and the scalar field as

$$ds^2=b^2(X)[(1+2A)dX^2+S_idx^idX+(\gamma_{ij}+h_{ij})dx^idx^j],$$
$$\phi=\phi_0(X)+\delta\phi. \tag{5}$$

and decompose S_i and h_{ij} as follows [16]:

$$h_{ij}=\frac{1}{3}h\gamma_{ij}+2\left(\nabla_i\nabla_j-\frac{\gamma_{ij}}{3}\Delta_3\right)E+2F_{(i|j)}+t_{ij},$$

$$S_i = B_{|i} + V_i . \tag{6}$$

Here Δ_3 is the Laplacian and $|j$ the covariant derivative on the three-sphere. With respect to reparametrizations of the three-sphere, h, B and E are scalars, V_i and F_i are divergenceless vectors and t_{ij} is a transverse traceless symmetric tensor.

One may expand the spatial part of each of these spin-r fields in terms of a complete set of harmonics, labeled by the eigenvalues $\lambda_p = p^2 + (r+1)$ of the Laplacian on S^3. Here $p = in$ and n is an integer. In general the decomposition of a metric perturbation h_{ij} into a scalar, vector and tensor part is unique. Hence one can write E, F_i and t_{ij} back in terms of h_{ij} [16,17]. For scalar $p_s^2 = -4$ and vector $p_v^2 = -4$ harmonics, however, the decomposition is not unique and there appears a degeneracy between scalar- or vector-type perturbations and $p_t^2 = 0$ and $p_t^2 = -1$ tensor modes respectively. Treatment of the former is complicated by the involvement of the scalar field [5], but the latter mode is unambiguously pure gauge. We will return to this point in Sec. IV.

The Euclidean action is

$$S = \frac{1}{2\kappa}\int d^4x \sqrt{g}\left(-R + \frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi + V(\phi)\right) - \frac{1}{\kappa}\int d^3x \sqrt{\gamma} K, \tag{7}$$

where the surface term is needed to remove second derivatives. Substituting the decomposition (6) into the action (7), we keep all terms to second order. The scalar, vector and tensor quantities decouple. The scalar perturbations are studied in Ref. [5]. The vector perturbations are uninteresting to first order in $\hbar$ since they are forced to be zero by the Einstein constraints. The tensor perturbations give the following second order positive Euclidean action:

$$S_2 = \frac{1}{8\kappa}\int d^4x \sqrt{\gamma} b^2 (t'^{ij} t'_{ij} + t^{ij|k} t_{ij|k} + 2 t^{ij} t_{ij}), \tag{8}$$

where the prime denotes differentiation with respect to the conformal coordinate X. If one performs the rescaling $\tilde{t}_{ij} = b(X) t_{ij}$ and integrates by parts, one obtains

$$S_2 = \frac{1}{8\kappa}\int d^4x \sqrt{\gamma} \tilde{t}_{ij} (\hat{K} + 3 - \Delta_3) \tilde{t}^{ij} - \frac{1}{8\kappa}\left[\int d^3x \sqrt{\gamma} \tilde{t}_{ij} \tilde{t}^{ij} \frac{b'}{b}(X)\right] \tag{9}$$

where the Schrödinger operator

$$\hat{K} = -\frac{d^2}{dX^2} + \frac{b''}{b} - 1 \equiv -\frac{d^2}{dX^2} + U(X). \tag{10}$$

The form of the potential $U(X)$ is shown in Fig. 1 for Euclidean de Sitter space (i.e. a four-sphere), as well as examples of a Coleman–De Luccia instanton and a Hawking-Turok instanton. The operator $\hat{K}$ has in all three cases a positive continuum starting at eigenvalue $p^2 = 0$, as well as a single bound state $\tilde{t}_{ij} = b(X) q_{ij}(\Omega)$ at $p = i$.

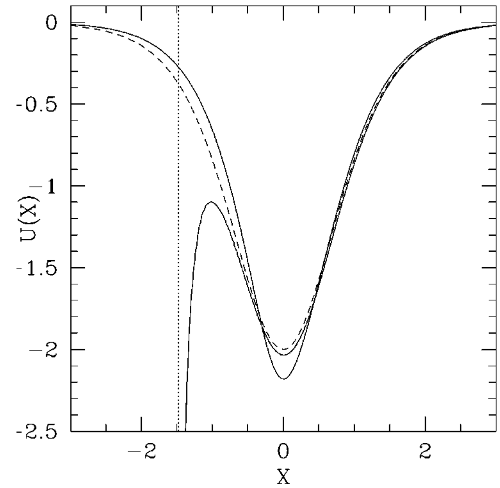

FIG. 1. Potential $U(X)$ occurring in the Schrödinger operator governing tensor perturbations about the various instanton solutions discussed in the text. The dashed line shows the potential for an S^4 instanton corresponding to perfect de Sitter space, where $U(X) = -2/\cosh^2(X)$. The upper solid line shows the potential for a Coleman–De Luccia instanton, where $-\infty < X < \infty$, and the lower solid line that for a Hawking-Turok instanton, with the singularity indicated by the vertical dotted line. The potentials have been shifted in X so their minima coincide. All three are very similar to the right of the minimum. To the left, the Hawking-Turok potential diverges as one approaches the singularity. The potential is reflectionless in the S^4 case, weakly reflecting in the Coleman–De Luccia case and totally reflecting in the Hawking-Turok case.

For singular instantons the surface terms in Eq. (9) play a crucial role. The potential $U(X) \to -1/4X^2$ as $X \to 0$. The eigenmodes of $\hat{K}$ thus behave as $X^{1/2}$ or $X^{1/2}\ln X$ near the singularity. The latter modes contribute positive infinity to the surface terms in Eq. (9), and are therefore suppressed in the path integral. Hence we see that as in the scalar case [5], the path integral unambiguously specifies the allowed fluctuation modes as those which vanish at the singularity.

