CMBR anisotropy with primordial magnetic fields

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Galactic magnetic fields are observed of order $\sim 10^{-6}$ G, but their origin is not definitely known yet. In this paper we consider the primordial magnetic fields generated in the early universe and analyze their effects on the density perturbations and the cosmic microwave background radiation (CMBR) anisotropy. We assume that the random magnetic fields have the power law spectrum and satisfy the force-free field condition. The peak heights of the CMBR anisotropy are shown to be shifted upward depending on the magnetic field strengths relative to the nonmagnetic field case.

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I. INTRODUCTION

Recently many possible generation mechanisms of primordial magnetic fields have been suggested to explain the observed galactic magnetic fields of order $\sim 10^{-6}$ G [1]. The dynamo mechanism explains the origin of the large scale galactic magnetic field with amplification of a small frozen-in seed field to the observed μG field through turbulence and differential rotation. The dynamo saturates when the growth enters the nonlinear regime. However the saturation might actually be too fast for a large scale field to form [2]. Without the dynamo mechanism to explain the galactic fields from the primordial fields, which get compressed when the protogalactic cloud collapses, the needed amplitude of the primordial magnetic fields is quite large, on the order of 10^{-10}–10^{-9} G. Cosmological phase transitions in the early universe may produce magnetic seed fields. If conformal invariance is broken during the inflationary period, magnetic seed fields are generated [3]. And also the electroweak phase transition [4,5] and QCD phase transition [6,7] can generate magnetic seed fields. Gasperini *et al.* [8] considered a generation mechanism in a stringy model with broken conformal invariance by a dilaton field. But the field amplitudes produced by several mechanisms are much too weak to explain the observations.

Primordial magnetic fields may generate density perturbations [9–11]. Tsagas and Barrow [12,13] considered the general relativistic density perturbations with magnetic fields. To treat the large scale cosmological perturbations we confront the gauge ambiguity problem. It is caused by regions larger than horizon size being causally disconnected. Bardeen formulated the gauge invariant method to solve the gauge ambiguity problem [14]. Details about the gauge invariant method of cosmological perturbations can be found in [14–16]. Cosmological perturbations can be classified according to how they transform under spatial coordinate transformations in the background space-time; scalar, vector, and tensor perturbations. They relate to density, vorticity and gravitational wave perturbations, respectively. In this paper we consider only scalar perturbations. From the observations that the magnetic field energy density is much less than the background radiation energy density, we can treat it within the linear perturbation theory.

The big bang nucleosynthesis (BBN) can constrain the amplitude of magnetic fields at early epochs. It is argued in [17] that the presence of magnetic fields affects BBN by changing the weak reaction rates, the electron density and the expansion rate of the universe. So they put constraint on the magnetic field amplitude $B < 2 \times 10^{9}$ G at $T = 0.01$ MeV. Barrow *et al.* [18] also derived an upper limit of the magnetic field amplitude at present $B_0 < 3.4 \times 10^{-9}$ G using microwave background anisotropy created by cosmological magnetic fields.

The cosmic microwave background (CMB) photons are polarized through the Thomson scattering of the photons and electrons during the decoupling time [19]. The upper limit on its degree of linear polarization large angular scales is $\Delta_P < 6 \times 10^{-5}$ [20]. We expect that the CMB radiation (CMBR) polarization on small angular scales would be observed with the future experiments, the microwave anisotropy probe (MAP) [21] and Planck [22]. If primordial magnetic fields exist at the decoupling time, they cause Faraday rotations which rotate the directions of polarization vectors. This effect can be imprinted on the cosmic background radiation and we may obtain information on the amplitude of primordial magnetic fields by measuring the polarizations of the CMBR [23,24].

In this paper we calculate the evolution of density perturbations with the primordial magnetic fields which have power law spectrum. We do not concern ourselves with the details of generation mechanism of magnetic seed fields, but assume that sometime during the radiation dominated era, large scale magnetic fields are generated instantaneously. We then investigate how they affect the temperature anisotropy and polarization of the CMBR using various spectral indices and field strengths of the magnetic field.

In Sec. II we derive, using the gauge invariant variable, the density perturbation equations with magnetic fields present. The equations are solved numerically and the effect of magnetic fields on the temperature anisotropy and polarizations of the CMBR are shown in Sec. III. Finally we summarize the results in Sec. IV.

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II. DENSITY PERTURBATIONS WITH MAGNETIC FIELDS

In this section we consider that the background space is homogeneous and isotropic. Cosmological perturbations are classified as scalar, vector and tensor perturbations according to how they transform under spatial coordinate transformations in the background space-time. We will treat here only scalar perturbations which are related to density perturbations. In the longitudinal gauge (conformal Newtonian gauge) the metric, including the scalar perturbations, is written by [16]

$$ds^2 = a(\eta)^2[-(1+2\Psi)d\eta^2 + (1+2\Phi)\gamma_{ij}dx^i dx^j], \quad (1)$$

where η is the conformal time defined by $dt = a(\eta)d\eta$ and γ_{ij} is the spatial metric tensor. Ψ and Φ are related to the gauge invariant quantities Φ_A and Φ_H of Bardeen [14] and the gauge invariant potentials Ψ and Φ of Kodama and Sasaki [15]. The physical meaning of Φ and Ψ are the curvature perturbation and Newtonian gravitational potential, respectively.

