# Semileptonic inclusive heavy meson decay: Duality in a nonrelativistic potential model in the Shifman-Voloshin limit

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The quark-hadron duality in the inclusive semileptonic decay $B \to X_c l \nu$ in the Shifman-Voloshin limit $\Lambda \ll \delta m = m_b - m_c \ll m_b, m_c$ is studied within a nonrelativistic potential model. The integrated semileptonic decay rate is calculated in two ways: first, by constructing the operator product expansion, and second by a direct summation of the exclusive channels. Sum rules (Bjorken, Voloshin, etc.) for the potential model are derived, providing a possibility to compare the two representations for $\Gamma(B \to X_c l \nu)$. An explicit difference between them referred to as the duality-violation effect is found. The origin of this effect is related to higher charm resonances which are kinematically forbidden in the decay process but are nevertheless picked up by the OPE. Within the considered $1/m_c^2$ order the OPE and the sum over exclusive channels match each other, up to the contributions of higher resonances, by virtue of the sum rules. In particular this is true for the terms of order $\delta m^2/m_c^2$ and $\Lambda\,\delta m/m_c^2$ which are present in each of the decay channels and cancel in the sum of these channels due to the Bjorken and Voloshin sum rules, respectively. The size of the duality violation effects is estimated to be of the order $O(\Lambda^{2+b}/m_c^2 \delta m^b)$ with $b > 0$ depending on the details of the potential. Constraints for a better accuracy are discussed.

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## I. INTRODUCTION

The interest in inclusive decays of heavy mesons is twofold: experimental study of such decays can provide important information on the weak mixing angles of heavy mesons, and a theoretical treatment of such processes which includes also nonperturbative effects is possible. The theoretical framework based on combining the operator product expansion (OPE) and heavy quark (HQ) expansion provides decay rates and differential distributions as series in inverse powers of the heavy quark mass with the coefficients proportional to the matrix elements of the operators of a proper dimension [1–4]. A remarkable property of this expansion is that in the leading-order this is just the free-quark decay, and the first correction appears only at order $1/m_Q^2$.

On the other hand, it is understood that the quark-hadron duality technically implemented through the OPE is an approximate framework [5]. For example, the calculation based on OPE does not take into account all the details of the hadron spectrum which lead to the dependence of the set of open decay channels on the momentum transfer. The OPE ignores this fact and this inevitably yields some errors in the OPE results [6].

The theoretical description based on the OPE represents the decay rate as a contour integral in the complex $q^0$-plane (for details see the next section). The OPE can be justified only in regions of the complex $q^0$-plane away from the physical region, whereas in the case of the calculation of the decay rate (both differential and integrated) the contour always involves a segment which is close to the physical region [1]. This can lead to duality-violating effects, i.e., the difference between the exact and the OPE based results.

However, it is not easy to estimate the errors arising in the OPE, since the exact hadron spectrum in QCD is complicated and not exactly known. So, testing directly the accuracy of the quark-hadron duality is only possible in few exceptional cases. Examples discussed in the literature are QCD in the Shifman-Voloshin (SV) limit [7], and the 't Hooft model [8].

In the 't Hooft model (2-dimensional QCD with $N_c \to \infty$) the spectrum is reduced to an infinite number of single bound states and known precisely so that the direct summation of exclusive channels is possible. First numerical analysis of the sum over exclusive channels reported the presence of the duality-violating $1/m_Q$ correction for the total width [9]. Later the summation was performed analytically for the case of a massless light quark [10]. The result of the OPE calculation agreed with the exact result in this case through $1/m_Q^4$ order.

Duality in QCD in the SV limit [7] has been studied in [11,12]. This limit requires $\Lambda_{QCD} \ll \delta m = m_b - m_c \ll m_Q$. A peculiar feature of the SV limit is that a summation over exclusive channels becomes possible due to kinematical reasons: the process occurs near the zero recoil and thus only few decay channels contribute in the leading $1/m_Q$ order. The expansion of the relevant transition form factors in this kinematical region is known and the sum over exclusive channels can be evaluated. The absence of $\Lambda_{QCD}/m_Q$ corrections to the free-quark result in the semileptonic (SL) decay rate has been demonstrated in [11]. However, to check the

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absence of $\Lambda_{\rm QCD}/m_Q$ and $\delta m/m_Q$ corrections within the SV kinematics is not enough to ensure duality in the $1/m_Q$ order in the general case, beyond the SV limit. Namely, one should also check that potentially large terms of order $O(\Lambda_{\rm QCD}\delta m^n/m_Q^{n+1})$ which are present in individual decay rates cancel in the sum over exclusive channels. The analysis of the $\Lambda_{\rm QCD}\delta m/m_Q^2$ terms in the exclusive sum was performed in [12] for QCD in the $V-A$ case. It was found that the duality within this accuracy requires a new sum rule. The full comparison to higher orders has not yet been performed.

We study the quark-hadron duality in the SV limit within a nonrelativistic potential model. The model has several features which make it especially suitable for this purpose: the model is self-consistent in the SV limit; the spectrum of bound states is relatively simple and can be calculated; the exact representations of the transition form factors in terms of the hadron wave functions are known. These features provide a possibility to calculate the exclusive sum. We adopt a technical simplification of a Lorentz scalar current instead of the $V-A$ current, like it is done in Ref. [6].

The main purpose of our analysis is to check whether or not the OPE result calculated to some order is equal to the sum over exclusive channels expanded to the same order. Both series are double expansions in powers of $\Lambda/m_c$ and $\Lambda/\delta m$. They are asymptotic series [10], and the question of their convergence is left for a later publication [13].

Our main results are as follows.

We construct the expansion of the $T$-product of the two currents in a series of local operators (the OPE) in the potential model for a general form of the quark potential. Technically this is done by the expansion of the Lippmann-Schwinger equation. We consider the expansion to all orders in $\Lambda/\delta m$ but neglect terms $\sim\Lambda^n/m_c^n$ with $n\geqslant 3$. This OPE series provides the expansion of the differential and integrated semileptonic decay rates in powers of $\Lambda/m_c$ and $\Lambda/\delta m$.

Let us point out that the OPE series in the potential model has an important distinction from the Wilsonian scheme in the field theory: Namely, in QCD (perturbative) contributions of small distances below the scale $1/\mu$ is referred to the Wilson coefficients while contributions of large distances above this scale is referred to the matrix elements of the local operators. As a result both the $c$-number Wilson coefficients and the matrix elements of the local operators acquire the $\mu$-dependence. In the potential model we also expand the average of the $T$-product of the two current operators over the $B$ meson in a series of local operators, but the resulting $c$-number coefficents as well as the average values of the local operators [see Eq. (8)] do not have a scale dependence.

The OPE and the sum over exclusive channels are related to each other by sum rules, similar to the Bjorken [14], the Voloshin [15], and the whole tower of higher moments [16]. We derive these sum rules. They involve an infinite sum of terms corresponding to all hadronic excitations, with each term having a well-defined heavy mass expansion. The question of the heavy mass expansion of the sum (in other words, of the uniform convergence of the series) has not been tackled in this paper. If the contribution of higher excitations vanishes rapidly enough the uniform convergence is expected.

The OPE provides a heavy mass expansion for the inclusive semileptonic decay rate. To compare it with the result of summation over exclusive semileptonic decay channels we make use of the sum rules. An explicit difference between the two expressions is found, both for the integrated and the differential rates. This difference corresponds to the contribution of the resonances kinematically forbidden in the decay process which are picked up by the OPE. This ''unphysical'' contribution is related to the poles in the complex $q^0$-plane outside the physical region which however contribute to the OPE result. The size of this duality-violation cannot be estimated in all generality since it depends on the potential and on the convergence properties of the sums over resonances.

For the integrated decay rate the OPE prediction and the sum over exclusive channels match, *up to the duality-violating contributions of higher resonances*, within the $1/m_c^2$ order: Terms of order $\delta m^2/m_Q^2$, $\delta m\Lambda/m_Q^2$, which are present in any individual decay rate cancel in the sum over all channels thanks to the Bjorken [14] and Voloshin [15] sum rules, respectively. For terms of order $\Lambda^2/m_Q^2$, $\Lambda^3/m_Q^2\delta m$, etc., the agreement (*again up to contributions of higher resonances*) is provided by the higher moment sum rules. The duality-violation induced by the kinematical truncation of these higher resonances in general has the order $O(\Lambda^{2+b}/m_c^2\delta m^b)$ where $b$ depends on the details of the potential $V(r)$ both at large and small $r$.

For the smeared differential distributions near maximal $q^2$ the violation of the *local duality* is found at the order $\Lambda\,\delta m/m_c^2$.

We make an explicit proof of the present results for the special case of the harmonic oscillator potential in Ref. [17]. This is important since some demonstrations given below are rather formal.

In the next section we present some details of the kinematics and discuss the analytical properties of the decay amplitude. In Sec. III the $1/m_Q$ expansion of the quark propagator is performed and the OPE series for the SL decay rate in nonrelativistic quantum mechanics is constructed. In Sec. IV we consider the HQ expansion of the exclusive form factors in the potential model, and derive the inclusive sum rules (Bjorken, Voloshin, etc.) which are crucial for comparing the exclusive sum and the OPE result. In Sec. V we provide an analytic expression for the duality-violation contribution and identify its origin. We estimate the accuracy of the OPE both for the integrated rate and the smeared distribution near zero recoil. A special emphasis is laid on discussing the role of different inclusive sum rules in establishing the relationship between the OPE and the sum over the exclusive channels. A conclusion summarizes our results.

