Off-shell structure of the anomalous Z and γ self-couplings

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We establish the general off-shell structure of the three neutral gauge boson self-couplings $V_1^* V_2^* V_3^*$, with applications to the $Z^*Z^*Z^*$, $Z^*Z^*\gamma^*$, $\gamma^*\gamma^*Z^*$ cases. New coupling forms appear which do not exist when two gauge bosons are on shell. We give the contribution arising from a fermionic triangle loop. It covers both the standard model (SM) and possible new physics (NP) contributions such as those arising in the minimal supersymmetric SM. For what concerns NP contributions with a high scale, we discuss the validity of an effective Lagrangian involving a limited set of parameters. Finally we write the general expression of the $V_1^* V_2^* V_3^*$-vertex contribution to the $e^+e^- \to (f\bar{f}) + (f'\bar{f}')$ amplitude.

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I. INTRODUCTION

The phenomenological description of neutral anomalous gauge couplings (NAGC) among the photon and Z was established in Refs. [1,2] and used for the discussion of their observability at various types of present and future colliders, see Refs. [3–7]. It has recently been reexamined and examples of new physics (NP) contributions have been discussed [8,9]. After the first events obtained at the Fermilab Tevatron [10], experimental data are now being collected at the CERN e^+e^- collider LEP2 [11] through the processes $e^+e^- \to ZZ$ and $Z\gamma$. New possibilities will be offered by linear e^+e^- colliders (LC) [12] and the CERN Linear Collider (CLIC) [13].

The description used in Refs. [1,8,9] applies to the case where one neutral gauge boson V^* is off shell[1] and coupled to e^+e^- ($V^* = \gamma^*$ or Z^*), while the other two neutral gauge bosons ZZ or $Z\gamma$ are on shell. However, a large set of events collected at LEP2 [14] consists of four-fermion states (such as $l\bar{l}q\bar{q}$), in which the invariant mass of the $l\bar{l}$ or $q\bar{q}$ pair varies from about 10 GeV up to the Z mass. For analyzing these events through the processes $e^+e^- \to Z^*Z^*$, $Z^*\gamma^*$, $\gamma^*\gamma^*$, taking into account[2] contributions from $V^* \to Z^*Z^*$, $Z^*\gamma^*$, $\gamma^*\gamma^*$, one needs a description of the off-shell $V_i^* V_j^* V_k^*$ vertex. The usual two-particle-on-shell vertices for Z^*ZZ, $Z^*Z\gamma$, γ^*ZZ, $\gamma^*Z\gamma$, which are forced by Bose statistics to vanish whenever V^* goes on shell, are not adequate to describe $V^*V^*V^*$, since additional q_i^2 dependences and new coupling forms may be generated, which cannot be ignored. Some attempts to treat these off-shell effects exist in the literature for the $V^* \to Z^*Z^*$ case [15], but a complete treatment is still lacking.

It is the purpose of this paper to present and discuss the general description of the $V_i^* V_j^* V_k^*$ off-shell couplings. We proceed in several steps.

In Sec. II and Appendixes A and B, we establish the most general form for a $V^*V^*V^*$ vertex involving three off shell neutral gauge bosons (NGB). For completeness, we also include the ''scalar'' qV terms, contributing in the case that one off-shell Z decays to a heavy fermion pair, through its axial coupling. The only assumptions used are Lorentz invariance, Bose statistics and $U(1)_{em}$ invariance; separately for the CP-conserving and CP-violating cases. We make explicit applications to the $Z^*Z^*Z^*$, $Z^*Z^*\gamma^*$, and $\gamma^*\gamma^*Z^*$ couplings, and we point out the new coupling forms which do not exist when two particles are on shell, thus making contact with the previous description [8,9]. These general vertices apply to any standard model (SM) or NP contribution.

In Sec. III we consider an effective Lagrangian parametrization which could apply to the case when the NP scale Λ is very high; i.e., $\Lambda \gg m_Z$. We show that the effective Lagrangian previously considered in Ref. [8] when two NGBs are on shell, already contains some of the off-shell forms; but new operators must be added in order to describe all possible ones. These operators involve higher dimensions, so a hierarchy may appear among the various possible off-shell effects, which is quite natural in this $\Lambda \gg m_Z$ case.

In Sec. IV we look for a possible dynamical origin of these couplings. Virtual SM or NP contributions may indeed generate various off-shell NAGC. We describe them by generalizing the procedure of Ref. [9] based on triangle fermionic loops, already considered in Refs. [16,17]. In Appendix C we give the complete expression of the off-shell $V_i^* V_j^* V_k^*$ vertices generated by such fermionic loops. This is useful for the computation of the SM and the MSSM or NP contributions, and it also allows to illustrate how the type of off-shell effects changes, as the NP scale increases from the 100 GeV level to the multi-TeV one. Typical figures are presented, illustrating the dependence of the various neutral gauge couplings on the off-shell masses, the relative size of these couplings as compared to their on-shell values, and the range of

[1]This off-shell state is indicated below by an asterisk.

[2]Note that electromagnetic gauge invariance prohibits any $\gamma^*\gamma^*\gamma^*$ vertex.

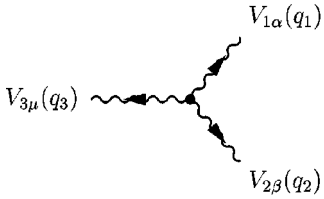

FIG. 1. The general neutral gauge boson vertex $V_1V_2V_3$.

the NP scales for which an effective Lagrangian description in terms of low dimension operators, is adequate.

In Sec. V, we write, for completeness, the general structure of the $V^*V^*V^*$ contribution to four fermion amplitude $e^+e^-\to(f\bar{f})+(f'\bar{f}')$, including all off-shell contributions. The results are summarized in Sec. VI, where the conclusions are also given.

II. DESCRIPTION OF OFF-SHELL NEUTRAL SELF-BOSON COUPLINGS

The general procedure for determining the off-shell $V_1^*V_2^*V_3^*$ couplings is described in Appendix A for the CP-conserving couplings and in Appendix B for the CP-violating ones. We use the notations of Fig. 1 for the general off-shell $V_1^\alpha(q_1)V_2^\beta(q_2)V_3^\mu(q3)$ vertex (all q_i being outgoing momenta[3]). The results can be summarized as follows.

A. $Z^*Z^*Z^*$ couplings

There are six CP-conserving independent forms listed in Appendix A, which are multiplied by six coupling functions denoted as

$$f_i^{Z^*Z^*Z^*}(s_1,s_2,s_3), \quad (i=1-3),$$

and

$$g_i^{Z^*Z^*Z^*}(s_1,s_2,s_3), \quad (i=1-3).$$

As in Eq. (A3) we write the vertex interaction as

$$\Gamma_{\alpha\beta\mu}^{Z^*Z^*Z^*}(q_1,q_2,q_3)=i\sum_{i=1}^{3} I_{\alpha\beta\mu}^{Z^*Z^*Z^*,i}f_i^{Z^*Z^*Z^*}(s_1,s_2,s_3)+i\sum_{i=1}^{3} J_{\alpha\beta\mu}^{Z^*Z^*Z^*,i}g_i^{Z^*Z^*Z^*}(s_1,s_2,s_3), \tag{1}$$

where the kinematics are defined in Fig. 1.

The three I^i and the three J^i forms are given in Eq. (A4). We note that the J^i forms, associated to g_i, involve at least one scalar qV factor, and they are thus are called ''scalar.'' In contrast to them, the three I^i terms associated to f_i, do not involve qV factors and they are called ''transverse.'' These f_i,g_i are functions of s_1,s_2,s_3 and satisfy the Bose symmetry relations presented in Eq. (A5). We note in particular from them, that $f_3(s_1,s_2,s_3)$ is fully antisymmetric.

In case two of the Z's are on-shell, say, e.g., $s_1=s_2=m_Z^2$, Bose statistics forces $f_2^{Z^*Z^*Z^*}$, $f_3^{Z^*Z^*Z^*}$ to vanish, leaving only one nonvanishing transverse coupling, corresponding to f_5^Z defined in Refs. [1,8,9] and satisfying

$$f_1^{Z^*Z^*Z^*}(m_Z^2,m_Z^2,s_3)\equiv\frac{s_3-m_Z^2}{m_Z^2}f_5^Z(s_3), \tag{2}$$

where we have emphasized the fact that generally f_5^Z is not necessarily constant, but rather a form factor depending on[4] s_3. In this on-shell case there remains also one ''scalar'' term

$$g_3^{Z^*Z^*Z^*}(m_Z^2,m_Z^2,s_3),$$

which contributes only when the off-shell Z^* couples to a heavy fermion pair (such as, e.g., $t\bar{t}$) at a ''mass'' squared s_3. Such terms had been previously neglected. Thus, comparing the on- and off-shell situations, we remark that in the off-shell case we have in addition two more ''transverse'' couplings and another two ''scalar'' ones.

In the CP violating case there exist 14 independent forms, listed in Appendix B. Defining the kinematics as before through Fig. 1, we write [compare Eq. (B1)]

$$\Gamma_{\alpha\beta\mu}^{Z^*Z^*Z^*}(q_1,q_2,q_3)=i\sum_{i=1}^{4} \tilde{I}_{\alpha\beta\mu}^{Z^*Z^*Z^*,i}\tilde{f}_i^{Z^*Z^*Z^*}(s_1,s_2,s_3)+i\sum_{i=1}^{10} \tilde{J}_{\alpha\beta\mu}^{Z^*Z^*Z^*,i}\tilde{g}_i^{Z^*Z^*Z^*}(s_1,s_2,s_3), \tag{3}$$

where the four $\tilde{I}^i$ are transverse, while the 10 $\tilde{J}^i$ are scalar. They are listed in Eqs. (B2),(B3) and imply the Bose constraints (B4),(B5) for the corresponding coupling functions $(\tilde{f}_i,\tilde{g}_j)$.

In case two of the Z's are onshell ($s_1=s_2=m_Z^2$), then $\tilde{f}_1$, $\tilde{f}_4$ vanish, while the other two transverse functions are opposite to each other, because of Bose symmetry. So only one transverse combination remains, related to the coupling constant f_4^Z defined in Refs. [1,8,9], through

$$\tilde{f}_2^{Z^*Z^*Z^*}(m_Z^2,m_Z^2,s_3)=-\tilde{f}_3^{Z^*Z^*Z^*}(m_Z^2,m_Z^2,s_3)=\frac{m_Z^2-s_3}{2m_Z^2}f_4^Z(s_3), \tag{4}$$

and the two scalar ones

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[3]In previous works [8,9] $P\equiv-q_3$ was used for the initial off-shell boson.

[4]A similar emphasis of their form-factor nature is made in this section for all NAGC defined in Refs. [1,8,9].

$$\tilde{g}_1^{Z^*Z^*Z^*}(m_Z^2,m_Z^2,s_3),\quad \tilde{g}_6^{Z^*Z^*Z^*}(m_Z^2,m_Z^2,s_3).$$

Comparing with the results of Appendix B and with those of the on-shell treatment of Refs. [1,8,9], we conclude that in the general CP-violating off-shell case, there are in addition two transverse and eight scalar terms.

B. $Z^*Z^*\gamma^*$ couplings

Now, there are five CP-conserving independent forms defined in Appendix A through [cf. Eqs. (A7),(A8)]

$$\Gamma_{\alpha\beta\mu}^{Z^*Z^*\gamma^*}(q_1,q_2,q_3)=i\sum_{i=1}^{3} I_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,i}f_i^{Z^*Z^*\gamma^*}(s_1,s_2,s_3) + i\sum_{i=1,2} J_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,i}g_i^{Z^*Z^*\gamma^*}(s_1,s_2,s_3). \tag{5}$$

Three of them $f_i^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)$ $(i=1,2,3)$ are transverse; while the (conserved vector current) (CVC) constraint $q_3^\mu\ \Gamma_{\alpha\beta\mu}^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=0$ reduces the number of the ''scalar'' terms to the two ones $g_i^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)$, $(i=1,2)$. These functions are submitted to the (Z^*Z^*) Bose symmetry relations appearing in Eq. (A9).

In case the two Z's are onshell $(s_1=s_2=m_Z^2)$, Bose symmetry forces two of the transverse functions to vanish, while the two ''scalar'' ones become inefficient, as they are proportional to q_1^α or q_2^β. Thus, we end up with only one (transverse) coupling, corresponding to f_5^γ defined in Refs. [1,8,9]:

$$f_1^{Z^*Z^*\gamma^*}(m_Z^2,m_Z^2,s_3)\equiv \frac{s_3}{m_Z^2} f_5^\gamma(s_3). \tag{6}$$

If only one Z and the photon are onshell (i.e., $s_1=m_Z^2$, $s_3=0$), we remain instead with two transverse combinations corresponding to the couplings $h_{3,4}^Z$ defined in Refs. [1,8,9],

$$f_2^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)=\frac{m_Z^2-s_2}{m_Z^2}\left[h_3^Z(s_2)+\frac{m_Z^2-s_2}{4m_Z^2}h_4^Z(s_2)\right],$$

$$f_3^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)=\frac{m_Z^2-s_2}{2m_Z^4}h_4^Z(s_2), \tag{7}$$

and one ''scalar'' term

$$g_2^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0),$$

since the other scalar term contains a factor q_1^α making it inefficient on shell.

Thus, in the general off-shell case, the three transverse functions can be considered as a generalization [due to the (s_1,s_2,s_3) dependence], of the three on-shell couplings f_5^γ,h_3^Z,h_4^Z. There are also two scalar functions, previously neglected.

In the CP-violating case, there are nine coupling forms, of which the four $\tilde{I}_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,i}$ $(i=1\text{–}4)$ are transverse, while the five $\tilde{J}_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,i}$ $(i=1\text{–}5)$ are scalar. They are listed in Eq. (B7). In terms of them, the corresponding neutral gauge self interactions is defined through [cf. Eq. (B6)]

$$\Gamma_{\alpha\beta\mu}^{Z^*Z^*\gamma^*}(q_1,q_2,q_3)=i\sum_{i=1}^{4} \tilde{I}_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,i}\tilde{f}_i^{Z^*Z^*\gamma^*}(s_1,s_2,s_3) + i\sum_{i=1}^{5} \tilde{J}_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,i}\tilde{g}_i^{Z^*Z^*\gamma^*}(s_1,s_2,s_3). \tag{8}$$

For $\gamma^*\rightarrow ZZ$ with the two Z's being on shell $(s_1=s_2=m_Z^2)$, $\tilde{f}_{1,3,4}$ vanish because of Bose symmetry; cf. Eq. (B8). In such a case the only remaining coupling is a transverse one related to f_4^γ defined in Refs. [1,8,9] through

$$\tilde{f}_2^{Z^*Z^*\gamma^*}(m_Z^2,m_Z^2,s_3)=-\frac{s_3}{2m_Z^2} f_4^\gamma(s_3). \tag{9}$$

No scalar term remains because q_1^α,q_2^β give no on-shell contribution.

For $Z^*\rightarrow Z\gamma$ with one real Z $(s_1=m_Z^2)$ and one real γ $(s_3=0)$, $\tilde{f}_1$ vanishes and $\tilde{f}_2$ is related to $\tilde{f}_4$ because of the CVC constraint. We thus end up with the two transverse functions related to the $h_{1,2}^Z$ couplings defined in Refs. [1,8,9] by

$$\tilde{f}_2^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)=-(s_2-m_Z^2)\tilde{f}_4^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0) =\frac{(s_2-m_Z^2)^2}{8m_Z^4}h_2^Z(s_2),$$

$$\tilde{f}_3^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)=\frac{s_2-m_Z^2}{2m_Z^2}\left[-h_1^Z(s_2) + \frac{s_2-m_Z^2}{4m_Z^2}h_2^Z(s_2)\right], \tag{10}$$

and the two scalar combinations

$$[\tilde{g}_1^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)-\tilde{g}_2^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)],$$

$$[\tilde{g}_3^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)-\tilde{g}_4^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)],$$

previously neglected. So the general off-shell case involves two more transverse couplings and three more scalar ones.