III. EUCLIDEAN GREEN FUNCTION

To evaluate the path integral (2), we first look for the Green function $G_E^{iji'j'}$ of the operator in Eq. (9). The Euclidean fluctuation correlator (2) will then be given by $b^{-1}(X) b^{-1}(X') G_E^{iji'j'}$. The Euclidean Green function satisfies

$$\frac{1}{4\kappa}(\hat{K} + 3 - \Delta_3) G^{ij}_{E i'j'}(X, X', \Omega, \Omega') = \delta(X - X') \gamma^{-1/2} \delta^{ij}{}_{i'j'}(\Omega - \Omega'). \tag{11}$$

If we think of the scalar product as defined by integration over S^3 and summation over tensor indices, then the right hand side is the normalized projection operator onto transverse traceless tensors on S^3. Since the eigenmodes of the Laplacian form a complete basis, we can write the last term as

$$\gamma^{-1/2}\delta^{ij}{}_{i'j'}(\Omega-\Omega')$$
$$=\sum_k \sum_{\mathcal{P}=e,o}\sum_{l=2}^{k}\sum_{m=-l}^{l} q^{(k)ij}_{\mathcal{P}lm}(\Omega)q^{(k),\mathcal{P}lm}_{i'j'}(\Omega')^* \quad (12)$$

where on S^3 we have

$$\Delta_3 q^{(k)ij}_{\mathcal{P}lm}=\lambda_k q^{(k)ij}_{\mathcal{P}lm} \quad (13)$$

with $\lambda_k=-k(k+2)+2$. Here $\mathcal{P}=\{e,o\}$ labels the parity, the angular momentum on S^3 takes the values $k=2,3,4,\ldots$ and $2\leqslant l\leqslant k$ and $-l\leqslant m\leqslant l$ are the usual quantum numbers on the two-sphere. Note that $(l\geqslant 2)$ because a spin-2 field has no monopole or dipole components. The eigenmodes are normalized by the condition

$$\int \sqrt{\gamma}d^3x q^{(k)ij}_{\mathcal{P}lm}q^{(k')*}_{\mathcal{P}'l'm'ij}=\delta^{kk'}\delta_{\mathcal{P}\mathcal{P}'}\delta_{ll'}\delta_{mm'}. \quad (14)$$

The set of eigenmodes forms a representation of the symmetry group $SO(4)$ of the manifold. It follows in particular that the sum over $\mathcal{P}$, l and m defines a maximally symmetric bitensor [18]

$$W^{ij}_{(k)i'j'}(\mu)=\sum_{\mathcal{P}lm} q^{(k)ij}_{\mathcal{P}lm}(\Omega)q^{(k)\mathcal{P}lm}_{i'j'}(\Omega')^* \quad (15)$$

which depends only on the geodesic distance $\mu(\Omega,\Omega')$ between the two points on S^3. The Green function $G^{ij}_{Ei'j'}$ can only be a function of $\mu(\Omega,\Omega')$ if it is to be invariant under isometries of the three-sphere. Note that the indices i,j lie in the tangent space over the point Ω while the indices i',j' lie in the tangent space over the point Ω'. The general form of the bitensor $W^{ij}_{(k)i'j'}$ appearing in tensor fluctuation correlators has been obtained by Allen [19] and is given in the Appendix below. Here we note already that in terms of the label $p=i(k+1)=in$, the bitensor on S^3 has precisely the same formal expression as the corresponding object on H^3. Since we would like to analytically continue our result for the Euclidean two-point correlator into the open universe, we will use the label $p=in$ from now on. We now return to Eq. (11) for the Euclidean Green function.

By substituting the following ansatz for the Green function,

$$G^{ij}_{Ei'j'}(\mu,X,X')=4\kappa\sum_{p=in} G_p(X,X')W^{ij}_{(p)i'j'}(\mu), \quad (16)$$

into Eq. (11) and noting that in terms of $p=in$, we have $\lambda_p=p^2+3$, we obtain an equation for the model-dependent part of the Green function:

$$(\hat{K}-p^2)G_p(X,X')=\delta(X-X'). \quad (17)$$

Let us first discuss the case of singular instantons. The solution to Eq. (17) is

$$G_p(X,X')=\frac{1}{\Delta_p}[\Psi^+_p(X)\Psi^-_p(X')\Theta(X-X')$$
$$+\Psi^-_p(X)\Psi^+_p(X')\Theta(X'-X)], \quad (18)$$

where $\Psi^-_p(X)$ is the solution to the Schrödinger equation that goes as $X^{1/2}$ as $X\rightarrow 0$ and $\Psi^+_p(X)$ is the solution going as $e^{ipX}=e^{-nX}$ as X tends to infinity. The factor Δ_p is the Wronskian $\Psi^{-\prime}_p\Psi^+_p-\Psi^{+\prime}_p\Psi^-_p$ of the two solutions.

We shall ultimately be interested in re-expressing this solution as an integral over real values of p in order to continue it to the open universe. To do so we must extend the solutions $\Psi^\pm_p$ defined above at $p=in$ into the complex p plane. $\Psi^-_p(X)$ becomes $\Psi_p(X)$, defined for all complex p to be the solution which tends to $X^{1/2}$ as $X\rightarrow 0$. Being a solution of a regular differential equation this is analytic for finite p in the complex p plane. On the other hand, $\Psi^+_p(X)$ is the analytic continuation of $g_p(X)$, defined on the real p axis to be the solution tending to e^{ipX} as $X\rightarrow\infty$. This is the Jost function, and is analytic in the upper half p plane [20]. The two solutions may be expressed in terms of each other as

$$\Psi_p(X)=a_p g_p(X)+a_{-p}g_{-p}(X), \quad (19)$$

and their Wronskian $\Delta_p=\Psi'_p g_p-g'_p\Psi_p=-2ipa_{-p}$, independent of X. This too is analytic in the upper half p plane. Zeros of a_{-p} in the upper half p plane correspond to normalizable bound states. They can only occur on the imaginary p axis, and in the case of interest here the only zero in the upper half p plane is at $p=i$. This zero corresponds to the bound state mentioned above. For $X>X'$ we have the Green function as a discrete sum:

$$G^{iji'j'}_E(\mu,X,X')$$
$$=4\kappa\sum_{p=3i}^{+i\infty}\frac{i}{2pa_{-p}}\Psi^+_p(X)\Psi^-_p(X')W^{iji'j'}_{(p)}(\mu). \quad (20)$$

For regular Coleman–De Luccia instantons a similar procedure may be followed. Here X ranges from $-\infty$ to $+\infty$ and we define the two linearly independent mode functions $g^{\rm left}_p(X)$, which tends to e^{-ipX} as $X\rightarrow-\infty$, and $g^{\rm right}_p(X)$, which tends to e^{ipX} as $X\rightarrow\infty$. These can be shown to be orthogonal and analytic in the upper half p plane. As $X\rightarrow+\infty$, we have $g^{\rm left}_p(X)\rightarrow c_pe^{ipX}+d_pe^{-ipX}$. Hence, the Wronskian $\Delta_p=g^{\rm left\prime}_p g^{\rm right}_p-g^{\rm right\prime}_p g^{\rm left}_p=-2ipd_p$ and the Green function $G^{iji'j'}_E(\mu,X,X')$ may be expressed in a form analogous to that for singular instantons.

