Cosmological perturbations in a Friedmann-Robertson-Walker model with a scalar field and false vacuum decay

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The unconstrained reduced action corresponding to the dynamics of scalar fluctuations about the Friedmann-Robertson-Walker (FRW) background is derived using Dirac's method of description of singular Lagrangian systems. The results are applied to the so-called negative mode problem in the description of tunneling transitions with gravity. With our special choice of physical variable, the kinetic term of the reduced action has a conventional signature for a wide class of models. In this representation, the existence of a negative mode justifying the false vacuum decay picture turns out to be manifest. We also explain how the present result becomes consistent with the previously proved ''no negative mode (supercritical supercurvature mode) theorem.''

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One of the interesting and widely exploited cosmological models is based on the theory of a scalar field coupled to gravity. Such remarkable phenomena as inflation and metastable (false) vacuum decay are usually discussed in the framework of this model. For a complete description of this process, it is very important to know the properties of the perturbations to the background configurations describing inflation or vacuum decay. The cosmological perturbations in the Lorentzian regime are related to the cosmic microwave background radiation and large scale structure formation [1–3]. Furthermore, perturbations in the Euclidean version of the theory define one-loop corrections to the bubble nucleation rate and determine the quantum state of the materialized bubble [4]. Since the model contains gauge degrees of freedom, the choice of physical variables to analyze the system is not unique. There are several known ways to derive and express unconstrained quadratic action solely written in terms of physical variables in the theory of a scalar field coupled to gravity in a nonspatially flat Friedmann-Robertson-Walker (FRW) universe [2,3,5–10]. However, none of them is completely satisfactory for the purpose of applying it to the issue of quntum tunneling, which we discuss mainly in this short paper. An extension of the convenient variable v discussed in the well-known review in [1] to the nonspatially flat case is found in Ref. [10]. However, the canonical transformation to arrive at this variable turns out to be singular in the case of quantum tunneling. In the reduction discussed in [5], the kinetic term does not have a definite signature. In the case of the reduction given in [6,2], the kinetic term has a definite signature (as long as the background scalar field is monotonic). However, the overall signature is unconventional and hence some analytic continuation similar to conformal rotation becomes necessary. Conformal rotation was introduced in the case of pure gravity to cure the conformal factor problem [11]. It was also found to be possible to treat this problem (at least in perturbative gravity) via careful gauge fixing [12].

In this short paper, we revisit the dynamics of small scalar-type perturbations of a scalar field coupled to a FRW-type background in the framework of the conventional theory of degenerate Lagrangian systems [13]. We obtain an expression for the reduced action which is more appropriate for discussing quantum tunneling with gravity. For a wide class of models, the signature of the obtained reduced action becomes the conventional one. The potential for the eigenvalue problem is also regular for the same class of models, and it is shown that there is a negative mode in the spectrum of small fluctuations about the Coleman–De Luccia bounce solution [15] in accordance with a consistent interpretation of metastable vacuum decay [4]. As will become clear later, this result is not in conflict with conclusions about the absence of a negative mode arrived at in the other reduction scheme [2,6–9].

The evolution of the system composed of a scalar matter field minimally coupled to gravity is determined by the conventional action

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}R-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi-V(\phi)\right], \qquad (1)$$

where $\kappa=8\pi G$ is the reduced Newton's constant and $V(\phi)$ is the scalar field potential.

We expand the metric and the scalar field over an FRW-type background

$$ds^2=a(\eta)^2[-[1+2A(\eta)Y]d\eta^2+2\mathcal{B}(\eta)Y_{|i}d\eta dx^i +\{\gamma_{ij}[1-2\Psi(\eta)Y]+2\mathcal{E}(\eta)Y_{|ij}\}dx^i dx^j],$$
$$\phi=\varphi(\eta)+\Phi(\eta)Y, \tag{2}$$

where γ_{ij} is the three-dimensional metric on the constant curvature space sections, a and φ are the background field values and A,Ψ, Φ, $\mathcal{B}$, and $\mathcal{E}$ are small perturbations. Y is a normalized eigenfunction of the three-dimensional Laplacian, $\Delta Y=-k^2 Y$, and the vertical line represents a covariant derivative with respect to γ_{ij}. To keep simplicity, we set $\mathcal{B}(\eta)=\mathcal{E}(\eta)=0$. These terms are absent for homogeneous perturbations from the beginning. Also for inhomogeneous perturbations, it is known that this choice of gauge is consistent.

The background fields a and φ satisfy the equations

$$\mathcal{H}^2-\mathcal{H}'+\mathcal{K}=\frac{\kappa}{2}\varphi'^2, \tag{3}$$

$$2\mathcal{H}'+\mathcal{H}^2+\mathcal{K}=\frac{\kappa}{2}[-\varphi'^2+2a^2V(\varphi)], \tag{4}$$

$$\varphi''+2\mathcal{H}\varphi'+a^2\frac{\delta V}{\delta\varphi}=0, \tag{5}$$

where a prime denotes a derivative with respect to conformal time η, $\mathcal{H}:=a'/a$, and $\mathcal{K}$ is the curvature parameter, which has the values 1, 0, and -1 for closed, flat, and open universes, respectively.