C. $\gamma^*\gamma^*Z^*$ couplings

There are four invariant forms in the CP-conserving case, listed in Appendix A [cf. Eqs. (A10),(A11)]

$$\Gamma_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*}(q_1,q_2,q_3)=i\sum_{i=1}^{3} I_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,i}f_i^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3) + iJ_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,1}g_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3), \tag{11}$$

including again the three transverse functions $f_i^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)$ $(i=1\text{–}3)$, but only one scalar $g_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)$. We note that this reduction of the number of scalar forms is due to the two CVC constraints

$$q_1^\alpha\Gamma_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=q_2^\beta\Gamma_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=0,$$

and the Bose symmetry between the two photons.

When one photon and one Z are on shell ($s_2=0$, $s_3=m_Z^2$), these forms reduce to two independent transverse ones corresponding to the couplings $h_{3,4}^\gamma$ defined in Refs. [1,8,9]:

$$f_1^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2)=\frac{s_1}{2m_Z^2}\left[h_3^\gamma(s_1)-\frac{s_1}{2m_Z^2}h_4^\gamma(s_1)\right],$$

$$f_2^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2)=\frac{s_1}{2m_Z^2}\left[h_3^\gamma(s_1)+\frac{m_Z^2-2s_1}{2m_Z^2}h_4^\gamma(s_1)\right],$$

$$f_3^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2)=-\frac{s_1}{2m_Z^4}h_4^\gamma(s_1), \tag{12}$$

and one previously neglected scalar term

$$g_1^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2).$$

So one sees that the general off-shell situation has one more (transverse) form than in the previously studied on-shell case.

In the CP-violating case there are only six forms

$$\Gamma_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*}(q_1,q_2,q_3)=i\sum_{i=1}^{4}\tilde{I}_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,i}\tilde{f}_i^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)$$
$$+i\sum_{i=1,2}\tilde{J}_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,i}\tilde{g}_i^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3), \tag{13}$$

four of which are transverse $\tilde{f}_i^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)$ $(i=1\text{–}4)$ and two scalar ones $\tilde{g}_i^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)$ (i=1,2); see Eqs. (B10), (B11) in Appendix B.

When one photon and one Z are on-shell, ($s_2=0$, $s_3=m_Z^2$, $q_2^\beta\equiv q_3^\mu\equiv 0$), one remains with only two independent transverse forms related to the couplings $h_{1,2}^\gamma$ defined in Refs. [1,8,9]:

$$\tilde{f}_1^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2)=\frac{s_1}{2m_Z^2}\left[h_1^\gamma(s_1)-\frac{s_1-m_Z^2}{2m_Z^2}h_2^\gamma(s_1)\right],$$

$$\tilde{f}_2^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2)=\tilde{f}_3^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2)=-\frac{s_1}{4m_Z^2}h_1^\gamma(s_1),$$

$$\tilde{f}_4^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2)=-\frac{s_1}{8m_Z^4}h_2^\gamma(s_1), \tag{14}$$

and no ''scalar'' term. Therefore, the general off-shell case for this vertex has two more transverse terms and two more scalar ones.

III. THE EFFECTIVE LAGRANGIAN DESCRIPTION

The effective Lagrangian is an adequate formalism to describe the NP effects generated at a scale Λ, which is much higher than the actual energy (or external mass) in the process considered ($\sqrt{s_i}$ or M_Z). In this case, it is natural to restrict the set of operators to those with the lowest possible dimensions (the higher dimension contributions being depressed by powers of s_i/Λ^2), and thus reducing somewhat the number of free parameters. Of course, the dimension of the operators needed to generate each specific form of interactions vertex, may strongly depend on it.

Below, for each NAGC type of vertex ($Z^*Z^*Z^*$, $Z^*Z^*\gamma^*$, $\gamma^*\gamma^*Z^*$), we first establish a set of operators, with the lowest possible dimension, which can generate the vertex forms established in Sec. II. Each such lowest dimensional operator generating a given vertex form ($I_i\cdots$ or $J_i\cdots$) produces a coupling function $[f_i(s_1,s_2,s_3)\cdots$ or $g_i(s_1,s_2,s_3)\cdots]$ characterized by the lowest power of s_i consistent with the corresponding Bose constraints presented in Appendixes A, B. Thus, a constant f_j appears in the case of a fully symmetric function, a factor (s_i-s_j) for a function antisymmetric in the exchange of s_i,s_j etc.

The lowest-dimensional operators contributing to NAGC have[5] mainly dim=6. We therefore start by enumerating all of them. It turns out though, that this list operators is not sufficient to generate all vertex forms. We therefore proceed to include also a minimal set of higher-dimensional operators which generate the missing vertices. This constitutes what we call the basic effective Lagrangian expressed as

$$\mathcal{L}=e\left(\sum_i l_i\mathcal{O}_i+\sum_i \tilde{l}_i\tilde{\mathcal{O}}_i\right), \tag{15}$$

where the operators $\mathcal{O}_i$ and $\tilde{\mathcal{O}}_i$ are CP conserving and CP violating, respectively, while l_i and $\tilde{l}_i$ are their corresponding (dimensional) coupling constants.

A. The $Z^*Z^*Z^*$ CP conserving operators $(i=1,6)$

Using the notation

$$\tilde{Z}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}Z^{\rho\sigma},\quad Z_{\mu\nu}=\partial_\mu Z_\nu-\partial_\nu Z_\mu, \tag{16}$$

and similarly for the photon tensor $F_{\mu\nu}$, the set of the $Z^*Z^*Z^*$ CP-conserving operators defined as above, is

$$\mathcal{O}_1^{Z^*Z^*Z^*}=\tilde{Z}_{\mu\nu}(\partial_\sigma Z^{\sigma\mu})Z^\nu,$$

$$\mathcal{O}_2^{Z^*Z^*Z^*}=\Box\tilde{Z}_{\mu\nu}Z^\mu\Box Z^\nu,$$

[5]For the CP-violating $Z^*Z^*Z^*$ case, there exists a single operator of dim=4 which is of course also included, see below.

$$\mathcal{O}_3^{Z^*Z^*Z^*}=(\Box^2\tilde{Z}_{\mu\nu})(\Box\partial^\sigma Z^{\mu\nu})Z_\sigma,$$

$$\mathcal{O}_4^{Z^*Z^*Z^*}=\tilde{Z}_{\mu\nu}(\partial^\mu Z^\nu)(\partial^\sigma Z_\sigma),$$

$$\mathcal{O}_5^{Z^*Z^*Z^*}=\tilde{Z}_{\mu\nu}(\partial^\mu Z^\nu)(\Box\partial^\sigma Z_\sigma),$$

$$\mathcal{O}_6^{Z^*Z^*Z^*}=\tilde{Z}_{\mu\nu}(\partial^\mu\Box Z^\nu)(\partial^\sigma Z_\sigma). \tag{17}$$

The transverse terms are given by $\mathcal{O}_1^{Z^*Z^*Z^*}$ (dim=6), $O_2^{Z^*Z^*Z^*}$ (dim=8), and $\mathcal{O}_3^{Z^*Z^*Z^*}$ (dim=12). We note in particular that the operator $\mathcal{O}_3^{Z^*Z^*Z^*}$ is required for generating the fully antisymmetric structure of $f_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)$, (see below). The scalar terms are $\mathcal{O}_4^{Z^*Z^*Z^*}$ (dim=6) and $O_{5,6}^{Z^*Z^*Z^*}$ (dim=8).

The corresponding coupling functions [see Eq. (1)]) are

$$\begin{aligned}f_1^{Z^*Z^*Z^*}(s_1,s_2,s_3)=&-\frac{1}{2}(s_1+s_2-2s_3)l_1^{Z^*Z^*Z^*}\\&+\frac{1}{2}(s_3(s_1+s_2)-2s_1s_2)l_2^{Z^*Z^*Z^*}\\&+\frac{1}{2}[s_1s_2(s_1-s_2)^2-s_3^2\{s_1(s_3-s_1)\\&+s_2(s_3-s_2)\}]l_3^{Z^*Z^*Z^*},\end{aligned}$$

$$\begin{aligned}f_2^{Z^*Z^*Z^*}(s_1,s_2,s_3)=&-\frac{3}{2}(s_1-s_2)l_1^{Z^*Z^*Z^*}\\&-\frac{3}{2}s_3(s_1-s_2)l_2^{Z^*Z^*Z^*}+\frac{s_2-s_1}{2}\\&\times\{s_3[s_1s_2-s_1^2-s_2^2+s_3(s_1+s_2)]\\&-2s_1s_2(s_1+s_2)\}l_3^{Z^*Z^*Z^*},\end{aligned}$$

$$\begin{aligned}f_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)=&[s_1^2(s_2-s_3)+s_3^2(s_1-s_2)\\&+s_2^2(s_3-s_1)]l_3^{Z^*Z^*Z^*},\end{aligned}$$

$$\begin{aligned}g_1^{Z^*Z^*Z^*}(s_1,s_2,s_3)=&2l_1^{Z^*Z^*Z^*}+2l_4^{Z^*Z^*Z^*}\\&-2s_1l_5^{Z^*Z^*Z^*}-(s_2+s_3)l_6^{Z^*Z^*Z^*}\\&+2(s_3^2s_2+s_2^2s_1-s_1^2s_2)l_3^{Z^*Z^*Z^*},\end{aligned}$$

$$\begin{aligned}g_2^{Z^*Z^*Z^*}(s_1,s_2,s_3)=&2l_1^{Z^*Z^*Z^*}+2l_4^{Z^*Z^*Z^*}\\&-2s_2l_5^{Z^*Z^*Z^*}-(s_1+s_3)l_6^{Z^*Z^*Z^*}\\&+2(s_3^2s_1+s_1^2s_2-s_2^2s_1)l_3^{Z^*Z^*Z^*},\end{aligned}$$

$$\begin{aligned}g_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)=&2l_1^{Z^*Z^*Z^*}+2l_4^{Z^*Z^*Z^*}-2s_3l_5^{Z^*Z^*Z^*}\\&-(s_2+s_1)l_6^{Z^*Z^*Z^*}-[s_1(s_3^2-s_2^2)\\&-s_3(s_1^2+s_2^2)+s_2(s_3^2-s_1^2)]l_3^{Z^*Z^*Z^*}.\end{aligned} \tag{18}$$

We also remark that the on-shell coupling f_5^Z defined in Refs. [1,8,9] for the CP-conserving Z^*ZZ vertex is related to the relevant three transverse couplings defined here for the off-shell case by

$$f_5^Z=m_Z^2[l_1^{Z^*Z^*Z^*}+m_Z^2(l_2^{Z^*Z^*Z^*}+s_3^2l_3^{Z^*Z^*Z^*})]. \tag{19}$$

Thus, going from the on-shell treatment of the CP-conserving Z^*ZZ NAGC case, to the present effective Lagrangian off-shell one, we have to increase the number of parameters from one to three.

B. The $Z^*Z^*Z^*$ CP-violating operators ($i=1,14$)

The relevant set of operators is

$$\tilde{\mathcal{O}}_1^{Z^*Z^*Z^*}=-Z_\sigma(\partial^\sigma Z_\nu)(\partial_\mu Z^{\mu\nu}),$$

$$\tilde{\mathcal{O}}_2^{Z^*Z^*Z^*}=(\Box Z_\alpha)(\partial^\alpha Z_\mu)(\Box Z^\mu),$$

$$\tilde{\mathcal{O}}_3^{Z^*Z^*Z^*}=Z_\alpha(\partial^\alpha Z_\mu)(\Box^2 Z^\mu),$$

$$\tilde{\mathcal{O}}_4^{Z^*Z^*Z^*}=(\Box^2\partial^\alpha Z_\beta)(\partial^\mu\Box Z_\alpha)(\partial^\beta Z_\mu),$$

$$\tilde{\mathcal{O}}_5^{Z^*Z^*Z^*}=Z^\mu Z_\mu(\partial^\sigma Z_\sigma),$$

$$\tilde{\mathcal{O}}_6^{Z^*Z^*Z^*}=(\Box Z^\mu)Z_\mu(\partial^\sigma Z_\sigma),$$

$$\tilde{\mathcal{O}}_7^{Z^*Z^*Z^*}=Z^\mu Z_\mu\Box(\partial^\sigma Z_\sigma),$$

$$\tilde{\mathcal{O}}_8^{Z^*Z^*Z^*}=(\partial^\sigma Z_\sigma)(\partial^\nu Z_\mu)(\partial^\mu Z_\nu),$$

$$\tilde{\mathcal{O}}_9^{Z^*Z^*Z^*}=(\partial^\sigma Z_\sigma)(\Box\partial^\alpha Z_\beta)(\partial^\beta Z_\alpha),$$

$$\tilde{\mathcal{O}}_{10}^{Z^*Z^*Z^*}=(\Box\partial^\sigma Z_\sigma)(\partial^\alpha Z_\beta)(\partial^\beta Z_\alpha),$$

$$\tilde{\mathcal{O}}_{11}^{Z^*Z^*Z^*}=\Box\partial^\alpha(\partial^\sigma Z_\sigma)(\partial^\beta Z_\beta)Z_\alpha,$$

$$\tilde{\mathcal{O}}_{12}^{Z^*Z^*Z^*}=\Box\partial^\alpha(\partial^\sigma Z_\sigma)(\partial^\beta Z_\beta)(\Box Z_\alpha),$$

$$\tilde{\mathcal{O}}_{13}^{Z^*Z^*Z^*}=\Box^2\partial^\alpha(\partial^\sigma Z_\sigma)(\partial^\beta Z_\beta)Z_\alpha,$$

$$\tilde{\mathcal{O}}_{14}^{Z^*Z^*Z^*}=(\partial^\sigma Z_\sigma)(\partial^\mu Z_\mu)(\partial^\nu Z_\nu). \tag{20}$$

The transverse terms are given by $\tilde{\mathcal{O}}_1^{Z^*Z^*Z^*}$ (dim=6), $\tilde{\mathcal{O}}_{2,3}^{Z^*Z^*Z^*}$ (dim=8), and $\tilde{\mathcal{O}}_4^{Z^*Z^*Z^*}$ (dim=12); while the scalar ones are generated by $\tilde{\mathcal{O}}_5^{Z^*Z^*Z^*}$ (dim=4), $\tilde{\mathcal{O}}_{6-8,14}^{Z^*Z^*Z^*}$ (dim=6), $\tilde{\mathcal{O}}_{9-11}^{Z^*Z^*Z^*}$ (dim=8) and $\tilde{\mathcal{O}}_{12,13}^{Z^*Z^*Z^*}$ (dim=10). Note the presence of a dim=4 operator, $\tilde{\mathcal{O}}_5^{Z^*Z^*Z^*}$, multiplied by a dimensionless coupling, which would induce CP violation when one Z has a scalar component coupled to a heavy quark pair.