Before proceeding to the analytic continuation, let us demonstrate that our Euclidean Green functions are regular at the regular pole. This is a nontrivial check because the coordinates σ and X are singular there, and the rescaling becomes divergent too, $b(X)\sim\sigma^{-1}\sim e^{+X}$. In the large X,X' limit, Eq. (20) becomes

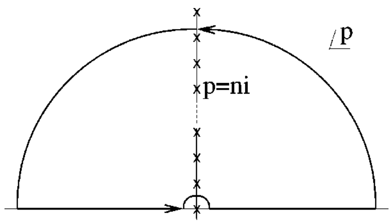

FIG. 2. Contour for the Euclidean correlator.

$$G_E^{iji'j'}(\mu,X,X')=2\kappa\sum_{n=3}^{\infty}\frac{1}{n}\left(e^{-n(X-X')}+\frac{a_{in}}{a_{-in}}e^{-n(X+X')}\right)\times W_{(in)}^{iji'j'}(\mu). \tag{21}$$

For $n\geqslant 3$ the Gaussian hypergeometric functions $F(3+n,3-n,7/2,z)$ that constitute the bitensor $W_{(n)}^{iji'j'}$ have a series expansion that terminates, and they essentially reduce to Gegenbauer polynomials $C_{n-3}^{(3)}(1-2z)$. Using then the identity [21]

$$\sum_{l=0}^{\infty} C_l^{\nu}(x)q^l=(1-2xq+q^2)^{-\nu} \tag{22}$$

with $q=e^{-(X\pm X')}$, one easily sees that the sum (21) indeed converges.

We have the Euclidean Green function defined as an infinite sum (20). We wish to represent it as an integral over p. To do so we must extend the summand into the upper half p plane. We have already defined the wave functions for all complex p but we need to extend the bitensor as well. When the Green function is expressed as a discrete sum, it involves the bitensor $W_{(p)}^{iji'j'}(\mu)$ evaluated at $p=in$ with n integral. At these values of p, the bitensor is regular at both coincident and opposite points on S^3, that is at $\mu=0$ and $\mu=\pi$. However, if we extend p into the complex plane, we lose regularity at $\mu=0$. This is clear from Eq. (11). For if we distort the p integral to run along the real axis, and use the completeness relation for the eigenfunctions $\Psi_p(X)$, it follows that $W_{(p)}^{iji'j'}(\mu)$ obeys a differential equation with a delta function source at $\mu=0$ (see the discussion of the scalar case in [5]). Similarly, when we extend $W_{(in)}^{iji'j'}(\mu)$ into the complex p plane, we must maintain regularity at $\mu=\pi$, since there is no delta function source there.

The condition of regularity at π imposed by the differential equation for the Green function is sufficient to uniquely specify the analytic continuation of $W_{(in)}^{iji'j'}(\mu)$ into the complex p plane. To see this, we note from the Appendix that the bitensor involves coefficient functions α and β which are hypergeometric functions of the variable $z=\cos^2(\mu/2)$. For coincident points, $z=1$ but for antipodal points $z=0$. There are two independent solutions of the hypergeometric equation, namely $\alpha(z)$ and $\alpha(1-z)$. They are related by the transformation formula (Eq. [15.3.6] in [27])

$$\begin{aligned}{}_2F_1\left(3+ip,3-ip,\frac{7}{2},z\right) &=(-\cosh p\pi)\,{}_2F_1\left(3+ip,3-ip,\frac{7}{2},1-z\right)\\ &+\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(3+ip)\Gamma(3-ip)}(1-z)^{-5/2}\,{}_2F_1\\ &\times\left(\frac{1}{2}-ip,\frac{1}{2}+ip,-\frac{3}{2},1-z\right).\end{aligned}$$

Notice that for the eigenvalues of the Laplacian on S^3, i.e. $p=in$ $(n\geqslant 3)$, the second term on the right-hand side vanishes. In this case the two choices are simply related by $(-1)^{n+1}$ and they are both regular for all μ. Since $F(1-z)\to 1$ for coincident points, we take this solution in Eq. (20). But when we express the discrete sum (20) as a contour integral, to maintain regularity of the integrand at $\mu=\pi$ we need to replace $F(1-z)$ by a term $F(z)(-1)^{n+1}$, and then continue the latter term to $-(\cosh p\pi)^{-1}{}_2F_1(3+ip,3-ip,\frac{7}{2},z)$.

Now we write the sum in Eq. (20) as an integral along a contour $\mathcal{C}_1$ encircling the points $p=3i,4i,\ldots,Ni$ on the imaginary p axis, where N tends to infinity. Using the analytic properties of the terms in the discrete sum extended into the complex p plane we have, for $X>X'$,

$$G_E^{iji'j'}(\mu,X,X')=\kappa\int_{\mathcal{C}_1}\frac{dp}{p\sinh p\pi}\frac{g_p(X)\psi_p(X')}{a_{-p}}W_{(p)}^{iji'j'}(\mu) \tag{23}$$

where $W_{(p)}^{iji'j'}(\mu)$ is defined in the Appendix, Eqs. (A6), but with $\alpha(z)$ replaced by $-\alpha(z)$. The explicit expressions for α and β may be obtained from Eqs. (A10) and (A11) by setting $\chi=i\mu$. To verify Eq. (23) note that $W_{(p)}^{iji'j'}(\mu)$ and the factors g_p, Ψ_p and a_{-p}^{-1} are analytic in the complex p plane in the required region. Introduce $1=\cosh p\pi/\cosh p\pi$ into the integral. Then note that $\coth p\pi$ has residue π^{-1} at every integer multiple of i. The remaining factor of $(\cosh p\pi)^{-1}$ is what is needed to convert $-\alpha(z)$ into $\alpha(1-z)$, according to Eq. (22). A similar formula relates $\beta(z)$ to $\beta(1-z)$.

We now distort the contour for the p integral to run along the real p axis (Fig. 2). At large imaginary p the integrand decays exponentially and the contribution vanishes in the limit of large N. However, as we deform the contour towards the real axis we encounter two poles in the $\sinh^{-1}p\pi$ factor, the latter at $p=i$ becoming a double pole due to the simple zero of a_{-p}. For the $p=2i$ pole, we note that it follows directly from the the normalization factor Q_p that $W_{(2i)}^{iji'j'}=0$. Indirectly, this is a consequence of the fact that spin-2 perturbations do not have a monopole or dipole component. At $p=i$ we have a double pole. However, the bound state wave function is just proportional to $b(X)$ and the metric tensor perturbation $t_{ij}=b^{-1}(X)\tilde{t}_{ij}$ is therefore independent of X. The latter coordinate continues to conformal time in the open universe, and it follows that the metric perturbation is time independent and will not contribute to the Sachs-Wolfe formula (37). However, to understand this mode more deeply, recall that for $p^2=-1$ a degeneracy appears between $p^2=-1$ tensor-type perturbations and $p_v^2=-4$ vector-type perturbations [11]. To be more precise, the tracelesss transverse tensors $q_{ij}^{(i)plm}$ may be constructed from the vector harmonics $V_i^{(2i)plm}$ by symmetrized covariant differentiation. One therefore has $q_{ij}^{plm}(p^2=-1)=V_{(i|j)}^{plm}(p_v^2=-4)$. This means that this discrete tensor mode is not invariant under (vector) gauge transformations. It may be generated by a purely spatial gauge transformation without disturbing the value of the scalar field [11]. We may therefore use the remaining gauge freedom in the decomposition (6) to set $W_{(i)}^{iji'j'}=0$. We conclude that up to a term involving a pure gauge mode, we can deform the contour $\mathcal{C}_1$ into the contour $\mathcal{C}$ shown in Fig. 2. Since the integrand involves a factor $(p\sinh p\pi)^{-1}$ which has a double pole at $p=0$, we leave the contour avoiding the origin on a small semicircle in the upper half p plane. We shall see that for the Coleman–De Luccia and Hawking-Turok cases the complete integrand is actually regular at $p=0$, but for perfect de Sitter space the double pole survives. In the latter case the contribution to the Green function from the small semicircle acts to regulate the integral $\int_0^\infty dp/p^2$ coming from the real axis. Thus, in our treatment, even in perfect de Sitter space the Green function is finite, in contradiction to the conclusion reached in treatments based on mode-by-mode matching.

