General primordial cosmic perturbation

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We consider the most general primordial cosmological perturbation in a universe filled with photons, baryons, neutrinos, and a hypothetical cold dark matter (CDM) component within the framework of linearized perturbation theory. We present a careful discussion of the different allowed modes, distinguishing modes which are regular at early times, singular at early times, or pure gauge. As well as the familiar growing and decaying adiabatic modes and the baryonic and CDM isocurvature modes, we identify two *neutrino isocurvature* modes. In the first, the ratio of neutrinos to photons varies spatially but the net density perturbation vanishes. In the second the photon-baryon plasma and the neutrino fluid have a spatially varying relative bulk velocity balanced so that the net momentum density vanishes. Possible mechanisms which could generate the two neutrino isocurvature modes are discussed. If one allows the most general regular primordial perturbation, all quadratic correlators of observables such as the microwave background anisotropy and matter perturbations are completely determined by a 5×5, real, symmetric matrix-valued function of comoving wave number. In a companion paper we examine prospects for detecting or constraining the amplitudes of the most general allowed regular perturbations using present and future CMB data.

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I. INTRODUCTION

A key challenge of modern cosmology is understanding the nature of the primordial fluctuations that eventually led to the formation of large scale structure in our universe. One possibility is that the fluctuations were generated during a period of inflation prior to the radiation dominated era of the hot big bang. As inflation ended the fluctuations would then have been imprinted as initial conditions for the radiation era on scales far beyond the Hubble radius. The second possibility is that the structure was generated through some causal mechanism operating within the standard big bang radiation and matter eras. In this paper we focus on the first option, that the fluctuations were imprinted early in the radiation era as linear fluctuations in the metric and in the matter and radiation content.

For several good reasons the possibility that the primordial perturbations were adiabatic has been the focus of most interest to date. If the relative abundances of different particle species were determined directly from the Lagrangian describing local physics, one would expect those abundance ratios to be spatially constant because all regions of the universe would share an identical early history, independent of the long wavelength perturbations. The stress-energy present in the universe would then be characterized on large scales by a single, spatially uniform equation of state. Such fluctuations are termed adiabatic and are the simplest possibility for perturbing the matter content and the geometry of the universe. They are also naturally predicted by the simplest inflationary models [1], although there also exist more complex models giving other types of perturbations [2]. For a recent discussion see Ref. [3].

Nevertheless, there is no *a priori* reason why the situation could not be more complicated with abundance ratios varying from place to place. Perturbations of this sort are known as *isocurvature*, or sometimes *entropy*, perturbations. Most studies of isocurvature perturbations have examined the possibility that the primordial perturbations were entirely isocurvature with a vanishing primordial adiabatic component and have sought to explore whether such pure isocurvature models could explain the observed structure in the universe [4–7].

In this paper we adopt a more phenomenological approach, which we believe is now warranted by the prospect of upcoming precision measurements of the cosmic microwave background (CMB) anisotropy on small scales. Ground based and balloon borne telescopes and the Microwave Anisotropy Probe (MAP) and Planck satellites will provide very detailed measurements of the primordial fluctuations [8,9]. Most work on how to interpret this data has focused on parameter estimation starting from the assumption that the initial perturbations were generated by an inflationary model specified by a small number of undetermined free parameters [10]. Adiabatic perturbations, characterized by an amplitude and spectral index, as well as tensor fluctuations also characterized by two parameters are usually assumed, and based on these assumptions a host of cosmological parameters, such as Ω_{total}, Ω_b, Ω_Λ, h, N_ν, are to be inferred from the observed CMB multipole moments. While the prospects of such measurements are beguiling, they rely heavily on assumptions regarding the form of the primordial perturbations. We feel that those assumptions are worth checking against the data using an approach that assumes neither inflation nor any other favorite theoretical model.

For learning about the fundamental physics responsible for structure formation, determining whether the primordial fluctuations were in fact Gaussian and adiabatic is at least as

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important and interesting as measuring the values of cosmological parameters. For this purpose the relevant question is not whether primordial isocurvature perturbations offer a viable alternative to adiabatic perturbations, which has been the focus of prior work, but rather how and to what extent can observations constrain the presence of isocurvature modes. Rather than considering a particular isocurvature model, it seems appropriate to consider the most general primordial perturbation possible without adding new physics to the hot big bang era. In other words, we assume a universe filled with neutrinos, photons, leptons and baryons, and a cold dark matter (CDM) component and then try to place constraints on the amplitudes of all possible perturbation modes.

As mentioned above, we shall need to make a simplifying assumption to limit the parameter space of possible perturbations to reasonable size. We shall assume that the fluctuations were indeed primordial—that is, that the perturbation modes were excited by physics operating at a very early epoch preceding the hot radiation dominated era and that no new dynamics influenced the perturbations at late times. Here ''late'' means well before decoupling, so that whatever decaying modes may have been excited earlier had a chance to decay before influencing the observed structure in our cosmic microwave sky. This assumption excludes cosmic defect models (i.e., strings, global textures, etc.) [11] of structure formation where a detailed understanding of the dynamics of the order parameter field is required to determine the unequal time correlations at late times [12,13].

How should the most general perturbation be characterized? This question has historically generated some confusion in the literature. The standard terminology refers to ''growing,'' ''decaying,'' and ''gauge'' modes, the latter referring to modes affected by general coordinate transformations preserving the chosen gauge condition. In this paper as in most work on cosmological perturbations we shall use synchronous gauge. The variables in other common gauges (e.g., Newtonian gauge) are merely linear combinations of the synchronous gauge variables and their time derivatives. In the synchronous gauge, the two gauge modes for scalar perturbations are easily identified. The first corresponds to time-independent spatial reparametrizations of the constant time hypersurfaces. In this mode the metric perturbation is constant and the matter is unperturbed. The second corresponds to deformations of these hypersurfaces through a spatially dependent shift in the time direction. For the latter gauge mode, which diverges at early times, both the matter variables and the metric are perturbed. Indeed this mode may be regarded as a shift in the time of the big bang singularity. The remaining modes are then defined modulo the two gauge modes.