The corresponding coupling functions are

$$\tilde{f}_1^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{2}(s_2-s_1)(\tilde{l}_1^{Z^*Z^*Z^*}+s_3\tilde{l}_2^{Z^*Z^*Z^*})-\frac{1}{2}(s_1^2-s_2^2)\tilde{l}_3^{Z^*Z^*Z^*},$$

$$\tilde{f}_2^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{2}(s_2-s_3)(\tilde{l}_1^{Z^*Z^*Z^*}+s_1\tilde{l}_2^{Z^*Z^*Z^*})-\frac{1}{2}(s_3^2-s_2^2)\tilde{l}_3^{Z^*Z^*Z^*},$$

$$\tilde{f}_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{2}(s_3-s_1)(\tilde{l}_1^{Z^*Z^*Z^*}+s_2\tilde{l}_2^{Z^*Z^*Z^*})-\frac{1}{2}(s_1^2-s_3^2)\tilde{l}_3^{Z^*Z^*Z^*},$$

$$\tilde{f}_4^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{8}(a_1-a_2)\tilde{l}_4^{Z^*Z^*Z^*},$$

$$\tilde{g}_1^{Z^*Z^*Z^*}(s_1,s_2,s_3)=-\frac{1}{2}(s_1+s_2)(\tilde{l}_1^{Z^*Z^*Z^*}+s_3\tilde{l}_2^{Z^*Z^*Z^*})-\frac{1}{2}(s_1^2+s_2^2)\tilde{l}_3^{Z^*Z^*Z^*}$$
$$+2\tilde{l}_5^{Z^*Z^*Z^*}-(s_1+s_2)\tilde{l}_6^{Z^*Z^*Z^*}-2s_3\tilde{l}_7^{Z^*Z^*Z^*},$$

$$\tilde{g}_2^{Z^*Z^*Z^*}(s_1,s_2,s_3)=-\frac{1}{2}(s_3+s_2)(\tilde{l}_1^{Z^*Z^*Z^*}+s_1\tilde{l}_2^{Z^*Z^*Z^*})-\frac{1}{2}(s_3^2+s_2^2)\tilde{l}_3^{Z^*Z^*Z^*}$$
$$+2\tilde{l}_5^{Z^*Z^*Z^*}-(s_3+s_2)\tilde{l}_6^{Z^*Z^*Z^*}-2s_1\tilde{l}_7^{Z^*Z^*Z^*},$$

$$\tilde{g}_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)=-\frac{1}{2}(s_1+s_3)(\tilde{l}_1^{Z^*Z^*Z^*}+s_2\tilde{l}_2^{Z^*Z^*Z^*})-\frac{1}{2}(s_1^2+s_3^2)\tilde{l}_3^{Z^*Z^*Z^*}$$
$$+2\tilde{l}_5^{Z^*Z^*Z^*}-(s_1+s_3)\tilde{l}_6^{Z^*Z^*Z^*}-2s_2\tilde{l}_7^{Z^*Z^*Z^*},$$

$$\tilde{g}_4^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{2}\tilde{l}_1^{Z^*Z^*Z^*}+\frac{1}{8}(a_1+a_2)\tilde{l}_4^{Z^*Z^*Z^*}-\frac{1}{2}\tilde{l}_8^{Z^*Z^*Z^*}$$
$$+\frac{1}{4}(s_2+s_3)\tilde{l}_9^{Z^*Z^*Z^*}+\frac{1}{2}s_1\tilde{l}_{10}^{Z^*Z^*Z^*},$$

$$\tilde{g}_5^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{2}\tilde{l}_1^{Z^*Z^*Z^*}+\frac{1}{8}(a_1+a_2)\tilde{l}_4^{Z^*Z^*Z^*}-\frac{1}{2}\tilde{l}_8^{Z^*Z^*Z^*}+\frac{1}{4}(s_1+s_3)\tilde{l}_9^{Z^*Z^*Z^*}$$
$$+\frac{1}{2}s_2\tilde{l}_{10}^{Z^*Z^*Z^*},$$

$$\tilde{g}_6^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{2}\tilde{l}_1^{Z^*Z^*Z^*}+\frac{1}{8}(a_1+a_2)\tilde{l}_4^{Z^*Z^*Z^*}-\frac{1}{2}\tilde{l}_8^{Z^*Z^*Z^*}+\frac{1}{4}(s_2+s_1)\tilde{l}_9^{Z^*Z^*Z^*}$$
$$+\frac{1}{2}s_3\tilde{l}_{10}^{Z^*Z^*Z^*},$$

$$\tilde{g}_7^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{8}(a_2-a_1)\tilde{l}_4^{Z^*Z^*Z^*}+\frac{1}{4}(s_2-s_1)\tilde{l}_9^{Z^*Z^*Z^*}+\frac{1}{2}(s_1-s_2)\tilde{l}_{10}^{Z^*Z^*Z^*}$$
$$+\frac{1}{2}(s_1-s_2)\tilde{l}_{11}+\frac{s_3}{2}(s_2-s_1)\tilde{l}_{12}^{Z^*Z^*Z^*}+\frac{1}{2}(s_2^2-s_1^2)\tilde{l}_{13}^{Z^*Z^*Z^*},$$

$$\tilde{g}_8^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{8}(a_1-a_2)\tilde{l}_4^{Z^*Z^*Z^*}+\frac{1}{4}(s_2-s_3)\tilde{l}_9^{Z^*Z^*Z^*}+\frac{1}{2}(s_3-s_2)\tilde{l}_{10}^{Z^*Z^*Z^*}+\frac{1}{2}(s_3-s_2)\tilde{l}_{11}^{Z^*Z^*Z^*}+\frac{s_1}{2}(s_2-s_3)\tilde{l}_{12}^{Z^*Z^*Z^*}+\frac{1}{2}(s_2^2-s_3^2)\tilde{l}_{13}^{Z^*Z^*Z^*},$$

$$\tilde{g}_9^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{8}(a_1-a_2)\tilde{l}_4^{Z^*Z^*Z^*}+\frac{1}{4}(s_3-s_1)\tilde{l}_9^{Z^*Z^*Z^*}+\frac{1}{2}(s_1-s_3)\tilde{l}_{10}^{Z^*Z^*Z^*}+\frac{1}{2}(s_1-s_3)\tilde{l}_{11}^{Z^*Z^*Z^*}+\frac{s_2}{2}(s_3-s_1)\tilde{l}_{12}^{Z^*Z^*Z^*}+\frac{1}{2}(s_3^2-s_1^2)\tilde{l}_{13}^{Z^*Z^*Z^*},$$

$$\tilde{g}_{10}^{Z^*Z^*Z^*}(s_1,s_2,s_3)=-\frac{3}{2}\tilde{l}_1^{Z^*Z^*Z^*}-\frac{1}{8}(a_1+a_2)\tilde{l}_4^{Z^*Z^*Z^*}-\frac{3}{2}\tilde{l}_8^{Z^*Z^*Z^*}+\frac{1}{2}(s_1+s_2+s_3)\tilde{l}_9^{Z^*Z^*Z^*}+\frac{1}{2}(s_1+s_2+s_3)\tilde{l}_{10}^{Z^*Z^*Z^*}-(s_1+s_2+s_3)\tilde{l}_{11}^{Z^*Z^*Z^*}+(s_1s_3+s_2s_3+s_1s_2)\tilde{l}_{12}^{Z^*Z^*Z^*}+(s_1^2+s_2^2+s_3^2)\tilde{l}_{13}^{Z^*Z^*Z^*}-\tilde{l}_{14}^{Z^*Z^*Z^*}, \quad (21)$$

with

$$a_1-a_2=s_1^2(s_2-s_3)+s_2^2(s_3-s_1)+s_3^2(s_1-s_2),$$
$$a_1+a_2=s_1^2(s_2+s_3)+s_2^2(s_3+s_1)+s_3^2(s_1+s_2). \quad (22)$$

We also remark that the on-shell single parameter f_4^Z, defined in [1,8,9], is related to the present ones by

$$f_4^Z=m_Z^2[\tilde{l}_1^{Z^*Z^*Z^*}+m_Z^2\tilde{l}_2^{Z^*Z^*Z^*}+(s_3+m_Z^2)\tilde{l}_3^{Z^*Z^*Z^*}]. \quad (23)$$

Thus, going from the on-shell treatment of the CP-violating Z^*ZZ NAGC case, to the present effective Lagrangian off-shell one, we have again to increase the number of parameters from 1 to 3.

C. The $Z^*Z^*\gamma^*$ CP-conserving operators $(i=1,5)$

The operator set is

$$\mathcal{O}_1^{Z^*Z^*\gamma^*}=-\tilde{F}_{\mu\nu}Z^\nu(\partial_\sigma Z^{\sigma\mu}),\quad \mathcal{O}_2^{Z^*Z^*\gamma^*}=\tilde{Z}^{\mu\nu}Z_\nu(\partial^\sigma F_{\sigma\mu}),$$
$$\mathcal{O}_3^{Z^*Z^*\gamma^*}=(\Box\partial^\sigma Z^{\rho\alpha})Z_\sigma\tilde{F}_{\rho\alpha},\quad \mathcal{O}_4^{Z^*Z^*\gamma^*}=\tilde{F}_{\mu\nu}Z^{\mu\nu}(\partial^\sigma Z_\sigma),$$
$$\mathcal{O}_5^{Z^*Z^*\gamma^*}=\tilde{F}_{\mu\nu}Z^{\mu\nu}\Box(\partial^\sigma Z_\sigma). \quad (24)$$

Here the transverse terms are given by $\mathcal{O}_{1,2}^{Z^*Z^*\gamma^*}$ (dim=6) and $\mathcal{O}_3^{Z^*Z^*\gamma^*}$ (dim=8); while the scalar ones are induced by $\mathcal{O}_4^{Z^*Z^*\gamma^*}$ (dim=6) and $\mathcal{O}_5^{Z^*Z^*\gamma^*}$ (dim=8).

The corresponding coupling functions are

$$f_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=s_3l_2^{Z^*Z^*\gamma^*}-\frac{1}{2}s_3(s_1+s_2)l_3^{Z^*Z^*\gamma^*},$$

$$f_2^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=(s_1-s_2)l_1^{Z^*Z^*\gamma^*}+\frac{1}{2}(s_2-s_1)(s_1+s_2)l_3^{Z^*Z^*\gamma^*},$$

$$f_3^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=(s_1-s_2)l_3^{Z^*Z^*\gamma^*},$$

$$g_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-l_1^{Z^*Z^*\gamma^*}+2s_2l_3^{Z^*Z^*\gamma^*}+2l_4^{Z^*Z^*\gamma^*}-2s_1l_5^{Z^*Z^*\gamma^*},$$

$$g_2^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-l_1^{Z^*Z^*\gamma^*}+2s_1l_3^{Z^*Z^*\gamma^*}+2l_4^{Z^*Z^*\gamma^*}-2s_2l_5^{Z^*Z^*\gamma^*}. \quad (25)$$

Comparing now to the parameters defined in Refs. [1,8,9], we remark that when two Z's are on shell one obtains

$$f_5^\gamma=m_Z^2(l_2^{Z^*Z^*\gamma^*}-m_Z^2l_3^{Z^*Z^*\gamma^*}), \quad (26)$$

while when one γ and one Z are on shell one obtains[6]

$$h_3^Z=m_Z^2(l_1^{Z^*Z^*\gamma^*}-m_Z^2l_3^{Z^*Z^*\gamma^*}),\quad h_4^Z=2m_Z^4l_3^{Z^*Z^*\gamma^*}. \quad (27)$$

[6]It is important to note that, contrary to the case of the form

$$I_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,3}=q_3^\beta[q_1q_2\mu\alpha]+q_3^\alpha[q_1q_2\mu\beta],$$

the form $q_3^\alpha[q_1q_2\mu\beta]$ associated to the $h_4^{Z,\gamma}$ couplings, defined in Refs. [1,8], has not a well-defined Bose symmetry property. In fact under Bose symmetry, $h_3^{Z,\gamma}$ and $h_4^{Z,\gamma}$ get mixed. The same remark applies to the CP-violating coupling $h_2^{Z,\gamma}$.

So when considering these two on-shell processes we have the same number of transverse parameters as in the general off-shell case.

D. The $Z^*Z^*\gamma^*$ *CP*-violating operators ($i=1,9$)

These operators are

$$\tilde{\mathcal{O}}_1^{Z^*Z^*\gamma^*}=-F^{\mu\beta}Z_\beta(\partial^\sigma Z_{\sigma\mu}),$$

$$\tilde{\mathcal{O}}_2^{Z^*Z^*\gamma^*}=-(\partial_\alpha\partial_\beta\Box Z_\mu)Z^\alpha F^{\mu\beta},$$

$$\tilde{\mathcal{O}}_3^{Z^*Z^*\gamma^*}=-(\partial_\mu F^{\mu\beta})Z_\alpha(\partial^\alpha Z_\beta),$$

$$\tilde{\mathcal{O}}_4^{Z^*Z^*\gamma^*}=\partial^\mu F_{\mu\nu}(\Box\partial^\nu Z_\alpha)Z^\alpha,$$

$$\tilde{\mathcal{O}}_5^{Z^*Z^*\gamma^*}=(\partial^\sigma Z_\sigma)F_{\mu\nu}(\partial^\mu Z^\nu),$$

$$\tilde{\mathcal{O}}_6^{Z^*Z^*\gamma^*}=(\partial^\sigma Z_\sigma)(\partial^\mu F_{\mu\nu})Z^\nu,$$

$$\tilde{\mathcal{O}}_7^{Z^*Z^*\gamma^*}=\Box(\partial^\sigma Z_\sigma)F_{\mu\nu}(\partial^\mu Z^\nu),$$

$$\tilde{\mathcal{O}}_8^{Z^*Z^*\gamma^*}=(\partial^\sigma Z_\sigma)F_{\mu\nu}(\Box\partial^\mu Z^\nu),$$

$$\tilde{\mathcal{O}}_9^{Z^*Z^*\gamma^*}=\Box\partial^\nu(\partial^\sigma Z_\sigma)(\partial^\beta Z_\beta)\partial^\mu F_{\mu\nu}. \tag{28}$$

The transverse terms are given by $\tilde{\mathcal{O}}_{1,3}^{Z^*Z^*\gamma^*}$ (dim=6), $\tilde{\mathcal{O}}_{2,4}^{Z^*Z^*\gamma^*}$ (dim=8); while the scalar ones are generated by $\tilde{\mathcal{O}}_{5,6}^{Z^*Z^*\gamma^*}$ (dim=6), $\tilde{\mathcal{O}}_{7,8}^{Z^*Z^*\gamma^*}$ (dim=8) and $\tilde{\mathcal{O}}_9^{Z^*Z^*\gamma^*}$ (dim =10).

The corresponding coupling functions are

$$\tilde{f}_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=\frac{s_3}{2}(s_1-s_2)\tilde{l}_4^{Z^*Z^*\gamma^*},$$

$$\tilde{f}_2^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\frac{1}{2}s_3\tilde{l}_3^{Z^*Z^*\gamma^*}-\frac{1}{8}[s_1(s_2-s_1-s_3)+s_2(s_1-s_2-s_3)]\tilde{l}_2^{Z^*Z^*\gamma^*},$$

$$\tilde{f}_3^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=\frac{1}{2}(s_1-s_2)\tilde{l}_1^{Z^*Z^*\gamma^*}+\frac{1}{8}[s_2(s_2+s_3)-s_1(s_1+s_3)]\tilde{l}_2^{Z^*Z^*\gamma^*},$$

$$\tilde{f}_4^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=\frac{1}{8}(s_1-s_2)\tilde{l}_2^{Z^*Z^*\gamma^*},$$

$$\begin{aligned}\tilde{g}_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=&\frac{1}{2}s_3\tilde{l}_1^{Z^*Z^*\gamma^*}-\frac{1}{2}s_3\tilde{l}_3^{Z^*Z^*\gamma^*}-\frac{1}{8}[s_1(s_2-s_1-s_3)+s_2(s_1-s_2-s_3)]\tilde{l}_2^{Z^*Z^*\gamma^*}\\&+\frac{1}{2}s_3\tilde{l}_5^{Z^*Z^*\gamma^*}-s_3\tilde{l}_6^{Z^*Z^*\gamma^*}+\frac{1}{4}[(s_2-s_1)^2-s_3(s_1+s_2)]\tilde{l}_7^{Z^*Z^*\gamma^*}\\&-\frac{1}{4}[(s_2-s_1)^2+s_3(s_1+s_2)]\tilde{l}_8^{Z^*Z^*\gamma^*},\end{aligned}$$

$$\begin{aligned}\tilde{g}_2^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=&\frac{1}{8}[s_2(s_2+s_3)-s_1(s_1+s_3)]\tilde{l}_2^{Z^*Z^*\gamma^*}\\&+\frac{1}{2}(s_2-s_1)\tilde{l}_5^{Z^*Z^*\gamma^*}\\&+\frac{1}{4}[(s_1^2-s_2^2)-s_3(s_1-s_2)]\tilde{l}_7^{Z^*Z^*\gamma^*}+\frac{1}{4}[(s_1^2-s_2^2)+s_3(s_1-s_2)]\tilde{l}_8^{Z^*Z^*\gamma^*},\end{aligned}$$

$$\begin{aligned}\tilde{g}_3^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=&-\frac{1}{4}\tilde{l}_1^{Z^*Z^*\gamma^*}+\frac{s_1+s_2}{8}\tilde{l}_2^{Z^*Z^*\gamma^*}+\frac{1}{4}\tilde{l}_5^{Z^*Z^*\gamma^*}-\frac{1}{8}(s_1+s_2)\tilde{l}_7^{Z^*Z^*\gamma^*}\\&-\frac{1}{8}(s_1+s_2)\tilde{l}_8^{Z^*Z^*\gamma^*},\end{aligned}$$

$$\tilde{g}_4^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\frac{1}{8}(s_1-s_2)\tilde{l}_7^{Z^*Z^*\gamma^*}+\frac{1}{8}(s_1-s_2)\tilde{l}_8,$$

$$\tilde{g}_5^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=\frac{s_2-s_1}{8}\tilde{l}_2^{Z^*Z^*\gamma^*}+\frac{1}{4}(s_2-s_1)\tilde{l}_7^{Z^*Z^*\gamma^*}+\frac{1}{4}(s_1-s_2)\tilde{l}_8^{Z^*Z^*\gamma^*}$$
$$-\frac{1}{2}s_3(s_1-s_2)\tilde{l}_9^{Z^*Z^*\gamma^*}. \tag{29}$$

Comparing to the parameters defined in Refs. [1,8,9], when two Z's are on shell, one obtains

$$f_4^{\gamma}=m_Z^2\left(\tilde{l}_3^{Z^*Z^*\gamma^*}-\frac{m_Z^2}{2}\tilde{l}_2^{Z^*Z^*\gamma^*}\right), \tag{30}$$

while when one γ and one Z are on shell, we get

$$h_1^{Z}=m_Z^2\left(\tilde{l}_1^{Z^*Z^*\gamma^*}-\frac{m_Z^2}{2}\tilde{l}_2^{Z^*Z^*\gamma^*}\right),\quad h_2^{Z}=m_Z^4\tilde{l}_2^{Z^*Z^*\gamma^*}. \tag{31}$$

So the off-shell case has one more transverse parameter ($\tilde{l}_4^{Z^*Z^*\gamma^*}$) than the on-shell one.