Expanding the total action, keeping terms of second order in perturbations, and using the background equations, we find

$$S=S^{(0)}+S^{(2)}, \tag{6}$$

where $S^{(0)}$ is the action for the background solution and $S^{(2)}$ is quadratic in perturbations with the Lagrangian for scalar perturbations:

$$^{(s)}\mathcal{L}=\frac{a^2\sqrt{\gamma}}{2\kappa}\Bigg[-6\Psi'^2+2(k^2-3\mathcal{K})\Psi^2 +\kappa\left\{\Phi'^2-\left(a^2\frac{\delta^2 V}{\delta\varphi\,\delta\varphi}+k^2\right)\Phi^2+6\varphi'\Psi'\Phi\right\} -\left\{2\kappa\varphi'\Phi'+2\kappa a^2\frac{\delta V}{\delta\varphi}\Phi +12\mathcal{H}\Psi'+4(k^2-3\mathcal{K})\Psi\right\} \times A-2(\mathcal{H}'+2\mathcal{H}^2-\mathcal{K})A^2\Bigg]. \tag{7}$$

To obtain the unconstrained system corresponding to the degenerate Lagrangian (7) we will follow Dirac's description of singular Lagrangian systems [13]. Performing the Legendre transformation with canonical momenta

$$\Pi_\Psi:=\frac{\delta^{(s)}\mathcal{L}}{\delta\Psi'}=\frac{6a^2\sqrt{\gamma}}{\kappa}\left(-\Psi'+\frac{\kappa}{2}\varphi'\Phi-\mathcal{H}A\right), \tag{8}$$

$$\Pi_\Phi:=\frac{\delta^{(s)}\mathcal{L}}{\delta\Phi'}=a^2\sqrt{\gamma}(\Phi'-\varphi' A), \tag{9}$$

$$\Pi_A:=\frac{\delta^{(s)}\mathcal{L}}{\delta A'}=0, \tag{10}$$

we find the primary constraint $C_1:=\Pi_A=0$. Thus the total Hamiltonian H_T is

$$H_T=H_C+u_1(\eta)C_1, \tag{11}$$

with arbitrary function $u_1(\eta)$ and canonical Hamiltonian

$$H_C=-\frac{\kappa}{12a^2\sqrt{\gamma}}\Pi_\Psi^2+\frac{1}{2a^2\sqrt{\gamma}}\Pi_\Phi^2+\frac{\kappa}{2}\varphi'\Pi_\Psi\Phi +a^2\sqrt{\gamma}\left[-\frac{k^2-3\mathcal{K}}{\kappa}\Psi^2+\frac{1}{2} \times\left(a^2\frac{\delta^2 V}{\delta\varphi\,\delta\varphi}-\frac{3}{2}\kappa\varphi'^2+k^2\right)\Phi^2\right]+AC_2, \tag{12}$$

where

$$C_2=\varphi'\Pi_\Phi-\mathcal{H}\Pi_\Psi+a^2\sqrt{\gamma} \times\left[\left(a^2\frac{\delta V}{\delta\varphi}+3\varphi'\mathcal{H}\right)\Phi+\frac{2(k^2-3\mathcal{K})}{\kappa}\Psi\right]. \tag{13}$$

Conservation of the primary constraint gives the secondary constraint $C_2=0$. The primary and the secondary constraints are first class and there are no ternary constraints.

The existence of constraints in the system usually means the presence of gauge degrees of freedom. To identify the physical degrees of freedom, we fix the gauge and solve the constraints. There are two simple strategies: either to eliminate perturbations of the scalar field (Π_Φ and Φ) or the gravitational degrees of freedom (Π_Ψ and Ψ). The approach developed in [6–9,2] is based on the first possibility.[1] Here we use the second possibility.

Thus we choose the following gauge fixing condition:

$$\chi_1:=\Pi_\Psi=0. \tag{14}$$

From the consistency condition $\chi_1'=0$, we obtain

[1]Note that the gauge invariant formulation in [2] corresponds to the gauge choice $\chi_{GMST}:=\Phi=0$.

$$\chi_2 = A - \Psi = 0, \tag{15}$$

which is the condition known as the Newton gauge. Then, the consistency condition $\Psi' - A' = 0$ will determine u_1, and the set of constraints closes.