## II. KINEMATICS AND THE ANALYTICAL PROPERTIES OF THE DECAY AMPLITUDE

We consider the inclusive SL decay $B\to X_c l\nu$. The rate of this process reads

$$\Gamma(B\to X_c l\nu)=\frac{1}{2M_B}\int\frac{d^4q}{2\pi}\theta(q^0>|\vec{q}|)L(q)W(q), \quad (1)$$

where $L$ is the leptonic tensor, and the hadronic tensor $W$ is defined as follows:

$$W=\sum_X \int d^4p_X\theta(p_X^0)\delta(p_X^2-M_X^2)\langle B|J|X(\vec{p})\rangle \times\langle X(\vec{p})|J^+|B\rangle\delta_4(p_B-p_X-q). \quad (2)$$

Here the relativistic normalization of states is implied:

$$\langle p|p'\rangle=2p^0(2\pi)^3\delta(\vec{p}-\vec{p}\,'). \quad (3)$$

For the sake of clarity we assume the technical simplification that the leptons are coupled to hadrons through the *scalar current*.[1] In this case the leptonic tensor is a scalar function of only one variable, $q^2$, and the hadronic tensor $W$ depends on the two invariant variables $\nu=P_B\cdot q/M_B$ and $q^2$. In the rest frame of the $B$-meson these are $q^0$ and $q^2$. At $q^0>0$ and fixed $q^2$ the sum in Eq. (2) runs over the hadronic states with masses $M_X<M_B-\sqrt{q^2}$. The decay rate can be written as follows:

$$\Gamma(B\to X_c l\nu)=\frac{1}{2M_B}\int dq^2dq^0|\vec{q}|\theta \times(q^0>|\vec{q}|)L(q^2)W(q^0,q^2), \quad (4)$$

with $q^2=(q^0)^2-\vec{q}^2$.

Equivalently, we can use $q^0$ and $\vec{q}^2$. Let us consider the $W(q^0,\vec{q}^2)$ as an analytical function of $q^0$ at fixed $\vec{q}^2$. One can write the following relation:

$$\frac{1}{2M_B}W(q^0,\vec{q}^2)=\frac{1}{2\pi i}\mathrm{disc}_{q^0}T(q^0,\vec{q}^2), \quad (5)$$

where

$$T(q^0,\vec{q}^2)=\frac{1}{2M_B}\int dx\exp^{-iqx}\langle B|T[J(x),J^+(0)]|B\rangle =\frac{1}{2M_B}\sum_X\frac{\langle B|J|X(-\vec{q})\rangle|^2}{M_X-E_X(-\vec{q})-q^0}, \quad (6)$$

$E_X(-\vec{q})=\sqrt{\vec{q}^2+M_X^2}$ is the energy of the state with the mass $M_X$ and the total 3-momentum $-\vec{q}$. The sum over $X$ in Eq. (6) for $T$ runs over *all* hadron states with the appropriate quantum numbers. The selection of the states kinematically allowed in the decay process is made by the proper choice of the integration contour in the complex $q^0$ plane. Namely, the

[1]Recall that for the case of the $V-A$ current and massless leptons, the leptonic tensor has the form $L_{\mu\nu}\sim g_{\mu\nu}q^2-q_\mu q_\nu$, and for the scalar current $L\sim q^2$. We consider throughout the paper the leptonic tensor of the generalized form $L=(q^2)^N$.

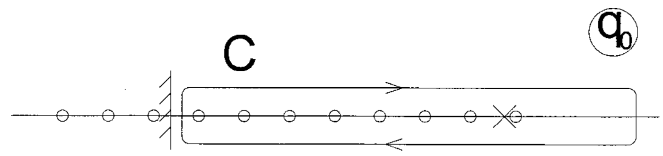


FIG. 1. Singularities of the amplitude $T(q^0,\vec{q}^2)$ in the complex $q^0$ plane. Circles are hadronic $(\bar{c}q)$ poles which are only singularities in the confined potential model, and the cross stands for the free $b\to c$ quark process. The vertical line $\mathrm{Re}\,(q^0)=|\vec{q}|$ separates the kinematically allowed region of the real axis from the kinematically forbidden region. The contour $C(\vec{q}^2)$ embraces all states in the allowed region. Poles at the left of the boundary correspond to kinematically forbidden bound states.

decay rate $\Gamma(B\to X_c l\nu)$ can be represented as the contour integral in the complex $q^0$ plane over the contour $C(\vec{q}^2)$ which depends on the value of $\vec{q}^2$ (Fig. 1) as follows:

$$\Gamma(B\to X_c l\nu)=\frac{1}{2\pi i}\int d\vec{q}^2|\vec{q}|\int_{C(\vec{q}^2)}dq^0\theta(q^0>|\vec{q}|) \times L((q^0)^2-\vec{q}^2)T(q^0,\vec{q}^2). \quad (7)$$

It is important that the left crossing of the contour $C(\vec{q}^2)$ with the real axis in the complex $q^0$ plane occurs at the point $q_0=|\vec{q}|$, otherwise the contour can be freely deformed in the region where the function $T_0(q_0,\vec{q}^2)$ is analytic. We symbolically mark this constraint with a $\theta$-function in the integrand. The integration over such contour selects at any given $\vec{q}^2$ only physical states which can be produced in the decay $B\to X_c l\nu$, i.e. states with the invariant masses such that $\sqrt{M_X^2+\vec{q}^2}<M_B-|\vec{q}|$. Notice that whereas the left crossing of the contour with the real axis is tightly fixed at the point $\mathrm{Re}\,(q^0)=|\vec{q}|$, $\mathrm{Im}\,(q^0)=0$, the right crossing of the contour with the real axis can be safely moved to the right. In the general relativistic case there are cuts which correspond to other physical processes. In the SV limit these cuts are separated from the physical decay cut by windows of the width $O(m_Q)$. In the potential quark model such cuts are absent.

The amplitude $T(q^0,\vec{q}^2)$ is given by the $T$-product of the two local current operators, which is the classical case for performing the OPE. Namely, one has

$$T(q^0,\vec{q}^2)=\frac{1}{2M_Bi}\int dx\exp(-iqx)\langle B|T(J(x),J^+(0))|B\rangle =\sum_n C_n(q^0,\vec{q}^2)\langle B|\hat{\mathcal{O}}_n|B\rangle, \quad (8)$$

where $\hat{\mathcal{O}}_n$ are local operators and $C_n(q^0,\vec{q}^2)$ are the $c$-number coefficients. Introducing the expansion (8) into Eq. (7) gives the integrated rate as a sum over various local operators.

We shall obtain the integrated SL decay rate within our model by two means: first, we construct the OPE series for $T(q^0,q^2)$, and second, we calculate directly the sum over exclusive channels.

### III. THE MODEL

We consider this decay in the SV limit,

$$\Lambda \ll \delta m = m_b - m_c \ll m_c, m_b. \tag{9}$$

Notice that in a non-relativistic model, $\Lambda$ refers to a fixed energy scale proportional to the light quark mass $m_d$, to the average quark momentum in the hadron rest frame $(\langle B|\vec{k}^2|B\rangle)^{1/2}$, to the parameter $\beta$ defined in Eq. (24) and $h_{bd}$ or $\epsilon_B$ to be defined in Eq. (11). These parameters may be strongly hierarchized, for example a genuine nonrelativistic situation implies $(\langle B|\vec{k}^2|B\rangle)^{1/2} \ll m_d$, but all these quantities remain constant as $m_c, m_b \to \infty$, they remain proportional to some fixed hadronic scale which we call $\Lambda$ by analogy with QCD. This is to be distinguished from $\delta m$ which is taken as an independent parameter. Thus we consider the double limit $\delta m/m_c \to 0$, and $\Lambda/\delta m \to 0$. Notice finally that $|\vec{q}|$ is of order $\delta m$.

To avoid confusion, it is important to stress that the standard OPE expansion assumes $\delta m/m_Q$ constant, even if small. So the order of a term $O(\Lambda^{n+m}/m_c^n(\delta m)^m)$ in this paper corresponds to the order $O(\Lambda^{n+m}/m_Q^{n+m})$ of the standard OPE expansion.

We treat the leptonic part relativistically, but for the calculation of the hadronic tensor we use the nonrelativistic potential model. The nonrelativistic treatment of the hadronic tensor is consistent within the SV kinematics and can be used as a tool for studying some of the aspects of quark-hadron duality. We shall make use the fact that in the nonrelativistic potential model we know the structure of the hadron spectrum and have an exact representation for the hadronic matrix elements of the quark currents.

It is convenient to use the nonrelativistic normalization of states (which is used hereafter)

$$\langle p|p'\rangle = (2\pi)^3 \delta(\vec{p} - \vec{p}\,'), \tag{10}$$

and consider the process in the rest frame of the decaying $B$-meson. The $B$ meson is the ground eigenstate of the Hamiltonian $\hat{H}_{bd}$,

$$\hat{H}_{bd}|B\rangle = M_B|B\rangle = (m_b + m_d + \epsilon_B)|B\rangle. \tag{11}$$

In the $B$-rest frame this Hamiltonian has the form

$$\hat{H}_{bd} = m_b + m_d + \frac{\vec{k}^2}{2m_b} + \frac{\vec{k}^2}{2m_d} + V_{bd}(r) \equiv m_b + m_d + h_{bd}. \tag{12}$$

For the $B \to X_c$ transition we need the $c\bar{d}$ bound states with the total 3-momentum $-\vec{q}$, which we denote $D_n(-\vec{q})$. These are eigenstates of the Hamiltonian

$$\hat{H}_{cd}(\vec{q}) = m_c + m_d + \frac{(\vec{k}+\vec{q})^2}{2m_c} + \frac{\vec{k}^2}{2m_d} + V_{cd}(r), \tag{13}$$

such that

$$\hat{H}_{cd}(\vec{q})|D_n(-\vec{q})\rangle = E_{D_n}(\vec{q})|D_n(-\vec{q})\rangle. \tag{14}$$

In this equation $E_{D_n}(\vec{q})$ is the nonrelativistic energy of the bound state $D_n$ with the 3-momentum $-\vec{q}$

$$E_{D_n}(\vec{q}) = M_{D_n} + \frac{\vec{q}^2}{2(m_c+m_d)}, \quad M_{D_n} = m_c + m_d + \epsilon_{D_n}. \tag{15}$$

The expression (6) for the decay amplitude $T$ now takes the form

$$T(q^0,\vec{q}) = \sum_n^{\infty} |F_n(\vec{q})|^2 \frac{1}{M_B - E_{D_n}(\vec{q}) - q^0}, \tag{16}$$

where $F_n(\vec{q})$ is the $B \to D_n$ transition form factor,

$$F_n(\vec{q}) = \langle B|J|D_n(-\vec{q})\rangle, \quad M_B = m_b + m_d + \epsilon_B,$$

and the sum runs over all $c\bar{d}$ resonances. The expression (16) can be also written as

$$T(q^0,\vec{q}) = \langle B|J \frac{1}{M_B - q^0 - \hat{H}_{cd}(\vec{q})} J^+|B\rangle, \tag{17}$$

where $[\hat{H}_{cd}(\vec{q}) - E]^{-1} = G_{cd}(\vec{q},E)$ is the full off-energy-shell Green function (propagator) of the $c\bar{d}$ system. The $B$ decay amplitude is thus given by an average of the Green function $\hat{G}_{cd}(\vec{q},E)$ at the point $E = M_B - q^0$ over the $B$ meson.