The Maxwell equations have the form

$$F^{\mu\nu}{}_{;\nu} = J^\mu, \quad (2)$$

$$F_{\mu\nu;\rho} + F_{\nu\rho;\mu} + F_{\rho\mu;\nu} = 0, \quad (3)$$

where $F_{\mu\nu}$ is the second-order antisymmetric Maxwell tensor which represent the electromagnetic field and J_μ is the four-vector current which generates the electromagnetic field. The Maxwell tensor splits into the electric and magnetic four vector [12], defined by

$$E_\mu = F_{\mu\nu}u^\nu, \quad (4)$$

$$B_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\lambda}u^\nu F^{\rho\lambda}, \quad (5)$$

where $\epsilon_{\mu\nu\rho\lambda}$ is Levi-Civita tensor and u^μ is the fluid four velocity. The background value of u^μ is taken to be $u^\mu = (1/a,0,0,0)$. Then the electric and magnetic four vectors are purely spatial, i.e., $E^\mu u_\mu = 0$ and $B^\mu u_\mu = 0$, so we denote the spatial components E^i and B^i by $\mathbf{E}$ and $\mathbf{B}$.

The generalized covariant Ohm's law is

$$J^\mu + J^\nu u_\nu u^\mu = \sigma F^{\mu\nu}u_\nu \quad (6)$$

where σ represents conductivity of the medium. The spatial components of Eq. (6) are reduced to $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{J}$ is the spatial component of J^μ. Assuming infinite conductivity of the medium in the Universe [3], we neglect the electric field so that $\mathbf{E} = 0$.

Now we can reduce Eq. (3), using the magnetic field three vector $\mathbf{B}(\eta,\mathbf{x})$ to

$$\frac{\partial(a^3\mathbf{B})}{\partial\eta} = 0, \quad \nabla\cdot\mathbf{B} = 0, \quad (7)$$

where ∇ is the covariant differentiation with respect to γ_{ij}. In this work, we consider only the case where $\gamma_{ij} = \delta_{ij}$ for simplicity. From the first of Eq. (7), we get $\mathbf{B}(\eta,\mathbf{x}) \propto a^{-3}$. The magnetic field energy density, $\frac{1}{2}B^2 (= \frac{1}{2}B^\mu B_\mu = \frac{1}{2}a^2\Sigma_{i=1}^3 B^i B^i)$, evolves the same as the radiation energy density $\sim a^{-4}$. The dimensionless quantity r is introduced in [3] defined by $r = B^2/2\rho_r$, which is the ratio of magnetic field energy density to the background radiation energy density. It is approximately constant in all the early history of the Universe.

Total energy momentum tensor is decomposed by

$$T^{\mu\nu} = T^{(\text{fluid})\mu\nu} + T^{(\text{em})\mu\nu}, \quad (8)$$

where the electromagnetic energy momentum tensor and fluid energy momentum tensor are given by

$$T^{(\text{em})\mu\nu} = F^\mu{}_\lambda F^{\nu\lambda} - \tfrac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \quad (9)$$

and

$$T^{(\text{fluid})\mu\nu} = (\rho + P)u^\mu u^\nu + Pg^{\mu\nu}. \quad (10)$$

Here we treat the matter as perfect fluid to neglect the anisotropic pressure perturbations and consider only adiabatic perturbations to neglect the entropy perturbations. The linearized perturbation equations are obtained from the Einstein equations up to first order, and are written as follows:

$$3\left(\frac{a'}{a}\right)^2\Psi - 3\frac{a'}{a}\Phi' + (\nabla^2 - 3K)\Phi = -4\pi G a^2\rho\left(\Delta + \frac{1}{2}\frac{B^2}{\rho}\right), \quad (11)$$

$$\nabla_i\left(\frac{a'}{a}\Psi - \Phi'\right) = -4\pi G a^2(\rho + P)v_i, \quad (12)$$

$$\Phi'' + \frac{a'}{a}(2\Phi' - \Psi') + \left[\left(\frac{a'}{a}\right)^2 - 2\frac{a''}{a}\right]\Psi + \frac{1}{3}\nabla^2\Psi - \frac{1}{3}(\nabla^2 + 3K)\Phi = -4\pi G a^2\rho\left(c_s^2\Delta + \frac{1}{6}\frac{B^2}{\rho}\right), \quad (13)$$

$$(\nabla^i\nabla_j - \tfrac{1}{3}\delta^i_j\nabla^2)(\Phi + \Psi) = -8\pi G a^2 \Pi^{(\text{em})i}{}_j, \quad (14)$$

where the prime denotes a derivative with respect to the conformal time η, and $\Pi^{(\text{em})i}{}_j = \frac{1}{3}\delta^i_j B^2 - B^i B_j$ corresponds to the magnetic field anisotropic pressure. Δ and v_i are the gauge invariant density contrast and velocity perturbation and c_s is the sound velocity. ∇^2 is the Laplace–Beltrami operator whose eigenvalue is $-k^2$. Equation (13) is the trace part of the spatial component of the perturbed energy-momentum tensor and Eq. (14) is the traceless part.