E. The $\gamma^*\gamma^*Z^*$ CP-conserving operators $(i=1,4)$

The four operators of this case are

$$\mathcal{O}_1^{\gamma^*\gamma^*Z^*}=-\tilde{F}_{\rho\alpha}(\partial_\sigma F^{\sigma\rho})Z^{\alpha},\quad \mathcal{O}_2^{\gamma^*\gamma^*Z^*}=\Box\tilde{F}^{\mu\nu}(\partial^{\sigma}F_{\sigma\mu})Z_{\nu},$$
$$\mathcal{O}_3^{\gamma^*\gamma^*Z^*}=(\Box\partial^{\sigma}F^{\rho\alpha})Z_{\sigma}\tilde{F}_{\rho\alpha},\quad \mathcal{O}_4^{\gamma^*\gamma^*Z^*}=\tilde{F}_{\mu\nu}F^{\mu\nu}(\partial^{\sigma}Z_{\sigma}). \tag{32}$$

The transverse terms are given by $\mathcal{O}_1^{\gamma^*\gamma^*Z^*}$ (dim=6) and $\mathcal{O}_{2,3}^{\gamma^*\gamma^*Z^*}$ (dim=8); while the scalar term by $\mathcal{O}_4^{\gamma^*\gamma^*Z^*}$ (dim=6).

The corresponding coupling functions are

$$f_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=\frac{1}{2}(s_1+s_2)l_1^{\gamma^*\gamma^*Z^*}+s_1s_2l_2^{\gamma^*\gamma^*Z^*}-\frac{1}{2}(s_1-s_2)^2l_3^{\gamma^*\gamma^*Z^*},$$

$$f_2^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=\frac{1}{2}(s_1-s_2)l_1^{\gamma^*\gamma^*Z^*}+\frac{1}{2}(s_1-s_2)(s_3-2s_1-2s_2)l_3^{\gamma^*\gamma^*Z^*},$$

$$f_3^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=(s_2-s_1)l_3^{\gamma^*\gamma^*Z^*},$$

$$g_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=(s_1+s_2)l_3^{\gamma^*\gamma^*Z^*}+4l_4^{\gamma^*\gamma^*Z^*}. \tag{33}$$

When one γ and one Z are on shell, one gets

$$h_3^{\gamma}=m_Z^2l_1^{\gamma^*\gamma^*Z^*},\quad h_4^{\gamma}=2m_Z^4l_3^{\gamma^*\gamma^*Z^*}, \tag{34}$$

when comparing to the parameters of Refs. [1,8,9], and one observes that there is one less transverse parameter ($l_2^{\gamma^*\gamma^*Z^*}$) than in the off-shell case.

F. The $\gamma^*\gamma^*Z^*$ CP-violating operators $(i=1,6)$

We now have

$$\tilde{\mathcal{O}}_1^{\gamma^*\gamma^*Z^*}=-(\partial^{\sigma}F_{\sigma\mu})Z_{\beta}F^{\mu\beta},\quad \tilde{\mathcal{O}}_2^{\gamma^*\gamma^*Z^*}=(\Box F^{\mu\nu})F_{\nu\alpha}(\partial_{\mu}Z^{\alpha}),$$

$$\tilde{\mathcal{O}}_3^{\gamma^*\gamma^*Z^*}=-(\partial_{\alpha}\partial_{\beta}\partial^{\rho}F_{\rho\mu})Z^{\alpha}F^{\mu\beta},\quad \tilde{\mathcal{O}}_4^{\gamma^*\gamma^*Z^*}=(\Box\partial_{\mu}F^{\mu\nu})(\partial^{\sigma}F_{\sigma\alpha})(\partial_{\nu}Z^{\alpha}),$$

$$\tilde{\mathcal{O}}_5^{\gamma^*\gamma^*Z^*}=(\partial_{\sigma}Z^{\sigma})F^{\mu\nu}F_{\mu\nu},\quad \tilde{\mathcal{O}}_6^{\gamma^*\gamma^*Z^*}=\partial_{\mu}(\partial_{\sigma}Z^{\sigma})F^{\mu\nu}(\partial^{\beta}F_{\beta\nu}). \tag{35}$$

The transverse terms are given by $\tilde{\mathcal{O}}_1^{\gamma^*\gamma^*Z^*}$ (dim=6), $\tilde{\mathcal{O}}_{2,3}^{\gamma^*\gamma^*Z^*}$ (dim=8) and $\tilde{\mathcal{O}}_4^{\gamma^*\gamma^*Z^*}$ (dim=10). The scalar terms are $\tilde{\mathcal{O}}_5^{\gamma^*\gamma^*Z^*}$ (dim=6) and $\tilde{\mathcal{O}}_6^{\gamma^*\gamma^*Z^*}$ (dim=8).

The corresponding coupling functions are

$$\begin{aligned}
\tilde{f}_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)&=\frac{1}{2}(s_1-s_2)\tilde{l}_1^{\gamma^*\gamma^*Z^*}+\frac{1}{4}[s_2(s_2+s_3)-s_1(s_1+s_3)]\tilde{l}_2^{\gamma^*\gamma^*Z^*}\\
&\quad+\frac{1}{4}(s_1-s_2)(s_3-s_1-s_2)\tilde{l}_3^{\gamma^*\gamma^*Z^*},\\
\tilde{f}_2^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)&=-\frac{1}{4}(s_1+s_2)\tilde{l}_1^{\gamma^*\gamma^*Z^*}+\frac{s_1^2+s_2^2}{4}\tilde{l}_2^{\gamma^*\gamma^*Z^*}-\frac{s_1s_2}{2}(s_1+s_2)\tilde{l}_4^{\gamma^*\gamma^*Z^*},\\
\tilde{f}_3^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)&=\frac{1}{4}(s_2-s_1)\tilde{l}_1^{\gamma^*\gamma^*Z^*}+\frac{s_1^2-s_2^2}{4}\tilde{l}_2^{\gamma^*\gamma^*Z^*}+\frac{s_1s_2}{2}(s_2-s_1)\tilde{l}_4^{\gamma^*\gamma^*Z^*},\\
\tilde{f}_4^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)&=\frac{1}{8}(s_1-s_2)\tilde{l}_2^{\gamma^*\gamma^*Z^*}-\frac{1}{8}(s_1-s_2)\tilde{l}_3^{\gamma^*\gamma^*Z^*},\\
\tilde{g}_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)&=\frac{1}{2}(s_1+s_2)\tilde{l}_1^{\gamma^*\gamma^*Z^*}+\frac{1}{4}[s_1(s_2-s_1-s_3)+s_2(s_1-s_2-s_3)]\tilde{l}_2^{\gamma^*\gamma^*Z^*}\\
&\quad-\frac{1}{4}(s_1+s_2)(s_3-s_1-s_2)\tilde{l}_3^{\gamma^*\gamma^*Z^*}-2(s_3-s_1-s_2)\tilde{l}_5^{\gamma^*\gamma^*Z^*}\\
&\quad+\frac{1}{2}[s_1(s_1-s_2-s_3)+s_2(s_2-s_1-s_3)]\tilde{l}_6^{\gamma^*\gamma^*Z^*},\\
\tilde{g}_2^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)&=\frac{1}{8}(s_1+s_2)\tilde{l}_2^{\gamma^*\gamma^*Z^*}+\frac{1}{8}(s_1+s_2)\tilde{l}_3^{\gamma^*\gamma^*Z^*}\\
&\quad+\tilde{l}_5^{\gamma^*\gamma^*Z^*}+\frac{1}{4}(s_1+s_2)\tilde{l}_6^{\gamma^*\gamma^*Z^*}.
\end{aligned}\tag{36}$$

When one γ and one Z are on shell, one obtains

$$\begin{aligned}
h_1^{\gamma}&=m_Z^2(\tilde{l}_1^{\gamma^*\gamma^*Z^*}-s_1\tilde{l}_2^{\gamma^*\gamma^*Z^*}),\\
h_2^{\gamma}&=-m_Z^4(\tilde{l}_2^{\gamma^*\gamma^*Z^*}-\tilde{l}_3^{\gamma^*\gamma^*Z^*}),
\end{aligned}\tag{37}$$

which express the on-shell parameters of Refs. [1,8,9], in terms of the present ones. We observe that $\tilde{l}_4^{\gamma^*\gamma^*Z^*}$ is not involved and that only two transverse parameters appear instead of four in the off-shell case.

G. Comments about the lowest-dimensional parametrization

As already said the effective Lagrangian of Eq. (15) is suitable for describing NP effects generated at a very high scale. If this occurs, then it may turn out to be adequate to restrict to operators of dim=6. Keeping only transverse terms (which is absolutely legitimate, provided that no events involving $Z\to t\bar{t}$ decays are considered) then we end up with just four CP-conserving and four CP-violating couplings; namely,

$$l_1^{Z^*Z^*Z^*},\quad \tilde{l}_1^{Z^*Z^*Z^*},\quad l_1^{Z^*Z^*\gamma^*},\quad l_2^{Z^*Z^*\gamma^*},\quad \tilde{l}_1^{Z^*Z^*\gamma^*},\quad \tilde{l}_3^{Z^*Z^*\gamma^*},\quad l_1^{\gamma^*\gamma^*Z^*},\quad \tilde{l}_1^{\gamma^*\gamma^*Z^*}.\tag{38}$$

If in addition, the higher dimensional operators above are also included, we have to add to this set of parameters the following:

$$\begin{aligned}
&l_2^{Z^*Z^*Z^*},\quad l_3^{Z^*Z^*Z^*},\quad \tilde{l}_2^{Z^*Z^*Z^*},\quad \tilde{l}_3^{Z^*Z^*Z^*},\quad \tilde{l}_4^{Z^*Z^*Z^*},\\
&l_3^{Z^*Z^*\gamma^*},\quad \tilde{l}_2^{Z^*Z^*\gamma^*},\quad \tilde{l}_4^{Z^*Z^*\gamma^*},\\
&l_2^{Z^*Z^*\gamma^*},\quad l_3^{Z^*Z^*\gamma^*},\quad \tilde{l}_2^{Z^*Z^*\gamma^*},\quad \tilde{l}_3^{Z^*Z^*\gamma^*},\quad \tilde{l}_4^{Z^*Z^*\gamma^*}.
\end{aligned}\tag{39}$$

Thus, within the context of the effective Lagrangian of this Sec. III, we need 21 parameters to describe the off-shell effects for all ''transverse'' NAGC. These parameters would be related to those defined on-shell in Refs. [1,8,9] by Eqs. (19),(23),(26),(27),(30),(31),(34),(37). Furthermore, if the NP scale is very high, then it is natural to expect that the dim=8 terms (which are proportional to $1/\Lambda^4$), should be strongly suppressed. The suppression should even be stronger for the higher dim=10,12 terms. In this case the set of eight parameters in Eq. (38) should be the dominant ones.

Let us insist on the merit of the effective Lagrangian (15) which allows through Eqs. (18),(21),(25),(29),(33),(36), to get the precise off-shell s_i dependence of the amplitudes consistent with Bose symmetry and CVC. Provided the NP scale is high, these should the suitable expressions for a model independent data analysis.

On the other hand, if the NP scale inducing NAGC is near the energy scale of the measurements, then the effective Lagrangian description becomes inadequate. In such a case, dynamical models such as those considered in the next section can be much more useful in providing hints for the description of the possible new physics.

Finally, if $Z\to t\bar{t}$ decays are also included in the NAGC analysis; then the ''scalar'' couplings should also be included. Altogether, there exist 23 such couplings in the effective Lagrangian listed above. Eleven of them correspond to dim=6 operators, and constitute a set of the three CP-conserving

$$l_4^{Z^*Z^*Z^*}, \quad l_4^{Z^*Z^*\gamma^*}, \quad l_4^{\gamma^*\gamma^*Z^*},$$

and the eight CP-violating

$$\tilde{l}_5^{Z^*Z^*Z^*}, \quad \tilde{l}_6^{Z^*Z^*Z^*}, \quad \tilde{l}_7^{Z^*Z^*Z^*}, \quad \tilde{l}_8^{Z^*Z^*Z^*}, \quad \tilde{l}_{14}^{Z^*Z^*Z^*},$$

$$\tilde{l}_5^{Z^*Z^*\gamma^*}, \quad \tilde{l}_6^{Z^*Z^*\gamma^*}, \quad \tilde{l}_5^{\gamma^*\gamma^*Z^*}$$

couplings, while the remaining 12 describe higher-dimensional scalar NAGC.

Before concluding this subsection we add a few comments concerning $SU(2)\times U(1)$ gauge invariance. Strictly speaking the NP vertices introduced to the effective Lagrangian by the NP operators in Eqs. (17),(20),(24),(28), (32),(35), should only be used in the unitary gauge.[7] This restriction can be easily cured though, by making the substitutions

$$Z_{\mu\nu}\to -s_W B_{\mu\nu}-\frac{2c_W}{v^2}(\Phi^\dagger\vec{\tau}\Phi)\cdot\vec{W}_{\mu\nu},$$

$$F_{\mu\nu}\to c_W B_{\mu\nu}-\frac{2s_W}{v^2}(\Phi^\dagger\vec{\tau}\Phi)\cdot\vec{W}_{\mu\nu},$$

$$Z_\mu\to i\,\frac{4s_W c_W}{v^2}(\Phi^\dagger D_\mu\Phi), \tag{40}$$

which transforms them to a gauge invariant form. In Eq. (40) Φ is the SM Higgs doublet, v its vacuum expectation value, and D_μ is the usual $SU(2)\times U(1)$ covariant derivative.

[7]We would like to thank E. Boos for discussions on this point.