Let us specify the transition current operator $\hat{J}_{b\to c}$. For the sake of argument we neglect the quark spin effect and consider spinless nonrelativistic quarks and choose the quark current in the form

$$\hat{J}_{b\to c} = \int d\vec{k} d\vec{k}' \hat{b}(\vec{k}')\hat{c}^+(\vec{k}), \tag{18}$$

where $\hat{c}(\hat{b})$ is the annihilation operator of the $c(b)$ quark.[2]

For the quark current (18) the $B \to D_n$ transition form factor in the rest frame of the $B$-meson reads

$$F_n(\vec{q}) = \int d\vec{k}_q \psi_B(\vec{k}_q)\psi_{D_n}\left(\vec{k}_q + \frac{m_d}{m_c+m_d}\vec{q}\right), \tag{19}$$

where $\vec{k}_q$ is the momentum of the light spectator.

[2]Notice that the standard scalar current reads $\hat{J}_{b\to c} = \int (d\vec{k}/2k^0)(d\vec{k}'/2k'^0)\hat{b}(\vec{k}')\hat{c}^+(\vec{k})$ and in the nonrelativistic limit takes the form $\simeq \int d\vec{k}\hat{c}^+(\vec{k})d\vec{k}'\hat{b}(\vec{k}')(1-\vec{k}^2/4m_c^2)(1-\vec{k}'^2/4m_b^2)$. Neglecting the factor $(1-\vec{k}^2/4m_c^2)(1-\vec{k}'^2/4m_b^2)$ as done in Eq. (18) leads to technical simplifications both in the OPE and in the exclusive sum. It can be easily realized that a particular choice of the current however does not touch any arguments related to duality.

Similarly, for the current (18) the expression (17) takes the form

$$T(q^0,\vec{q}) = \langle B| \frac{1}{M_B - q^0 - \hat{H}_{cd}(\vec{q})} |B\rangle. \tag{20}$$

### IV. THE OPE OF THE DECAY RATE

The main idea in constructing the OPE series for $T$, Eq. (8), is to single out $\hat{H}_{bd}$ from $\hat{H}_{cd}(\vec{q})$ in the denominator in Eq. (20) and to use the eigenvalue equation (11). First, let us introduce the operator $\delta H(\vec{q})$ which measures the difference of the denominator of Eq. (20) from the inverse Green function of the free-quark transition

$$M_B - q^0 - \hat{H}_{cd}(\vec{q}) = \left( \delta m - \frac{\vec{q}^2}{2m_c} - q^0 \right) - \delta H(\vec{q}). \tag{21}$$

Explicitly, one finds

$$\delta H(\vec{q}) = \hat{H}_{cd}(\vec{q}) - m_c - m_d - \frac{\vec{q}^2}{2m_c} - \epsilon_B. \tag{22}$$

Next, isolating $h_{bd}$ in $\delta H(\vec{q})$ we obtain

$$\delta H(\vec{q}) = (h_{bd} - \epsilon_B) + \frac{1}{2}\left( \frac{1}{m_c} - \frac{1}{m_b} \right)(\vec{k}^2 + V_1) + \frac{\vec{k}\cdot\vec{q}}{m_c} + O\left( \frac{\beta^3 \delta m}{m_c^3} \right), \tag{23}$$

where the scale $\beta$ is provided by the hadronic matrix elements

$$\beta^2 \simeq \langle B|\vec{k}^2|B\rangle \simeq \langle B|\hat{V}_1|B\rangle. \tag{24}$$

As already mentioned, $\beta$ is of the order of $\Lambda$. The quantity $V_1$ here is a part of the expansion of the potential $V_{Qq}$ in powers of $1/m_Q$

$$V_{Qq} = V_0 + \frac{1}{2m_Q} V_1 + \frac{1}{2m_Q^2} V_2 + O\left( \frac{\Lambda^4}{m_Q^3} \right). \tag{25}$$

Equations (20) and (21) allow us to construct the expansion of $T(q^0,\vec{q}^2)$ in inverse powers of $\delta m - \vec{q}^2/2m_c - q^0$ as follows:

$$T(q_0,\vec{q}^2) = \frac{1}{\delta m - \frac{\vec{q}^2}{2m_c} - q^0} \sum_{i=0}^{\infty} \frac{\langle B|(\delta H(\vec{q}))^i|B\rangle}{\left( \delta m - \frac{\vec{q}^2}{2m_c} - q^0 \right)^i}. \tag{26}$$

Making use of Eq. (23) we obtain

$$T(q_0,\vec{q}^2) = \langle B|B\rangle \frac{1}{\delta m - \frac{\vec{q}^2}{2m_c} - q^0} + \langle B|(h_{bd} - \epsilon_B)|B\rangle \frac{1}{\left( \delta m - \frac{\vec{q}^2}{2m_c} - q^0 \right)^2} + \langle B| \left[ \frac{\delta m}{2m_c^2}(\vec{k}^2 + V_1) - \frac{\vec{k}\cdot\vec{q}}{m_c} \right] |B\rangle \times \frac{1}{\left( \delta m - \frac{\vec{q}^2}{2m_c} - q^0 \right)^2} + \sum_{i,j=1}^{3} \sum_{n=0}^{\infty} q_i q_j \langle B| \left[ \frac{k_i (h_{bd} - \epsilon_B)^n k_j}{m_c^2} \right] |B\rangle \frac{1}{\left( \delta m - \frac{\vec{q}^2}{2m_c} - q^0 \right)^{3+n}} + O(\Lambda^2/m_c^3). \tag{27}$$

The remainder has the order $O(\Lambda^2/m_c^3)$ if we keep $\vec{q}^2 \simeq \delta m^2$ and $q_0$ fixed.

Finally, using Eq. (11) and the relations $\langle B|k_i|B\rangle = 0$ and $\langle B|k_i k_j|B\rangle = \frac{1}{3}\delta_{ij}\langle B|\vec{k}^2|B\rangle$ we find the following OPE series:

$$T(q^0,\vec{q}^2) = \frac{1}{\delta m - \frac{\vec{q}^2}{2m_c} - q^0} + \frac{\delta m}{2m_c^2}\langle B|\vec{k}^2 + V_1|B\rangle \frac{1}{\left( \delta m - \frac{\vec{q}^2}{2m_c} - q^0 \right)^2} + \sum_{n=2}^{\infty} \frac{\langle B|\hat{O}_{n-2}|B\rangle}{3m_c^2} \vec{q}^2 \frac{1}{\left( \delta m - \frac{\vec{q}^2}{2m_c} - q^0 \right)^{n+1}} + O(\Lambda^2/m_c^3), \tag{28}$$

where $\hat{O}_n = \Sigma_{j=1}^3 k_j (h_{bd} - \epsilon_B)^n k_j$. Hereafter the $\Sigma$ symbol is omitted. We denote $\langle B|\vec{k}^2 + V_1|B\rangle = \beta_0^2$, $\beta_0 \simeq \Lambda$.

The series (28) is a double expansion of $T(q^0,q^2)$ in $\Lambda/m_c$ and $\Lambda/(\delta m-\vec{q}^2/2m_c-q^0)$, limited to second order in $\Lambda/m_c$ and expanded to all orders in $\Lambda/(\delta m-\vec{q}^2/2m_c-q^0)$. The poles are at

$$q_c^0(\vec{q}^2)=\delta m-\frac{\vec{q}^2}{2m_c}. \tag{29}$$

The *first term* in Eq. (28) gives *the free quark decay amplitude*. A remarkable feature of this series is that the $\Lambda/\delta m$ and $\Lambda/m_c$ corrections to the free-quark decay are absent thanks to Eq. (11) and the relation $\langle B|k_i|B\rangle=0$. The expansion (28) substitutes the whole set of hadron poles by a complicated quark singularity at the point $q^0=q_c^0(\vec{q}^2)$.

Let us treat the series (28) *formally* and calculate the integrated rate which is obtained as a double expansion in $\Lambda/m_c$ and $\Lambda/\delta m$.