Finally, since the Green equation (17) only depends on p^2, we would like to write our Euclidean Green function as a sum of its symmetric and antisymmetric part. Denoting the integrand in Eq. (23) by I_p we then have

$$G_E^{iji'j'}=\frac{1}{2}\int dp(I_p+I_{-p})+\frac{1}{2}\int dp(I_p-I_{-p}), \quad (24)$$

where the integral is taken from $p=-\infty$ to ∞ along a path avoiding the origin above. But $\int dpI_{-p}$ along this contour is equal to the integral of I_p taken along a contour avoiding the origin below. The second term is therefore equal to the integral of I_p along a contour around the origin. Hence we have

$$\frac{1}{2}\int dp(I_p-I_{-p})=-\pi i\,\mathbf{Res}(I_p;\,p=0). \quad (25)$$

It has been known that a degeneracy appears between $p^2=0$ tensor modes and $p_s^2=-4$ scalar harmonics. As a consequence of this, the $p^2=0$ tensor perturbation couples to the inflaton field, and is not represented by a simple action of the form (8). Hence this part of the correlator should be treated as a scalar perturbation, as was done in [5].

In the p-symmetric part of the correlator, we can leave the integrand as a sum of I_p and I_{-p}. We henceforth denote the path from $-\infty$ to $+\infty$ avoiding the origin above by $\mathcal{R}$. This shall turn out to be a regularized version of the integral over the real axis. Our final result for the Euclidean tensor Green function then reads:

$$G^E{}_{iji'j'}(\mu,X,X')=\frac{\kappa}{2}\int_{\mathcal{R}}\frac{dp}{p\sinh p\pi}W^{(p)}_{iji'j'}(\mu)\times\left(\frac{g_p(X)\Psi_p(X')}{a_{-p}}+\frac{g_{-p}(X)\Psi_{-p}(X')}{a_p}\right). \quad (26)$$

IV. TWO-POINT TENSOR CORRELATOR IN AN OPEN UNIVERSE

The analytic continuation into the open universe is given by setting $\Omega=-i\chi$ and $\sigma=it$ (see [5]) and letting $a(t)\to b(\sigma)\equiv -ia(i\sigma)$. Here Ω, is the polar angle on the three-sphere. For our correlator, without loss of generality we may take one of the two points, say Ω', to be at the north pole of the three-sphere. Then $\mu=\Omega$, and μ continues to $-i\chi$. We then obtain the correlator in the open universe where one point has been chosen as the origin of the radial coordinate χ.

The background line element of the Lorentzian region is

$$ds^2=-dt^2+a^2(t)(d\chi^2+\sinh^2\chi d\Omega_2^2). \quad (27)$$

The conformal coordinate X continues to conformal time τ as follows:

$$X\equiv\int_{it}^{\sigma_m}\frac{d\sigma}{b(\sigma)}=-\tau-\frac{i\pi}{2} \quad (28)$$

where the conformal time τ is defined via

$$\tau\equiv\lim_{\epsilon\to 0}\left(\int_\epsilon^{\sigma_m}\frac{d\sigma}{b(\sigma)}-\int_\epsilon^{t}\frac{dt'}{a(t')}\right). \quad (29)$$

We now wish to make the substitutions $\mu=-i\chi$, where χ is the comoving separation on H^3, in the open universe, and $X=-i\pi/2-\tau$. The first continuation may be done immediately. We use the explicit formula for the bitensor regular at $\mu=\pi$, given in the Appendix, Eqs. (A6), (A10) and (A11), to write the following p integral for the Euclidean Green function:

$$G_E^{iji'j'}(\mu,X,X')=\frac{\kappa}{2}\int_{\mathcal{R}}\frac{dp}{p\sinh p\pi}\left(g_p(X)g_{-p}(X')+\frac{a_p}{a_{-p}}g_p(X)g_p(X')\right)W_{(p)}^{iji'j'}(\chi)+(p\to -p), \quad (30)$$

where we have used the formula (19) to re-express Ψ_p in terms of the Jost functions $g_p(X)$. The obstacle to setting $X=-\tau-i\pi/2$ is that the integrand of Eq. (30) contains a term $g_p(X)g_p(X')\sim e^{ip(X+X')}$. If we simply make the substitution $X=-i\pi/2-\tau$, this would produce a term going as $e^{p\pi}$. But the bitensor defined in Eqs. (A10) and (A11) involves terms which behave as $e^{+p(\pi+i\chi)}$, and the two factors of $e^{p\pi}$ would lead to a meaningless divergent integral. To circumvent the problem, we use the following identity. For $X-X'>0$, we have

$$\int_{\mathcal{R}} \frac{dp}{p} \frac{g_p(X)\psi_p(X')}{a_{-p}} e^{ip\chi} F(p) = 0 \tag{31}$$

where $F(p)$ are the p-dependent coefficients occurring in the final (Lorentzian) form for the bitensor given in Eq. (A12). This identity holds up to the constant $p=i$ gauge mode. It follows from the analyticity properties of the integrand explained above and the fact that, despite first appearances, formulas (A12) are actually analytic at $p=i$. We now insert $1=\sinh p\pi/\sinh p\pi$ under the integral, to show that the integral (31) with a factor $e^{p\pi}/\sinh p\pi$ inserted equals that with a factor $e^{-p\pi}/\sinh p\pi$ inserted. The resulting identity allows us to replace the dangerous terms in the bitensor $e^{+p(\pi+i\chi)}$ by $e^{-p\pi+ip\chi}$, and similarly in the $(p\to -p)$ term of (30).