To write down the perturbation equations for a given wave mode $\mathbf{k}$, we define the Fourier transform of the perturbed quantities and random magnetic fields. In this paper we consider density perturbations in flat space $K=0$, so the spatial dependence of the Fourier transform is just the plane wave $e^{i\mathbf{k}\cdot\mathbf{x}}(=e^{ik_ix^i})$,

$$\Delta(\mathbf{x})=\int d^3k \exp(i\mathbf{k}\cdot\mathbf{x})\ \Delta(\mathbf{k}) \tag{15}$$

and also $v(\mathbf{x})$, $\Phi(\mathbf{x})$,$\Psi(\mathbf{x})$ and $B(\mathbf{x})$ are defined similarly. We assume that the force-free field condition ($\mathbf{B}\times\nabla\times\mathbf{B}=0$) is satisfied to treat the magnetic field. Then Eqs. (11)–(14) can be written by

$$3\left(\frac{a'}{a}\right)^2\Psi-3\frac{a'}{a}\Phi'-k^2\Phi=-4\pi Ga^2\rho\left(\Delta+\frac{1}{2}\frac{F(k)}{\rho a^{-4}}\right), \tag{16}$$

$$k\left(\frac{a'}{a}\Psi-\Phi'\right)=-4\pi Ga^2(\rho+P)v, \tag{17}$$

$$\Phi''+\frac{a'}{a}(2\Phi'-\Psi')+\left[\left(\frac{a'}{a}\right)^2-2\frac{a''}{a}\right]\Psi+\frac{k^2}{3}\Psi+\frac{1}{3}k^2\Phi$$
$$=-4\pi Ga^2\rho\left(c_s^2\Delta+\frac{1}{6}\frac{F(k)}{\rho a^{-4}}\right), \tag{18}$$

$$k^2(\Phi+\Psi)=-8\pi Ga^{-2}\left(\frac{1}{4}F(k)\right), \tag{19}$$

where $F(k)$ is defined by

$$F(k)=\int d^3k' B^l(\mathbf{k}')B_l(\mathbf{k}-\mathbf{k}'), \tag{20}$$

which represents the spectral dependence of magnetic fields. The fact that the magnetic field energy density decays as $\sim a^{-4}$ is used. In Appendix A, we calculate the Fourier transform of the magnetic field anisotropic pressure using force-free field conditions and derived the expression $F(k)$, Eq. (20).

To investigate the spectral dependence of perturbed quantities, we need to take ensemble average of $|F(k)|^2$ due to random magnetic field. For a homogeneous and isotropic random magnetic field, $\mathbf{B}(\mathbf{k})$ satisfies the relation [10,25]

$$\langle B^i(\mathbf{k})B^{j*}(\mathbf{k}')\rangle=\delta^3(\mathbf{k}-\mathbf{k}')\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)\frac{B^2(k)}{2}, \tag{21}$$

and then

$$\langle B_0^2\rangle=\int d^3kB^2(k), \tag{22}$$

where angular brackets denote a statistical average over an ensemble of possible magnetic field configurations and $\langle B_0^2\rangle^{1/2}$ is the average field strength observed today.

From Eq. (20)

$$|F(k)|^2=\int d^3k' d^3k'' \mathbf{B}(\mathbf{k}')\cdot\mathbf{B}(\mathbf{k}-\mathbf{k}')\mathbf{B}(\mathbf{k}'')^*\cdot\mathbf{B}(\mathbf{k}-\mathbf{k}'')^*. \tag{23}$$

Taking ensemble average of both sides, we obtain

$$\langle|F(k)|^2\rangle=\int d^3k' d^3k''\langle\mathbf{B}(\mathbf{k}')\cdot\mathbf{B}(\mathbf{k}-\mathbf{k}')\mathbf{B}(\mathbf{k}'')^*\cdot\mathbf{B}(\mathbf{k}-\mathbf{k}'')^*\rangle$$
$$=\int d^3k'\ d^3k''[\langle B^l(\mathbf{k}-\mathbf{k}')B^{*m}(\mathbf{k}-\mathbf{k}'')\rangle\langle B_l(\mathbf{k}')B_m^*(\mathbf{k}'')\rangle+\langle B^l(\mathbf{k}-\mathbf{k}')B_m^*(\mathbf{k}'')\rangle\langle B_l(\mathbf{k}')B^{*m}(\mathbf{k}-\mathbf{k}'')\rangle]. \tag{24}$$

Using Eq. (21), and integrating the δ function, we can get

$$\langle|F(k)|^2\rangle=2\int d^3k'\left[1+\frac{\{(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k}'\}^2}{|\mathbf{k}-\mathbf{k}'|^2k'^2}\right]\frac{B^2(|\mathbf{k}-\mathbf{k}'|)}{2}\ \frac{B^2(k')}{2}$$
$$=\frac{A^2V}{8\pi^2}\int_0^{k_{\max}}dk' k'^{q+2}\int_{-1}^{1}d\mu[k^2(1+\mu^2)+2k'^2-4kk'\mu][k^2+k'^2-2kk'\mu]^{(q/2)-1}, \tag{25}$$

where $\mathbf{k}\cdot\mathbf{k}'=kk'\mu$, $\mu=\cos\theta$ and $\delta^3(k=0)=V/(2\pi)^3$ is used. V is the volume factor. We used the power-law spectrum

$$B^2(k)=Ak^q. \tag{26}$$

We define the average field on scale λ by

$$\langle B^2\rangle_\lambda=\int d^3kB^2(k)\exp\left(\frac{-k^2\lambda^2}{2}\right). \tag{27}$$

Then we can determine the coefficient A in Eq. (26)

$$A=\frac{1}{4\pi}\frac{\lambda^{q+3}}{2^{(n+1)/2}\Gamma(n+\frac{3}{2})}\langle B^2\rangle_\lambda . \tag{28}$$