Let us rewrite the expression (28) as follows:

$$T(q^0,\vec{q}^2)=\left(1-\frac{\langle B|\vec{k}^2+V_1|B\rangle}{2m_c^2}\delta m\frac{\partial}{\partial\delta m}+\vec{q}^2\sum_{n=2}^{\infty}\frac{\langle B|\hat{O}_{n-2}|B\rangle}{3m_c^2}\frac{1}{n!}\left(-\frac{\partial}{\partial\delta m}\right)^n\right)\frac{1}{\left(\delta m-\frac{\vec{q}^2}{2m_c}-q^0\right)}. \tag{30}$$

This representation is very convenient for the calculation of the decay rate Eq. (7): the integration over $q^0$ is now easily performed since

$$\int_{C(\vec{q}^2)}dq^0L((q^0)^2-\vec{q}^2)\theta(q^0>|\vec{q}|)\frac{1}{\left(\delta m-\frac{\vec{q}^2}{2m_c}-q^0\right)}=L(q_c^0(\vec{q}^2),\vec{q}^2)\,\theta(|\vec{q}|<-m_c+\sqrt{m_c^2+2m_c\delta m}), \tag{31}$$

where the $\theta$-function $\theta(|\vec{q}|<-m_c+\sqrt{m_c^2+2m_c\delta m})$ reflects the fact that the left crossing of the contour with the real axis in the complex $q^0$ plane should always happen at the point $\mathrm{Re}(q^0)=|\vec{q}|$. The integrated rate is then given by the expression[3]

$$\Gamma^{OPE}(B\to X_cl\nu)=\left(1-\frac{\langle B|\vec{k}^2+V_1|B\rangle}{2m_c^2}\delta m\frac{\partial}{\partial\delta m}\right)I_1(\delta m,m_c)+\sum_{n=2}^{\infty}\frac{\langle B|\hat{O}_{n-2}|B\rangle}{3m_c^2}\frac{1}{n!}\left(-\frac{\partial}{\partial\delta m}\right)^nI_3(\delta m,m_c), \tag{32}$$

where

$$I_n(\delta m,m_c)=\int_0^{-m_c+\sqrt{m_c^2+2m_c\delta m}}d\vec{q}^2|\vec{q}|^nL([q_c^0(\vec{q}^2)]^2-\vec{q}^2). \tag{33}$$

For the free quark decay one finds

$$\Gamma(b\to cl\nu)=I_1(\delta m,m_c). \tag{34}$$

Let us consider the leptonic tensor of the general form $L(q^2)=(q^2)^N$. For semileptonic decays to massless spin 1/2 leptons $N=1$. The case $N=0$ corresponds to scalar leptons. Since the leptonic tensor is proportional to $q^2$, it is now convenient to introduce a new integration variable $q^2$ as follows:

$$q^2=[q_c^0(\vec{q}^2)]^2-\vec{q}^2. \tag{35}$$

Then the integrated rate takes the form

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[3]A remark is in order here. When computing the decay rate in Eq. (32) we have interchanged the derivation with respect to $\delta m$ and the integration over $dq^0$. We can also directly integrate the expression Eq. (28) over $dq^0$. In this case we should take into account that the contour $C(\vec{q}^2)$ in the complex $q^0$-plane always lies on the right-hand side (RHS) of the line $\mathrm{Re}(q^0)=|\vec{q}|$. If we erroneously do not take this condition into account then multiple poles in Eq. (28) do not contribute at all since the complex integral vanishes when the multiple poles are inside the contour as well as when they are outside (see also discussion in [2]). To proceed correctly, one should replace a multiple pole, say a double pole, by an equivalent set of two neighboring poles. Then the crossing of the boarder gives a nonvanishing result when one of the two poles is inside the contour and the other one is outside. Integrating before taking the derivative with respect to $\delta m$ as in Eq. (32) corresponds to treating in a specific way the crossing of the boarder by the multiple poles. We show elsewhere [13] that both treatments lead to the same result.

$$\Gamma^{OPE}(B\to X_c l\nu)=\Gamma(b\to cl\nu)-\frac{\langle B|\vec{k}^2+V_1|B\rangle}{2m_c^2}\delta m\frac{\partial}{\partial\delta m}I_1(\delta m,m_c)+\sum_{n=2}^{\infty}\frac{\langle B|\hat{O}_{n-2}|B\rangle}{3m_c^2}\frac{1}{n!}\left(-\frac{\partial}{\partial\delta m}\right)^n I_3(\delta m,m_c), \quad (36)$$

where we have taken into account that $I_1(\delta m,m_c)$ gives the exact free-quark decay rate. A simple algebraic exercise gives to the $1/m_c^2$ accuracy

$$\begin{aligned} I_1(\delta m,m_c)&=(\delta m)^{2N+3}\left[A_{1/2}^N\left(1-\frac{3}{2}\frac{\delta m}{m_c}+\frac{15}{8}\frac{\delta m^2}{m_c^2}\right)+\frac{5}{8}A_{3/2}^N\frac{\delta m^2}{m_c^2}+O\left(\frac{\delta m^3}{m_c^3}\right)\right],\\ I_3(\delta m,m_c)&=(\delta m)^{2N+5}\left[A_{3/2}^N+O\left(\frac{\delta m}{m_c}\right)\right], \end{aligned} \quad (37)$$

where

$$A_m^N=\int_0^1 dx x^m(1-x)^N=B(N+1,m+1), \quad A_{3/2}^N=\frac{3}{2N+5}A_{1/2}^N, \quad (38)$$

$B(p,q)$ being the Euler function. Finally, we come to the relation

$$\frac{\Gamma(B\to X_c l\nu)}{\Gamma(b\to cl\nu)}=1+\frac{\langle B|\vec{k}^2|B\rangle}{2m_c^2}-(2N+3)\frac{\langle B|V_1|B\rangle}{2m_c^2}+\sum_{n=3}^{2N+5}\frac{(-1)^n C_{2N+5}^n}{2N+5}\frac{\langle B|\hat{O}_{n-2}|B\rangle}{m_c^2\delta m^{n-2}}+O\left(\frac{\Lambda^2\delta m}{m_c^3}\right), \quad (39)$$

with $C_n^k=n!/k!(n-k)!$. Notice that the coefficient of the term $\langle B|\vec{k}^2|B\rangle$ does not depend on $N$, i.e. it does not depend on the form of the leptonic tensor.

Summing up, the OPE predicts the following features of the inclusive SL decay rate.

The LO term reproduces the rate of the free-quark decay process $b\to c$.

The $1/m_c$ and $1/\delta m$ corrections are absent. This is due to the fact that the average over the $B$-state of the operator of the relevant dimension vanishes.

Lowest-order corrections to the free-quark process emerge in the $1/m_c^2$ order. A main part of these corrections is due to the average values of the dimension-2 operators $\langle B|\vec{k}^2|B\rangle$ and $\langle B|\vec{k}^2+V_1|B\rangle$. Also the operators $\hat{O}_n=k_j(h_{bd}-\epsilon_B)^n k_j$ contribute in the $1/m_c^2$ order. Their contribution is however suppressed with the additional powers of $\delta m$.

In the next section we shall analyze the accuracy of the OPE predictions.

## V. HEAVY QUARK EXPANSION AND THE HADRONIC SUM RULES

Before proceeding with the direct summation of the exclusive channels one by one we derive hadronic sum rules which are important for the comparison of the exact result with the OPE analysis.

### A. Heavy quark expansion of the form factors in the potential model

The wave function of the $Q\bar{q}$ bound state has the form

$$\begin{aligned}\Psi_{\vec{p}}(\vec{k}_Q,\vec{k}_q)&=\delta(\vec{p}-\vec{k}_Q-\vec{k}_q)\psi\left(\frac{m_Q m_q}{m_Q+m_q}\left(\frac{\vec{k}_q}{m_q}-\frac{\vec{k}_Q}{m_Q}\right)\right)\\ &=\delta(\vec{p}-\vec{k}_Q-\vec{k}_q)\psi\left(\vec{k}_q-\frac{m_q}{m_Q+m_q}\vec{p}\right), \end{aligned} \quad (40)$$

where $\vec{p}$ is the momentum of the bound state.

The $B\to D_n$ transition form factor is the average over the meson states of the operator $\Omega_{bc}(\vec{q})$ given by the following kernel:

$$\langle\vec{k}_b|\Omega_{bc}(\vec{q})|\vec{k}_c\rangle=\delta(\vec{k}_b-\vec{k}_c-\vec{q}). \quad (41)$$

So the transition form factor is defined by the following expression:

$$\langle B(\vec{p}_B)|\Omega_{bc}(\vec{q})|D_n(\vec{p}_n)\rangle=\delta(\vec{p}_B-\vec{p}_n-\vec{q})F_n((\vec{v}_B-\vec{v}_n)^2), \quad (42)$$

with $\vec{v}_B=\vec{p}_B/(m_b+m_q)$ and $\vec{v}_n=\vec{p}_n/(m_c+m_q)$ and

$$F_n(\vec{v}_B-\vec{v}_n)=\int d\vec{k}_q\psi_B\left(\vec{k}_q-\frac{m_q}{m_b+m_q}\vec{p}_B\right)\times\psi_{D_n}\left(\vec{k}_q-\frac{m_q}{m_c+m_q}\vec{p}_n\right). \quad (43)$$

A simple change of variables $\vec{k}_q\to\vec{k}_q+m_q/(m_b+m_q)\vec{p}_B$ makes it obvious that the decay form factor depends on the square of the relative 3-velocities of the initial and final mesons, and not on the relative 3-momentum squared as the elastic form factor. Nevertheless in the $B$ rest frame we write

$$F_n(\vec{q})=\int d\vec{k}_q \psi_B(\vec{k}_q)\psi_{D_n}\left(\vec{k}_q+\frac{m_q}{m_c+m_q}\vec{q}\right). \quad (44)$$

The wave function $\psi_{Q\bar{q}}$ is an eigenstate of the Hamiltonian

$$h_{Q\bar{q}}=\frac{\vec{k}^2}{2}\left(\frac{1}{m_Q}+\frac{1}{m_q}\right)+V_{Qq}(r), \quad (45)$$

where $\vec{k}$ is the conjugate variable to $\vec{r}$. The transition form factors $F_n$ has some general properties independent of the details of the potential $V_{Q\bar{q}}$. Such properties of the transition form factors are derived by performing the HQ expansion of the Hamiltonian. To this end we apply the usual quantum mechanical perturbation theory.

For our purposes it is convenient to consider $h_{bd}$ as the full Hamiltonian, $h_{cd}$ as a nonperturbed Hamiltonian, and $\hat{U}=h_{bd}-h_{cd}$ as the perturbation. The perturbation has the form

$$\hat{U}=\frac{1}{2}\left(\frac{1}{m_b}-\frac{1}{m_c}\right)(\vec{k}^2+V_1)+O\left(\beta^3\frac{\delta m}{m_c^3}\right)$$
$$=-\frac{\delta m}{2m_c^2}(\vec{k}^2+V_1)+O\left(\beta^2\frac{\delta m^2}{m_c^3}\right), \quad (46)$$

where we assume the following expansion of the $Q\bar{q}$ potential

$$V_{Q\bar{q}}=V_0+\frac{1}{2m_Q}V_1+\frac{1}{2m_Q^2}V_2+\cdots. \quad (47)$$

The perturbation has the order $\delta m/m_c^2$ such that we can construct the HQ expansion of the wave functions and the binding energies. Let us remind the standard formulas: Let $\{\psi_{D_n}\}$ be the full system of eigenstates of the $h_{cd}$, and the $\{\epsilon_{D_n}\}$ the corresponding eigenvalues. Then, the mass of the $n$th excitation in the $c\bar{d}$ system reads $M_{D_n}=m_c+m_d+\epsilon_{D_n}$. Let $\{\psi_{B_n}\}$ be the full system of eigenstates of the $h_{bd}$, and the $\{\epsilon_{B_n}\}$ the corresponding eigenvalues.