We now perform the X continuation. The analytic continuation of the Euclidean mode functions is given by

$$g_{\pm p}(X) \to e^{\pm p\pi/2} g^L_{\pm p}(\tau) \tag{32}$$

where the Lorentzian Jost function $g^L_p(\tau)$ is the solution to the Lorentzian perturbation equation $\hat{K} g^L_p(\tau) = p^2 g^L_p(\tau)$ obeying $g^L_p(\tau) \to e^{-ip\tau}$ as $\tau \to -\infty$. Equation (32) follows by matching at large X. We finally obtain the Lorentzian tensor Feynman (time-ordered) correlator, for $\tau' - \tau > 0$:

$$G_L^{iji'j'}(\chi,\tau,\tau') = \frac{\kappa}{2}\int_{\mathcal{R}} \frac{dp}{p\sinh p\pi}\left(e^{-p\pi} g^L_p(\tau) g^L_{-p}(\tau') + \frac{a_p}{a_{-p}} g^L_p(\tau) g^L_p(\tau')\right) W^{iji'j'}_{L(p)}(\chi) + (p\to -p), \tag{33}$$

where the Lorentzian bitensor $W^{L(p)}_{iji'j'}(\chi)$ of relevance in the hyperbolic universe is defined in the Appendix, Eq. (A12). The factor a_p/a_{-p} is simply a phase, since for real p the Euclidean wave function is real, so $a^*_p = a_{-p}$.

Now we would like to represent the result (33) as an integral over real p. The term $(p\sinh p\pi)^{-1}$ in the integrand seems to produce a double pole at $p=0$. However, for either the Coleman–De Luccia or Hawking-Turok instantons, the reflection term in Eq. (33) turns out to precisely cancel the first term as $p\to 0$. This cancellation was first discovered in Refs. [10,8]. The reason for the cancellation is that for any potential except a perfectly reflectionless one, at very low momenta (i.e. very long wavelengths) the wave function is completely reflected. This means that in the small p limit both a_p/a_{-p} and c_p/d_p tend to -1 [22]. This makes the integrand of Eq. (33) analytic as $p\to 0$. It is, however, clear from the form of the potentials (Fig. 1) that the Coleman–De Luccia instantons are much closer to the perfect S^4 nonreflecting solution. Therefore we may expect the regime $c_p/d_p \to -1$ to set in at much lower p than in the Hawking-Turok case. This will lead to a larger contribution to the large angle microwave anisotropies. As mentioned above, a virtue of our treatment seems to be that even the de Sitter result is finite.

In the cases of interest therefore there is no singularity at $p=0$, and we may take the contour to run along the real p axis. Using the symmetry $p\to -p$, the right hand side of Eq. (33) becomes

$$\frac{\kappa}{2}\int_{-\infty}^{\infty} \frac{dp}{p} W^{L(p)}_{iji'j'}(\chi)\Bigg(\coth p\pi [g^L_p(\tau) g^L_{-p}(\tau') + g^L_{-p}(\tau) g^L_p(\tau')] - [g^L_p(\tau) g^L_{-p}(\tau') - g^L_{-p}(\tau) g^L_p(\tau')] + \frac{1}{\sinh p\pi}\left[\frac{a_p}{a_{-p}} g^L_p(\tau) g^L_p(\tau') + \frac{a_{-p}}{a_p} g^L_{-p}(\tau) g^L_{-p}(\tau')\right]\Bigg). \tag{34}$$

For real p, $g^L_{-p}(\tau)$ is the complex conjugate of $g^L_p(\tau)$ and a_{-p} of a_p. So the second term is imaginary but the first and third terms are real. In fact it is straightforward to see that if we apply the Lorentzian version of the perturbation operator $\hat{K}$ to Eq. (34) with an appropriate Heaviside function of $\tau - \tau'$, the imaginary term will produce the Wronskian of $g^L_{-p}(\tau)$ and $g^L_p(\tau)$, which is proportional to p, times $\delta(\tau - \tau')$. Then the integral over p produces a spatial delta function. From this one sees that our Feynman correlator obeys the correct second order partial differential equation, with a delta function source. The delta function source term in Eq. (11) goes from being real in the Euclidean region to imaginary in the Lorentzian region because the factor $\sqrt{g}$ continues to $i\sqrt{-g}$.

For cosmological applications, we are usually interested in the expectation of some quantity squared, such as the microwave background multipole moments. For this purpose, all that matters is the symmetrized correlator $\langle\{t_{ij}(x), t_{i'j'}(x')\}\rangle$ which is just the real part of the Feynman correlator. It also represents the "classical" piece, which in the situations of interest, where occupation numbers of modes are large, is much larger than the quantum piece.

For the tensor correlator we also need to restore the factor $ia^{-1}(\tau)$ to t_{ij}. It is convenient to define the eigenmodes $\Phi^L_p(\tau) = g^L_p(\tau)/a(\tau)$. The extra minus sign hereby introduced in the correlator is cancelled by a change in sign of the normalization factor Q_p, which then becomes $Q_p = +(p^2+4)/(30\pi^2)$. These two sign changes are naturally related, as is seen by considering the behavior of the line element (27). Under continuation the line element on S^3 changes to minus that on H^3, but the change in sign of the a^2 coefficient compensates, keeping the spatial line element positive. The cancellation of these signs ensures that the Lorentzian correlator has the correct positivity properties. The symmetrized correlator is then given by

$$\langle\{t_{ij}(x), t_{i'j'}(x')\}\rangle = 2\kappa\,\Re\int_0^{\infty} \frac{dp}{p}\left(\coth p\pi\, \Phi^L_p(\tau)\Phi^L_{-p}(\tau') + \frac{a_p}{a_{-p}} \frac{\Phi^L_p(\tau)\Phi^L_p(\tau')}{\sinh p\pi}\right) W^{L(p)}_{iji'j'}(\chi) \tag{35}$$

where $W^{L(p)}_{iji'j'}(\chi)$ is defined in the Appendix, Eqs. (A3) and (A12). In this integral it may be written as

$$W^{iji'j'}_{(p)}(\chi)=\sum_{\mathcal{P}lm} q^{(p)ij}_{\mathcal{P}lm}(\Omega)q^{(p)i'j'}_{\mathcal{P}lm}(\Omega')^* \tag{36}$$

where $q^{(p)ij}_{\mathcal{P}lm}(\Omega)$ are the rank-2 tensor eigenmodes with eigenvalues $\lambda_p=-(p^2+3)$ of the Laplacian on H^3. For $\chi\to 0$ the bitensor converges and it exponentially decays at large geodesic distance. At large p, its coefficient functions $w^{(p)}_j$ (see the Appendix) behave like $p\sin p\chi$. Hence the above integral converges at large p for both timelike and spacelike separations. Equation (35) is our final result for the tensor spectrum from singular instantons. As in the scalar calculation [5], and as mentioned above, for Coleman–De Luccia instantons the phase a_p/a_{-p} gets replaced by c_p/d_p, which is the reflection amplitude for waves incident from $X=+\infty$ in the Euclidean region.