We characterize the remaining modes as either ''regular'' or ''singular,'' according to their behavior as the time since the big bang tends to zero. This terminology is preferable to the standard one because it includes constant modes such as the neutrino velocity mode we shall discuss. In this paper we shall treat only the ''regular'' modes (i.e., those regular up to the gauge modes). There are several reasons for this. Singular modes are necessarily decaying as one proceeds forward in time away from the big bang, so if they are present at some very early time with small amplitude so that a perturbative treatment is valid, they quickly become irrelevant. A second reason is that even if the perturbation amplitude is small, the perfect fluid approximation breaks down at early times for the singular modes. Higher moments in the Boltzmann hierarchy become progressively more important as one goes back in time, and specifying the initial conditions involves specifying an infinite number of constants. The perfect fluid approximation seems essential to a simple specification of primordial initial perturbations. Of course, decaying modes might be produced by some late time physics (such as cosmic defects) operating well after the big bang, but our point here is that it would be very difficult to characterize them without introducing explicit source terms. In contrast, the ''regular'' modes are completely characterized by specifying the leading terms in a power series expansion in conformal time of the low moments of the phase space density—that is, the density and velocity of the fluids.

Having characterized the regular perturbations by the leading terms in the power series for the metric and fluid densities and velocities, we note that any quadratic observable (e.g., the matter power spectrum or the cosmic microwave anisotropy power spectrum) is then completely determined by a primordial power spectral matrix, which rather than a single function of k as it would be for growing mode adiabatic perturbations is a 5×5, real, symmetric matrix function of k. The off-diagonal elements establish correlations between the modes. As long as only quadratic observables are considered, no assumption of Gaussianity is required.

The counting arises as follows. Each cosmological fluid is described by two first order equations, so each fluid introduces two new perturbation modes. However, in the synchronous gauge, as mentioned, there is one gauge mode affecting the fluid perturbations. For a single fluid there is therefore just one regular, growing mode perturbation, and no physical (i.e., non-gauge) decaying mode. A convenient way to deal with the gauge mode is to identify it with the velocity of the cold dark matter. By a coordinate choice this may be chosen initially to be zero. If it is assumed that the cold dark matter couples to the other fluids only via gravity, there is no scattering term to consider, and with coordinates chosen so that the velocity is initially zero, it remains so for all times. If we now introduce photons and baryons, four new modes arise. The first is an adiabatic decaying mode. There are also the baryon isocurvature and cold dark matter isocurvature modes, where the initial conditions contain equal and opposite perturbations of the photon density and the baryon or cold dark matter densities, respectively. The fourth new mode is that in which the photon and baryon fluids have a relative velocity that diverges at early times and the fluid approximation breaks down. So far we have three regular modes. Let us now introduce neutrinos into the picture. We shall imagine we are setting up the perturbations after neutrino decoupling ($T\sim$MeV, $t\sim$ seconds). For our purposes there is no distinction between the different neutrino species or between neutrinos and antineutrinos, since we are only interested in how perturbations in the neutrino fluid affect cosmological observations of the density and microwave background today. Two new perturbation modes are introduced, the first a neutrino isocurvature density perturbation and the second a neutrino isocurvature velocity perturbation. In the latter, we can arrange the neutrino and photon-baryon fluids to have equal and opposite momentum density. In the approximation that we ignore the collision term coupling neutrinos to the photon-baryon fluid, which is valid after neutrino decoupling, we shall show there is no divergence in this mode at early times.

The neutrino isocurvature velocity perturbation may be considered as primordial as long as one takes primordial to mean ''generated after one second'' but well before photon-baryon decoupling. Of course similar remarks apply to the cold dark matter, baryon isocurvature or neutrino isocurvature perturbations. Namely, if we go back far enough in cosmic history, where the various conservation laws for baryon or lepton number break down or where the cold dark matter was initially generated or reached thermal equilibrium, then a description of the form used here would also break down. Since one second is still quite early, and certainly well before photon-baryon decoupling, we regard it admissible to consider the possibility of ''primordial'' neutrino velocity perturbations. Possible mechanisms for generating shall be discussed in a separate paper [28].

A primeval baryon isocurvature (PBI) model was introduced by Peebles [4] in which a universe with just baryons, radiation, and neutrinos is assumed and primordial perturbations in the baryon-to-photon ratio are assumed. Since at early times the baryons contribute negligibly to the total density, such perturbations lack an adiabatic, or curvature, component at early times. Compared to an $\Omega=1$ CDM model, the PBI model gives (1) lower small-scale relative peculiar velocities, (2) greater large-scale flow velocities, (3) earlier re-ionization, and (4) earlier galaxy and star formation [6]. The consequences of the PBI model and comparison with observations have been studied by a number of authors [15–20,22].

Bond and Efstathiou [7,5] have considered a CDM isocurvature model in which the CDM-to-photon ratio varies spatially. A possible mechanism for generating such perturbations arises in models with axion dark matter in which scale invariant fluctuations of the axion field $A(x)$ are converted into density fluctuations after the axion field acquires a mass in the QCD phase transition [21]. Under the assumption that quantum fluctuations during inflation impart a scale free spectrum of fluctuations to the axion field, so that $\delta A(k) \sim k^{-3/2}$, and $\delta A(k) \ll \bar{A}$, the mean axion field displacement from the vacuum, a scale-free spectrum of Gaussian fluctuations in the axion-to-photon ratio is generated. The resulting spectrum of density fluctuations today has the same power law on large scales as for adiabatic fluctuations with a scale free spectrum $P_\rho(k) \sim k^1$, but compared to adiabatic, scale-free perturbations the turnover to $P_\rho(k) \sim k^{-3}$ power law behavior on small scales occurs on a larger scale for the isocurvature case. For the amplitude of density perturbations normalized on small scales (for example, using σ_8), this gives about 30 times more power in the matter perturbations on large scales and entails an excessive CMB anisotropy on large angles. This work has been extended to consider mixtures of CDM isocurvature and adiabatic fluctuations, low density universes, and the predictions compared with more recent CMB data [23,24].