The standard formulas give

$$\psi_{B_n}=\psi_{D_n}+\sum_{m\neq n}\frac{U_{mn}}{\epsilon_{D_n}-\epsilon_{D_m}}\psi_{D_m}+O(\delta m^2\beta^2/m_c^4) \quad (48)$$

and

$$\epsilon_{B_n}=\epsilon_{D_n}+U_{nn}+\sum_{m\neq n}\frac{|U_{mn}|^2}{\epsilon_{D_n}-\epsilon_{D_m}}+\cdots, \quad (49)$$

where[4]

[4] $\epsilon_{B_0}$ is just $\epsilon_B$ defined in Eq. (11). We also usually write $B$ instead of $B_0$.

$$U_{mn}=-\frac{\delta m}{2m_c^2}\langle\psi_{D_n}|\vec{k}^2+V_1|\psi_{D_m}\rangle+O\left(\beta^2\frac{\delta m^2}{m_c^3}\right). \quad (50)$$

The excitation energies satisfy the relation

$$\epsilon_{D_n}-\epsilon_{D_m}\simeq(n-m)\lambda_{nm}, \quad \lambda_{mn}\simeq\beta. \quad (51)$$

In terms of the wave functions, the transition form factor (44) takes a simple form:

$$F_n(\vec{q})=\langle\psi_{B_0}|\psi_{D_n}(-\vec{q})\rangle. \quad (52)$$

The expansion of the wave function $\psi_{B_0}\equiv\psi_B$ reads

$$\psi_{B_0}=\psi_{D_0}+\sum_{m\neq 0}\frac{1}{2}\left(\frac{1}{m_b}-\frac{1}{m_c}\right)\frac{\langle\psi_{D_0}|\vec{k}^2+V_1|\psi_{D_m}\rangle}{\epsilon_{D_n}-\epsilon_{D_m}}\psi_{D_m}$$
$$+O(\delta m^2\beta^2/m_c^4), \quad (53)$$

such that

$$F_n(\vec{q})=\langle\psi_{D_0}+\sum_{m\neq 0}\frac{1}{2}\left(\frac{1}{m_b}-\frac{1}{m_c}\right)\frac{\langle\psi_{D_0}|\vec{k}^2+V_1|\psi_{D_m}\rangle^*}{\epsilon_{D_0}-\epsilon_{D_m}}$$
$$\times\psi_{D_m}|\psi_{D_n}(\vec{q})\rangle+O(\delta m^2\beta^2/m_c^4)$$
$$=f_{0n}(\vec{q})+\sum_{m\neq 0}\left(-\frac{\delta m}{2m_c^2}\right)\frac{\langle\psi_{D_0}|\vec{k}^2+V_1|\psi_{D_m}\rangle^*}{\epsilon_{D_0}-\epsilon_{D_m}}f_{mn}(\vec{q})$$
$$+O(\delta m^2\beta^2/m_c^4), \quad (54)$$

where $f_{mn}(\vec{q})=\langle\psi_{D_m}|\psi_{D_n}(\vec{q})\rangle$. By virtue of Eq. (44) one obtains

$$f_{nn}^2(\vec{q})=1-r_{nn}^2\frac{\vec{q}^2}{m_c^2}+O(\vec{q}^4/m_c^4),$$

$$f_{nm}^2(\vec{q})=r_{nm}^2\frac{\vec{q}^2}{m_c^2}+O(\vec{q}^4/m_c^4), \quad m\neq n, \quad (55)$$

with $r_{mn}$ being numbers of order unity plus higher order $1/m_c$ corrections. We shall use the notation $r_n=r_{n0}$. Notice that the radii $r_n$ describe the form factors of the transitions between different levels in the $c\bar{d}$ system ($D_0\to D_n$) and so know nothing about $\delta m$.

We now rewrite Eq. (54) as follows:

$$F_0(\vec{q})=1-r_0^2\frac{\vec{q}^2}{2m_c^2}+\sum_{m\neq 0}\left(-\frac{\delta m}{2m_c^2}\right)\frac{\langle\psi_{D_0}|\vec{k}^2+V_1|\psi_{D_m}\rangle^*}{\epsilon_{D_0}-\epsilon_{D_m}}$$
$$\times f_{m0}(\vec{q})+O\left(\frac{\delta m^2\beta^2}{m_c^4}\right). \quad (56)$$

At $\vec{q}^2=0$ we thus come to the relation

$$F_0(0)=1+O(\delta m^2\beta^2/m_c^4). \tag{57}$$

One can see that the $O(\delta m\beta/m_c^2)$ term in $F_n(0)$ is absent. This is a nonrelativistic analog of the Luke theorem [18].

For the squares of the form factors we obtain the following important relations:[5]

$$F_0^2(\vec{q})=1-\rho_0^2\frac{\vec{q}^2}{m_c^2}+O\left(\frac{\delta m^2\beta^2}{m_c^4}\right),\quad \rho_0^2=r_0^2+O\left(\frac{\beta\delta m}{m_c^2}\right),$$

$$F_n^2(\vec{q})=\rho_n^2\frac{\vec{q}^2}{m_c^2}+O\left(\frac{\delta m^2\beta^2}{m_c^4}\right),\quad \rho_n^2=r_n^2+O\left(\frac{\beta\delta m}{m_c^2}\right). \tag{58}$$

As we shall see later the radii $\rho_n$ (as well as $r_n$) are not independent and satisfy certain sum rules. The relations (58) are the main result of this section. They are necessary for the calculation of the decay rates.

### B. Inclusive hadronic sum rules

To obtain a nonrelativistic equivalent of the whole tower of sum rules [16], i.e., the Bjorken sum rule, the Voloshin and the higher moments, we consider the following set of quantities ($i=0,1,\dots$):

$$S_i(\vec{q})=\langle B|(\delta H(\vec{q}))^i|B\rangle, \tag{59}$$

where $\delta H(\vec{q})$ is defined in Eq. (21). Notice that $S_i(\vec{q})$ appear in the expansion for $T(q^0,\vec{q}^2)$, Eq. (26). We shall derive two different representations for $S_i(\vec{q})$ and obtain sum rules equating these representations.

The first representation is obtained by inserting the full system of the eigenstates $|D_n(-\vec{q})\rangle$ of the Hamiltonian $H_{cd}(\vec{q})$ in Eq. (59). The $|D_n(-\vec{q})\rangle$ are also eigenstates of the operator $\delta H(\vec{q})$ that is made obvious using $\delta H_{cd}(\vec{q})$ in the form Eq. (22):

$$\delta H(\vec{q})|D_n(-\vec{q})\rangle=\delta_n(\vec{q})|D_n(-\vec{q})\rangle,$$

$$\delta_n(\vec{q})=\epsilon_{D_n}-\epsilon_B+\frac{\vec{q}^2}{2(m_c+m_d)}-\frac{\vec{q}^2}{2m_c}. \tag{60}$$

As a result of inserting the full system we find

$$S_i(\vec{q})=\sum_{n=1}^{\infty}|F_n(\vec{q})|^2(\delta_n(\vec{q}))^i. \tag{61}$$

Equation (49) gives the following expansion for $\delta_n(\vec{q})$:

$$\delta_n(\vec{q})=\Delta_n+\frac{1}{2}\left(\frac{1}{m_c}-\frac{1}{m_b}\right)\langle D_0|\vec{k}^2+V_1|D_0\rangle -\frac{m_d\vec{q}^2}{2m_c(m_c+m_d)}+O\left(\frac{\delta m^2\beta^3}{m_c^4}\right), \tag{62}$$

where

$$\Delta_n\equiv\epsilon_{D_n}-\epsilon_{D_0}. \tag{63}$$

Notice that within the leading-order accuracy we can replace $\langle D_0|\vec{k}^2+V_1|D_0\rangle$ with $\langle B|\vec{k}^2+V_1|B\rangle$.

Another representation for $S_i(\vec{q})$ is obtained by using $\delta H(\vec{q})$ in the form (23):

$$S_i(\vec{q})=\langle B|(\delta H(\vec{q}))^i|B\rangle =\langle B|\left(\hat{h}_{bd}-\epsilon_B+\frac{\vec{k}^2+V_1}{2}\left(\frac{1}{m^c}-\frac{1}{m^b}\right) +\frac{\vec{k}\cdot\vec{q}}{m_c}+O\left(\frac{\beta^3\delta m}{m_c^3}\right)\right)^i|B\rangle. \tag{64}$$

This formula gives $S_i(\vec{q})$ in terms of the matrix elements of various operators over the $B$-meson.

The representations (61) and (64) for $S_i(\vec{q})$ provide the LHS and the RHS of the sum rules, respectively. Let us notice that terms denoted by $O(\beta^3\delta m/m_c^3)$ in Eqs. (62) and (64) do not depend on $\vec{q}$. All $\vec{q}$-dependent terms are shown explicitly.

