

Resummation of nonalternating divergent perturbative expansions

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A method for the resummation of a nonalternating divergent perturbation series is described. The procedure constitutes a generalization of the Borel-Padé method. Of crucial importance is a special integration contour in the complex plane. Nonperturbative imaginary contributions can be inferred from the purely real perturbative coefficients. A connection is drawn from the quantum field theoretic problem of resummation to divergent perturbative expansions in other areas of physics.

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In view of the probable divergence of quantum field theory in higher order [1,2], the resummation of the perturbation series is necessary for obtaining finite answers to physical problems. While the divergent expansions probably constitute an asymptotic series [3], it is unclear whether unique answers can be inferred from perturbation theory [4,5]. Significant problems in the resummation are caused by infrared (IR) renormalons. These are contributions corresponding to a nonalternating divergent perturbation series. The IR renormalons are responsible for the Borel nonsummability of a number of field theories including quantum chromodynamics (QCD) and quantum electrodynamics (QED) [4,6].

Here I advocate a modification of the resummation method proposed in [5,7] for a nonalternating divergent perturbation series. The method starts with a given input series,

$$f(g) \sim \sum_{n=0}^{\infty} c_n g^n, \quad c_n > 0, \quad g > 0, \quad (1)$$

where g is the coupling parameter and the perturbative coefficients c_n are expected to diverge as follows [8]:

$$c_n \sim K \frac{n! n^\gamma}{S^n}, \quad n \rightarrow \infty, \quad (2)$$

with K , γ and S being constant. The Borel transform f_B of the perturbation series (1),

$$f_B(g) = \sum_{n=0}^{\infty} \frac{c_n}{n!} g^n, \quad (3)$$

has a finite radius of convergence about the origin. For the evaluation of the Borel integral, $f_B(g)$ has to be continued analytically beyond the radius of convergence. Strictly speaking, this analytic continuation has to be done on the branch cut in view of the nonalternating character of the series (1). This requirement can be relaxed slightly by performing the analytic continuation into regions where g acquires at least an infinitesimal imaginary part $g \rightarrow g \pm i \epsilon$. In

this case, the analytic continuation can be achieved by evaluating Padé approximants [9]. The first $n+1$ terms of the Borel transformed series (3) can be used to construct a diagonal or off-diagonal Padé approximant (for the notation see [10,11]):

$$\mathcal{P}_n(z) = [[n/2] / [(n+1)/2]]_{f_B}(z), \quad (4)$$

where $[[x]]$ denotes the integral part of x . The resummation is accomplished by constructing the sequence of transforms $\{\mathcal{T}f_n(g)\}_{n=0}^{\infty}$ where

$$\mathcal{T}f_n(g) = \int_{C_j} dt \exp(-t) \mathcal{P}_n(gt), \quad (5)$$

and the integration contour C_j (where $j = -1, 0, +1$) is as shown in Fig. 1 (for $j = -1$ and $j = +1$). The result obtained along C_{-1} is the complex conjugate of the result along C_{+1} . The arithmetic mean of the results of the integrations along C_{-1} and C_{+1} is associated with C_0 . Therefore, the result along C_0 is real rather than complex. The limit of the sequence $\{\mathcal{T}f_n(g)\}_{n=0}^{\infty}$ (provided it exists),

$$\lim_{n \rightarrow \infty} \mathcal{T}f_n(g) = f(g), \quad (6)$$

is a plausible complete nonperturbative result inferred from the perturbative expansion (1). Which of the contours C_j

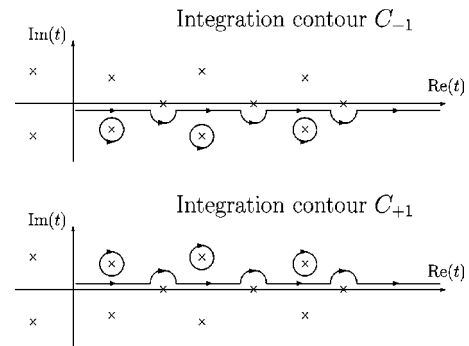


FIG. 1. Integration contours for the evaluation of the generalized Borel integral in Eq. (5).

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(where $j = -1, 0, +1$) is chosen has to be decided on the basis of additional considerations which do not follow from perturbation theory alone.