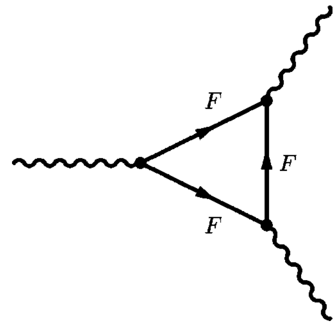

FIG. 2. The fermionic triangle.

The substitutions (40) generally change the dimensionality of the various operators. If after performing them, we make the further restriction that only the lowest dim=8 operators are retained, then we just end up with the two operators

$$\mathcal{O}_{SU(2)\times U(1)}=i\tilde{B}_{\mu\nu}(\partial_\sigma B^{\sigma\mu})(\Phi^\dagger D^\nu\Phi),$$

$$\tilde{\mathcal{O}}_{SU(2)\times U(1)}=iB_{\mu\nu}(\partial_\sigma B^{\sigma\mu})(\Phi^\dagger D^\nu\Phi). \tag{41}$$

These are the only dim=8 $SU(2)\times U(1)$ invariant operators which in the unitary gauge only involve either purely neutral *triple* gauge couplings, or couplings affecting three neutral gauge bosons and a Higgs field. They are closely related to the $\mathcal{O}_1^{V_1V_2V_3}$ and $\tilde{\mathcal{O}}_1^{V_1V_2V_3}$ defined in the various subsections above.

IV. A TOY MODEL: THE FERMIONIC TRIANGLE LOOP

In Ref. [9], we have discussed the possible dynamical origin of the triple neutral gauge boson interactions, when two of the gauge bosons are on shell. The first conclusion there was that, at the one-loop level of any fundamental renormalizable gauge theory, nonvanishing contributions could only arise if fermions run along the loop; the bosonic loop always giving a vanishing result. The second point was that no CP-violating NAGC couplings are generated in such a context. Here we explore the consequences of this model when all three neutral gauge bosons are taken off shell.

A. General structure of one-loop couplings

The triangle diagram is depicted in Fig. 2. The fermion couplings are defined through the gauge Lagrangian [9]

$$\mathcal{L}=-eQ_F A^\mu\bar{F}\gamma_\mu F-\frac{e}{2s_W c_W}Z^\mu\bar{F}(\gamma_\mu g_{vF}-\gamma_\mu\gamma_5 g_{aF})F. \tag{42}$$

The complete expressions of the resulting off-shell CP-conserving NAGC are given in Appendix C, where for simplicity we take a single fermion running along the triangular loop. These expressions are directly applicable to any fermionic contributions. Thus, e.g., the SM prediction for the neutral gauge boson self-interactions is obtained by summing the contributions of the leptons and of the quarks.

To present these results, we first observe that the one-loop fermionic diagrams strongly reduce the six independent forms that could exist in the general case (compare the most general type of such forms in Appendix A). More explicitly, the only nonvanishing coupling functions contained in the one-loop diagrams are the two nonvanishing transverse ones called $f_{1,2}(s_1,s_2,s_3)$, and a single scalar function[8] called $g_1(s_1,s_2,s_3)$. In particular, no h_4-type of coupling (compare Refs. [1,8]), is allowed by such diagrams. This has already been noticed in the on-shell case [9]; where it has been remarked that higher order or nonperturbative effects are required for generating h_4 couplings.

To establish contact with the effective Lagrangian of Sec. III, we consider the heavy fermion limit of the above functions. In such a limit (retaining only the dominant $1/M_F^2$ contributions) the heavy fermion loop predictions are identical to those of a CP-conserving effective Lagrangian in which the only nonvanishing couplings are

$$l_1^{Z^*Z^*Z^*},\quad l_4^{Z^*Z^*Z^*},\quad l_1^{Z^*Z^*\gamma^*},\quad l_2^{Z^*Z^*\gamma^*},\quad l_4^{Z^*Z^*\gamma^*},$$

$$l_1^{\gamma^*\gamma^*Z^*},\quad l_4^{\gamma^*\gamma^*Z^*}.$$

Of course, if the mass of the fermion in the loop of Fig. 2 is comparable to (or lighter than) the energies considered, additional structures appear in the $f_{1,2}$ and g_1 functions, that cannot be described by the above effective Lagrangian. If NAGC are ever observed, then the experimental search for such structures, will provide a very important means for identifying the responsible NP degrees of freedom.

1. The $Z^*Z^*Z^*$ couplings at one-loop

Following the results in Appendix C, the fermionic triangle contribution is written as

$$f_1^{Z^*Z^*Z^*}(s_1,s_2,s_3)=-\frac{e^2g_{aF}}{32\pi^2s_W^3c_W^3}\times\{(3g_{vF}^2+g_{aF}^2)\mathcal{G}_1(s_1,s_2,s_3)-(g_{aF}^2-g_{vF}^2)\mathcal{G}_3(s_1,s_2,s_3)\},$$

$$f_2^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{e^2g_{aF}}{32\pi^2s_W^3c_W^3}\times\{(3g_{vF}^2+g_{aF}^2)\mathcal{G}_2(s_1,s_2,s_3)-(g_{aF}^2-g_{vF}^2)\mathcal{G}_4(s_1,s_2,s_3)\},$$

$$g_1^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{e^2}{8\pi^2s_W^3c_W^3}\times g_{aF}(3g_{vF}^2+g_{aF}^2)\mathcal{G}_1'(s_1,s_2,s_3),\tag{43}$$

where the functions $\mathcal{G}_i(s_1,s_2,s_3)$ and $\mathcal{G}_1'$ are given in Appendix C in terms of Passarino-Veltman B_0 and C_0 functions [18].

As required by the anomaly cancellation (and explained in Appendix C), all the $\mathcal{G}_j$ and $\mathcal{G}_j'$ functions vanish in the large M_F limit. Moreover, at the $1/M_F^2$ level, they satisfy

$$\mathcal{G}_1\simeq 3\mathcal{G}_3\simeq\frac{s_1+s_2-2s_3}{40M_F^2},\quad \mathcal{G}_2\simeq 3\mathcal{G}_4\simeq\frac{3(s_2-s_1)}{40M_F^2},$$

$$\mathcal{G}_1'\simeq\frac{1}{24M_F^2},\tag{44}$$

from which the leading contributions to $f_{1,2}$ and g_1 are calculated using Eq. (43). As expected, these large M_F results coincide with those of the effective Lagrangian description, with the only nonzero parameters being

$$l_1^{Z^*Z^*Z^*}=\left(\frac{g_{aF}}{30M_F^2}\right)\left(\frac{e^2}{32\pi^2s_W^3c_W^3}\right)(5g_{vF}^2+g_{aF}^2),$$

$$l_4^{Z^*Z^*Z^*}=\left(\frac{g_{aF}}{60M_F^2}\right)\left(\frac{e^2}{32\pi^2s_W^3c_W^3}\right)(5g_{vF}^2+3g_{aF}^2).\tag{45}$$

Combining this with Eq. (18) for the on-shell case $Z^*\to ZZ$ $(s_1=s_2=m_Z^2)$, for which Eq. (44) implies $\mathcal{G}_{2,4}=0$, we obtain

$$f_1^{Z^*Z^*Z^*}(m_Z^2,m_Z^2,s_3)=(s_3-m_Z^2)l_1^{Z^*Z^*Z^*}=\frac{s_3-m_Z^2}{m_Z^2}f_5^Z(s_3),$$

$$f_2^{Z^*Z^*Z^*}(m_Z^2,m_Z^2,s_3)=0,\tag{46}$$

which agrees with the expression given in Ref. [9].

2. The $Z^*Z^*\gamma^*$ couplings at one loop

The formalism in Appendix C leads to

$$f_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\frac{e^2Q_Fg_{aF}g_{vF}}{8\pi^2s_W^2c_W^2}[\mathcal{G}_1(s_1,s_2,s_3)+\mathcal{G}_5(s_1,s_2,s_3)],$$

$$f_2^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=\frac{e^2Q_Fg_{aF}g_{vF}}{8\pi^2s_W^2c_W^2}\left[\mathcal{G}_2(s_1,s_2,s_3)+\frac{1}{3}\mathcal{G}_4(s_1,s_2,s_3)\right],$$

[8]Depending on the NAGC coupling considered, there may by additional scalar functions such as $g_2(s_1,s_2,s_3)$ and/or $g_3(s_1,s_2,s_3)$; but these functions are related to $g_1(s_1,s_2,s_3)$ by equations such as Eq. (C7), since $f_3(s_1,s_2,s_3)\equiv 0$ for the diagram in Fig. 2.

$$g_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=g_2^{Z^*Z^*\gamma^*}(s_2,s_1,s_3)$$
$$=\frac{e^2Q_Fg_{aF}g_{vF}}{2\pi^2s_W^2c_W^2}\mathcal{G}_1'(s_1,s_2,s_3), \quad (47)$$

where the needed $\mathcal{G}_j$ functions are again given there. To derive the leading contribution to these couplings in the large M_F^2 limit, we need first the leading contributions to the $\mathcal{G}_j$ defined in Appendix C. Keeping terms only up to the $1/M_F^2$ order [as in the derivation of Eq. (44)] this is given by

$$\mathcal{G}_1+\mathcal{G}_5\simeq-\frac{s_3}{12M_F^2}, \quad \mathcal{G}_2+\frac{1}{3}\mathcal{G}_4\simeq\frac{(s_2-s_1)}{12M_F^2}, \quad \mathcal{G}_1'\simeq\frac{1}{24M_F^2}, \quad (48)$$

which, substituted to Eq. (47), result to values of the couplings functions consistent with those obtained in Eq. (25), provided

$$-l_1^{Z^*Z^*\gamma^*}=l_2^{Z^*Z^*\gamma^*}=2l_4^{Z^*Z^*\gamma^*}=\left(\frac{1}{12M_F^2}\right)\frac{e^2Q_Fg_{aF}g_{vF}}{8\pi^2s_W^2c_W^2}, \quad (49)$$

while all other $l_j^{Z^*Z^*\gamma^*}$ should vanish.

Comparing to the on-shell cases:

(a) $\gamma^*\to ZZ$, $s_1=s_2=m_Z^2$, $\mathcal{G}_2+\mathcal{G}_4/3=0$ leads to

$$f_1^{Z^*Z^*\gamma^*}(m_Z^2,m_Z^2,s_3)=s_3l_2^{Z^*Z^*\gamma^*}=\frac{s_3}{m_Z^2}f_5^{\gamma}(s_3),$$
$$f_2^{Z^*Z^*\gamma^*}(m_Z^2,m_Z^2,s_3)=0; \quad (50)$$

(b) $Z^*\to Z\gamma$, $s_3=0$, $s_1=m_Z^2$, $\mathcal{G}_1+\mathcal{G}_5=0$ implies $h_4^Z=0$, and

$$f_1^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)=0,$$
$$f_2^{Z^*Z^*\gamma^*}(m_Z^2,s_2,0)=(m_z^2-s_2)l_1^{Z^*Z^*\gamma^*}=\frac{m_z^2-s_2}{m_Z^2}h_3^Z(s_1), \quad (51)$$

which agree with the expressions given in Ref. [9].

3. The $\gamma^*\gamma^*Z^*$ couplings at one loop

The results of Appendix C give

$$f_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=-\frac{e^2Q_F^2g_{aF}}{8\pi^2s_Wc_W}[\mathcal{G}_6(s_1,s_2,s_3)+\mathcal{G}_7(s_1,s_2,s_3)],$$

$$f_2^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=-\frac{e^2Q_F^2g_{aF}}{8\pi^2s_Wc_W}[\mathcal{G}_6(s_1,s_2,s_3)-\mathcal{G}_7(s_1,s_2,s_3)],$$

$$g_3^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=\frac{e^2Q_F^2g_{aF}}{2\pi^2s_Wc_W}\mathcal{G}_1'(s_1,s_2,s_3). \quad (52)$$

At the $1/M_F^2$ level, the leading heavy fermion values of the $\mathcal{G}_j$-combinations appearing in Eq. (52) are

$$\mathcal{G}_6(s_1,s_2,s_3)+\mathcal{G}_7(s_1,s_2,s_3)\simeq\frac{s_2+s_1}{12M_F^2},$$
$$\mathcal{G}_6(s_1,s_2,s_3)-\mathcal{G}_7(s_1,s_2,s_3)\simeq\frac{(s_1-s_2)}{12M_F^2},$$
$$\mathcal{G}_1'\simeq\frac{1}{24M_F^2}, \quad (53)$$

which as expected coincide with the effective Lagrangian results of (33) provided the only nonvanishing couplings are

$$-l_1^{\gamma^*\gamma^*Z^*}=4l_4^{\gamma^*\gamma^*Z^*}=\left(\frac{1}{6M_F^2}\right)\frac{e^2Q_F^2g_{aF}}{8\pi^2s_Wc_W}. \quad (54)$$

When only one photon and one Z are on-shell (i.e., $s_2=0$, $s_3=m_Z^2$, $\mathcal{G}_7=0$), we obtain

$$f_1^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2)=f_2^{\gamma^*\gamma^*Z^*}(s_1,0,m_Z^2)$$
$$=\frac{s_1}{2}l_1^{\gamma^*\gamma^*Z^*}=\frac{s_1}{2m_Z^2}h_3^{\gamma}(s_1), \quad (55)$$

which agree with the expressions given in Ref. [9].

B. Quantitative discussion of the one-loop off-shell effects

After having shown the structure of the NAGC generated at one loop, we now make a quantitative discussion of the off-shell effects. These effects are described below by three sets of ratios which quantify the following features: (a) The ratios R_1^{5Z}, R_3^{5Z}, $R^{5\gamma}$, R^{3Z}, $R^{3\gamma}$ are sensitive to the s_i dependences of the type of couplings existing already on shell; (b) the ratios R'^{5Z}_1, R'^{5Z}_3, $R'^{5\gamma}$, R'^{3Z}, $R'^{3\gamma}$ study the relative size (versus s_i) of new types of couplings as compared to those already existing on shell; (c) the ratios R_1^{ZZZ}, R_2^{ZZZ}, $R^{ZZ\gamma}$, $R^{Z\gamma Z}$, $R^{\gamma\gamma Z}$ aim to quantify the range of the mass M_F of the fermion running along the loop, for which the effective Lagrangian structure (which already contains some s_i dependence) is adequate.

For each ratio, we indicate below their value in the large M_F limit. As shown in the previous section, these values agree with the predictions of the effective Lagrangian. We have compared these values to a numerical computation done with the exact expressions for finite M_F values and for some choices of s_i values falling inside the range accessible at LEP2 (0.2 TeV) or at LC (0.5 TeV). In Figs. 3 –6 we have selected some typical examples of the s_i and M_F behaviors. The above three points are discussed in turn for each NAGC vertex.