To determine $k_{\rm max}$ we use the argument in [11]. Small scales reach nonlinear variance ($\Delta\geqslant 1$) earlier than large scales, and the first scales to become nonlinear have $k\approx 2k_{\rm max}$. If we choose the scale that corresponds to the formation of galaxies at $z_{\rm nl}=6$, $k_{\rm max}=\pi\,{\rm Mpc}^{-1}$. If instead we require that magnetic fields form clusters of galaxies at $z_{\rm nl}=1$, that would correspond to $k_{\rm max}=\pi/2\ {\rm Mpc}^{-1}$. As we shall see below, the CMBR anisotropy depends mostly on the $k_{\rm max}$ value. Treating $k/k_{\rm max}$ as a small parameter, the leading term of the result of integrating Eq. (25) is

$$\langle|F(k)|^2\rangle\simeq\frac{\pi V}{(2\pi)^3}A^2\frac{4}{2q+3}k_{\rm max}^{2q+3}. \tag{29}$$

For the case of $q\leqslant -3$, the integration of Eq. (25) diverge as $k'\to 0$. So we only consider $q>-3$.

III. CMBR ANISOTROPY

Cosmological density perturbations cause the temperature fluctuations when the photons decouple from the thermal bath at the last scattering surface. Furthermore small metric perturbations induce bulk velocities of the fluid, and the resulting anisotropies in the photon distribution will induce polarization when the photons scatter off charged particles (Thomson scattering) [26]. After decoupling, the photons freely propagate along the geodesics, and any polarization produced through the Thomson scattering will remain fixed. The evolution of the CMB anisotropies is described by a set of radiative transfer equations. Temperature and polarization anisotropies are expressed in terms of Stokes parameters I,Q,U and V. The parameter I gives the radiation intensity which is positive definite, Q and U represent the linearly polarized light and V describes the circular polarization. The degree of linear polarization Δ_P is defined in terms of Q and U, $\Delta_P=(Q^2+U^2)^{1/2}$, and the temperature anisotropy is denoted by $\Delta_T(\equiv\Delta T/T)$. Δ_T and Δ_P can be expanded in multipole moments defined such that $\Delta(\eta,k,\mu)=\Sigma_l(2l+1)\Delta_l(\eta,k)P_l(\mu)$, where P_l is the Legendre polynomial of order l and μ is the cosine angle between the wave vector and the direction of observation. Their evolution equations are given by [27]

$$\Delta_T'+ik\mu(\Delta_T+\Psi)=-\Phi'-\kappa'[\Delta_T-\Delta_{T_0}-\mu v_b+\tfrac{1}{2}P_2(\mu)S_p], \tag{30}$$

$$\Delta_P'+ik\mu\Delta_P=-\kappa'[\Delta_P-\tfrac{1}{2}(1-P_2(\mu))S_P], \tag{31}$$

where $S_P\equiv-\Delta_{T_2}-\Delta_{P_2}+\Delta_{P_0}$. κ' is the differential optical depth defined by $\kappa'=x_e n_e\sigma_T a/a_0$ with x_e the ionization fraction, n_e the electron number density and σ_T the Thomson scattering cross section. The Thomson scattering cannot produce any net circular polarization [19] and thus we expect $V=0$ for the microwave background. v_b is the velocity perturbations of the baryon component, which is affected by the existence of primordial random magnetic fields. Metric perturbations Φ and Ψ, which evolve according to the equations in Sec. II under the influence of random magnetic fields, act as the source terms in Eq. (30) which governs the evolution of the temperature anisotropy.

Equations (30) and (31) are formally integrated to yield [27]

$$\begin{aligned}\Delta_T(\eta_0)=&\int_0^{\eta_0}d\eta e^{ix\mu}g(\eta)[\Delta_{T_0}(\eta)+\mu v_b(\eta)\\&-\tfrac{1}{2}P_2(\mu)S_P(\eta)]\\&+\int_0^{\eta_0}d\eta e^{ix\mu}e^{-\kappa(\eta_0,\eta)}(\Psi'-\Phi'),\end{aligned} \tag{32}$$

$$\Delta_P(\eta_0)=\int_0^{\eta_0}d\eta e^{ix\mu}g(\eta)\tfrac{1}{2}[1-P_2(\mu)]S_P(\eta), \tag{33}$$

where

$$g(\eta)\equiv\kappa' e^{-\kappa(\eta_0,\eta)} \tag{34}$$

is the visibility function and

$$\kappa(\eta_0,\eta)=\int_\eta^{\eta_0}\kappa' d\eta \tag{35}$$

is the optical depth to photons emitted at the conformal time η. The visibility function represents the probability that a photon observed at η_0 last scattered within $d\eta$ of a given η. For the standard recombination this function has a sharp peak at the conformal time of decoupling η_D. And $x=k(\eta_0-\eta)$.

Under a clockwise rotation in the plane perpendicular to the direction of observation, $\hat{\mathbf{n}}$, the temperature is invariant while Q and U transform as

$$\begin{aligned}Q'&=Q\cos 2\psi+U\sin 2\psi,\\U'&=-Q\sin 2\psi+U\cos 2\psi,\end{aligned} \tag{36}$$

or

$$(Q\pm iU)'=e^{\mp 2i\psi}(Q\pm iU), \tag{37}$$

where ψ is the rotation angle. Therefore the quantities can be expanded in terms of the spin-2 spherical harmonics [28]

$$(Q\pm iU)(\hat{\mathbf{n}})=\sum_{lm}a_{\pm 2,lm\pm 2}Y_{lm}(\hat{\mathbf{n}}), \tag{38}$$

where ${}_{\pm 2}Y_l^m(\hat{\mathbf{n}})$ is the spin-2 spherical harmonics whose properties are summarized briefly in Appendix B. The expansion coefficients are

$$a_{\pm 2,lm}=\int d\Omega_{\pm 2}Y^{*}_{lm}(Q\pm iU)(\hat{\mathbf{n}}). \tag{39}$$