The zeros of the denominator polynomial of the Padé approximant [see Eq. (4)] correspond to the poles of the integrand in Eq. (5). Denote by t the integration variable for the evaluation of the generalized Borel integral in Eq. (5), then the poles lie at $t = z_i$ (where the index i numbers the poles) along the positive real axis ($\text{Im } z_i = 0$) and in the complex plane ($\text{Im } z_i \neq 0$). The poles lying on the positive real axis are treated as half-poles encircled in the mathematically positive sense for C_{-1} and as half-poles encircled in the mathematically negative sense for C_{+1} . The contour C_{-1} encircles all poles at $t = z_i$ in the lower right quadrant of the complex plane ($\text{Re } z_i > 0, \text{Im } z_i < 0$) in the positive sense (see Fig. 1). The contribution of these poles should be added to the final result. The contour C_{+1} is understood to encircle all poles in the upper right quadrant of the complex plane in the mathematically negative sense. In general, the integrations along C_{-1} and C_{+1} lead to a nonvanishing imaginary part in the final result for $f(g)$ [see Eq. (6)], although all the perturbative coefficients c_n are by assumption real and positive [see Eq. (1)]. It might be interesting to note that, as with any complex integration, it is permissible to deform the integration contours shown in Fig. 1 in accord with the Cauchy theorem as long as all pole contributions are properly taken into account.

This paper represents a continuation of previous work on the subject [5,7,9]. The resummation method defined in Eqs. (1)–(6) differs from [5] in the combination of Borel and Padé techniques and, if compared to the remarkable investigations in [7,9] on the resummation of QCD perturbation series, in the integration contour used for the evaluation of the generalized Borel integral. It is argued here that, when the Borel transform (3) is analytically continued with Padé approximants (4), the contribution of poles lying off the positive real axis has to be taken into account in order to obtain consistent results in the resummation (see Fig. 1). In [7,9] it is argued that the Borel integral should be evaluated by principal value. It could appear that the C_0 contour corresponds to the principal-value prescription. However, this is not necessarily the case, if there are poles present which lie off the positive real axis (i.e., at $t = z_i$ with $\text{Re } z_i > 0, \text{Im } z_i \neq 0$). The contribution of these poles modifies not only the imaginary, but also the real part of the final nonperturbative result. Of course, when there are no poles lying off the positive real axis, as is the case for the problems discussed in [7,9], then the principal-value prescription used in [7,9] is equivalent to the C_0 contour. Because the result obtained along C_0 is real, this contour should be used whenever the existence of an imaginary part is discouraged by physical reasons.

It is important to mention that the method presented here is not the only prescription currently available for the resummation of divergent perturbative expansions in quantum field theory. For example, the δ transformation [see Eq. (4) in [11]] is a very useful method for the resummation of divergent perturbation series. The δ transformation has a number of appealing mathematical properties, including rapid and

numerically stable convergence, and it has been shown to yield consistent results in many cases, including applications from quantum field theory [11] and from other areas of physics [12]. Because the δ transformation fulfills an accuracy-through-order relation [see Eq. (9) in [11]], it can be used to predict perturbative coefficients. The δ transformation is primarily useful for alternating series. It fails, in general, in the resummation of the nonalternating series discussed here. *The δ transformation and the resummation method introduced here complement each other.*

Three applications of the resummation method defined in Eqs. (1)–(6) are considered below: (i) the QED effective action in the presence of a constant background electric field, (ii) a mathematical model series which simulates the expected large-order behavior of perturbative coefficients in quantum field theory, (iii) the perturbation series for the energy shift of an atomic level in a constant background electric field (including the auto-ionization width). The nonperturbative imaginary contributions obtained along C_{-1} and C_{+1} find a natural physical interpretation in all cases considered.