Using Eq. (11) and the relations $\langle B|k_i|B\rangle=0$ and $\langle B|k_ik_j|B\rangle=\frac{1}{3}\delta_{ij}\langle B|\vec{k}^2|B\rangle$ we come to the set of sum rules. In fact each of these sum rules is equivalent to an infinite number of relations at different powers of $\vec{q}^2$ and $1/m_c$.

i=0:

$$S_0=\sum_n^{\infty}|F_n(\vec{q})|^2=1. \tag{65}$$

Obviously the RHS does not depend on $\vec{q}$. At $\vec{q}^2=0$ this is an identity. Using the definition (58) and comparing the term linear in $\vec{q}^2$ we find the *NR Bjorken sum rule* [14]

$$\rho_0^2=\sum_{n=1}^{\infty}\rho_n^2. \tag{66}$$

i=1: The RHS of this sum rule reads

$$S_1=\frac{1}{2}\left(\frac{1}{m_c}-\frac{1}{m_b}\right)\langle B|\vec{k}^2+V_1|B\rangle+O\left(\frac{\beta^3\delta m}{m_c^3}\right), \tag{67}$$

where we have used Eqs. (11) and (12). The RHS of this sum rule is also independent of $\vec{q}$. From the definition (59) and using Eq. (62) as well as the SR (65), we rewrite Eq. (67) as follows:

[5] At any $n$ states with angular momenta $L=0,\dots,n$ exist. The form factors $f_n(\vec{q}^2)$ and $F_n(\vec{q}^2)$ are thus understood as properly normalized sums $\Sigma_{L=0}^n f_{nL}(\vec{q})$ and $\Sigma_{L=0}^n F_{nL}(\vec{q})$, respectively.

$$\sum_{n=0}^{\infty} F_n^2(\vec{q})\Delta_n = \frac{\vec{q}^{\,2} m_d}{2m_c(m_c+m_d)} + O\left(\frac{\beta^3 \delta m}{m_c^3}\right). \tag{68}$$

Notice that the terms $O(\beta^2 \delta m/m_c^2)$ cancel between RHS amd LHS. Comparing the linear in $\vec{q}^{\,2}$ term yields the *NR Voloshin sum rule* [15]

$$\sum_{n=1}^{\infty} \rho_n^2 \Delta_n = \frac{m_d}{2} \frac{1}{1+\frac{m_d}{m_c}}. \tag{69}$$

Let us notice that the RHS of the Eq. (69) does not contain higher-order $1/m_c$ corrections.

Combining the Bjorken and the Voloshin sum rules provides a simple constraint on the parameter $\rho_0^2$ which is in fact the slope of the Isgur-Wise function. Namely,

$$\rho_0^2 = \sum_{n=1}^{\infty} \rho_n^2 = \frac{1}{\Delta_1}\sum_{n=1}^{\infty} \rho_n^2 \Delta_1 < \frac{1}{\Delta_1}\sum_{n=1}^{\infty} \rho_n^2 \Delta_n$$
$$= \frac{m_d}{2\Delta_1}\frac{1}{1+m_d/m_c} < \frac{m_d}{2\Delta_1}, \tag{70}$$

where $\Delta_n$ are defined in Eq. (63)

i=2: The RHS of this sum rule reads

$$S_2 = \frac{\vec{q}^{\,2}\langle B|\vec{k}^2|B\rangle}{3m_c^2} + O\left(\frac{\beta^4 \delta m^2}{m_c^4}\right). \tag{71}$$

Using Eqs. (65) and (67) yields for the LHS

$$\sum_{n=0}^{\infty} F_n^2(\vec{q})\Delta_n^2 = \frac{\vec{q}^{\,2}\langle B|\vec{k}^2|B\rangle}{3m_c^2}\left(1+O\left(\frac{\beta^2 \delta m}{m_c^3}\right)\right)$$
$$+ \frac{\vec{q}^{\,4} m_d^2}{4m_c^2(m_c+m_d)^2} + O\left(\frac{\beta^4 \delta m^2}{m_c^4}\right). \tag{72}$$

The linear $\vec{q}^{\,2}$ term yields

$$\sum_{n=1}^{\infty} \rho_n^2 \Delta_n^2 = \frac{1}{3}\langle B|\vec{k}^2|B\rangle\left(1+O\left(\frac{\beta^2 \delta m}{m_c^3}\right)\right). \tag{73}$$

$i \geqslant 3$:

For i=3 we find for the RHS

$$S_3 = \frac{1}{3}\frac{\vec{q}^{\,2}\langle B|k_j(h_{bd}-\epsilon_B)k_j|B\rangle}{m_c^2}\left(1+O\left(\frac{\beta\delta m}{m_c^2}\right)\right)$$
$$+O\left(\frac{\beta^6 \delta m^3}{m_c^6}\right). \tag{74}$$

Using Eqs. (65)–(71) yields for the LHS

$$\sum_{n=0}^{\infty} F_n^2(\vec{q})\Delta_n^3 = \frac{\vec{q}^{\,2}\langle B|k_j(h_{bd}-\epsilon_B)k_j|B\rangle}{3m_c^2}\left(1+O\left(\frac{\beta\delta m}{m_c^2}\right)\right)$$
$$+\frac{\vec{q}^{\,4}}{m_c^4}O\left(\frac{m_d^2\beta^2 \delta m}{m_c^2}\right)+\frac{\vec{q}^{\,6}}{m_c^6}O(m_d^3)$$
$$+O\left(\frac{\beta^5 \delta m^2}{m_c^4}\right). \tag{75}$$

The linear $\vec{q}^{\,2}$ term yields

$$\sum_{n=1}^{\infty} \rho_n^2 \Delta_n^3 = \frac{\langle B|k_j(h_{bd}-\epsilon_B)k_j|B\rangle}{3}\left(1+O\left(\frac{\beta\delta m}{m_c^2}\right)\right)$$
$$= \frac{1}{3}\langle B|\hat{O}_1|B\rangle\left(1+O\left(\frac{\beta\delta m}{m_c^2}\right)\right). \tag{76}$$

Similarly at higher $i \geqslant 3$ one obtains at the $\beta\delta m/m_c^2$ accuracy

$$\sum_{n=1}^{\infty} \rho_n^2 \Delta_n^i = \frac{1}{3}\langle B|k_j(h_{bd}-\epsilon_B)^{i-2}k_j|B\rangle = \frac{1}{3}\langle B|\hat{O}_{i-2}|B\rangle. \tag{77}$$

These sum rules are used in the next section for comparison of the exact decay rate with the OPE result and for analyzing the duality-violation effects.

## VI. SUMMATION OVER THE EXCLUSIVE CHANNELS

We now proceed to the summation of the exclusive channels. As the first step, let us show that there is an explicit difference between the exclusive sum and the OPE series.

### A. The origin of duality violation

Proceeding with the sum over the exclusive channels we write

$$\Gamma(B\to X_c l\nu)=\frac{1}{2\pi i}\sum_n^{\infty}\int dq^2L(q^2)\int_{C(q^2)}dq^0|\vec{q}|\frac{|F_n(\vec{q})|^2}{M_b-q^0-E_n(\vec{q})}$$

$$=\frac{1}{2\pi i}\sum_n^{\infty}\int dq^2L(q^2)\int_{C(q^2)}dq^0|\vec{q}|\frac{|F_n(\vec{q})|^2}{\delta m-\frac{\vec{q}^2}{2m_c}-q^0+\delta_n(\vec{q})}$$

$$=\frac{1}{2\pi i}\sum_n^{\infty}\int dq^2L(q^2)\int_{C(q^2)}dq^0|\vec{q}|\frac{|F_n(\vec{q})|^2}{\delta m-\frac{\vec{q}^2}{2m_c}-q^0}\left[1-\frac{\delta_n(\vec{q})}{\delta m-\frac{\vec{q}^2}{2m_c}-q^0}+\cdots\right]$$

$$=\frac{1}{2\pi i}\int dq^2L(q^2)\sum_n^{n(q^2)}\int_{C(q^2)}dq^0|\vec{q}|\frac{|F_n(\vec{q})|^2}{\delta m-\frac{\vec{q}^2}{2m_c}-q^0}\left[1-\frac{\delta_n(\vec{q})}{\delta m-\frac{\vec{q}^2}{2m_c}-q^0}+\cdots\right], \quad (78)$$

where in the RHS $|\vec{q}|=\sqrt{(q^0)^2-q^2}$. Notice that the sum is truncated at the proper $n(q^2)$ which is the maximal number of hadron resonances kinematically allowed at a given value of $q^2$, i.e., resonances satisfying the relation $M_n<M_B-\sqrt{q^2}$. The contour $C(q^2)$ is responsible for this selection, since only the resonances enclosed by the contour contribute into the sum. All states which are beyond this contour do not contribute.

Finally, the series (78) can be written in the form

$$\Gamma(B\to X_c l\nu)=\int dq^2L(q^2)\theta(q^2)\int dq^0d\vec{q}^2|\vec{q}|\delta((q^0)^2-q^2-\vec{q}^2)\sum_n^{n(q^2)}|F_n(\vec{q})|^2$$
$$\times\left(1+\delta_n(\vec{q}^2)\frac{\partial}{\partial\delta m}+\frac{1}{2}\delta_n^2(\vec{q}^2)\frac{\partial^2}{\partial\delta m^2}+\cdots\right)\delta\left(q^0-\frac{\vec{q}^2}{2m_c}-\delta m\right). \quad (79)$$

On the other hand, the sum rules (67)–(74) allow us to rewrite the decay rate (32) in the form

$$\Gamma^{OPE}(B\to X_c l\nu)=\int dq^2L(q^2)\theta(q^2)\int dq^0d\vec{q}^2|\vec{q}|\delta((q^0)^2-q^2-\vec{q}^2)\sum_n^{\infty}|F_n(\vec{q})|^2$$
$$\times\left(1+\delta_n(\vec{q}^2)\frac{\partial}{\partial\delta m}+\frac{1}{2}\delta_n^2(\vec{q}^2)\frac{\partial^2}{\partial\delta m^2}+\cdots\right)\delta\left(q^0-\frac{\vec{q}^2}{2m_c}-\delta m\right). \quad (80)$$

It is easy to see that the exact result and the result of the OPE are different due to contributions of highly excited states: *at any $q^2$ the OPE picks up also the contribution of the resonances forbidden kinematically at this $q^2$*. Thus the accuracy of duality is determined by the accuracy of violating the sum rules connected with the truncation of the exclusive sum, and is therefore connected with the convergence of these sums.

### B. Sum of the exclusive channels and the accuracy of the OPE

We now calculate the individual decay rates keeping terms of order $(\Lambda^2/m_c^2)(\delta m/\Lambda)^n$ in the decay rates but neglecting higher orders $(\Lambda^3/m_c^3)(\delta m/\Lambda)^n$. The necessary expressions with the relevant accuracy are given below.