As a next step one can introduce the Dirac brackets, or equivalently we can identify the Hamiltonian for the physical degrees of freedom H^* by the relation

$$\Pi_\Psi \Psi' + \Pi_\Phi \Phi' + \Pi_A A' - H_C|_{\chi_i=0, C_i=0} = \Pi_\Phi \Phi' - H^*. \tag{16}$$

After some algebra, assuming $k^2 \neq 3\mathcal{K}$, we obtain

$$H^* = \frac{Q}{2a^2\sqrt{\gamma}}\Pi_\Phi^2 - \frac{\kappa\varphi'}{2(k^2-3\mathcal{K})}\left(3\varphi'\mathcal{H} + a^2\frac{\delta V}{\delta\varphi}\right)\Pi_\Phi\Phi + \frac{1}{2}a^2\sqrt{\gamma}\left[-\frac{\kappa}{2(k^2-3\mathcal{K})}\left(3\varphi'\mathcal{H} + a^2\frac{\delta V}{\delta\varphi}\right)^2 + \left(a^2\frac{\delta^2 V}{\delta\varphi\delta\varphi} - \frac{3}{2}\kappa\varphi'^2 + k^2\right)\right]\Phi^2, \tag{17}$$

with

$$Q := 1 - \frac{\kappa\varphi'^2}{2(k^2-3\mathcal{K})}. \tag{18}$$

To proceed further it is useful to introduce new canonical coordinates f and π_f defined by

$$\pi_f - \left[Q\mathcal{H} + \frac{\kappa\varphi'}{6\mathcal{K}}\frac{1+Q}{Q}\left(2\varphi'\mathcal{H} + a^2\frac{\delta V}{\delta\varphi}\right)\right]f = \sqrt{\frac{Q}{a^2\sqrt{\gamma}}}\Pi_\Phi, \tag{19}$$

$$f = \sqrt{\frac{a^2\sqrt{\gamma}}{Q}}\Phi. \tag{20}$$

One finds that the dynamics of the physical variable f is governed by the simple harmonic oscillator Hamiltonian

$$H^* = \frac{1}{2}\pi_f^2 + \frac{1}{2}w^2[a(\eta), \varphi(\eta)]f^2, \tag{21}$$

with frequency whose time dependence is determined by the background solutions

$$w^2[a(\eta), \varphi(\eta)] = \frac{\sqrt{Q}}{\varphi'}\left(\frac{\varphi'}{\sqrt{Q}}\right)'' - (k^2 - 3\mathcal{K})Q. \tag{22}$$

Thus, the unconstrained quadratic action becomes

$$S^{(2)} = \int\left(\frac{1}{2}f'^2 - \frac{1}{2}w^2[a(\eta), \varphi(\eta)]f^2\right)d\eta. \tag{23}$$

Now we apply the derived reduced action to the investigation of the negative mode problem in quantum tunneling with gravity. The false vacuum decay is described by the bounce solution of Euclidean equations [14,15]. The value of the Euclidean action at the bounce gives the leading exponential factor in the decay rate. The quadratic action defines one-loop corrections. It is remarkable that in the spectrum of small perturbations about a bounce in the absence of gravity there is exactly one negative mode [4]. This mode is responsible for making a correction to the ground-state energy imaginary, i.e., justifying the decay interpretation. The relevant object for tunneling transitions is the Euclidean action which can be obtained from the action, Eq. (23), by the analytic continuation $\eta = -i\tau$. Defining the Euclidean action as usual by $S^{(2)} = iS_E^{(2)}$ and specifying $\mathcal{K} = +1$, we obtain

$$S_E^{(2)} = 2\pi^2\int\left(\frac{1}{2}\dot{f}^2 + \frac{1}{2}U[a(\tau), \varphi(\tau)]f^2\right)d\tau, \tag{24}$$

where $\dot{} = d/d\tau$ and

$$U = \frac{\sqrt{Q_E}}{\dot{\varphi}}\left(\frac{d^2}{d\tau^2}\frac{\dot{\varphi}}{\sqrt{Q_E}}\right) + [l(l+2) - 3]Q_E, \tag{25}$$

with

$$Q_E = 1 + \frac{\kappa\dot{\varphi}^2}{2[l(l+2)-3]}. \tag{26}$$

Here we used the fact that the eigenvalues of the Laplacian Δ on a unit sphere take discrete values, i.e., $k^2 = l(l+2)$ with $l = 0,1,2, \ldots$.

The equation for the mode functions, which diagonalize the action, Eq. (24), has the form of the Schrödinger equation

$$-\frac{d^2}{d\tau^2}f + U[a(\tau), \varphi(\tau)]f = Ef. \tag{27}$$

Let us first show that for the $l=0$ case Eq. (27) has at least one negative mode for the Coleman–De Luccia background bounce solution. We define a new potential

$$\tilde{U} := \frac{\sqrt{Q_E}}{\dot{\varphi}}\left(\frac{d^2}{d\tau^2}\frac{\dot{\varphi}}{\sqrt{Q_E}}\right) > U. \tag{28}$$

The eigenvalue problem with this potential manifestly has a zero eigenvalue state with $f \propto \dot{\varphi}/\sqrt{Q_E}$. Since $\tilde{U} > U$, the eigenvalue problem with the potential U must have at least one negative eigen mode. □

A similar discussion leads to the conclusion that there are no negative modes for $l > 1$ (compare [7]). The $l=1$ case needs separate consideration. As was explained in Ref. [6], there are no physical degrees of freedom in this sector.