Before moving on to the observational consequences of Eq. (35) we would like to make one more technical comment. We mentioned already that a degeneracy appears between $p^2=0$ tensor modes and $p_s^2=-4$ scalar perturbations. These discrete modes were initially interpreted as bubble wall fluctuations [23,24]. However, in our approach they do not contribute in the scalar calculation (for $l\geqslant 2$) because the corresponding spherical harmonics are singular and overcomplete on the Euclidean three-sphere. More recently the wall fluctuations were argued to have re-appeared as a long-wavelength continuum contribution on top of the usual continuous spectrum of even parity gravitational wave modes [10]. In this way, the bubble wall fluctuations were found to regularize the tensor spectrum, thought to be infrared divergent in pure de Sitter space [10]. Our result for the correlator for a Coleman–De Luccia model is indeed infrared finite and the cancellation caused by total reflection of low momentum modes allowed us to represent the result as an integral starting at $p=0$. However, we do not agree that the presence of the bubble was needed to regularize the spectrum. In our method, even in perfect de Sitter space we obtain a finite result, because the contribution of the small semicircle on the contour $\mathcal{C}$ shown in Fig. 2 regularizes the final answer. So in our approach the tensor spectrum in perfect de Sitter space appears to be infrared finite, contrary to the findings of earlier works.

V. IMPLICATIONS FOR THE CMB ANISOTROPY

Gravitational waves provide an extra source of time dependence in the background in which the cosmic microwave background photons propagate. The contribution of gravitational waves to the CMB anisotropy is given by the integral in the Sachs-Wolfe formula [25]

$$\frac{\delta T_{SW}}{T}(\theta,\phi)=-\frac{1}{2}\int_{\tau_e}^{\tau_0} d\tau\, t_{\chi\chi,\tau}(\tau,\chi,\theta,\phi)|_{\chi=\tau_0-\tau} \tag{37}$$

where τ_0 and τ_e are respectively the observing and last scattering time for the photons and χ is the comoving radial coordinate. The anisotropy is characterized by the two-point angular correlation function $C(\gamma)$, where γ is the angle between two points on the celestial sphere. It is customary to expand the correlation function in terms of Legendre polynomials as

$$C(\gamma)=\left\langle \frac{\delta T}{T}(0)\frac{\delta T}{T}(\gamma)\right\rangle=\sum_{l=2}^{\infty}\frac{2l+1}{4\pi}C_l P_l(\cos\gamma), \tag{38}$$

where in standard notation $C_l=\langle|a_{lm}|^2\rangle$. Hence, inserting the Sachs-Wolfe integral into Eq. (38) and substituting Eq. (35) for the two-point fluctuation correlator yields

$$C(\gamma)=\frac{1}{4}\int_{\tau_e}^{\tau_0} d\tau\int_{\tau_e}^{\tau_0} d\tau'\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'}\langle t_{\chi\chi}(\tau,0)t_{\chi'\chi'}(\tau',\gamma)\rangle. \tag{39}$$

In order to obtain C_l we write the bitensor back in terms of its defining tensor eigenmodes on H^3, Eq. (36). Since $q^{(p)olm}_{\chi\chi}=0$, only the even parity modes contribute to the CMB anisotropy. The normalized eigenfunctions $q^{(p)elm}_{\chi\chi}(\chi,\theta,\phi)$ can be written as $Q^{pl}_{\chi\chi}(\chi)Y_{lm}(\theta,\phi)$, where [26]

$$Q^{pl}_{\chi\chi}(\chi)=\frac{N_l(p)}{p^2(p^2+1)}(\sinh\chi)^{l-2}\left(\frac{-1}{\sinh\chi}\frac{d}{d\chi}\right)^{l+1}(\cos p\chi) \tag{40}$$

and

$$N_l(p)=\left[\frac{(l-1)l(l+1)(l+2)}{\pi\prod_{j=2}^{l}(j^2+p^2)}\right]^{1/2}. \tag{41}$$

Hence we obtain for the power spectrum of multipole moments,

$$C_l=\kappa\Re\int_0^{+\infty}\frac{dp}{2p}\int_{\tau_e}^{\tau_0}d\tau\int_{\tau_e}^{\tau_0}d\tau'\left(\coth p\pi[\dot\Phi^L_p(\tau)\dot\Phi^L_{-p}(\tau')]+\frac{1}{\sinh p\pi}\left[\frac{a_p}{a_{-p}}\dot\Phi^L_p(\tau)\dot\Phi_p(\tau')\right]\right)Q^{pl}_{\chi\chi}Q^{pl}_{\chi'\chi'}. \tag{42}$$

The contribution to the multipole moments due to the second reflection term falls exponentially with increasing wave number. However, in contrast with the scalar fluctuations the long-wavelength tensor perturbations do give a substantial contribution to the CMB anisotropies. Hence the dependence of the tensor spectrum on the boundary conditions for the perturbations defined by the instanton background—Dirichlet for Hawking-Turok, free boundary conditions for Coleman–De Luccia—may provide a way to observationally distinguish different versions of open inflation. From the discussion above, we expect a larger contribution at low p for regular instantons. We shall perform the numerical computation of the needed reflection coefficients in future work [7].

In addition, for a complete calculation of the C_l one must evolve the Lorentzian mode functions $\Phi_p^L(\tau)$ forward from the beginning $\tau=-\infty$ of inflation inside the open universe up to the present time $\tau=\tau_0$. In the inflationary phase of the open universe the mode functions closely follow perfect de Sitter evolution in which they tend to a constant after the physical wavelength has been stretched outside the Hubble radius. The amplitude and phase of this constant define initial conditions for the radiation and matter dominated eras in which the modes of interest re-enter the Hubble radius. The radiation and matter evolution is straightforward to study numerically, and from this one can compute the Sachs-Wolfe integral (42) and the multipole moments C_l.

VI. CONCLUSION

We have computed the spectrum of tensor perturbations predicted in open inflation, according to Euclidean no boundary initial conditions. The Euclidean path integral unambiguously specifies the tensor correlators with no additional assumptions. We feel that the present work places earlier results on a substantially firmer footing. Our final result for the correlator Eq. (35), and the cosmic microwave multipole moments (42) is given in terms of scattering amplitudes in the Euclidean region and mode functions in the Lorentzian region. Both are straightforward to compute numerically, and we shall do so in future work [7].

Note added in proof. We have given an analogous treatment of gravitational waves in de Sitter space in [28].