$|\vec{q}|$ in the free-quark decay $b\to cl\nu$ at $q^2$ has the form

$$|\vec{q}|=\sqrt{\delta m^2-q^2}\left(1-\frac{\delta m}{2m_c}+\frac{3}{8}\frac{\delta m^2}{m_c^2}+\frac{\delta m^2-q^2}{8m_c^2}\right). \quad (81)$$

The general expression for $|\vec{q}|$ in the $B\to D_n l\nu$ transition at $q^2$ reads

$$|\vec{q}|_n=\sqrt{\delta M_n^2-q^2}\left(1-\frac{\delta M_n}{2(m_c+m_d)}+\frac{3}{8}\frac{\delta M_n^2}{(m_c+m_d)^2}+\frac{\delta M_n^2-q^2}{8(m_c+m_d)^2}\right), \quad (82)$$

where $\delta M_n=M_B-M_n\simeq\delta m-\Delta_n-\delta m\beta_0/(2m_c)$ from Eqs. (50) and (63).

The necessary accuracy for the transition into the ground state is

$$|\vec{q}|_{n=0}=\sqrt{\delta m^2\left(1-\frac{\beta_0^2}{2m_c^2}\right)^2-q^2}\left(1-\frac{\delta m}{2(m_c+m_d)}+\frac{3}{8}\frac{\delta m^2}{m_c^2}+\frac{\delta m^2-q^2}{8m_c^2}\right). \tag{83}$$

Recall that $\beta_0^2=\langle B|\vec{k}^2+V_1|B\rangle$.

For $|\vec{q}|_{n\neq 0}$ less accuracy is enough since the contribution of the $D_n, n\neq 0$ into the SL decay rate is suppressed by the additional factor $\vec{q}^2/m_c^2$:

$$|\vec{q}|_{n\neq 0}=\sqrt{(\delta m-\Delta_n)^2-q^2}. \tag{84}$$

With these formulas for $|\vec{q}|$ the decay rates of the exclusive channels take the following form.

Free quark decay $b\to cl\nu$:

$$\frac{1}{L(q^2)}\frac{d\Gamma(b\to cl\nu)}{dq^2}=|\vec{q}|\left(1-\frac{\delta m}{m_c}+\frac{3}{2}\frac{\delta m^2}{m_c^2}-\frac{q^2}{2m_c^2}\right)=\sqrt{\delta m^2-q^2}\left(1-\frac{3}{2}\frac{\delta m}{m_c}+\frac{15}{8}\frac{\Delta m^2}{m_c^2}+\frac{5}{8}\frac{\delta m^2-q^2}{m_c^2}\right). \tag{85}$$

The $B\to D_0 l\nu$ channel

$$\frac{1}{L(q^2)}\frac{d\Gamma(B\to D_0 l\nu)}{dq^2}=|\vec{q}|_{n=0}\left(1-\frac{\delta m}{m_c+m_d}+\frac{3}{2}\frac{\delta m^2}{m_c^2}-\frac{q^2}{2m_c^2}\right)-\frac{\rho_0^2}{m_c^2}|\vec{q}|^3_{n=0}, \tag{86}$$

using the definition (56).

The $B\to D_n l\nu$ $(n\neq 0)$ channel:

$$\frac{1}{L(q^2)}\frac{d\Gamma(B\to D_n l\nu)}{dq^2}=\frac{\rho_n^2}{m_c^2}((\delta m-\Delta_n)^2-q^2)^{3/2}. \tag{87}$$

Now everything is ready for the calculation of the integrated SL decay rate. We again consider $L=(q^2)^N$.

### *1. The integrated rate and the global duality*

It is convenient to represent the results for the partial decay rates in terms of their ratios to the free quark decay rate. The latter has the form

$$\Gamma(b\to cl\nu)=(\delta m)^{2N+3}\left[A^N_{1/2}\left(1-\frac{3}{2}\frac{\delta m}{m_c}+\frac{15}{8}\frac{\delta m^2}{m_c^2}\right)+\frac{5}{8}A^N_{3/2}\frac{\delta m^2}{m_c^2}+O\left(\frac{\delta m^3}{m_c^3}\right)\right]. \tag{88}$$

Making use of the relation (38) we find

$$\frac{\Gamma(B\to D_0 l\nu)}{\Gamma(b\to cl\nu)}=1-\frac{3\rho_0^2}{2N+5}\frac{\delta m^2}{m_c^2}+\frac{3}{2}\frac{m_d}{1+m_d/m_c}\frac{\delta m}{m_c^2}-(2N+3)\frac{\beta_0^2}{2m_c^2}, \tag{89}$$

$$\frac{\Gamma(B\to D_n l\nu)}{\Gamma(b\to cl\nu)}=\frac{\delta m^2}{m_c^2}\frac{3\rho_n^2}{2N+5}\left(1-\frac{\Delta_n}{\delta m}\right)^N=\frac{3\rho_n^2}{2N+5}\frac{\delta m^2}{m_c^2}-3(\rho_n^2\Delta_n)\frac{\delta m}{m_c^2}+\frac{1}{m_c^2}\times\sum_{k=2}^{2N+5}(-1)^k\frac{1}{2N+5}C^k_{2N+5}\frac{(3\rho_n^2\Delta_n^k)}{\delta m^{k-2}}. \tag{90}$$

Some remarks are in order.

(1) The main part of the OPE (i.e., the free quark decay) is reproduced by the $\Gamma(B\to D_0 l\nu)$, within the leading and the subleading $1/m_c$ orders. The excited states contribute only within the $(\delta m)^2/m_c^2$ and $\Lambda\delta m/m_c^2$ orders in the SV limit.

(2) Nevertheless, each of the individual exclusive channels contains potentially large terms of the order $\delta m^2/m_c^2$ and $\Lambda\delta m/m_c^2$ which are absent in the OPE series.

Now summing over all exclusive channels we find

$$\frac{\Gamma(B\to X_c l\nu)}{\Gamma(b\to cl\nu)}=1-\frac{\delta m^2}{m_c^2}\left(\rho_0^2-\sum_{n=1}^{n_{max}}\rho_n^2\right)\frac{3}{2N+5}+3\frac{\delta m}{m_c^2}\left(\frac{1}{2}\frac{m_d}{1+m_d/m_c}-\sum_{n=1}^{n_{max}}\rho_n^2\Delta_n\right)-(2N+3)\frac{\langle B|\vec{k}^2+V_1|B\rangle}{2m_c^2}+(2N+4)\frac{\left(\sum_{n=1}^{n_{max}}\rho_n^2\Delta_n^2\right)}{2m_c^2}+\frac{1}{m_c^2\delta m}\sum_{k=3}^{2N+5}\frac{(-1)^k C^k_{2N+5}}{2N+5}\frac{\left(3\sum_{n=1}^{n_{max}}\rho_n^2\Delta_n^k\right)}{\delta m^{k-3}}. \tag{91}$$

The sum over the charm resonance levels is truncated at $n_{max}$, which is the number of the resonance levels opened at $q^2=0$. For the confining potential and in the SV limit $n_{max}$ is found from the relation $\Delta_{n_{max}} \simeq \delta m$.

Using the sum rules (66)–(77) to rewrite the OPE result (39) as the sum over hadronic resonances, the difference between the OPE and the exclusive sum (the duality-violating contribution) explicitly reads

$$\frac{\Gamma^{OPE}(B\to X_c l\nu)-\Gamma(B\to X_c l\nu)}{\Gamma(b\to cl\nu)} = \frac{\delta m^2}{m_c^2}\sum_{k=0}^{2N+5}\frac{(-1)^k C_{2N+5}^k}{2N+5}\frac{\delta^{(k)}}{\delta m^k}+O(\Lambda^2\delta m/m_c^3), \tag{92}$$

where

$$\delta^{(k)} \equiv \sum_{n=n_{max}+1}^{\infty}\rho_n^2(\Delta_n)^k = \sum_{n=n_{max}+1}^{\infty}[r_n^2+O(\Lambda\,\delta m/m_c^2)](\Delta_n)^k. \tag{93}$$

As expected, this duality-violating contribution is connected with the charm resonance states forbidden kinematically in the decay process. This kinematical truncation of the higher resonances induces a violation of duality equal to $[(\delta m)^{2-k}/m_c^2]\delta^{(k)}$ for every $k<2N+5$.

To estimate the error induced by the truncation and thus the size of the duality-violation effects, we need to know the behavior of the excitation energies and the transition radii at large $n$.

(1) For quite a general form of the confining potential we can write the following relations for $\Delta_n$ for large $n$ (recall that in the SV limit $\Delta_{n_{max}} \simeq \delta m$):

$$\Delta_{n_{max}}=\Lambda C(n_{max})^a=\delta m, \qquad \Delta_n \geqslant \Lambda C n^a, \quad n>n_{max}, \tag{94}$$

with $C$ and $a$ some positive numbers. In particular, this estimate is valid for the confining potentials with a power behavior at large $r$.