The neutrino isocurvature modes discussed here allow for a spatially varying relative density of photons and neutrinos and a relative velocity between the photon and neutrino components as well. For the neutrino isocurvature density mode, the total density perturbation vanishes but the relative density of neutrinos and photons varies spatially. On superhorizon scales, the neutrinos and photons evolve similarly but upon entering the horizon the neutrinos free stream, developing nonvanishing higher moments of the neutrino phase space density $F_{\nu l}(k)$, while because of Thomson scattering the photons continue to behave much as a perfect fluid. These distinct behaviors subsequently generate perturbations in the total density. For the neutrino isocurvature velocity mode, the rest frames of the neutrino and the photon fluids do not coincide. The relative velocities are such that initially the perturbation in the total momentum density vanishes. If this last condition were not satisfied, the metric perturbation generated by this mode would diverge at early times, rendering the mode a singular mode, which according to the discussion above should be excluded. But the lack of divergence owing to this cancellation makes this an admissible regular perturbation mode. The neutrino isocurvature mode solutions are implicit in the work of Rebhan and Schwarz [24] and of Challinor and Lasenby [25], but their implications were not explored.

A possible obstacle to exciting neutrino isocurvature modes arises at early times from processes in which neutrinos are generated or scattered. If the neutrino chemical potentials vanish so that there are in each generation precisely as many neutrinos as antineutrinos, a spatially varying relative density can only be established at a temperature sufficiently low that the processes turning photons into neutrino-antineutrino pairs and vice versa had already been frozen out—that is, below a few MeV. Nonvanishing chemical potentials for the various neutrino species can protect variations in the ratio (ρ_ν/ρ_γ) from erasure by processes involving $\nu\bar{\nu}$ annihilation, and observational constraints that would rule out neutrino chemical potentials of the required magnitude are lacking. L-violating processes mediated through sphalerons, unsuppressed at temperatures above the electroweak phase transition, can readjust the neutrino chemical potentials μ_{ν_e}, μ_{ν_μ}, and μ_{ν_τ} and, moreover, can convert lepton asymmetries in the neutrino sector into baryon asymmetry, but to the extent that the neutrino chemical potentials satisfy $\mu_{\nu_e}+\mu_{\nu_\mu}+\mu_{\nu_\tau}=0$, no tendency favoring sphalerons over anti-sphalerons is introduced and the net effect of these electroweak processes vanishes. Recall that the neutrino overdensity is proportional to ${\mu_{\nu_e}}^2+{\mu_{\nu_\mu}}^2+{\mu_{\nu_\tau}}^2$. Similarly, for the neutrino isocurvature velocity mode, scattering of neutrinos with other components dampens this mode at very early times, and for this mode to be relevant there must exist a process capable of exciting it after this dampening effect has frozen out.

There exist many candidate mechanisms that might generate these neutrino isocurvature modes. The neutrino density isocurvature mode could be generated during inflation if the theory included a light scalar field carrying lepton number, with mass much smaller than the Hubble constant during inflation [26]. During inflation this field would be excited but would contribute negligibly to the density of the universe. After inflation, when the Hubble constant fell below the mass of the scalar field, it would oscillate and decay, producing a lepton asymmetry. We would expect the neutrino chemical potential to be proportional to the scalar field value, so the most natural possibility would be Gaussian, scale invariant perturbations in the neutrino chemical potential, with the density perturbation in the neutrinos being proportional to the square of the chemical potential.

The neutrino velocity mode could have been excited by the decay of relics such as cosmic strings, walls, or superstrings, surviving until after the neutrinos had decoupled and then decaying into neutrinos. The neutrino fluid produced in such processes would be perturbed both in its density and velocity, and these perturbations would be isocurvature in character [28]. Another possibility for exciting the modes arises from magnetic fields frozen into the plasma whose stress gradients impart a velocity to the photon-lepton-baryon plasma but not to the neutrino component.

II. IDENTIFYING THE MODES

We now turn to identifying the possible perturbation modes using synchronous gauge with the line element

$$ds^2=a^2(\tau)[-d\tau^2+(\delta_{ij}+h_{ij})dx^i dx^j]. \tag{1}$$

For spatial dependence of a given wave number $\mathbf{k}$, we define

$$h_{ij}(\mathbf{k},\tau)=e^{i\mathbf{k}\cdot\mathbf{x}}\left[\hat{k}_i\hat{k}_j h(\mathbf{k},\tau)+\left(\hat{k}_i\hat{k}_j-\frac{1}{3}\delta_{ij}\right)6\,\eta(\mathbf{k},\tau)\right]. \tag{2}$$

Our discussion generally follows the notation of Ref. [27].

The linearized Einstein equations are

$$k^2\eta-\frac{1}{2}\mathcal{H}\dot{h}=(4\pi Ga^2)\,\delta T^0{}_0,$$

$$k^2\dot{\eta}=(4\pi Ga^2)(\overline{\rho}+\overline{p})\,\theta,$$

$$\ddot{h}+2\mathcal{H}\dot{h}-2k^2\eta=-(8\pi Ga^2)\,\delta T^i{}_i,$$

$$\ddot{h}+6\,\ddot{\eta}+2\mathcal{H}(\dot{h}+6\,\dot{\eta})-2k^2\eta=-(24\pi Ga^2)(\overline{\rho}+\overline{p})\sigma \tag{3}$$

where we define $(\overline{\rho}+\overline{p})\theta=ik^jT^0{}_j$ and $(\overline{\rho}+\overline{p})\sigma=-(\hat{k}_i\hat{k}_j-\frac{1}{3}\delta_{ij})T_i{}^j$. We define $\mathcal{H}=\dot{a}/a$. With only a single fluid, θ is simply the divergence $(\nabla\cdot\mathbf{v})$.