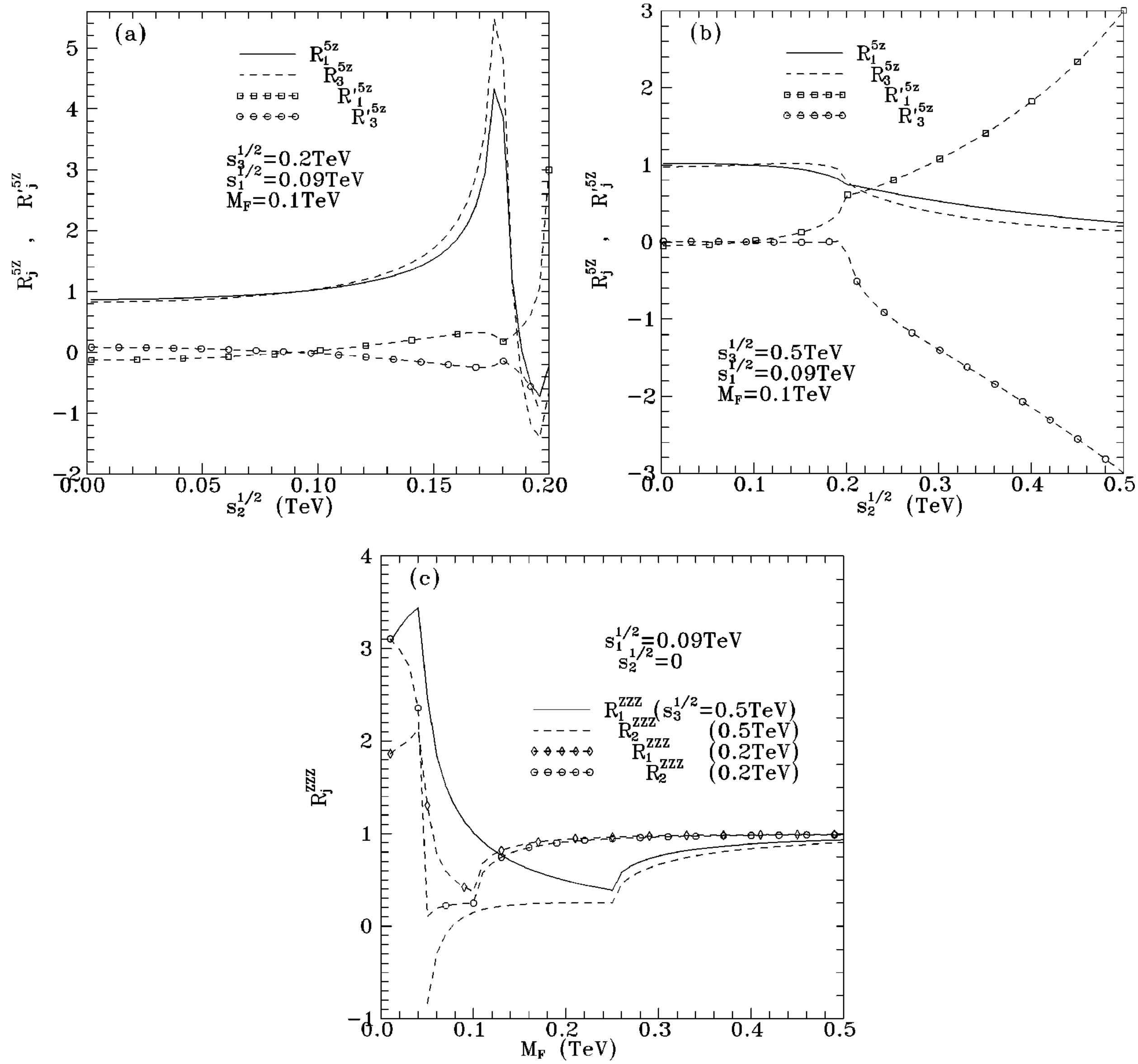

FIG. 3. $Z^*Z^*Z^*$ off-shell effects compared to $Z^*\to ZZ$: Ratios R_1^{5Z} and R_3^{5Z} show the $\sqrt{s_2}$ dependence of the contributions to the f_5^Z type of coupling; ratios R'^{5Z}_1 and R'^{5Z}_3 give the relative size, versus $\sqrt{s_2}$, of the new contributions as compared to the ones already existing on shell; (a) at $\sqrt{s_3}=0.2$ TeV, (b) at $\sqrt{s_3}=0.5$ TeV. Ratios R_1^{ZZZ} and R_2^{ZZZ} show the departure versus M_F of the exact 1-loop contribution, as compared to the effective Lagrangian prediction at $\sqrt{s_3}=0.2$ TeV and 0.5 TeV, (c). The definitions of s_1,s_2,s_3 are given in the text.

1. The off-shell one-loop effects in $Z^*Z^*Z^*$ compared to $Z^*\to ZZ$

(a) The ratios R_1^{5Z} and R_3^{5Z} show the evolution of the contributions to the $f_1^{Z^*Z^*Z^*}(s_1,s_2,s_2)$ type of coupling as defined in Eq. (43), from $s_1=s_2=m_Z^2$ up to some off-shell value

$$R_1^{5Z}=\frac{\mathcal{G}_1(s_1,s_2,s_3)}{\mathcal{G}_1(m_Z^2,m_Z^2,s_3)}\to\frac{2s_3-s_1-s_2}{2(s_3-m_Z^2)},$$

$$R_3^{5Z}=\frac{\mathcal{G}_3(s_1,s_2,s_3)}{\mathcal{G}_3(m_Z^2,m_Z^2,s_3)}\to\frac{2s_3-s_1-s_2}{2(s_3-m_Z^2)}. \tag{56}$$

Note from Eq. (2) the way that $f_1^{Z^*Z^*Z^*}(s_1,s_2,s_2)$ is related to the on-shell f_5^Z coupling of Refs. [1,8].

(b) The ratios R'^{5Z}_1 and R'^{5Z}_3 give the relative size of the new $f_2^{Z^*Z^*Z^*}$ coupling as compared to $f_1^{Z^*Z^*Z^*}$ already existing on shell:

$$R'^{5Z}_1=-\frac{\mathcal{G}_2(s_1,s_2,s_3)}{\mathcal{G}_1(s_1,s_2,s_3)}\to\frac{3(s_1-s_2)}{s_1+s_2-2s_3},$$

$$R'^{5Z}_3=\frac{\mathcal{G}_4(s_1,s_2,s_3)}{\mathcal{G}_3(s_1,s_2,s_3)}\to-\frac{3(s_1-s_2)}{s_1+s_2-2s_3}. \tag{57}$$

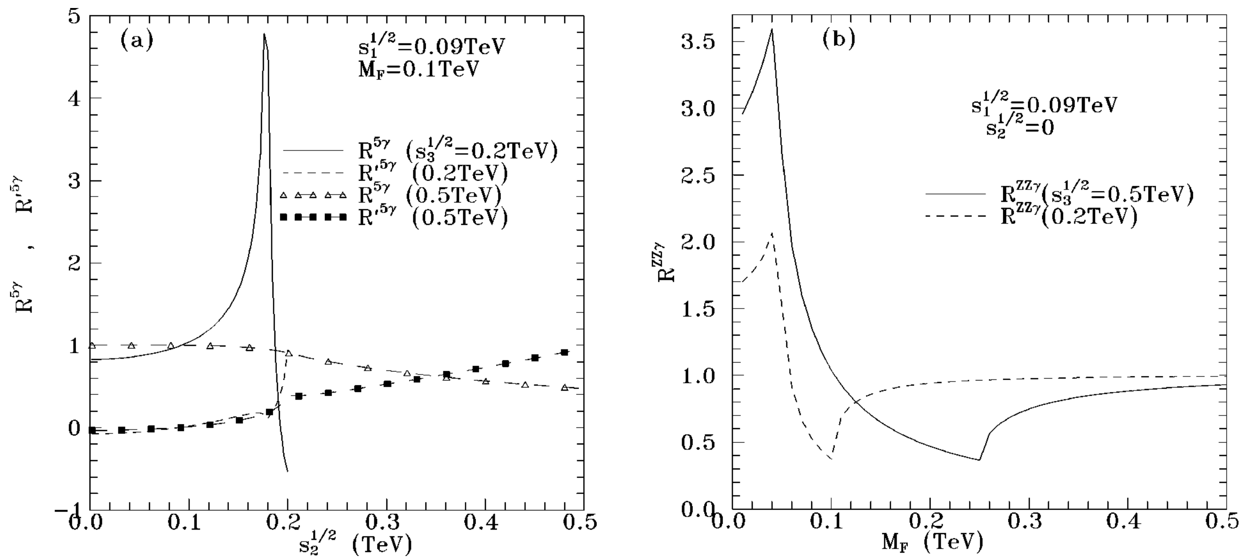

FIG. 4. $Z^*Z^*\gamma^*$ off-shell effects compared to $\gamma^*\to ZZ$: Ratio $R^{5\gamma}$ show the $\sqrt{s_2}$ dependence of the contributions to the f_5^γ type of coupling, while $R'^{5\gamma}$ gives the relative size, versus $\sqrt{s_2}$, of the new contributions as compared to the ones already existing on shell; at $\sqrt{s_3}=0.2$ TeV and at $\sqrt{s_3}=0.5$ TeV (a). Ratios $R^{ZZ\gamma}$ show the departure versus M_F of the exact one-loop contribution, as compared to the effective Lagrangian prediction at $\sqrt{s_3}=0.2$ TeV and 0.5 TeV, (b). The definitions of s_1,s_2,s_3 are given in the text.

The four ratios in Eqs. (56), (57) are plotted versus $\sqrt{s_2}$ in Figs. 3(a) and 3(b), for $\sqrt{s_3}=0.2$ and 0.5 TeV, respectively. The fixed values of $\sqrt{s_1}$ and M_F are indicated in the figures. It can be seen there, that the quadratic s_2-dependence predicted for the large M_F limit, starts to be valid already at a rather low M_F, apart from threshold violations at $s_2 \sim 4M_F^2$.

(c) The ratios R_1^{ZZZ} and R_2^{ZZZ},

$$R_1^{ZZZ}=\frac{(2s_3-s_1-s_2)\mathcal{G}_2}{3\mathcal{G}_1(s_1-s_2)}\to 1,$$

$$R_2^{ZZZ}=\frac{(2s_3-s_1-s_2)\mathcal{G}_4}{3\mathcal{G}_3(s_1-s_2)}\to 1, \tag{58}$$

which are equal to 1 at large M_f, show how much the exact 1-loop contribution at finite values of M_F differs from the effective Lagrangian prediction. They are presented in Fig. 3(c) versus M_F, for $\sqrt{s_3}=0.2$, 0.5 TeV, and fixed typical values of $\sqrt{s_{1,2}}$. For these ratios also, we observe that they are close to their large M_F limits, provided that M_F is away from the threshold $\sqrt{s_3}/2$. Similar ratios are next constructed for the other NAGC processes.

2. The off-shell one-loop effects in $Z^*Z^*\gamma^*$ compared to $\gamma^*\to ZZ$

The corresponding ratios are

$$R^{5\gamma}=\frac{\mathcal{G}_1(s_1,s_2,s_3)+\mathcal{G}_5(s_1,s_2,s_3)}{\mathcal{G}_1(m_Z^2,m_Z^2,s_3)+\mathcal{G}_5(m_Z^2,m_Z^2,s_3)}\to 1, \tag{59}$$

$$R'^{5\gamma}=-\frac{3\mathcal{G}_2(s_1,s_2,s_3)+\mathcal{G}_4(s_1,s_2,s_3)}{3\mathcal{G}_1(s_1,s_2,s_3)+3\mathcal{G}_5(s_1,s_2,s_3)}\to\frac{s_2-s_1}{s_3}, \tag{60}$$

illustrated versus $\sqrt{s_2}$ in Fig. 4(a) for $\sqrt{s_3}=0.2$, 0.5 TeV and fixed $\sqrt{s_1}$, M_F and the ratio

$$R^{ZZ\gamma}=\frac{s_3(3\mathcal{G}_2+\mathcal{G}_4)}{3(s_1-s_2)(\mathcal{G}_1+\mathcal{G}_5)}\to 1, \tag{61}$$

presented versus M_F in Fig. 4(b), for $\sqrt{s_3}=0.2$, 0.5 TeV, and typical values of $\sqrt{s_{1,2}}$.

3. The off-shell one-loop effects in $Z^*Z^*\gamma^*$ compared to $Z^*\to Z\gamma$

The relevant ratios (together with their large M_F limits) are

$$R^{3Z}=\frac{3\mathcal{G}_2(s_1,s_2,s_3)+\mathcal{G}_4(s_1,s_2,s_3)-3\mathcal{G}_1(s_1,s_2,s_3)-3\mathcal{G}_5(s_1,s_2,s_3)}{3\mathcal{G}_2(m_Z^2,s_2,0)+\mathcal{G}_4(m_Z^2,s_2,0)-3\mathcal{G}_1(m_Z^2,s_2,0)-3\mathcal{G}_5(m_Z^2,s_2,0)}\to\frac{s_2+s_3-s_1}{s_2-m_Z^2}, \tag{62}$$

$$R'^{3Z}=-\frac{3\mathcal{G}_1(s_1,s_2,s_3)+3\mathcal{G}_5(s_1,s_2,s_3)}{3\mathcal{G}_2(s_1,s_2,s_3)+\mathcal{G}_4(s_1,s_2,s_3)-3\mathcal{G}_1(s_1,s_2,s_3)-3\mathcal{G}_5(s_1,s_2,s_3)}\to\frac{s_3}{s_2-s_1+s_3}, \tag{63}$$

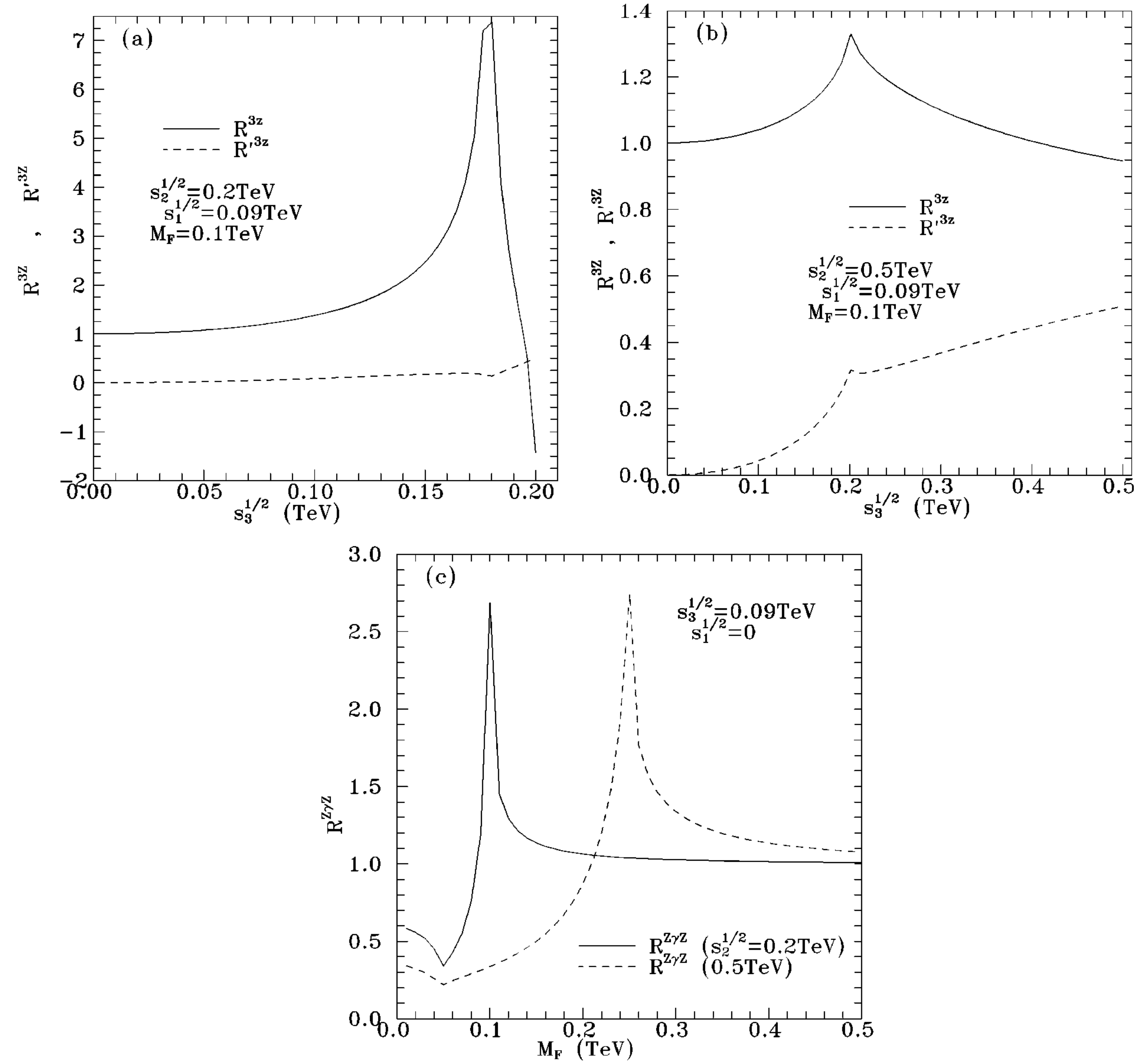

FIG. 5. $Z^*Z^*\gamma^*$ off-shell effects as compared to $Z^*\to Z\gamma$: Ratio R^{3Z} shows the $\sqrt{s_3}$ dependence of the contributions to the h_3^Z type of coupling, and ratio R'^{3Z} gives the relative size, versus $\sqrt{s_3}$, of the new contributions as compared to the ones already existing on shell; (a) at $\sqrt{s_2}=0.2$ TeV, (b) at $\sqrt{s_2}=0.5$ TeV. Ratio $R^{Z\gamma Z}$ shows the departure versus M_F of the exact one-loop contribution, as compared to the effective Lagrangian prediction at $\sqrt{s_2}=0.2$ TeV and 0.5 TeV, (c). The definitions of s_1, s_2, s_3 are given in the text.

presented versus $\sqrt{s_3}$ in Figs. 5(a),5(b), for $\sqrt{s_2}=0.2$, 0.5 TeV, and fixed values of $\sqrt{s_1}$, M_F. On the other hand, the ratio

$$R^{Z\gamma Z}=\frac{s_3[3\mathcal{G}_2(s_1,s_3,s_2)+\mathcal{G}_4(s_1,s_3,s_2)]}{3(s_1-s_2)[\mathcal{G}_1(s_1,s_3,s_2)+\mathcal{G}_5(s_1,s_3,s_2)]}\to 1, \tag{64}$$

is shown versus M_F in Fig. 5(c) for $\sqrt{s_2}=0.2$, 0.5 TeV, and fixed values of $\sqrt{s_{1,3}}$.