In [28], the authors introduce the following linear combinations of $a_{\pm 2,lm}$ to circumvent the difficulty that the Stokes parameters are not invariant under rotations:

$$a_{E,lm}=-\frac{1}{2}(a_{2,lm}+a_{-2,lm}),$$

$$a_{B,lm}=\frac{i}{2}(a_{2,lm}-a_{-2,lm}). \tag{40}$$

These newly defined variables are expanded in terms of ordinary spherical harmonics Y_{lm}

$$E(\hat{\mathbf{n}})=\sum_{lm} a_{E,lm}Y_{lm}(\hat{\mathbf{n}}),$$

$$B(\hat{\mathbf{n}})=\sum_{lm} a_{B,lm}Y_{lm}(\hat{\mathbf{n}}). \tag{41}$$

The spin-zero spherical harmonics Y_{lm} is free from the ambiguity with the rotation of the coordinate system, and therefore E and B are rotationally invariant quantities. The E mode has $(-1)^l$ parity and the B mode $(-1)^{(l+1)}$ parity in analogy with electric and magnetic fields. Scalar perturbations generate only the E mode of the polarizations [29]. The power spectra of temperature and polarization anisotropies are defined as $C_{Tl}\equiv\langle|a_{T,lm}|^2\rangle$ for $\Delta_T=\Sigma_{lm}a_{T,lm}Y_{lm}$ and analogously for C_{El}. So if we get the evolution of the temperature and polarization anisotropy amplitude from Eqs. (32) and (33), the amplitudes for each mode of power spectra are given by

$$C_{T,l}=(4\pi)^2\int k^2dkP_\delta(k)[\Delta_{T,l}(k)]^2, \tag{42}$$

$$C_{E,l}=(4\pi)^2\int k^2dkP_\delta(k)[\Delta_{E,l}(k)]^2, \tag{43}$$

$$C_{Cl}=(4\pi)^2\int k^2dkP_\delta(k)\Delta_{Tl}(k)\Delta_{El}(k), \tag{44}$$

where $P_\delta(k)$ is the initial power spectrum and Δ_{Tl} and Δ_{El} are given by [28]

$$\Delta_{Tl}(k)=\int_0^{\eta_0}d\eta S_T(k,\eta)j_l(x), \tag{45}$$

$$\Delta_{El}(k)=\sqrt{\frac{(l+2)!}{(l-2)!}}\int_0^{\eta_0}d\eta S_E(k,\eta)j_l(x), \tag{46}$$

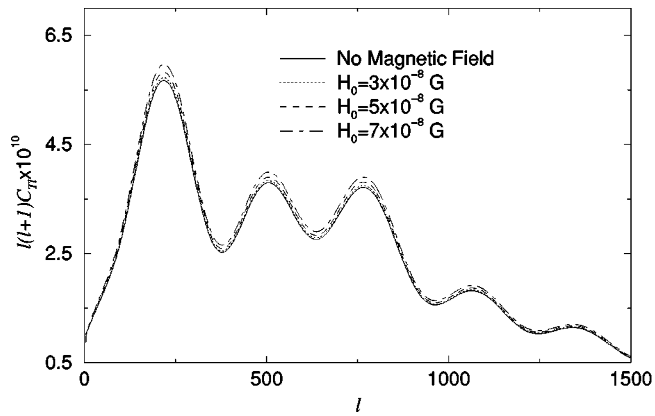

FIG. 1. The angular power spectrum of temperature fluctuations with the magnetic field strength $B_\lambda=3\times10^{-8}$ G, 5×10^{-8} G, 7×10^{-8} G with spectral index $q=1$ for $\lambda=0.1h^{-1}$ Mpc.

$$S_T(k,\eta)=g\left(\Delta_{T0}+\Psi-\frac{v_b'}{k}-\frac{S_P}{4}-\frac{3S_P''}{4k^2}\right)+e^{-\kappa}(\Phi'+\Psi')$$
$$-g'\left(\frac{v_b}{k}+\frac{3S_P'}{4k^2}\right)-\frac{3g''S_P}{4k^2}, \tag{47}$$

$$S_E(k,\eta)=\frac{3gS_P}{4x^2}. \tag{48}$$

We here concern ourselves with the flat cold dark matter (CDM) universe with adiabatic initial conditions. We use the CMBFAST code [30] to calculate numerically the CMBR anisotropy. During this calculation we put h (Hubble constant divided by 100 km/sec/Mpc) $=0.5$ and assume three species of massless neutrinos. In Fig. 1, we plot the angular power spectrum of temperature fluctuations $l(l+1)C_{Tl}$ with the magnetic field strengths, 3×10^{-8} G, 5×10^{-8} G, and 7×10^{-8} G, for a given magnetic field spectrum index, $q=1$. Observed amplitude of galactic magnetic fields is of order $\sim10^{-6}$ G. The BBN can constrain the amplitude of magnetic fields, $B_0<10^{-7}$ G [17], and also derived an upper limit of the magnetic field amplitude $B_0<10^{-9}$ G using the CMB anisotropy [18]. Another constraint on magnetic field intensity can be obtained from $r(\equiv B^2/2\rho_r)\leqslant\Delta_H$, where Δ_H is the horizon crossing amplitude. The cosmic background explorer (COBE) 4 yr data gives $\Delta_H\sim10^{-5}$ on large angular scales. Considering magnetic fields on the order of $\sim10^{-8}$ G does not violate too much current bounds on magnetic field amplitudes by the observational and theoretical considerations. The figure shows that the spectral curves of the CMBR temperature anisotropy are shifted upward with increasing magnetic field strengths. We can conclude from the numerical calculations that the presence of the magnetic fields which have field strength on the order of $\leqslant10^{-9}$ G at present do not affect significantly the temperature fluctuations. The density perturbations with magnetic fields on the order of 10^{-8} G result in the deviations of the angular power spectrum C_{Tl} of up to 14%.