Assuming N fluids, labeled $(J=1,\ldots,N)$, one may rewrite the above as

$$k^2\eta-\frac{1}{2}\mathcal{H}\dot{h}=-\frac{3}{2}\mathcal{H}^2\sum_J\,\Omega_J\delta_J,$$

$$k^2\dot{\eta}=\frac{3}{2}\mathcal{H}^2\sum_J\,\Omega_J(1+w_J)\theta_J,$$

$$\ddot{h}+2\mathcal{H}\dot{h}-2k^2\eta=-9\mathcal{H}^2\sum_J\,\Omega_J c_{sJ}^2\delta_J,$$

$$\ddot{h}+6\,\ddot{\eta}+2\mathcal{H}(\dot{h}+6\,\dot{\eta})-2k^2\eta=-12\mathcal{H}^2\Omega_\nu\sigma_\nu, \tag{4}$$

where $w_J=p_J/\rho_J$ and $c_s^2=\partial p_J/\partial\rho_J$. In the last equation only the neutrino contribution to the anisotropic stress is included. At early times only the neutrino contribution is relevant, but later when the photons and baryons begin to decouple the photon quadrupole moment must also be included.

For the photons, the equations of motion at early times before the baryonic component of the fluid was significant are

$$\dot{\delta}_\gamma+\frac{4}{3}\theta_\gamma+\frac{2}{3}\dot{h}=0,$$

$$\dot{\theta}_\gamma-\frac{1}{4}k^2\delta_\gamma=0. \tag{5}$$

Similarly, for the neutrinos, after neutrino decoupling at temperatures of $\sim$1 MeV, we have

$$\dot{\delta}_\nu+\frac{4}{3}\theta_\nu+\frac{2}{3}\dot{h}=0,$$

$$\dot{\theta}_\nu-\frac{1}{4}k^2\delta_\nu+k^2\sigma_\nu=0,$$

$$\dot{\sigma}_\nu=\frac{2}{15}[2\,\theta_\nu+\dot{h}+6\,\dot{\eta}]-\frac{3}{10}kF_{\nu3}, \tag{6}$$

where $\sigma_\nu=F_{\nu2}/2$ is the quadrupole moment of the neutrino phase space density and $F_{\nu l}$ is the lth multipole, in the notation used below.

The photon and neutrino evolution equations above differ only in the presence of an anisotropic stress term σ_ν. Photons, because of their frequent scattering by charged leptons and baryons, at early times are unable to develop a quadrupole moment in their velocity distribution. Neutrinos, on the other hand, develop significant anisotropic stresses upon crossing the horizon. The addition of an extra degree of freedom that would result from the equation for $\dot{\sigma}_\nu$ is avoided by setting $\sigma_\nu=0$ at $\tau=0$.[1]

[1]If we were to consider the physical decaying mode with $\eta\propto\tau^{-1}$, we would find this condition cannot be satisfied. As discussed above, decaying modes are inevitably associated with the breakdown of the fluid approximation at early times.

For a CDM component, the equations of motion are

$$\dot{\delta}_c+\theta_c+\frac{1}{2}\dot{h}=0,$$

$$\dot{\theta}_c+\mathcal{H}\theta_c=0. \tag{7}$$

At later times, when Ω_b becomes comparable to Ω_γ, the photon equations of motion are modified as follows:

$$\dot{\delta}_\gamma+\frac{4}{3}\theta_\gamma+\frac{2}{3}\dot{h}=0,$$

$$\dot{\theta}_\gamma-\frac{k^2}{4}\delta_\gamma+an_e\sigma_T(\theta_\gamma-\theta_b)=0. \tag{8}$$

Here a is the scale factor, n_e is the electron density, and σ_T is the Thomson scattering cross section. The baryon equations of motion are

$$\dot{\delta}_b+\theta_b+\frac{1}{2}\dot{h}=0,$$

$$\dot{\theta}_b+\mathcal{H}\theta_b+\frac{4}{3}\frac{\Omega_\gamma}{\Omega_b}an_e\sigma_T(\theta_b-\theta_\gamma)=0. \tag{9}$$

At early times, the characteristic time for the synchronization of the photon and baryon velocities $t_{b\gamma}\approx 1/(n_e\sigma_T)$ is small compared to the expansion time $t_{exp}\approx a\tau$ and to the oscillation period $t_{osc}\approx(a\tau/k)$. (We use units where the speed of light $c=1$.) Any deviation of $(\theta_\gamma-\theta_b)$ from zero rapidly decays away. This may be seen by subtracting the second of Eqs. (9) from the second of Eqs. (8) and regarding $\mathcal{H}\theta_b+\frac{1}{3}k^2\delta_\gamma$ as a forcing term. In the limit $\sigma_T\to\infty$, one obtains $\theta_b=\theta_\gamma$—in other words, a tight coupling between the baryons and the photons. Therefore, at early times we may set $\theta_b=\theta_\gamma=\theta_{\gamma b}$.

In the tight coupling approximation the evolution equation for $\theta_{\gamma b}$ is obtained by adding the second equation of Eqs. (8) multiplied by $\frac{4}{3}\Omega_\gamma$ to the second equation of Eqs. (9) multiplied by Ω_b, so that the scattering terms cancel, giving

$$\left(\frac{4}{3}\Omega_\gamma+\Omega_b\right)\dot{\theta}_{\gamma b}=-\Omega_b\mathcal{H}\theta_{\gamma b}+\frac{1}{3}\Omega_\gamma k^2\delta_\gamma \tag{10}$$

and the baryon and photon density contrasts evolve according to

$$\dot{\delta}_b=-\theta_{\gamma b}-\frac{1}{2}\dot{h},$$

$$\dot{\delta}_\gamma=-\frac{4}{3}\theta_{\gamma b}-\frac{2}{3}\dot{h}. \tag{11}$$

In the absence of baryon isocurvature perturbations, $\delta_\gamma=\frac{4}{3}\delta_b$, making one of these equations redundant, but both equations are required for the most general type of perturbation.