4. The off-shell one-loop effects in $\gamma^*\gamma^*Z^*$ compared to $\gamma^*\to Z\gamma$

We now have

$$R^{3\gamma}=\frac{\mathcal{G}_6(s_1,s_2,s_3)}{\mathcal{G}_6(s_1,0,m_Z^2)}\to 1, \tag{65}$$

$$R'^{3\gamma}=\frac{\mathcal{G}_7(s_1,s_2,s_3)}{\mathcal{G}_6(s_1,s_2,s_3)}\to\frac{s_2}{s_1}, \tag{66}$$

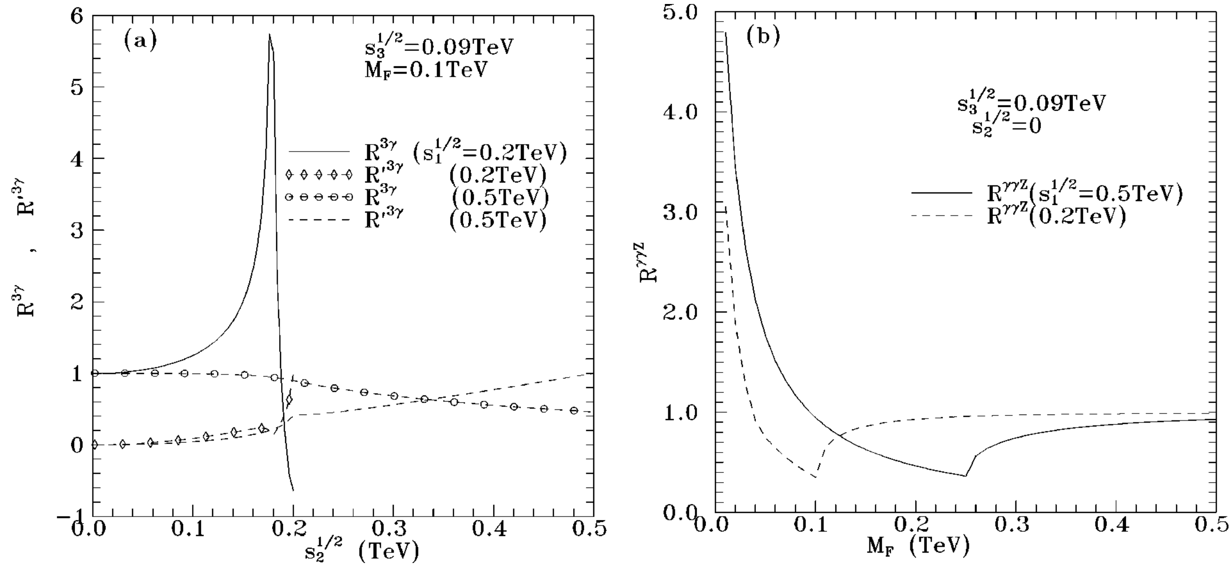

FIG. 6. $\gamma^*\gamma^*Z^*$ off-shell effects as compared to $\gamma^*\to Z\gamma$: Ratio $R^{3\gamma}$ shows the $\sqrt{s_2}$ dependence of the contributions to the h_3^γ type of coupling, and ratio $R'^{3\gamma}$ gives the relative size, versus $\sqrt{s_3}$, of the new contributions as compared to the ones already existing on shell; (a) at $\sqrt{s_1}=0.2$ TeV and at $\sqrt{s_1}=0.5$ TeV. Ratio $R^{\gamma\gamma Z}$ shows the departure versus M_F of the exact one-loop contribution, as compared to the effective Lagrangian prediction at $\sqrt{s_1}=0.2$ TeV and 0.5 TeV, (b). The definitions of s_1,s_2,s_3 are given in the text.

presented versus $\sqrt{s_2}$ in Fig. 6(a) for $\sqrt{s_1}=0.2$, 0.5 TeV, and $\sqrt{s_3}$, M_F; while the ratio

$$R^{\gamma\gamma Z}=\frac{s_1\mathcal{G}_7(s_1,s_2,s_3)}{s_2\mathcal{G}_6(s_1,s_2,s_3)}\to 1, \tag{67}$$

versus M_F in Fig.6(b) for $\sqrt{s_1}=0.2$, 0.5 TeV and fixed typical values of $\sqrt{s_{2,3}}$.

5. General comments

We have made many other runs with different s_i and M_F values. The following are the general conclusions we draw from these.

The first is that the off-shell effects cannot be ignored in detail experiments such as those performed at LEP2, where data with a fermion-pair invariant mass ranging from very low values up to about m_Z, have been collected. That will be even more true at a linear collider in the future.

Our one-loop calculations indicate that the large M_F predictions are quite adequate, even at low M_F values, so long as M_F is not too close to a threshold. This is the same situation as in the previous on-shell analysis [9]. It is furthermore a welcome situation, since it encourages us to analyze the data, by using the effective Lagrangian formalism, in which only operators of dim$\leqslant 6$ are retained. Ignoring $Z\to t\bar{t}$ events, this means that the eight parameters in Eq. (38) may be adequate, provided of course that we are not too close to an NP threshold.

If on the other hand we are close to an NP threshold, then we might even have direct production of new particles. In such a case, the study of NAGC will provide useful complementary information on their nature. Particularly because the set of new particle parameters entering their loop NAGC contribution, is certainly different from the one determining, e.g., their decay. This is obviously true, e.g., for NP of the SUSY type.

V. GENERAL OFF-SHELL NAGC CONTRIBUTION TO $e^-e^+\to f\bar{f}f'\bar{f}'$

The NAGC contribution to the $e^+e^-\to(f\bar{f})+(f'\bar{f}')$ process is depicted in Fig. 7. The complete Feynman amplitude has the general form

$$\mathcal{A}=-\frac{e}{m_Z^2}\sum_{ijk}\frac{\mathcal{V}_i^\sigma(f\bar{f})}{D_i}\frac{\mathcal{V}_j^\tau(f'\bar{f}')}{D_j}\Gamma_{\sigma\tau\rho}^{ijk}\frac{\mathcal{V}_k^\rho(e^+e^-)}{D_k}, \tag{68}$$

where the summation over ijk covers all possible off-shell

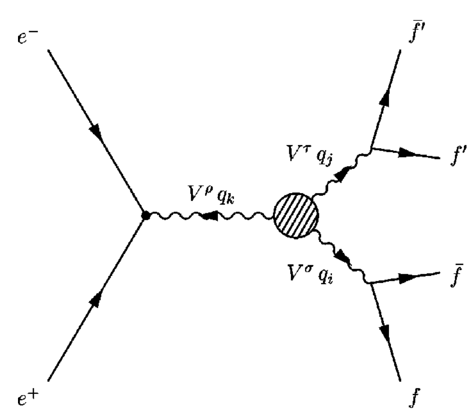

FIG. 7. The VVV contribution to the $e^+e^-\to(f\bar{f})(f'\bar{f}')$ process.

combinations of γ^* and Z^*, namely, $Z^*Z^*Z^*$, $Z^*Z^*\gamma^*$, $Z^*\gamma^*Z^*$, $Z^*\gamma^*\gamma^*$, $\gamma^*Z^*Z^*$, $\gamma^*Z^*\gamma^*$, and $\gamma^*\gamma^*Z^*$, with the propagators

$$D_{i,j,k}=q^2_{i,j,k} \quad \text{for} \quad \gamma^* \quad \text{or} \quad q^2_{i,j,k}-m_Z^2+im_Z\Gamma_Z \quad \text{for} \quad Z^*$$

and the initial and final fermionic vertices

$$\mathcal{V}^\sigma_i(f\overline{f})=\overline{u}(f)\gamma^\sigma(g^i_{vf}-g^i_{af}\gamma^5)v(\overline{f}),$$

$$\mathcal{V}^\tau_j(f'\overline{f'})=\overline{u}(f')\gamma^\tau(g^j_{vf'}-g^j_{af'}\gamma^5)v(\overline{f'}),$$

$$\mathcal{V}^\rho_k(e^+e^-)=\overline{v}(e^+)\gamma^\rho(g^k_{ve}-g^k_{ae}\gamma^5)u(e^-) \tag{69}$$

with g^i_{vf}, g^i_{af} being the vector and axial vector, photon or Z, couplings to the fermion f (including the factor $-e$ or $-e/2s_Wc_W$).[9] In Eq. (69), $\Gamma^{ijk}_{\sigma\tau\rho}$ are the general vertices given in Appendixes A,B and discussed throughout the paper.

One should be careful in reordering the indices and momenta in the various (i,j,k) combinations in order to use the formulas written for $Z^*Z^*\gamma^*$ and $\gamma^*\gamma^*Z^*$ in Appendixes A and B, so for clarity we list them explicitly:

$$\Gamma^{Z^*\gamma^*\gamma^*}_{\sigma\tau\rho}(q_1,q_2,q_3=-P)=\Gamma^{\gamma^*\gamma^*Z^*}_{\rho\tau\sigma}(q_3=-P,q_2,q_1), \tag{70}$$

$$\Gamma^{\gamma^*Z^*\gamma^*}_{\sigma\tau\rho}(q_1,q_2,q_3=-P)=\Gamma^{\gamma^*\gamma^*Z^*}_{\sigma\rho\tau}(q_1,q_3=-P,q_2), \tag{71}$$

$$\Gamma^{Z^*\gamma^*Z^*}_{\sigma\tau\rho}(q_1,q_2,q_3=-P)=\Gamma^{Z^*Z^*\gamma^*}_{\sigma\rho\tau}(q_1,q_3=-P,q_2), \tag{72}$$

$$\Gamma^{\gamma^*Z^*Z^*}_{\sigma\tau\rho}(q_1,q_2,q_3=-P)=\Gamma^{Z^*Z^*\gamma^*}_{\rho\tau\sigma}(q_3=-P,q_2,q_1). \tag{73}$$

The basic SM (or MSSM) contributions are assumed to be included in the Γ vertices expressed in terms of f_i and $\tilde{f}_i$ defined in Sec. II, using the analytic expressions given in Appendix C.

For an experimental determination of possible unknown additional contributions, a simple parametrization of the $f_i(s_1,s_2,s_3)$ and $g_i(s_1,s_2,s_3)$ is needed. If the NP effects arise at a high scale, then the the effective Lagrangian of Sec. III, in which only the lowest-dimensional operators are retained, may be adequate.

[9]We mention for completeness that conventions are such that the effective Lagrangian for a gauge boson fermion interaction is

$$\mathcal{L}=V_{i\mu}\mathcal{V}^\mu_i(f\overline{f}).$$

VI. CONCLUSIONS

We have established the general Lorentz and $U(1)_{em}$ invariant form of the off-shell three neutral gauge boson self-couplings $V_1^*V_2^*V_3^*$, with applications to $Z^*Z^*Z^*$, $Z^*Z^*\gamma^*$, and $\gamma^*\gamma^*Z^*$. In it, we have kept all types of transverse and scalar off-shell vector boson components and considered both CP-conserving and CP-violating couplings. They are given in Appendix A and B, respectively. We have pointed out the new coupling forms which do not exist when two particles are on shell, thus making contact with the previous description valid only when two gauge bosons are on-shell [1,8,9].

In the $Z^*Z^*Z^*$ case, we have found (three transverse + three scalar) CP-conserving and (four transverse + ten scalar) CP-violating coupling forms; which reduce in the previously considered $Z^*\to ZZ$ on-shell case to $(1+1)+(1+3)$. In the $Z^*Z^*\gamma^*$ case we have found $(3+2)+(4+5)$ coupling forms. They reduce to $(1+0)+(1+0)$ in $\gamma^*\to ZZ$, and to $(2+1)+(2+2)$ in $Z^*\to Z\gamma$. Finally in the $\gamma^*\gamma^*Z^*$ case we found $(3+1)+(4+2)$ coupling forms, which reduce to $(2+1)+(2+0)$ in $\gamma^*\to Z\gamma$.

These vertex forms apply to any kind of standard or nonstandard dynamics [SM, minimal supersymmetric standard model (MSSM), . . .]. In general the functions which multiply these coupling forms depend on the three off-shell masses (s_1,s_2,s_3). If the NP scale inducing NAGC is very high $(\Lambda\gg m_Z)$, then we have found that an effective Lagrangian involving a minimal set of operators should be adequate for generating all possible vertex forms consistent with Bose symmetry and CVC. Some of these vertex forms can be generated by dim=6 operators, while other ones require higher (dim=8, 10, 12) operators. So a hierarchy is obtained among the various possible off-shell effects. In each of the $Z^*Z^*Z^*$, $Z^*Z^*\gamma^*$, and $\gamma^*\gamma^*Z^*$ cases, this allows us a simple description in terms of a limited set of constant parameters. This should constitute a useful tool for data analysis. For that purpose we have explicitly written the vertices with both the complete set as well as with the set restricted to the dim=6 operators. They are given in Eqs. (18), (21),(25),(29),(33),(36).

As an illustration of the SM and NP contributions, we have considered the neutral anomalous gauge couplings generated by a fermionic triangle loop. In Appendix C we have given the complete analytic expression of the coupling functions arising at one loop, using general gauge couplings to any fermion. The use of this is twofold. First, it allows us to make an exact computation of the SM contribution. And second, it provides an illustration of what type of off-shell effects can appear for any kind of NP fermion generating NAGC.

To this aim we have quantitatively discussed through Figs. 3–6, the dependence of the neutral anomalous couplings on the off-shell masses; as well as the relative size of the new NAGC as compared to those already existing in the on-shell case. The $1/M_F^2$ limit of the heavy fermion contribution appears to coincide with the effective Lagrangian description restricted to the dim=6 operators. Thus, we have found that the effective Lagrangian description is also valid, so long the fermion mass M_F is not too close to M_Z or the energy threshold $\sqrt{s}/2$ of the process considered.

We emphasize though, that the one-loop results should also be very useful in analyzing possible NAGC data *close to the threshold* for actually producing the new particles responsible for these NAGC. In such a case the effective Lagrangian formalism is not applicable, and the NAGC analysis must be done taking into account the above one-loop predictions; thus, providing important complementary information on the nature of the responsible NP particles. Finally we have written the complete structure of the off-shell $V_1^* V_2^* V_3^*$ contribution to the $e^+e^- \to (f\overline{f}) + (f'\overline{f'})$ amplitude, which should be used in the analysis of the events observable at present and future e^+e^- colliders.