The QED effective action, or vacuum-to-vacuum amplitude, in the presence of a constant background electric field has been treated nonperturbatively in [13,14], and the result is proportional to the integral

$$S(g_E) = - \int_{0-i\epsilon}^{\infty-i\epsilon} \frac{ds}{s^2} \left\{ \cot s - \frac{1}{s} + \frac{s}{3} \right\} \exp \left[- \frac{1}{\sqrt{g_E}} s \right], \quad (7)$$

where g_E is a coupling parameter proportional to the square of the electric field strength, $g_E = e^2 E^2 / m_e^4$. Here, m_e is the electron mass, and e is the elementary charge. The natural unit system ($\hbar = c = 1$) is used. The imaginary part of $S(g_E)$ is proportional to the electron-positron pair-production amplitude per space-time interval [there is, of course, also a muon-antimuon pair-production amplitude, obtained by the imaginary part of Eq. (7) under the replacement $m_e \rightarrow m_\mu$, which is not discussed here]. $S(g_E)$ has the following asymptotic expansion in the coupling parameter,

$$S(g_E) \sim 16 \left[\sum_{n=0}^{\infty} \frac{4^n |\mathcal{B}_{2n+4}|}{(2n+4)(2n+3)(2n+2)} g_E^{n+1} \right], \quad (8)$$

where \mathcal{B}_{2n+4} is a Bernoulli number. In view of the asymptotics

$$\frac{4^n |\mathcal{B}_{2n+4}|}{n^3} \sim \frac{\Gamma(2n+2)}{\pi^{2n+4}} \left[1 + \mathcal{O}\left(\frac{1}{4^n}\right) \right], \quad n \rightarrow \infty, \quad (9)$$

the perturbative coefficients, which are nonalternating in sign, diverge factorially in absolute magnitude. The asymptotic series (8) for $S(g_E)$ is taken as the input series for the resummation process [Eq. (1)], and a sequence of transforms $\mathcal{T}S_n(g_E)$ is evaluated using the prescription (5). The results have to be compared to the exact nonperturbative expression (7). This is done in Table I for $g_E = 0.05$. The partial sums of the asymptotic series (8) are listed in the second column.

TABLE I. Resummation of the asymptotic series for the QED effective action (8) in a constant background electric field for $g_E=0.05$. Results in the third column are obtained by the method indicated in Eq. (5) along the integration contour C_{-1} . The partial sums in the second column are obtained from the asymptotic series (8).

n	Partial sum	$\mathcal{T}S_n(g_E)$
2	0.001 146 032	$0.001\ 144\ 848 - i\ 7.70 \times 10^{-17}$
3	0.001 146 705	$0.001\ 146\ 639 - i\ 8.22 \times 10^{-11}$
4	0.001 146 951	$0.001\ 147\ 113 - i\ 3.54 \times 10^{-8}$
5	0.001 147 087	$0.001\ 147\ 264 - i\ 1.93 \times 10^{-8}$
6	0.001 147 195	$0.001\ 147\ 173 - i\ 3.15 \times 10^{-7}$
7	0.001 147 310	$0.001\ 147\ 113 - i\ 2.58 \times 10^{-7}$
8	0.001 147 469	$0.001\ 147\ 162 - i\ 2.30 \times 10^{-7}$
9	0.001 147 743	$0.001\ 147\ 165 - i\ 2.63 \times 10^{-7}$
10	0.001 148 327	$0.001\ 147\ 144 - i\ 2.53 \times 10^{-7}$
11	0.001 149 825	$0.001\ 147\ 157 - i\ 2.46 \times 10^{-7}$
12	0.001 154 375	$0.001\ 147\ 155 - i\ 2.56 \times 10^{-7}$
13	0.001 170 560	$0.001\ 147\ 151 - i\ 2.51 \times 10^{-7}$
14	0.001 237 137	$0.001\ 147\ 156 - i\ 2.51 \times 10^{-7}$
15	0.001 550 809	$0.001\ 147\ 153 - i\ 2.53 \times 10^{-7}$
16	0.003 228 880	$0.001\ 147\ 154 - i\ 2.51 \times 10^{-7}$
17	0.013 345 316	$0.001\ 147\ 154 - i\ 2.52 \times 10^{-7}$
18	0.081 610 937	$0.001\ 147\ 153 - i\ 2.52 \times 10^{-7}$
19	0.594 142 371	$0.001\ 147\ 154 - i\ 2.52 \times 10^{-7}$
20	4.852 426 276	$0.001\ 147\ 154 - i\ 2.52 \times 10^{-7}$
exact	0.001 147 154	$0.001\ 147\ 154 - i\ 2.52 \times 10^{-7}$

Numerical results from perturbation theory are normally obtained by (optimal) truncation of the perturbation series. For the example considered, (i) the partial sums do not account for the imaginary part and (ii) due to the divergence of the perturbative expansion, no improvement in the final result could be obtained by adding more than the first seven perturbative terms. It requires a valid resummation procedure to go beyond the accuracy obtainable by optimal truncation of the perturbation series. The transforms $\mathcal{T}S_n(g_E)$ displayed in the third column of Table I apparently converge to the full nonperturbative result given in Eq. (7), and the nonperturbative imaginary part, which corresponds to the pair-production amplitude, is reproduced although the input series (8) has purely real perturbative coefficients.