Although an excellent approximation early on, the tight coupling assumption breaks down at later times as the photons and baryons decouple, and for a more accurate description the two equations in Eqs. (8) for the time derivatives of the monopole and dipole moments of the velocity of the photon phase space distribution must be replaced by the following infinite hierarchy of equations for the time derivatives of higher order moments of the photon distribution function as well:

$$\dot{\delta}_\gamma=-\frac{4}{3}\theta_\gamma-\frac{2}{3}\dot{h},$$

$$\dot{\theta}_\gamma=k^2\left(\frac{\delta_\gamma}{4}-\frac{F_{\gamma 2}}{2}\right)+an_e\sigma_T(\theta_b-\theta_\gamma),$$

$$\dot{F}_{\gamma 2}=\frac{8}{15}\theta_\gamma-\frac{3}{5}kF_{\gamma 3}+\frac{4}{15}\dot{h}+\frac{8}{5}\dot{\eta}-\frac{9}{10}an_e\sigma_T F_{\gamma 2},$$

$$\dot{F}_{\gamma l}=\frac{k}{2l+1}[lF_{\gamma(l-1)}-(l+1)F_{\gamma(l+1)}]-an_e\sigma_T F_{\gamma l} \tag{12}$$

where $l\geqslant 3$ for the last equation. Initially, as $\tau\to 0$, $F_{\gamma l}=0$ for $l\geqslant 2$. If this condition were relaxed, an infinite number of decaying modes would appear. The fluid approximation for the photons, Eqs. (8), is obtained using only the first two of the above equations combined with the approximation $F_{\gamma 2}=0$.

To describe the neutrinos during and after horizon crossing requires a Boltzman hierarchy for δ_ν, θ_ν, $F_{\nu l}$ identical to the one above except with the Thomson scattering terms omitted. This assumes that neutrino masses are irrelevant.

Before solving the Einstein equations (4), we first identify the two gauge modes resulting from the residual gauge freedom remaining within synchronous gauge. A general infinitesimal gauge transformation considered to linear order is a coordinate transformation $x^\mu\to x'^\mu(x)$ where $x^\mu=x'^\mu+\epsilon^\mu(x)$. From the transformation rule for tensors

$$g'_{\mu\nu}(x')=\frac{\partial x^\xi}{\partial x'^\mu}\frac{\partial x^\eta}{\partial x'^\nu}g_{\xi\eta}(x), \tag{13}$$

it follows that

$$\delta g_{\mu\nu}=\epsilon^\xi\frac{\partial g^{(0)}_{\mu\nu}}{\partial x^\xi}+\epsilon^\xi{}_{,\mu}g^{(0)}_{\xi\nu}+g^{(0)}_{\mu\xi}\epsilon^\xi{}_{,\nu} \tag{14}$$

where $g^{(0)}_{\mu\nu}$ is the unperturbed, zeroth-order metric. Applying Eq. (14) to the metric $ds^2=a^2(\tau)[-(1-h_{00})d\tau^2+2h_{0i}d\tau dx^i+(\delta_{ij}+h_{ij})dx^i dx^j]$, where

$$\tau=\tau'+T(\mathbf{x},\tau),\quad \mathbf{x}=\mathbf{x}'+\mathbf{S}(\mathbf{x},\tau) \tag{15}$$

with S and T regarded as infinitesimal, one obtains

$$\delta h_{00}=-\frac{2\dot{a}}{a}T(\mathbf{x},\tau)-2\dot{T}(\mathbf{x},\tau),$$

$$\delta h_{0i} = -T_{,i}(\mathbf{x},\tau) + \dot{S}_i(\mathbf{x},\tau),$$

$$\delta h_{ij} = S_{i,j} + S_{j,i} + \frac{2\dot{a}}{a}\delta_{ij}T. \quad (16)$$

For the density contrast for a perfect fluid component labeled by α, one obtains $\delta(\delta_\alpha) = -3(1+w_\alpha)\mathcal{H}T$. After a gauge transformation the velocity is shifted by $\delta\mathbf{v} = -\dot{S}(\mathbf{x},\tau)$ for all components.

The synchronous gauge condition $\delta h_{00}=0$ implies that $T(\mathbf{x},\tau) = A(\mathbf{x})/a(\tau)$, and $\delta h_{0i}=0$ implies that

$$\mathbf{S}(\mathbf{x},\tau) = B(\mathbf{x}) + \nabla A(\mathbf{x}) \int \frac{d\tau}{a(\tau)}. \quad (17)$$

Therefore

$$\delta h_{ij} = \frac{2\dot{a}}{a^2}A(\mathbf{x})\delta_{ij} + 2A_{,ij}\int \frac{d\tau}{a(\tau)} + B_{i,j} + B_{j,i}. \quad (18)$$

For a radiation-dominated universe [with $a(\tau)=\tau$], considering for the moment only a single wave number, we find that[2]

$$\delta h_{ij} = \left[\left(\frac{2}{\tau^2}\delta_{ij} - 2k_ik_j \ln(\tau)\right)A(\mathbf{k}) + 2Bk_ik_j\right]e^{i\mathbf{k}\cdot\mathbf{x}} \quad (19)$$

or

$$h = A\left[\frac{6}{\tau^2} - 2k^2\ln(\tau)\right] + 2B, \quad \eta = A\left[\frac{-1}{\tau^2}\right]. \quad (20)$$

To obtain a more accurate power series expansion close to the time of matter-radiation equality and to consider the baryon and CDM isocurvature modes, we assume a scale factor evolution for a universe filled with matter and radiation: $a(\tau) = \tau + \tau^2$. At matter-radiation equality $a_{eq} = 1/4$ and $\tau_{eq} = (\sqrt{2}-1)/2$. Since $\mathcal{H} = (2\tau+1)/(\tau^2+\tau)$ has a pole at $\tau = -1$, the resulting power series solutions are expected to diverge beyond the unit circle.