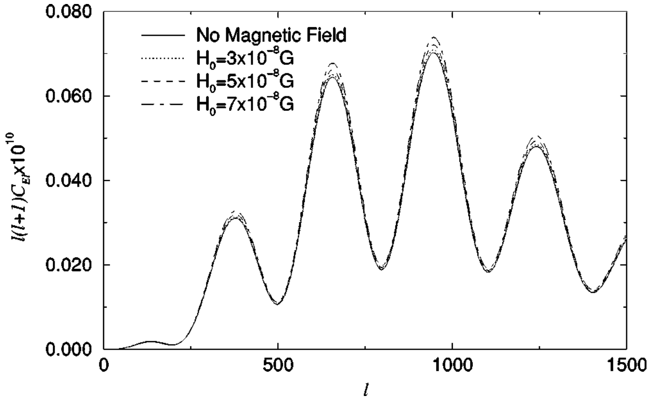

FIG. 2. The E mode polarization spectrum $l(l+1)C_{El}$ for $q=1$.

The E polarization spectrum $l(l+1)C_{El}$ and temperature-polarization correlation spectrum $l(l+1)C_{Cl}$ are shown in Figs. 2 and 3 for $q=1$. Also in these figures we can see that the spectrum curves are shifted upward with increasing magnetic field strengths relative to the nonmagnetic case. The current bound on the degrees of linear polarizations of the CMBR on large angular scales is $\Delta_P<6\times10^{-5}$ [20]. As we discussed in the previous section, we plot the temperature anisotropy with $k_{\max}=\pi/2$ Mpc^{-1} and π Mpc^{-1} in Fig. 4. In this figure, we can see that there is a strong dependence of the spectrum curves on the cutoff $k_{\max}$.

In Fig. 5 we plot the temperature anisotropy with the spectral index of magnetic field $q=1$, 2 and 3 for $B_\lambda=5\times10^{-8}$ G with $\lambda=0.1h^{-1}$ Mpc. The spectrum curves are nearly independent of the spectral index. We probe the vicinity of the acoustic oscillation peak, $l\simeq200$, to investigate the dependence of spectral index more closely. The result is that the spectrum curves are shifted downward with the increasing spectral index. In Ref. [31] recently, the authors derive an expression for the angular power spectrum of CMBR anisotropies due to gravity waves generated by a stochastic magnetic field. They show that, for $n>-3/2$ (n is magnetic field spectral index in their notations), the induced C_l spectrum from gravity waves is independent of n, but only the amplitude depends on the spectral index, $l^2C_l \sim(\lambda k_{\max})^{2n+3}l^3$. They also derive an upper bound of B_λ for $n>-3/2$ and $\lambda=0.1h^{-1}$ Mpc

$$B_\lambda<9.5\times10^{-8}e^{-0.37n}\ \mathrm{G}. \tag{49}$$

Here we do not consider the Faraday rotation due to the magnetic field which can change the polarization spectrum because we restrict our calculations in linear perturbation theory. The authors in [23,24,32] studied the effect on the CMBR anisotropy with the uniform primordial magnetic field causing Faraday rotations in the homogeneous background universe. They argued that the presence of magnetic fields depolarizes the CMBR anisotropy [24] and proposed that the temperature and B mode polarization correlation, which are generated by Faraday rotations, can constrain the magnetic field [32].

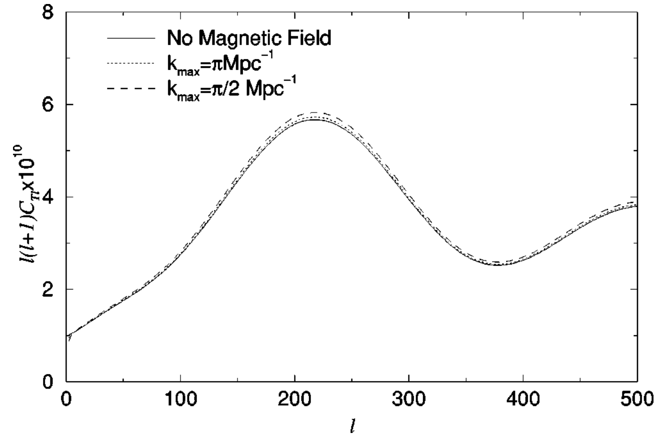

FIG. 4. The angular power spectrum of temperature fluctuations with the value of the magnitude of cutoff wave vector $k_{\max}=\pi$ Mpc^{-1}, $\pi/2$ Mpc^{-1} for $B_\lambda=5\times10^{-8}$ G for $\lambda=0.1h^{-1}$ Mpc.