Two specific mathematical model series are considered next. The series

$$\mathcal{N}(g) \sim \sum_{n=0}^{\infty} n! g^n$$

has been used as a paradigmatic example for nonalternating divergent series in the literature [3,15,16]. This series can be resummed by the method (5). Moreover, this resummation is even exact for all transformation orders $n \geq 2$. This can be seen as follows. The Borel transform

TABLE II. Resummation of the model series (10) for $\gamma=2.3, g=0.1$ by the method indicated in Eq. (5) along the integration contour C_{+1} . The partial sums are obtained from the asymptotic series (10).

n	Partial sum	$\mathcal{T}\mathcal{M}_n(g)$
2	0.445 451	$0.393\ 554 + i\ 0.373\ 912$
3	0.559 685	$0.840\ 561 + i\ 0.446\ 830$
4	0.640 410	$0.764\ 942 + i\ 0.274\ 640$
5	0.703 981	$0.765\ 339 + i\ 0.218\ 156$
6	0.759 669	$0.763\ 012 + i\ 0.219\ 638$
7	0.813 594	$0.762\ 186 + i\ 0.219\ 197$
8	0.870 909	$0.762\ 196 + i\ 0.219\ 126$
9	0.937 322	$0.762\ 224 + i\ 0.219\ 123$
10	1.020 707	$0.762\ 225 + i\ 0.219\ 127$
11	1.133 528	$0.762\ 223 + i\ 0.219\ 127$
12	1.297 220	$0.762\ 223 + i\ 0.219\ 127$
exact	0.762 223	$0.762\ 223 + i\ 0.219\ 127$

$$\mathcal{N}_B(g) = \sum_{n=0}^{\infty} g^n = 1/(1-g)$$

is a geometric series. The summation of geometric series inside and outside of the circle of convergence by Padé approximants is exact in all transformation orders $n \geq 2$. So, for all $n \geq 2$ the transforms $\mathcal{T}\mathcal{N}_n(g)$ fulfill the equality $\mathcal{T}\mathcal{N}_n(g) = -(1/g)\exp(-1/g)\Gamma(0, -1/g) = \mathcal{N}(g)$, where $\Gamma(0, x)$ is the incomplete gamma function (see [17]), and the choice of the contour (C_{-1} or C_{+1}) determines on which side of the branch cut the incomplete gamma function is evaluated.

The asymptotic series,

$$\mathcal{M}(g) \sim \sum_{n=0}^{\infty} \frac{\Gamma(n+\gamma)}{\Gamma(n)} n! g^n, \quad (10)$$

constitutes a more interesting application of the resummation method than $\mathcal{N}(g)$. On account of the asymptotics,

$$\frac{\Gamma(n+\gamma)}{\Gamma(n)} \sim n^\gamma \left(1 + \mathcal{O}\left(\frac{1}{n}\right) \right), \quad n \rightarrow \infty, \quad (11)$$

the series $\mathcal{M}(g)$ serves as a model for the expected large-order behavior of perturbative coefficients in quantum field theory [see Eq. (2)]. The analytic summation of Eq. (10) leads to

$$\mathcal{M}(g) = \Gamma(\gamma) \left(g \frac{\partial}{\partial g} \right) {}_2F_0(1, \gamma; g), \quad (12)$$

where the hypergeometric ${}_2F_0$ function has a branch cut along the positive real axis (see [17]). The imaginary part of Eq. (12) for $g > 0$ as a function of g and γ is $\text{Im } \mathcal{M}(g) = \pi(1-g\gamma)g^{-\gamma-1}\exp(-1/g)$, where the integration is assumed to have been performed along the contour C_{+1} . For C_{-1} , the sign of the imaginary part is reversed. The numerical example considered here is $\gamma=2.3, g=0.1$. In Table II,