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APPENDIX: MAXIMALLY SYMMETRIC BITENSORS

A maximally symmetric bitensor T is one for which $\sigma^* T=0$ for any isometry σ of the maximally symmetric manifold. Any maximally symmetric bitensor may be expanded in terms of a complete set of ''fundamental'' maximally symmetric bitensors with the correct index symmetries. For instance,

$$\begin{aligned}T_{iji'j'}=&t_1(\mu)g_{ij}g_{i'j'}+t_2(\mu)[n_i g_{ji'}n_{j'}+n_j g_{ii'}n_{j'}\\&+n_i g_{jj'}n_{i'}+n_j g_{ij'}n_{i'}]+t_3(\mu)[g_{ii'}g_{jj'}+g_{ji'}g_{ij'}]\\&+t_4(\mu)n_i n_j n_{i'}n_{j'}+t_5(\mu)[g_{ij}n_{i'}n_{j'}+n_i n_j g_{i'j'}]\end{aligned} \tag{A1}$$

where the coefficient functions $t_j(\mu)$ depend only on the distance $\mu(\Omega,\Omega')$ along the shortest geodesic from Ω to Ω'. Here $n_{i'}(\Omega,\Omega')$ and $n_i(\Omega,\Omega')$ are unit tangent vectors to the geodesics joining Ω and Ω' and $g_{ij'}(\Omega,\Omega')$ is the parallel propagator along the geodesic; $V^i g_i^{j'}$ is the vector at Ω' obtained by parallel transport of V^i along the geodesic from Ω to Ω' [18].

The bitensor

$$W^{ij}_{(p)i'j'}(\mu)=\sum_{\mathcal{P}lm} q^{(p)ij}_{\mathcal{P}lm}(\Omega)q^{(p)\mathcal{P}lm}_{i'j'}(\Omega')^* \tag{A2}$$

appearing in our Green function (15) has some additional properties arising from its construction in terms of the transverse and traceless tensor harmonics $q_{ij}^{(p)\mathcal{P}lm}$ on S^3 (or H^3). The tracelessness of $W^{(p)}_{iji'j'}$ allows one to eliminate two of the coefficient functions in Eq. (A1). It may then be written as [19]

$$\begin{aligned}W^{(p)}_{iji'j'}(\mu)=&w_1^{(p)}[g_{ij}-3n_i n_j][g_{i'j'}-n_{i'}n_{j'}]\\&+w_2^{(p)}[n_i g_{ji'}n_{j'}+n_j g_{ii'}n_{j'}+n_i g_{jj'}n_{i'}\\&+n_j g_{ij'}n_{i'}+4n_i n_j n_{i'}n_{j'}]+w_3^{(p)}[g_{ii'}g_{jj'}\\&+g_{ji'}g_{ij'}-2n_i g_{i'j'}n_j-2n_{i'}g_{ij}n_{j'}\\&+6n_i n_j n_{i'}n_{j'}].\end{aligned} \tag{A3}$$

The requirement that the bitensor be transverse $\nabla^i W^{(p)}_{iji'j'}=0$ and the eigenvalue condition $(\Delta_3-\lambda_p)W^{iji'j'}_{(p)}=0$ impose additional constraints on the remaining coefficient functions $w_j^{(p)}(\mu)$. To solve these constraint equations it is convenient to introduce the new variables [19]

$$\alpha(\mu)=w_1^{(p)}(\mu)+w_3^{(p)}(\mu)$$

$$\beta(\mu)=\frac{7}{(p^2+9)\sin\mu}\frac{d\alpha(\mu)}{d\mu}, \tag{A4}$$

where μ is the geodesic distance on S^3. In terms of a new argument $z=\cos^2(\mu/2)$ the transversality and eigenvalue conditions imply, for $\alpha(z)$,

$$z(1-z)\frac{d^2\alpha(z)}{d^2z}+\left[\frac{7}{2}-7z\right]\frac{d\alpha(z)}{dz}=(p^2+9)\alpha(z) \tag{A5}$$

and then, for the coefficient functions,

$$\begin{aligned}w_1=&Q_p[2(\lambda_p r^2-6)z(z-1)-2]\alpha(z)\\&+\tfrac{4}{7}[(\lambda_p r^2+6)z(z-\tfrac{1}{2})(z-1)]\beta(z)\\w_2=&Q_p 2(1-z)[(\lambda_p r^2-6)z+3]\alpha(z)\\&-\tfrac{4}{7}[(\lambda_p r^2+6)z(z-1)(z-\tfrac{3}{2})]\beta(z)\\w_3=&Q_p[-2(\lambda_p r^2-6)z(z-1)+3]\alpha(z)\\&-\tfrac{4}{7}[(\lambda_p r^2+6)z(z-\tfrac{1}{2})(z-1)]\beta(z)\end{aligned} \tag{A6}$$

with $\lambda_p=(p^2+3)$ on S^3 and Q_p a normalization constant.

To fix the normalization constant Q_p we contract the indices in the coincident limit $z\to 1$. This yields [19]

$$W^{(p)ij}_{ij}(\Omega,\Omega)=\sum_{\mathcal{P}lm} q_{ij}^{(p)\mathcal{P}lm}(\Omega)q^{(p)\mathcal{P}lm\,ij}(\Omega)^*=30Q_p\alpha(1). \tag{A7}$$

By integrating over the three-sphere and using the normalization condition (14) on the tensor harmonics one obtains $Q_p = -p^2 + 4/30\pi^2\alpha(1)$.

Notice that Eq. (A5) is precisely the hypergeometric differential equation, which has a pair of independent solutions $\alpha(z) = {}_2F_1(3+ip,3-ip,7/2,z)$ and $\alpha(1-z) = {}_2F_1(3+ip,3-ip,7/2,1-z)$. The former of these solutions is singular at $z=1$, i.e. for coincident points on the three-sphere, and the latter is singular for opposite points. The solution for $\beta(z)$ follows from Eqs. (A4) and is given by

$$\beta(z) = {}_2F_1(4-ip,4+ip,9/2,z). \tag{A8}$$

The hypergeometric functions are related by the transformation formula (Eq. [15.3.6] in [27])

$$\begin{aligned} {}_2F_1(a,b,c,z) = {} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a,b,a+b-c,1-z) \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} {}_2 \\ & \times F_1(c-a,c-b,c-a-b,1-z). \end{aligned} \tag{A9}$$

Only for the eigenvalues of the Laplacian on S^3, i.e. $p = in$ $(n \geqslant 3)$, does the term on the second line vanish for ${}_2F_1(3+ip,3-ip,7/2,z)$. In this case the functions are related by $(-1)^{n+1}$ and they are both regular for any angle on the three-sphere. But since $F(1-z) \to 1$ for coincident points, it is convenient to take $\alpha(1-z)$ in the bitensor appearing in the Euclidean Green function (20). In fact, the hypergeometric series terminates for these parameter values and the hypergeometric functions reduce to Gegenbauer polynomials $C^{(3)}_{n-3}(1-2z)$.

We conclude that the above properties required of the bitensor completely determine its form. Notice that in terms of the label p we have obtained a ‘‘unified’’ functional description of the bitensor $W^{iji'j'}_{(p)}$ on S^3 and H^3 although its explicit form is very different in both cases. In fact it is precisely this which allowed us in Sec. IV to analytically continue the angular part of the Green function from the Euclidean region into the open universe.