*This estimate for $\Delta_n$ is only depending on the behavior of the potential at large distances (the infrared region).*

(2) The transition radii $r_n^2$ satisfy sum rules similar to sum rules for $\rho_n^2$ in Sec. V, namely[6]

$$\sum_{n=1}^{\infty} r_n^2 = r_0^2, \qquad \sum_{n=1}^{\infty} r_n^2\Delta_n = \frac{m_d}{2(1+m_d/m_c)},$$

$$\sum_{n=1}^{\infty} r_n^2(\Delta_n)^{(k+2)} = \frac{1}{3}\langle D_0|k_j(h_{cd}-\epsilon_{D_0})^k k_j|D_0\rangle, \qquad k=0,1,2,\ldots . \tag{95}$$

Hence, the behavior of the radii $r_n^2$ at large $n$ are connected with the finiteness of the RHS of the sum rules. We can guarantee this for the Bjorken and Voloshin sum rules, where finite values stand in the RHS (the ground state radius $r_0$ is finite for the confining potential). In general, the finiteness of the matrix elements of the operators $k_j(h_{cd}-\epsilon_0)^k k_j$ (such as, e.g., the kinetic energy of quarks in the ground state) depend on the properties of the potential at small $r$ (*the ultraviolat behavior*) and probably also at large $r$ (*the infrared behavior*).[7] We have assumed throughout the paper that the average kinetic energy of the light spectator quark in the ground state is finite, i.e., $\langle D_0|\vec{k}^2|D_0\rangle \simeq \Lambda^2$. This already restricts some properties of the potential at small $r$ and provides convergency of one more sum rule in Eq. (95). If, in addition to this, we assume that the average values of the operators $k_j(h_{cd}-\epsilon_{D_0})^k k_j$ for $k=1,\ldots,K$ over the ground state are finite, then combining with the behavior of the energies at large $n$ we come to the following estimate:

$$r_n^2 \lesssim \frac{1}{n^{1+\varepsilon}}\left(\frac{1}{n^a}\right)^{2+K}, \quad \varepsilon>0. \tag{96}$$

This allows us to obtain the duality-violation originating from the truncation of the various sum rules:

$$\text{Bjorken:}\quad \frac{\delta m^2}{m_c^2}\delta^{(0)} = \frac{\delta m^2}{m_c^2}\sum_{n_{max}} r_n^2 \lesssim \frac{\delta m^2}{m_c^2}\left(\frac{1}{n_{max}^a}\right)^{K+2} \simeq \frac{\Lambda^2}{m_c^2}\left(\frac{\Lambda}{\delta m}\right)^K,$$

$$\text{Voloshin:}\quad \frac{\delta m}{m_c^2}\delta^{(1)} = \frac{\delta m}{m_c^2}\sum_{n_{max}} r_n^2\Delta_n \lesssim \frac{\delta m}{m_c^2}\Lambda\left(\frac{1}{n_{max}^a}\right)^{K+1} \simeq \frac{\Lambda^2}{m_c^2}\left(\frac{\Lambda}{\delta m}\right)^K. \tag{97}$$

Similar estimates can be done for higher moment sum rules. One can see that the truncation error in any of the sum rules leads to the duality-violation of the same order $O(\Lambda^{2+K}/m_c^2\delta m^K)$. An interesting feature about these esti-

[6] Notice that these relations are exact and do not have any $1/m_c$ corrections.

[7] We do not have a classical Wilsonian scheme where the ultraviolet region is referred to the Wilson coefficients and the infrared region is referred to the matrix elements of the operators, so we can have these regions mixed.

mates is that the dependence on $a$ has disappeared from the final result. Hence, *the estimates are independent of the details of the potential at large r*, provided the potential guarantees the confinement, i.e., $a$ is positive.

These are however rather crude estimates which do not take into account further possible suppressions (due, e.g., to the orthogonality of the wave functions of the ground $n=0$ and the excited states $n>0$). In such a case the real accuracy is better, and might depend on the details of the potential also at large $r$. In general, we can state that the truncation (duality-violaiton) error occurs at the order

$$\frac{\Lambda^2}{m_c^2}\left(\frac{\Lambda}{\delta m}\right)^b, \tag{98}$$

where the exponent $b>0$ depends on the properties of the potential (in general, both at short and long distances). A more detailed analysis of which potentials satisfy the above requirements is beyond the scope of this paper and is left for another publication [13].

If we would like to have the truncation error of a higher order in $1/m_c$, e.g. in $O(\Lambda^3/m_c^3)$, this is not so straight. Namely, in this case we need

$$\frac{\delta^{(k)}}{\delta m^k}\simeq O\left(\frac{\Lambda^3}{\delta m^2 m_c}\right). \tag{99}$$

As we have noticed, the series for $\delta^{(k)}$ in the main part does not depend on $m_c$, so the only possibility to have the relation (99) fulfilled in the SV limit, is to have $r_n^2=0$ starting from some number $n$. (Exactly this situation takes place for the HO potential where all $D_0\to D_n$ transition radii for $n>1$ are equal zero [17]). In this case for large enough values of $\delta m$, the term proportional to $r_n^2$ in Eq. (93) disappears and the second term provides the truncation error of the order $O(\delta m^2\Lambda^2/m_c^4)$.

As we are going to show elsewhere, the accuracy of duality of order $\Lambda^3/m_c^3$ can be achieved if we keep a fixed ratio $\delta m/m_c$ when $m_c\to\infty$. One can proceed exactly along the same lines, but technically a bit different treatment is necessary: namely, at several places throughout the paper terms of the order $\delta m^3/m_c^3$ have been omitted, and they should be kept if the limit $\delta m/m_c=$const is considered. This analysis will be presented in [13].

Finally, it is interesting to notice that all resonance levels opened at $q^2=0$ are contributing on equal footing to the sum rules and therefore to the decay rate. So, a considerable delay in opening channels with large $n$ compared to the channels with small $n$ with the increasing recoil does not matter at all. This is a very important feature which basically determines a high accuracy of the OPE calculation of the integrated decay rate (cf. [6,12]).

#### 2. The smeared $q^2$ distribution and the local duality

The situation however differs considerably if we consider the differential decay widths. We find it more physical to use here the four vector $q^2$ variable. The region near $q^2_{max}$ (zero recoil) is special: as we move to higher $q^2$, the excited channels close one after another leaving ultimately only the $D_0$ ground state opened.

Let us consider a partially integrated decay rate in the $q^2$-region above the threshold of the $D_{n=1}$ channel. In this case the relation between the OPE and the exact result (which is reduced in this case to the exclusive $B\to D_0 l\nu$ decay) reads

$$\int_{(\delta m-\Delta_1)^2}^{\delta m^2} dq^2\frac{d\Gamma(B\to X_c l\nu)}{dq^2} = \int_{(\delta m-\Delta_1)^2}^{\delta m^2} dq^2\frac{d\Gamma(b\to c l\nu)}{dq^2}\left[1+O\left(\frac{\Delta_1\delta m}{m_c^2}\right)\right]. \tag{100}$$

In this formula we have neglected a difference between the upper boundaries of the quark and hadron channels of the order $\Lambda^2/m_c^2$. Equation (100) means that local duality near maximal $q^2$ is violated at order $O(\Lambda_1\delta m/m_c^2)$. As we have seen, the dangerous terms of this order are cancelled in the integrated rate against similar contributions of other channels due to the Voloshin sum rule. However, the $O(\Lambda\delta m/m_c^2)$ violation of the local duality might have negative consequences for the application of the method to the analysis of the experimental results. For example this happens if one observes only a small part of the phase space near maximal $q^2$ [6].

## VII. CONCLUSION

We have studied quark-hadron duality in decays of heavy mesons in the SV limit using the nonrelativistic potential model for the description of mesons as $q\bar{q}$ bound states. Our main results are as follows:

(1) The OPE is constructed and the following $\Lambda/m_c$ and $\Lambda/\delta m$ double series is found for the integrated decay rate:

$$\frac{\Gamma^{OPE}(B\to X_c l\nu)}{\Gamma(b\to c l\nu)}=1+C_0\frac{\langle B|\vec{k}^2+V_1|B\rangle}{2m_c^2}+(1-C_0)\frac{\langle B|\vec{k}^2|B\rangle}{2m_c^2}+\sum_{k=1}^{k_0}C_k\frac{\langle B|\vec{k}(h_{bd}-\epsilon_B)^k\vec{k}|B\rangle}{2m_c^2(\delta m)^k}+O\left(\frac{\Lambda^2\delta m}{m_c^3}\right),$$

where $C_k$ are calculable constants and $k_0$ depends on the leptonic tensor.

(2) The HQ expansion of the transition form factors in the nonrelativistic potential model is performed. A nonrelativistic analog of the Luke theorem for the exclusive transition form factor between the ground states is obtained.

It is shown that the sum of the squares of the $B \to D_n l \nu$ transition form factors are expressed through the expectation values of the operators emerging in the OPE series. These nonrelativistic analogs of the Bjorken, Voloshin, and higher order sum rules provide a bridge between the sum over exclusive channels and the OPE series.

(3) The integrated decay rate is calculated by direct summation of exclusive channels. For the comparison of this directly calculated $\Gamma(B \to X_c l \nu)$ and the corresponding $\Gamma^{OPE}(B \to X_c l \nu)$ the sum rules are necessary. A difference (duality-violation) between the two expressions is observed. As shown explicitly by the use of the sum rules, this difference is connected with the higher $c\bar{d}$ resonances which are forbidden kinematically in the decay process but are implicitly taken into account in the OPE approach. Therefore the accuracy of the OPE is directly related to the error induced by the kinematical truncation in the sum rules (Bjorken, Voloshin, etc). The actual error depends on the convergence of the series, i.e., on the nature of the potential. We have discussed the constraints on the latter convergence which lead to the duality violation of order $O(\Lambda^{2+b}/m_c^2 \delta m^b)$ with $b$ depending on the behavior of the potential both at the short and long distances.

*Up to the mentioned duality-violation*, the agreement between the OPE and the exclusive sum is achieved within different $1/m_c$ orders due to different reasons:

The leading order and the subleading $\delta m/m_c$ and $\Lambda/m_c$ orders the free quark integrated decay rate $\Gamma(b \to c l \nu)$ is equal to the rate of the transition into the ground state $D_0$. This is due to the specific behavior of the transition form factor between the ground states near the zero recoil (Luke theorem). Also part of the $\Lambda^2/m_Q^2$ correction in the OPE result proportional to the $\langle B|\vec{k}^2 + V_1|B\rangle$ matches the contribution of the ground state $D_0$ in the exclusive sum.

For higher order terms the agreement between the OPE and the exclusive sum is a collective effect due to subtle cancellations in the sum over exclusive channels:

Namely, each of the individual decay rates $\Gamma(B \to D_n l \nu)$ contain potentially large terms of the order $\delta m^2/m_c^2$ and $\Lambda \delta m/m_c^2$. These terms cancel in the exclusive sum due to the Bjorken and Voloshin sum rules, respectively. The higher order sum rules allow us to represent the contribution of exclusive channels in terms of the average values of the operators $O_i$ over the $B$-meson state.

(4) If the differential semileptonic decay widths are considered near maximum $q^2$, the violation of the local duality occurs at order $O(\Lambda \delta m/m_c^2)$.

Clearly, in QCD the situation is more complicated because of the multiparticle $X_c$ states, pion emission, hybrid and multiquark exotic $D$ mesons, radiative corrections. Nevertheless the duality violation due to the kinematical truncation of the series should be quite similar to the case of nonrelativistic quantum mechanics. Also similar is the role of the inclusive sum rules in obtaining the duality relations.

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