numerical results are displayed for the n th partial sums of the asymptotic series (10) and the transforms $\mathcal{T}\mathcal{M}_n(g)$ calculated according to Eq. (5) in the range $n=2, \dots, 12$. While the partial sums eventually diverge, the transforms $\mathcal{T}\mathcal{M}_n(g)$ exhibit apparent convergence to about six significant figures in ($n=12$)th transformation order, and the transforms reproduce the imaginary part although the coefficients of the series (10) are all real rather than complex. The integration is performed along the contour C_{+1} . The exact result in the last row of Table II is obtained from Eq. (12). For the evaluation of the transforms $\mathcal{T}\mathcal{M}_n(g)$ it is crucial to use the contour C_{+1} rather than a contour infinitesimally above the real axis. For example, in order to obtain consistent numerical results, it is necessary to take into account the poles at $t=9.99 \pm i0.578$ in ($n=11$)th transformation order, encountered in the evaluation of the transform $\mathcal{T}\mathcal{M}_{11}(g)$ according to Eq. (5), and the pole at $t=9.99 \pm i0.495$ in ($n=12$)th order for the evaluation of $\mathcal{T}\mathcal{M}_{12}(g)$. These poles approximately correspond to the triple pole at $t=1/(0.1)=10$ which would be expected in the case $\gamma=2$.

When an atom is brought into an electric field, the levels become unstable against auto-ionization, i.e., the energy levels \mathcal{E} acquire a width Γ (that is to say, $\mathcal{E} \rightarrow \text{Re } \mathcal{E} - i\Gamma/2$ where Γ is the width). Perturbation theory cannot account for the width. The coefficients are real, not complex [18]. An established method for the determination of the width is by numerical diagonalization of the Hamiltonian matrix [19–21]. It is argued here that the full complex energy eigenvalue, including the width, can also be inferred from the divergent perturbation series by the resummation method defined in Eqs. (1)–(6), where the appropriate integration contour is C_{+1} . Perturbative coefficients for the energy shift in arbitrarily high order can be inferred from the Eqs. (9), (13)–(15), (28)–(33), (59)–(67), (73) in [18].

The symmetry of the problem suggests the introduction of the parabolic quantum numbers n_1, n_2 , and m [22] (the principal quantum number is $n=n_1+n_2+m+1$). Here, calculations are performed for the ground state with parabolic quantum numbers $n_1=0, n_2=0, m=0$ and two L shell states, both of which are coherent superpositions of the $2S$ and $2P$ states. One of the L shell states investigated here has the parabolic quantum numbers $n_1=1, n_2=0, m=0$, and the other L shell state has the quantum numbers $n_1=0, n_2=1, m=0$. The Stark effect is interesting because, depending on the atomic state, the perturbation series are either completely nonalternating in sign (e.g., for the ground state), or they constitute nonalternating divergent series with a subleading divergent alternating component (e.g., for $n_1=0, n_2=1, m=0$), or the series are alternating with a subleading divergent nonalternating component (e.g., for $n_1=1, n_2=0, m=0$). The perturbation series for the Stark effect do not strictly fulfill the assumptions of Eq. (1), and *the successful resummation of these series might indicate that the method introduced here is in fact more generally applicable*. The large-order asymptotics of the perturbative coefficients for the Stark effect are given in Eqs. (4,5) in [23]. In quantum field theory, the alternating and nonalternating components correspond to ultraviolet (UV) and IR renormalons. Using the first

20 coefficients of the perturbation series for the energy and evaluating the first 20 transforms according to Eq. (5), estimates for the real part of the energy (Stark energy shift) and the imaginary part of the energy (decay width of the state) may be obtained. The apparent convergence of the first 20 transforms for the real part of the energy extends to 6–8 significant figures, whereas the convergence of the imaginary part is much slower (2–3 significant figures). In all cases considered, both the real and the imaginary part of the energy obtained by resummation compare favorably with values for the decay width obtained by numerical diagonalization of the Hamiltonian matrix [19–21]. Here we concentrate on the decay width, the full calculation will be described in detail elsewhere. The atomic unit system is used in the sequel, as is customary for this type of calculation [18–21]. In the atomic unit system, the unit of energy is $\alpha^2 m_e c^2 = 27.211 \text{ eV}$, where α is the fine structure constant, and the unit for the electric field is the field strength felt by an electron at a distance of one Bohr radius a_{Bohr} to a nucleus of elementary charge, which is $1/(4\pi\epsilon_0)(e/a_{\text{Bohr}}^2) = 5.142 \times 10^{11} \text{ V/m}$ (here, ϵ_0 is the permittivity of the vacuum).