Note that the present result does not contradict the no negative mode theorem [9]. The argument of the no negative mode theorem is that, when we consider a bounce solution such that it realizes the minimum value of the action among all nontrivial $O(4)$-symmetric configurations, there is no negative mode for the specific variable q defined by

$$q := \frac{8a}{\sqrt{\kappa}\dot{\varphi}}\Psi^N, \tag{29}$$

where Ψ^N represents Ψ evaluated in the Newton gauge [2]. In terms of this variable q, the kinetic term stays negative for $l=0$ modes. Hence, as mentioned earlier, we need to do some analytic continuation similar to conformal rotation to perform the path integration. Although it is not fully justified, this procedure of analytic continuation is thought to produce the required imaginary factor (see [6,9]). Now we are working with a different variable f for which the kinetic term takes the conventional signature. Therefore there should be one and only one negative mode.

Once we accept the original no negative mode theorem, we can give indirect proof of the uniqueness of the negative mode for f under the same condition that the absence of a negative mode for q was proved. It is easy to find the relation between f and q because our gauge condition corresponds to the Newton gauge (15). Eliminating Π_Φ from the constraint equation $C_2=0$ and the Hamiltonian equation of motion for Φ derived from Eq. (17), we obtain

$$\frac{d}{d\tau}\left(\frac{\sqrt{Q_E}}{\sqrt[4]{\gamma}\dot{\varphi}}f\right) = \frac{l(l+2)-3\mathcal{K}}{4\sqrt{\kappa}\dot{\varphi}}Q_E q, \tag{30}$$

where we used the definitions (20) and (29). Suppose that f has two negative modes for $l=0$, and let us try to derive a contradiction by using the fact that q does not have any negative mode. By assumption, the zero eigenvalue solution $(E=0)$ of f which satisfies the boundary condition on one side must have two nodes. One may think that this implies that there are at least two zeros of $z := (\sqrt{Q_E}/\sqrt[4]{\gamma}\dot{\varphi})f$. But it is not true because the regularity condition for f just requires it to behave like $e^{-(l+1)|\tau|}$ on boundaries. Since $\dot{\varphi}$ behaves like $\sim e^{-2|\tau|}$, even the regular solution of z does not go to zero on the boundary for $l=0$. Hence, we arrive at the conclusion that there is at least one point where the derivative of z vanishes. With the aid of Eq. (30), this implies that the zero eigenvalue solution of q has a node, which means the existence of a negative mode in q and contradicts with the no negative mode theorem. □

We also note that with the present choice of variable there is no manifest correspondence between the no negative mode theorem and the no supercritical supercurvature mode theorem [8] as there was in terms of the variable q defined in [2]. The equation which determines the perturbation spectrum in the context of one-bubble open inflation in terms of the present variable f is not the standard Schrödinger-type equation and the existence of a mode with a negative value of E does not imply the existence of a supercritical supercurvature mode. So far we considered background solutions with a positive definite factor Q_E. If this factor becomes zero or negative for some region(s) of τ, the Euclidean analogue of the canonical transformation, Eqs. (19),(20), becomes singular. This will not immediately indicate a certain physical meaning because nothing special happens as long as our discussion is in terms of q in Ref. [2]. However, if there exists some class of bounce solutions for which the signature of the kinetic term cannot be set to be positive definite without passing through a singular canonical transformation, then it might suggest some physical meaning. In this case, we might have to reconsider the possibility of catastrophic particle creation discussed in Ref. [5].

To conclude we have investigated the dynamics of small perturbations in a nonflat FRW model coupled to a scalar field. Using the gauge conditions, Eq. (14), we reduced the system of coupled perturbations, Eq. (7), to the dynamical system, Eq. (23), with one physical degree of freedom. The reduced quadratic action, Eq. (23), has the conventional overall signature. Investigating the Euclidean quadratic action, Eq. (24), we proved that there is exactly one negative eigenvalue mode about the Coleman–De Luccia bounce solution. The result is consistent with the so-called ''no negative mode'' theorem [9] in false vacuum decay with gravity which was proved in the other reduction scheme. The treatment discussed here is restricted to background solutions which satisfy the condition that the quantity Q_E be positive. Hence, another question arises as to whether we can always find a variable for which the kinetic term of a perturbation in the reduced action becomes positive definite. This issue needs further investigation.

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