For the matter-radiation universe, with $a(\tau) = \tau + \tau^2$, Eq. (20) is modified to become

$$h = A\left[\frac{6(1+2\tau)}{\tau^2(1+\tau)^2} - 2k^2\ln\left(\frac{\tau}{1+\tau}\right)\right] + 2B,$$

$$\eta = -A\left[\frac{(1+2\tau)}{\tau^2(1+\tau)^2}\right]. \quad (21)$$

We now present the power series solutions. R_γ and R_ν represent the fractional contribution of photons and neutrinos to the total density at early times deep within the radiation dominated epoch. (We ignore possible effects due to nonvanishing neutrino masses.) We also assume that we are considering the perturbations after the annihilation of electrons and positrons, so that the latter have dumped their energy into the photon background. For N_ν species of massless neutrinos we define $R = \frac{7}{8}N_\nu(\frac{4}{11})^{4/3}$ and we have $R_\gamma = (1+R)^{-1}$ and $R_\nu = R(1+R)^{-1}$.

Adiabatic growing mode:

$$h = \frac{1}{2}k^2\tau^2,$$

$$\eta = 1 - \frac{5+4R_\nu}{12(15+4R_\nu)}k^2\tau^2,$$

$$\delta_c = -\frac{1}{4}k^2\tau^2,$$

$$\delta_b = -\frac{1}{4}k^2\tau^2,$$

$$\delta_\gamma = -\frac{1}{3}k^2\tau^2,$$

$$\delta_\nu = -\frac{1}{3}k^2\tau^2,$$

$$\theta_c = 0,$$

$$\theta_{\gamma b} = -\frac{1}{36}k^4\tau^3,$$

$$\theta_\nu = -\frac{1}{36}\left[\frac{23+4R_\nu}{15+4R_\nu}\right]k^4\tau^3,$$

$$\sigma_\nu = \frac{2}{3(12+R_\nu)}k^2\tau^2. \quad (22)$$

Baryon isocurvature mode:

$$h = 4\Omega_{b,0}\tau - 6\Omega_{b,0}\tau^2,$$

$$\eta = -\frac{2}{3}\Omega_{b,0}\tau + \Omega_{b,0}\tau^2,$$

$$\delta_c = -2\Omega_{b,0}\tau + 3\Omega_{b,0}\tau^2,$$

$$\delta_b = 1 - 2\Omega_{b,0}\tau + 3\Omega_{b,0}\tau^2,$$

$$\delta_\gamma = -\frac{8}{3}\Omega_{b,0}\tau + 4\Omega_{b,0}\tau^2,$$

$$\delta_\nu = -\frac{8}{3}\Omega_{b,0}\tau + 4\Omega_{b,0}\tau^2,$$

$$\theta_c = 0,$$

$$\theta_{\gamma b} = -\frac{1}{3}\Omega_{b,0}k^2\tau^2,$$

[2]Note that the gauge mode A disagrees with Eqs. (94) and (95) of Ref. [27], which are incorrect.

$$\theta_\nu = -\frac{1}{3}\Omega_{b,0}k^2\tau^2,$$

$$\sigma_\nu = \frac{-2\Omega_{b,0}}{3(2R_\nu+15)}k^2\tau^3. \tag{23}$$

There is no regular baryon velocity mode because of the tight coupling of the baryons to the photons.

CDM isocurvature mode:

$$h = 4\Omega_{c,0}\tau - 6\Omega_{c,0}\tau^2,$$

$$\eta = -\frac{2}{3}\Omega_{c,0}\tau + \Omega_{c,0}\tau^2,$$

$$\delta_c = 1 - 2\Omega_{c,0}\tau + 3\Omega_{c,0}\tau^2,$$

$$\delta_b = -2\Omega_{c,0}\tau + 3\Omega_{c,0}\tau^2,$$

$$\delta_\gamma = -\frac{8}{3}\Omega_{c,0}\tau + 4\Omega_{c,0}\tau^2,$$

$$\delta_\nu = -\frac{8}{3}\Omega_{c,0}\tau + 4\Omega_{c,0}\tau^2,$$

$$\theta_c = 0,$$

$$\theta_{\gamma b} = -\frac{1}{3}\Omega_{c,0}k^2\tau^2,$$

$$\theta_\nu = -\frac{1}{3}\Omega_{c,0}k^2\tau^2,$$

$$\sigma_\nu = \frac{-2\Omega_{c,0}}{3(2R_\nu+15)}k^2\tau^3. \tag{24}$$

The CDM velocity mode may be identified with the gauge mode. This can be seen from the equation for the CDM velocity, which is decoupled from the other equations. θ_{CDM} behaves as τ^{-1} at early times. If there were several species of CDM, however, a new physical, non-gauge relative velocity mode would arise, which would be divergent at early times.