Evaluations have been performed for all atomic levels and field strengths of Table III in [23]. Three examples are presented here. For the ground state, at an electric field strength of $E=0.1$ in atomic units, the imaginary part of the first 20 transforms calculated according to Eq. (5) exhibits apparent convergence to $\Gamma = 1.46(5) \times 10^{-2}$, which has to be compared to $\Gamma = 1.45 \times 10^{-2}$ obtained from numerical diagonalization of the Hamiltonian matrix [19]. For the L shell state with quantum numbers $n_1=0, n_2=1, m=0$, at a field strength of $E=0.004$, the first 20 transforms exhibit apparent convergence to an imaginary part of $\Gamma = 4.46(5) \times 10^{-6}$ which compares favorably to $\Gamma = 4.45 \times 10^{-6}$ from [21]. The most interesting case is the state $n_1=1, n_2=0, m=0$, for which the nonalternating component of the perturbation series is subleading. At $E=0.006$, resummation of the complete perturbation series (including the leading alternating part) leads to a decay width of $\Gamma = 6.08(5) \times 10^{-5}$, which is again consistent with the result of $\Gamma = 6.09 \times 10^{-5}$ from [21]. The contour C_{+1} is crucial, due to poles lying off the real axis.

With the help of Carleman's theorem [24] it is possible to formulate a criterion which guarantees that there is a one-to-one correspondence between a function and its associated asymptotic series (see for example [25], Theorems XII.17 and XII.18 and the definition on p. 43 in [26], p. 410 in [27], or the comprehensive and elucidating review [4]). Let $f(z)$ be a function which is analytic in the interior and continuous on a sectorial region $\mathcal{S} = \{z \mid |\arg(z)| \leq k\pi/2 + \epsilon, 0 < |z| < R\}$ of the complex plane for some $\epsilon > 0$. Let the function f have an asymptotic expansion $f(z) \sim \sum_{n=0}^{\infty} c_n z^n$ (for $z \rightarrow 0$). The function f obeys a strong asymptotic condition (of order k) if there are suitable positive constants C and σ such that $|f(z) - \sum_{n=0}^m c_n z^n| \leq C\sigma^{m+1} [k(m+1)]! |z|^{m+1}$ holds for all m and for all $z \in \mathcal{S}$. The validity of such a condition implies that the function $f(z)$ is uniquely determined by its asymptotic series (see Theorem XII.19 of [26]). Typically, series which entail nonperturbative (imaginary) contributions do not fulfill the Carleman condition. The resulting ambigu-

ity is reflected in the three integration contours in Fig. 1, only one of which gives the physically correct result.

It has not escaped our attention that specialized variants of the method introduced here can be constructed in those cases where additional information about the perturbative coefficients (large-order asymptotics, location of poles in the Borel plane, etc.) is available.

Finite and consistent answers in quantum field theory are obtained after regularization, renormalization and resummation. Using a resummation method, as shown in Tables I and II, it is possible to go beyond the accuracy obtainable by optimal truncation of the perturbation series. The purpose of resummation is to eventually reconstruct the full nonperturbative result from the divergent perturbation series (see also [5]). I have examined two physical examples, the QED effective action in a constant background electric field [Eq. (8)] and the Stark energy shift. The perturbation series for the Stark effect contains nonalternating and alternating divergent contributions, which correspond in their mathematical

structure to IR and UV renormalons in quantum field theory, respectively. It has been shown in each case that complete nonperturbative results, including the pair-production amplitude for electron-positron pairs and the atomic decay width, can be inferred from the divergent nonalternating perturbation series by the resummation method defined in Eqs. (1)–(6). A mathematical model series (10), which simulates the expected large-order growth of perturbative coefficients in quantum field theory [see Eq. (2)], can also be resummed by the proposed method (see Table II). In all cases considered, the full nonperturbative result involves an imaginary part, whereas the perturbative coefficients are real. The advocated method of resummation makes use of the Padé approximation applied to the Borel transform of the divergent perturbation series. Advantage is taken of the special integration contours C_j (with $j = -1, 0, 1$) shown in Fig. 1.

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