To perform the continuation we note that the Euclidean geodesic separation μ continues to $-i\chi$ where χ is the comoving geodesic separation on H^3. We apply the relation (A9) in an intermediate step of the calculation, the continuation of the bitensor into the complex p plane. In this step the functions $-\alpha(z),\beta(z)$ rather than $\alpha(1-z)$ and $\beta(1-z)$ enter. The hypergeometric functions on H^3 are defined by analytic continuation (Eq. [15.3.7] in [27]) and may be expressed in terms of associated Legendre functions as

$$\alpha(z) = 15\sqrt{\frac{\pi}{2}}(-\sinh\chi)^{-5/2} P^{-5/2}_{-1/2+ip}(-\cosh\chi),$$

$$\beta(z) = 15\sqrt{\frac{\pi}{2}}(-\sinh\chi)^{-7/2} P^{-7/2}_{-1/2+ip}(-\cosh\chi). \tag{A10}$$

Using the relation $-\cosh(\chi) = \cosh(\chi - i\pi)$, the Legendre functions may be expressed as

$$\begin{aligned} P^{-5/2}_{-1/2+ip}(-\cosh\chi) = {} & \sqrt{\frac{2}{-\pi\sinh\chi}}(1+p^2)^{-1}(4+p^2)^{-1} \\ & \times\Bigg[-3\coth\chi\cosh p(\pi+i\chi) \\ & - \frac{i\sinh p(i\chi+\pi)}{2p}[(2-p^2)(1 \\ & +\coth^2\chi) + (4+p^2)\mathrm{cosech}^2\chi]\Bigg] \end{aligned}$$

$$\begin{aligned} P^{-7/2}_{-1/2+ip}(-\cosh\chi) = {} & \sqrt{\frac{2}{-\pi\sinh\chi}}(1+p^2)^{-1}(4 \\ & +p^2)^{-1}(9+p^2)^{-1}\Bigg[\cosh p(\pi+i\chi) \\ & \times(p^2-11-15\,\mathrm{cosech}^2\chi) \\ & -6\frac{i\sinh p(i\chi+\pi)}{p}[(1-p^2)\coth^3\chi \\ & +(p^2+\tfrac{3}{2})\coth\chi\,\mathrm{cosech}^2\chi]\Bigg]. \end{aligned} \tag{A11}$$

The factors $e^{\pm p\pi}$ in these expressions combine with similar factors from the continuation of the conformal spatial coordinate X to produce our final result (35). The coefficient functions of the bitensor $W^{L(p)}_{iji'j'}(\chi)$ in our final result (35) for the tensor correlator are

$$\begin{aligned} w_1 = {} & \frac{\mathrm{cosech}^5\chi}{4\pi^2(p^2+1)}\Bigg[\frac{\sin p\chi}{p}[3+(p^2+4)\sinh^2\chi \\ & -p^2(p^2+1)\sinh^4\chi] \\ & -\cos p\chi[3/2+(p^2+1)\sinh^2\chi]\sinh 2\chi\Bigg] \end{aligned}$$

$$\begin{aligned} w_2 = {} & \frac{\mathrm{cosech}^5\chi}{4\pi^2(p^2+1)}\Bigg[\frac{\sin p\chi}{p}[3+12\cosh\chi-3p^2 \\ & \times(1+2\cosh\chi)\sinh^2\chi+p^2(p^2+1)\sinh^4\chi] \\ & +\cos p\chi[-12-3\cosh\chi+2(p^2-2)\sinh^2\chi \\ & +2(p^2+1)\cosh\chi\sinh^2\chi]\sinh\chi\Bigg] \end{aligned}$$

$$\begin{aligned} w_3 = {} & \frac{\mathrm{cosech}^5\chi}{4\pi^2(p^2+1)}\Bigg[\frac{\sin p\chi}{p}[3-3p^2\sinh^2\chi \\ & +p^2(p^2+1)\sinh^4\chi]+\cos p\chi[-3/2 \\ & +(p^2+1)\sinh^2\chi]\sinh 2\chi\Bigg]. \end{aligned} \tag{A12}$$

As mentioned before, for $\chi \to 0$ these functions converge and they exponentially decay at large geodesic distances. We also note that in this form one should take the normalization factor Q_p to be positive, as explained in the text.

Finally, let us mention that the scalar Green function [5] may also be described in terms of hypergeometric functions. In terms of the variable z, the equation for the angular part $C_p(\mu)$ of the scalar Euclidean Green function [Eq. (35) in [5]] reads

$$z(1-z)\frac{d^2C_p(z)}{d^2z}+\left[\frac{3}{2}-3z\right]\frac{dC_p(z)}{dz}=(p^2+1)C_p(z). \tag{A13}$$

If we express the Green function as an infinite sum [Eq. (38) in [5]], the appropriate solution regular at $\mu=0$ and $\mu=\pi$ is

$$C_p(z)=Q_pF(1+ip,1-ip,3/2,1-z)=\frac{Q_p\sinh p\mu}{p\sin\mu}. \tag{A14}$$

As for the tensor correlator, the normalization constant Q_p is determined by the normalization of the scalar harmonics on S^3. However, because of the extra factor (Δ_3+3) in the scalar Green equation [Eq. (35) in [5]], we must also divide by $4+p^2$ in this case. This reproduces precisely the angular part of the scalar Green function [Eq. (38) in [5]].

When expressing the Euclidean Green function as an integral, we continue $C_p(z)$ into the complex p plane, and again need to express it in terms of the hypergeometric function regular at $z=0$. We re-express $F(1+ip,1-ip,3/2,1-z)$ using the relation (A9) and obtain

$$\coth p\pi\frac{\sinh p\mu}{p\sin\mu}=\frac{\sinh p(\pi-\mu)}{p\sinh p\pi\sin\mu}+\frac{\cosh p\mu}{p\sin\mu}. \tag{A15}$$

The factor $\coth p\pi$ is needed in converting the sum into a contour integral. The first term is regular for opposite points and leads exactly to the angular part of the Lorentzian correlator [Eq. (46) in [5]] in the same way as described above for tensor fluctuations. The second term is a bit more subtle. Its analogue in the tensor correlator did not contribute to the contour integral because it had no poles within the contour. However, in the scalar case we need to take into account the extra normalization factor $1/(p^2+4)$ which has a pole at $p=2i$. This is the underlying reason for the presence of the extra term in the integral representation of the scalar Euclidean Green function [second term in Eq. (37) in [5]]. As explained in [5], the $(\pi-\Omega)$ factor in front of it arises from matching the delta function in the Green equation, which unlike the tensor Green equation is fourth order in derivatives. This is also the reason why we had to include the extra factor $1/(p^2+4)$. Nevertheless, it is clear that the scalar and tensor cases are very closely parallel.

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