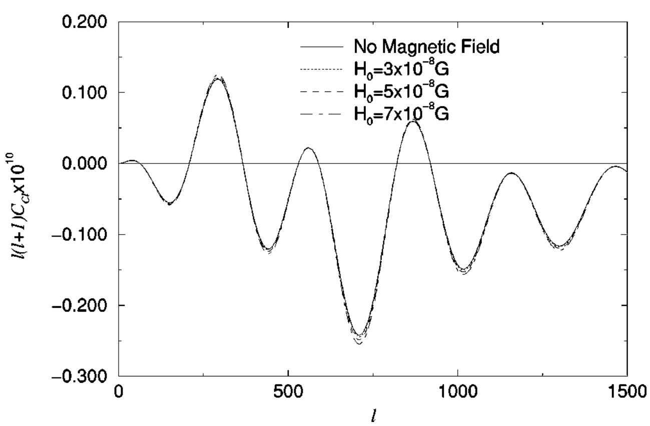

FIG. 3. Temperature and polarization cross-correlations, $l(l+1)C_{cl}$, for $q=1$.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper we consider the density perturbations with primordial magnetic fields are present using gauge invariant formalism. While magnetic field generation mechanism is

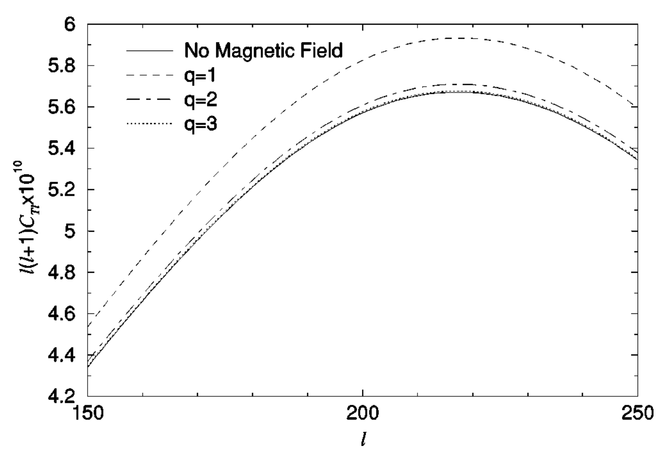

FIG. 5. Temperature anisotropy with the spectral index of magnetic field energy density $q=1,2,3$ for $B_\lambda=5\times10^{-8}$ G for $\lambda=0.1h^{-1}$ Mpc.

not yet known, we assume that the magnetic fields smear out all over the universe randomly in the radiation dominated era. Using the CMBFAST code [30] we solve numerically the coupled density perturbation equations for flat CDM universe with adiabatic initial conditions. We investigate the CMBR anisotropies for the magnetic field permeated universe. With the temperature anisotropy spectrum we cannot fully determine the cosmological parameters and the information about the cosmological perturbations. CMB photons are polarized due to Thomson scattering during the decoupling time. Small cosmological perturbations induce the polarization at the last scattering surface through Thomson scattering. The linear polarization relates to the quadrupole anisotropy in the photons. So if we investigate the polarization as well as the temperature anisotropy, we can get enough information at the time of decoupling. The information can constrain the cosmological parameters and the cosmological perturbations. Next we study the effect of the random magnetic field on the CMBR temperature anisotropies and polarizations. We consider the several scale magnetic fields with the assumption of power-law magnetic spectrum. To get the polarization power spectra, we use the rotation invariant scalar quantities E and B, which are introduced in [28]. B vanishes for scalar perturbations.

For a given spectral index the temperature anisotropy and polarization spectrum are shifted upward with increasing magnetic field strengths. The density perturbations with magnetic fields on the order of 10^{-8} G result in the deviations of temperature anisotropy power spectrum of up to 14%. The fluctuations of this order due to the primordial magnetic field are sufficiently large to be observed in future satellite experiments. Further, if $B_\lambda < \sim 10^{-9}$ G, magnetic field energy density does not affect noticeably the CMBR anisotropy. We assume that the magnetic fields have the power-law spectrum. The spectrum curves are nearly independent of the magnetic field spectral index.

Here we assume that the magnetic field energy density evolves as $\sim a^{-4}$. But in the early era, when the magnetic fields are generated, their evolution behaviors may be different depending on generation mechanism. If so, temperature fluctuations due to magnetic field may be shown.

In the early part of the next century, the new satellite experiments, MAP [21] and PLANCK [22], will be set forth with better accuracy than the COBE satellite. They are expected to detect the imprint of the polarization as well as the gravitational wave. If it is possible, we can constrain the magnetic field strength and the spectral index and be informed about the magnetic field generation mechanism.

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APPENDIX A: CALCULATION OF TRACELESS PART OF PRESSURE PERTURBATION

Here we calculate the Fourier transform of random magnetic fields with the assumption of the force-free magnetic field ($\mathbf{B}\times\nabla\times\mathbf{B}=0$) condition. From the perturbed Einstein equations, we can write the traceless part of pressure perturbations as follows in real space:

$$(\nabla^i\nabla_j - \tfrac{1}{3}\delta^i_j)(\Phi+\Psi) = -8\pi G a^2 \Pi^{(\mathrm{em})i}{}_j, \tag{A1}$$

where $\Pi^{(\mathrm{em})i}{}_j = \frac{1}{3}\delta^i_j B^2 - B^i B_j$. Using the Fourier transform of $\mathbf{B}(\mathbf{x})$, $\Pi^{(\mathrm{em})i}{}_j(\mathbf{x})$ can be written as

$$\Pi^{(\mathrm{em})i}{}_j(\mathbf{x}) = \int d^3k d^3k' [\tfrac{1}{3}\delta^i_j B^l(\mathbf{k}) B_l(\mathbf{k}-\mathbf{k}') - B^i(\mathbf{k}) B_j(\mathbf{k}-\mathbf{k}')] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A2}$$

where we omit the time dependence for brevity. Then we differentiate $\Pi^{(\mathrm{em})i}{}_j(\mathbf{x})$ with respect to x^i to get

$$\nabla_i \Pi^{(\mathrm{em})i}{}_j(\mathbf{x}) = \int d^3k d^3k' \, i k_i [\tfrac{1}{3}\delta^i_j B^l(\mathbf{k}) B_l(\mathbf{k}-\mathbf{k}') - B^i(\mathbf{k}) B_j(\mathbf{k}-\mathbf{k}')] e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{A3}$$