Neutrino isocurvature density mode:

$$h = \frac{\Omega_{b,0}R_\nu}{10R_\gamma}k^2\tau^3,$$

$$\eta = -\frac{R_\nu}{6(15+4R_\nu)}k^2\tau^2,$$

$$\delta_c = \frac{-\Omega_{b,0}R_\nu}{20R_\gamma}k^2\tau^3,$$

$$\delta_b = \frac{1}{8}\frac{R_\nu}{R_\gamma}k^2\tau^2,$$

$$\delta_\gamma = -\frac{R_\nu}{R_\gamma} + \frac{1}{6}\frac{R_\nu}{R_\gamma}k^2\tau^2,$$

$$\delta_\nu = 1 - \frac{1}{6}k^2\tau^2,$$

$$\theta_c = 0,$$

$$\theta_{\gamma b} = -\frac{1}{4}\frac{R_\nu}{R_\gamma}k^2\tau + \frac{3\Omega_{b,0}R_\nu}{4R_\gamma^2}k^2\tau^2,$$

$$\theta_\nu = \frac{1}{4}k^2\tau,$$

$$\sigma_\nu = \frac{1}{2(15+4R_\nu)}k^2\tau^2. \tag{25}$$

Physically, one starts with a uniform energy density, with the sum of the neutrino and photon densities unperturbed. When a mode enters the horizon, the photons behave as a perfect fluid while the neutrinos free stream, creating nonuniformity in the energy density, pressure, and momentum density, in turn generating metric perturbations.

Neutrino isocurvature velocity mode:

$$h = \frac{3}{2}\Omega_{b,0}\frac{R_\nu}{R_\gamma}k\tau^2,$$

$$\eta = -\frac{4R_\nu}{3(5+4R_\nu)}k\tau + \left(\frac{-\Omega_{b,0}R_\nu}{4R_\gamma} + \frac{20R_\nu}{(5+4R_\nu)(15+4R_\nu)}\right)k\tau^2,$$

$$\delta_c = -\frac{3\Omega_{b,0}}{4}\frac{R_\nu}{R_\gamma}k\tau^2,$$

$$\delta_b = \frac{R_\nu}{R_\gamma}k\tau - \frac{3\Omega_{b,0}R_\nu(R_\gamma+2)}{4R_\gamma^2}k\tau^2,$$

$$\delta_\gamma = \frac{4}{3}\frac{R_\nu}{R_\gamma}k\tau - \frac{\Omega_{b,0}R_\nu(R_\gamma+2)}{R_\gamma^2}k\tau^2,$$

$$\delta_\nu = -\frac{4}{3}k\tau - \frac{\Omega_{b,0}R_\nu}{R_\gamma}k\tau^2,$$

$$\theta_c = 0,$$

$$\theta_{\gamma b} = -\frac{R_\nu}{R_\gamma}k + \frac{3\Omega_{b,0}R_\nu}{R_\gamma^2}k\tau + \frac{R_\nu}{R_\gamma}\left(\frac{3\Omega_{b,0}}{R_\gamma} - \frac{9\Omega_{b,0}{}^2}{R_\gamma^2}\right)k\tau^2 + \frac{R_\nu}{6R_\gamma}k^3\tau^2,$$

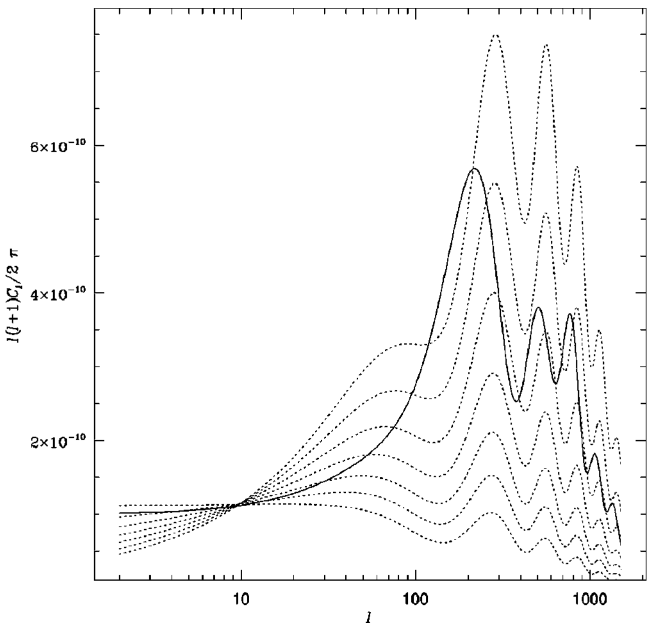

FIG. 1. CMB anisotropy for the neutrino isocurvature density mode. We plot $l(l+1)c_l/2\pi$ for the neutrino isocurvature density mode (dashed curves) for initial power spectra $P_{\delta_\nu}\sim k^\alpha$ where $\alpha=-3.0,\ldots,-2.4$, increasing in increments of 0.1 from bottom to top at large l. The adiabatic growing mode (solid curve) with a scale-invariant spectrum is included for comparison. All curves are normalized to COBE. For the lowest curve the variations in the photon-to-neutrino ratio obey a scale-invariant initial power spectrum.

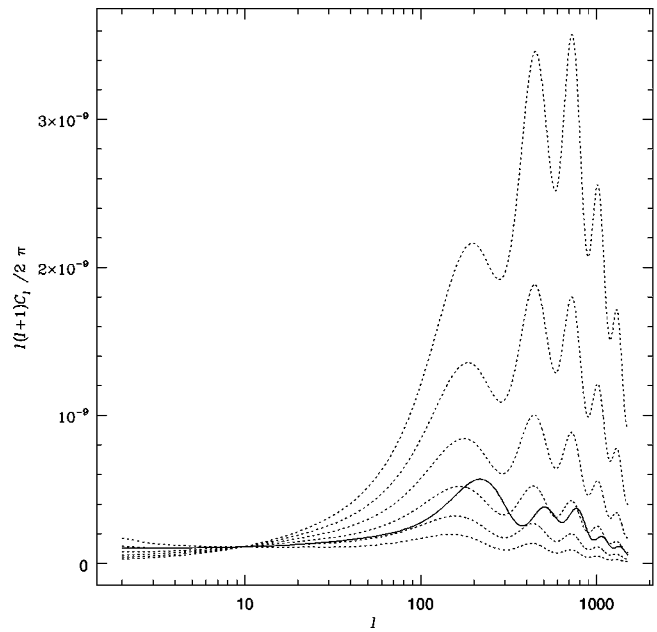

FIG. 2. CMB anisotropy for the neutrino isocurvature velocity mode. We plot $l(l+1)c_l/2\pi$ for the neutrino isocurvature velocity mode (dashed curves) for initial power spectra $P_{v_\nu}\sim k^\alpha$ with $\alpha=-3.0,\ldots,-2.0$, increasing in increments of 0.2 from bottom to top at large l. Here $\alpha=-3.0$ is the ''scale invariant'' distribution for v_ν, and $\alpha=-2.0$ corresponds to a white noise initial power spectrum in the divergence of the velocity field.