We can assume

$$\int d^3k' B^i(\mathbf{k}) B_j(\mathbf{k}-\mathbf{k}') = \left(A\delta^i_j + B\frac{k^i k_j}{k^2}\right) k^2 F(k), \tag{A4}$$

which is obvious from the fact that the tensorial component of scalar perturbations is split into the trace and traceless part.

Next, differentiating $B^i B_j$ with respect to x^i yields

$$\nabla_i (B^i B_j) = (\mathbf{B}\times\nabla\times\mathbf{B})_j + B^i \nabla_j B_i. \tag{A5}$$

The first part of the right-hand side is the magnetic force due to the current density ($\mathbf{J}=\nabla\times\mathbf{B}$), and we neglect this term assuming that the force-free condition is satisfied in the early universe. We take the Fourier transform of the $\Pi^{(\mathrm{em})i}{}_j$ and then again differentiate with respect to x^i using the force-free field condition to obtain

$$\nabla_i \Pi^{(\mathrm{em})i}{}_j(\mathbf{x}) = \int d^3k \int d^3k' \, i k_j \left[\frac{1}{3} B^l(\mathbf{k}') B_l(\mathbf{k}-\mathbf{k}') - \frac{1}{2} B^l(\mathbf{k}') B_l(\mathbf{k}-\mathbf{k}') \right] e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{A6}$$

Comparing Eqs. (A3) and (A6), we can find the relations

$$A+B = \frac{1}{2},$$

$$F(k) = \int d^3k' B^l(\mathbf{k}') B_l(\mathbf{k}-\mathbf{k}'). \tag{A7}$$

Then Eq. (A1) is written by

$$\int d^3k \left(\frac{1}{3}\delta^i_j - \frac{k^i k_j}{k^2}\right) k^2 [\Phi(\mathbf{k}) + \Psi(\mathbf{k})] e^{i\mathbf{k}\cdot\mathbf{x}} = -8\pi G a^2 \int d^3k \left(\frac{1}{3}\delta^i_j - A\delta^i_j - B\frac{k^i k_j}{k^2}\right) F(k) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{A8}$$

Further we only treat the scalar density perturbations and traceless component of pressure perturbations, so the right hand side should be proportional to $[\frac{1}{3}\delta^i_j-(k^i k_j/k^2)]$. Then we find that A must have value of $\frac{1}{4}$. Finally, we can write the traceless part for a given mode k,

$$k^2(\Phi+\Psi)=-8\pi G a^2(\tfrac{1}{4}F(k)). \qquad \text{(A9)}$$

APPENDIX B: SPIN-s HARMONICS

In this appendix we summarize the definition of spin-s function and the property of spin-s harmonics. We mainly refer to [28] and [33].

A function ${}_sf(\theta,\phi)$ defined on the sphere is said to have spin s if under a right-handed rotation of $(\widehat{\mathbf{e}_1},\widehat{\mathbf{e}_2})$ by an angle ψ it transforms as ${}_sf'(\theta,\phi)=e_s^{-is\psi}f(\theta,\phi)$. A spin-$s$ function can be expanded in spin-s spherical harmonics, ${}_sY_{lm}(\theta,\phi)$, which form a complete and orthonormal basis. The spin-s harmonics are expressed as

$$\begin{aligned}{}_sY_{lm}(\theta,\phi)&=e^{im\phi}\left[\frac{(l+m)!(l-m)!}{(l+s)!(l-s)!}\,\frac{2l+1}{4\pi}\right]^{1/2}\\&\times\sin^{2l}(\theta/2)\sum_r\binom{l-s}{r}\binom{l+s}{r+s-m}\\&\times(-1)^{l-r-s+m}\cot^{2r+s-m}(\theta/2). \qquad \text{(B1)}\end{aligned}$$

These set of functions satisfy the conjugation, completeness and orthogonality relations:

$${}_sY^*_{lm}=(-l)^{m+s}_{-s}Y_{l-m}, \qquad \text{(B2)}$$

$$\int_0^{2\pi}d\phi\int_{-1}^{1}d\cos\theta_sY^*_{l'm'}(\theta,\phi)_sY_{lm}(\theta,\phi)=\delta_{l'l}\delta_{m'm}, \qquad \text{(B3)}$$

$$\sum_{lm}{}_sY^*_{lm}(\theta,\phi)_sY_{lm}(\theta',\phi')=\delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta'). \qquad \text{(B4)}$$

Finally the harmonics are related to the ordinary spherical harmonics as

$$\begin{aligned}{}_{\pm}Y_{lm}=\left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\Big[&\partial_\theta^2-\cot\theta\partial_\theta\\&\pm\frac{2i}{\sin\theta}(\partial_\theta-\cot\theta)\partial_\phi-\frac{1}{\sin^2\theta}\partial_\phi^2\Big]Y_{lm}. \qquad \text{(B5)}\end{aligned}$$

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