$$\theta_\nu=k-\frac{(9+4R_\nu)}{6(5+4R_\nu)}k^3\tau^2,$$

$$\sigma_\nu=\frac{4}{3(5+4R_\nu)}k\tau+\frac{16R_\nu}{(5+4R_\nu)(15+4R_\nu)}k\tau^2,$$

$$F_{\nu 3}=\frac{4}{7(5+4R_\nu)}k^2\tau^2. \qquad (26)$$

Here the neutrinos and photons start with uniform total density and uniform density ratio but with relative velocities matched so that initially the total momentum density vanishes. If the relative momenta were not perfectly matched, the metric perturbation generated would diverge at early times, as in the adiabatic decaying mode. But because of the perfect match, a divergence at early times is avoided.

It is also possible, at least in principle, to consider regular modes with higher moments of $F_{\nu l}$ with $l\geqslant 3$ excited initially, as was considered in Ref. [24]; however, it is difficult to envision plausible mechanisms for exciting these higher moment modes.

Newtonian potentials. In this paper we have used synchronous gauge, but for completeness we give the form of the Newtonian potentials for the regular modes presented above. The Newtonian potentials ϕ and ψ are related to the synchronous variables as follows:

$$\phi=\frac{1}{2k^2}[\ddot{h}+6\ddot{\eta}+\mathcal{H}(\dot{h}+6\dot{\eta})],$$
$$\psi=\eta-\frac{\mathcal{H}}{2k^2}[\dot{h}+6\dot{\eta}]. \qquad (27)$$

We define the Newtonian potentials according to the convention $ds^2=a^2(\tau)[-d\tau^2(1+2\phi)+dx^i dx^j\delta_{ij}(1-2\psi)]$. We now give the Newtonian potentials to leading order. For the growing adiabatic mode,

$$\phi=\frac{10}{(15+4R_\nu)},$$
$$\psi=\frac{10}{(15+4R_\nu)}. \qquad (28)$$

For the neutrino isocurvature density mode,

$$\phi=\frac{-2R_\nu}{(15+4R_\nu)},$$
$$\psi=\frac{R_\nu}{(15+4R_\nu)}. \qquad (29)$$

For the neutrino isocurvature velocity mode,

$$\phi = \frac{-4R_\nu}{(15+4R_\nu)} k^{-1}\tau^{-1},$$

$$\psi = \frac{4R_\nu}{(15+4R_\nu)} k^{-1}\tau^{-1}. \quad (30)$$

The potentials for the baryon isocurvature mode are

$$\phi = \frac{(4R_\nu - 15)\Omega_{b,0}}{2(15+2R_\nu)}\tau,$$

$$\psi = \frac{-(4R_\nu - 15)\Omega_{b,0}}{6(15+2R_\nu)}\tau, \quad (31)$$

and for the CDM isocurvature mode are

$$\phi = \frac{(4R_\nu - 15)\Omega_{c,0}}{2(15+2R_\nu)}\tau,$$

$$\psi = \frac{-(4R_\nu - 15)\Omega_{c,0}}{6(15+2R_\nu)}\tau. \quad (32)$$

It is curious that the Newtonian potential diverges at early times for the neutrino isocurvature velocity mode while in the synchronous gauge there is no singularity. It appears that the synchronous gauge is a more physical gauge and that the Newtonian gauge is inadequate for modes based on anisotropic stresses. The dimensionless Ricci curvature $a^2(\tau)\tau^2 R$ (i.e., the Ricci curvature scaled to the horizon size) is nonsingular at early times. In any case, the synchronous gauge is more physical in the sense that its evolution is *local* whereas in the Newtonian gauge the evolution of the shape of the constant cosmic time hypersurfaces depends on how matter behaves infinitely far away (because the gauge choice is defined in terms of the *scalar-vector-tensor* decomposition, which is nonlocal). The divergence of the Newtonian potentials arises from the need to warp the surfaces of constant cosmic time so as to put the metric in a spatially isotropic form; however, there is nothing at all physical about this particular form. The neutrino isocurvature velocity mode was excluded in Ref. [25] because of the behavior of the Newtonian potentials at early times; however, when a physical phenomemon can be described in a nonsingular manner in a certain gauge, its singularity in another gauge must be regarded as a coordinate singularity.

III. DISCUSSION

For each wave number, we have identified five nondecaying modes: an adiabatic growing mode, a baryon isocurvature mode, a CDM isocurvature mode, a neutrino density isocurvature mode, and a neutrino velocity isocurvature mode. Under the assumption that the primordial perturbations are small enough so that the linear theory suffices, two-point derived observables are completely determined by the generalization of the power spectrum given by the symmetric, positive definite correlation matrix

$$\langle A_m(k) A_n(-k)\rangle \quad (33)$$

where the indices $(m,n=1,\ldots,5)$ label the modes, independently of whether or not the primordial fluctuations were Gaussian. In Figs. 1 and 2 the shapes of the CMB moments for the neutrino isocurvature modes are indicated. In this computation the cosmological parameters $H_0 = 50\ \mathrm{km\,s^{-1}}$, $\Omega_b = 0.05$, $\Omega_c = 0.95$ were assumed. In a companion paper [14], we examine prospects for constraining the amplitudes of these modes using upcoming MAP and Planck data.

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