As an overall conclusion we should stress that the off-shell effects in the neutral gauge boson self-interactions cannot be ignored in detail experiments such as those performed at LEP2, and will be performed in the future at a linear e^-e^+ collider. This is certainly related to the fact that these couplings have to vanish whenever all three gauge bosons participating in the vertex are on shell.

ACKNOWLEDGMENTS

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APPENDIX A: THE CP-CONSERVING $V_1^* V_2^* V_3^*$ VERTEX

The general interaction among three, possibly off-shell neutral gauge bosons (NAGC), is defined following the notation of Fig. 1 and $s_i \equiv q_i^2$. Note that all q_i momenta are outgoing, so that $q_1+q_2+q_3=0$. Since a vertex involving three neutral gauge bosons is necessarily C violating, the construction of CP-conserving couplings requires the use of P-violating forms involving the $\epsilon^{\mu\nu\rho\sigma}$ tensor, conveniently denoted as

$$\epsilon^{\mu\nu\rho\sigma} A_\mu B_\nu C_\rho D_\sigma = [ABCD]. \tag{A1}$$

The most general Lorentz-invariant CP-conserving $V_1^* V_2^* V_3^*$ vertex involves at most six independent forms, two of which are linear in the q_i momenta, while the rest are cubic. For an easy comparison with the forms written in the on-shell case [1,8,9] we choose the basis

$$\begin{gathered}[q_1 - q_2 \mu\alpha\beta], \quad [q_3 \mu\alpha\beta], \\ q_3^\beta [q_1 q_2 \mu\alpha] + q_3^\alpha [q_1 q_2 \mu\beta], \\ q_1^\alpha [\beta q_3 \mu q_2], \quad q_2^\beta [\alpha q_3 \mu q_1], \quad q_3^\mu [\beta q_1 \alpha q_2]. \end{gathered} \tag{A2}$$

The last three forms in Eq. (A2) imply at least one scalar $q_\mu V^\mu$ term and they are called ''scalar,'' in contrast to the other forms called ''transverse.''

The $Z^*Z^*Z^*$ case. Here the additional constraint of full Bose symmetry among the quantum numbers (q_1,α), (q_2,β), (q_3,μ), describing the three off-shell Z^* should be imposed. Writing thus

$$\begin{aligned}\Gamma_{\alpha\beta\mu}^{Z^*Z^*Z^*}(q_1,q_2,q_3) = {}& i\sum_{j=1}^{3} I_{\alpha\beta\mu}^{Z^*Z^*Z^*,j} f_j^{Z^*Z^*Z^*}(s_1,s_2,s_3) \\ &+ i\sum_{j=1}^{3} J_{\alpha\beta\mu}^{Z^*Z^*Z^*,j} g_j^{Z^*Z^*Z^*}(s_1,s_2,s_3),\end{aligned} \tag{A3}$$

with

$$\begin{aligned}&I_{\alpha\beta\mu}^{Z^*Z^*Z^*,1} = [q_1 - q_2 \mu\alpha\beta], \quad I_{\alpha\beta\mu}^{Z^*Z^*Z^*,2} = [q_3 \mu\alpha\beta], \\ &I_{\alpha\beta\mu}^{Z^*Z^*Z^*,3} = q_3^\beta [q_1 q_2 \mu\alpha] + q_3^\alpha [q_1 q_2 \mu\beta], \\ &J_{\alpha\beta\mu}^{Z^*Z^*Z^*,1} = q_1^\alpha [\beta q_3 \mu q_2], \quad J_{\alpha\beta\mu}^{Z^*Z^*Z^*,2} = q_2^\beta [\alpha q_3 \mu q_1], \\ &J_{\alpha\beta\mu}^{Z^*Z^*Z^*,3} = q_3^\mu [\beta q_1 \alpha q_2],\end{aligned} \tag{A4}$$

we obtain that $f_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)$ is a fully *antisymmetric* function of $(s_1,\ s_2,\ s_3)$, while the other transverse and scalar functions satisfy the Bose relations

$$f_1^{Z^*Z^*Z^*}(s_1,s_2,s_3) = f_1^{Z^*Z^*Z^*}(s_2,s_1,s_3), \tag{A5}$$

$$f_2^{Z^*Z^*Z^*}(s_1,s_2,s_3) = -f_2^{Z^*Z^*Z^*}(s_2,s_1,s_3),$$

$$\begin{aligned}f_1^{Z^*Z^*Z^*}(s_1,s_3,s_2) = \frac{1}{2}\Bigg[& -f_1^{Z^*Z^*Z^*}(s_1,s_2,s_3) \\ & + f_2^{Z^*Z^*Z^*}(s_1,s_2,s_3) \\ & - \frac{s_2+s_1-s_3}{2} f_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)\Bigg],\end{aligned}$$

$$\begin{aligned}f_2^{Z^*Z^*Z^*}(s_1,s_3,s_2) = {}& \frac{3}{2} f_1^{Z^*Z^*Z^*}(s_1,s_2,s_3) \\ & + \frac{1}{2} f_2^{Z^*Z^*Z^*}(s_1,s_2,s_3) \\ & + \frac{s_2-s_3-3s_1}{4} f_3^{Z^*Z^*Z^*}(s_1,s_2,s_3),\end{aligned}$$

$$g_1^{Z^*Z^*Z^*}(s_1,s_2,s_3) = g_2^{Z^*Z^*Z^*}(s_2,s_1,s_3),$$

$$g_3^{Z^*Z^*Z^*}(s_1,s_2,s_3) = g_3^{Z^*Z^*Z^*}(s_2,s_1,s_3),$$

$$\begin{aligned}g_1^{Z^*Z^*Z^*}(s_1,s_2,s_3) = {}& g_1^{Z^*Z^*Z^*}(s_1,s_3,s_2) \\ & + 2 f_3^{Z^*Z^*Z^*}(s_1,s_3,s_2),\end{aligned}$$

$$\begin{aligned}g_2^{Z^*Z^*Z^*}(s_1,s_2,s_3) = {}& g_3^{Z^*Z^*Z^*}(s_1,s_3,s_2) \\ & - f_3^{Z^*Z^*Z^*}(s_1,s_3,s_2),\end{aligned}$$

$$g_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)=g_2^{Z^*Z^*Z^*}(s_1,s_3,s_2)-f_3^{Z^*Z^*Z^*}(s_1,s_3,s_2).$$

Note that Eq. (A5) together with the antisymmetry of $f_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)$ imply

$$g_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{1}{2}[g_1^{Z^*Z^*Z^*}(s_3,s_2,s_1)+g_2^{Z^*Z^*Z^*}(s_1,s_3,s_2)]. \quad \text{(A6)}$$

The $Z^*Z^*\gamma^*$ case. Restarting from the general $V_1^*V_2^*V_3^*$ vertex in Eq. (A2), with (q_3,μ) corresponding to the photon and imposing the CVC constraint $q_3^\mu\Gamma_{\alpha\beta\mu}^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=0$ and Bose symmetry for Z^*Z^*, we end up with general vertex containing the five independent forms: namely,

$$\Gamma_{\alpha\beta\mu}^{Z^*Z^*\gamma^*}(q_1,q_2,q_3)=i\sum_{j=1}^{3} I_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,j}f_j^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)+i\sum_{j=1,2} J_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,j}g_j^{Z^*Z^*\gamma^*}(s_1,s_2,s_3), \quad \text{(A7)}$$

where

$$I_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,1}=[q_1-q_2\mu\alpha\beta]+\frac{2q_3^\mu}{s_3}[q_1q_2\alpha\beta],$$

$$I_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,2}=[q_3\mu\alpha\beta],$$

$$I_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,3}=q_3^\beta[q_1q_2\mu\alpha]+q_3^\alpha[q_1q_2\mu\beta],$$

$$J_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,1}=q_1^\alpha[\beta q_3\mu q_2],$$

$$J_{\alpha\beta\mu}^{Z^*Z^*\gamma^*,2}=q_2^\beta[\alpha q_3\mu q_1]. \quad \text{(A8)}$$

Bose symmetry imposes the constraints

$$f_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=f_1^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),$$

$$f_2^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-f_2^{Z^*Z^*\gamma^*}(s_2,s_1,s_3)$$

$$f_3^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-f_3^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),$$

$$g_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=g_2^{Z^*Z^*\gamma^*}(s_2,s_1,s_3). \quad \text{(A9)}$$

The $\gamma^*\gamma^*Z^*$ case. In the general $V_1^*V_2^*V_3^*$ vertex of Eq. (A2), (q_3,μ) corresponds now to Z^*. Imposing then the two CVC constraints $q_1^\alpha\ \Gamma_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=q_2^\beta\Gamma_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=0$ and Bose symmetry for $\gamma^*\gamma^*$, we end up with the four independent vertex forms

$$\Gamma_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*}(q_1,q_2,q_3)=i\sum_{j=1}^{3} I_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,j}f_j^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)+iJ_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,1}g_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3), \quad \text{(A10)}$$

with

$$I_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,1}=[q_1-q_2\mu\alpha\beta]-\frac{q_1^\alpha}{s_1}[\beta q_3\mu q_2]-\frac{q_2^\beta}{s_2}([\alpha q_3\mu q_1]),$$

$$I_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,2}=\Bigg([q_3\mu\alpha\beta]-\frac{q_1^\alpha}{s_1}[\beta q_3\mu q_2]+\frac{q_2^\beta}{s_2}([\alpha q_3\mu q_1]),$$

$$I_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,3}=q_3^\beta[q_1q_2\mu\alpha]+q_3^\alpha[q_1q_2\mu\beta]+\frac{s_2-s_1-s_3}{2s_1}q_1^\alpha[\beta q_3\mu q_2]-\frac{s_1-s_2-s_3}{2s_2}q_2^\beta[\alpha q_3\mu q_1],$$

$$J_{\alpha\beta\mu}^{\gamma^*\gamma^*Z^*,1}=q_3^\mu[\beta q_1\alpha q_2], \quad \text{(A11)}$$

and the Bose symmetry constraints

$$f_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=f_1^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3),$$

$$f_2^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=-f_2^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3),$$

$$f_3^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=-f_3^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3),$$

$$g_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=g_1^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3). \quad \text{(A12)}$$

APPENDIX B: THE *CP*-VIOLATING FORMS FOR THE $V_1^*V_2^*V_3^*$ VERTEX

These vertices are *P*-conserving and *C*-violating, and can most generally be expressed in terms of the following 14 independent Lorentz invariant forms (indices *i*, *j*, *k* run from 1 to 3): three terms similar to $(V_i.V_j)[V_k.(q_i-q_j)]$; three terms similar to $(V_i.V_j)(V_k.q_k)$; eight terms similar to $[V_k.(q_i-q_j)$ or $V_k.q_k]\cdot[V_j.(q_k-q_i)$ or $V_j.q_j]\cdot[V_i.(q_j-q_k)$ or $V_i.q_i]$. Four of these terms are ''transverse,'' while the other 10 contain at least one ''scalar'' qV coefficient.

The $Z^*Z^*Z^*$ case. Applying full Bose symmetry among the three Z^*, we obtain the structure

$$\Gamma^{Z^*Z^*Z^*}_{\alpha\beta\mu}(q_1,q_2,q_3)=i\sum_{j=1}^{4}\tilde{I}^{Z^*Z^*Z^*,j}_{\alpha\beta\mu}\tilde{f}_j^{Z^*Z^*Z^*}(s_1,s_2,s_3)+i\sum_{j=1}^{10}\tilde{J}^{Z^*Z^*Z^*,j}_{\alpha\beta\mu}\tilde{g}_j^{Z^*Z^*Z^*}(s_1,s_2,s_3), \tag{B1}$$

where the transverse forms are

$$\begin{aligned}
&\tilde{I}^{Z^*Z^*Z^*,1}_{\alpha\beta\mu}=g^{\alpha\beta}(q_1-q_2)^{\mu}, \quad \tilde{I}^{Z^*Z^*Z^*,2}_{\alpha\beta\mu}=g^{\beta\mu}(q_3-q_2)^{\alpha},\\
&\tilde{I}^{Z^*Z^*Z^*,3}_{\alpha\beta\mu}=g^{\alpha\mu}(q_1-q_3)^{\beta},\\
&\tilde{I}^{Z^*Z^*Z^*,4}_{\alpha\beta\mu}=(q_2-q_3)^{\alpha}(q_1-q_3)^{\beta}(q_1-q_2)^{\mu},
\end{aligned} \tag{B2}$$

while the scalar ones are

$$\begin{aligned}
&\tilde{J}^{Z^*Z^*Z^*,1}_{\alpha\beta\mu}=g^{\alpha\beta}q_3^{\mu}, \quad \tilde{J}^{Z^*Z^*Z^*,2}_{\alpha\beta\mu}=g^{\beta\mu}q_1^{\alpha},\\
&\tilde{J}^{Z^*Z^*Z^*,3}_{\alpha\beta\mu}=g^{\alpha\mu}q_2^{\beta},\\
&\tilde{J}^{Z^*Z^*Z^*,4}_{\alpha\beta\mu}=q_1^{\alpha}(q_1-q_3)^{\beta}(q_1-q_2)^{\mu},\\
&\tilde{J}^{Z^*Z^*Z^*,5}_{\alpha\beta\mu}=q_2^{\beta}(q_2-q_3)^{\alpha}(q_2-q_1)^{\mu},\\
&\tilde{J}^{Z^*Z^*Z^*,6}_{\alpha\beta\mu}=q_3^{\mu}(q_3-q_1)^{\beta}(q_3-q_2)^{\alpha},\\
&\tilde{J}^{Z^*Z^*Z^*,7}_{\alpha\beta\mu}=q_1^{\alpha}q_2^{\beta}(q_1-q_2)^{\mu},\\
&\tilde{J}^{Z^*Z^*Z^*,8}_{\alpha\beta\mu}=q_3^{\mu}q_2^{\beta}(q_3-q_2)^{\alpha},\\
&\tilde{J}^{Z^*Z^*Z^*,9}_{\alpha\beta\mu}=q_1^{\alpha}q_3^{\mu}(q_1-q_3)^{\beta},\\
&\tilde{J}^{Z^*Z^*Z^*,10}_{\alpha\beta\mu}=q_1^{\alpha}q_2^{\beta}q_3^{\mu}.
\end{aligned} \tag{B3}$$

The Bose relations obtained from them for the transverse forms are

$$\begin{aligned}
\tilde{f}_1^{Z^*Z^*Z^*}(s_1,s_2,s_3)&=-\tilde{f}_1^{Z^*Z^*Z^*}(s_2,s_1,s_3)\\
&=\tilde{f}_2^{Z^*Z^*Z^*}(s_3,s_2,s_1)\\
&=-\tilde{f}_2^{Z^*Z^*Z^*}(s_3,s_1,s_2)\\
&=\tilde{f}_3^{Z^*Z^*Z^*}(s_1,s_3,s_2)\\
&=-\tilde{f}_3^{Z^*Z^*Z^*}(s_2,s_3,s_1),\\
\tilde{f}_4^{Z^*Z^*Z^*}(s_1,s_2,s_3)&=-\tilde{f}_4^{Z^*Z^*Z^*}(s_2,s_1,s_3)\\
&=-\tilde{f}_4^{Z^*Z^*Z^*}(s_1,s_3,s_2)\\
&=\tilde{f}_4^{Z^*Z^*Z^*}(s_3,s_2,s_1),
\end{aligned} \tag{B4}$$

while for the scalar ones we get

$$\begin{aligned}
\tilde{g}_1^{Z^*Z^*Z^*}(s_1,s_2,s_3)&=\tilde{g}_1^{Z^*Z^*Z^*}(s_2,s_1,s_3)\\
&=\tilde{g}_2^{Z^*Z^*Z^*}(s_3,s_2,s_1)\\
&=\tilde{g}_2^{Z^*Z^*Z^*}(s_3,s_1,s_2)\\
&=\tilde{g}_3^{Z^*Z^*Z^*}(s_1,s_3,s_2)\\
&=\tilde{g}_3^{Z^*Z^*Z^*}(s_2,s_3,s_1),\\
\tilde{g}_6^{Z^*Z^*Z^*}(s_1,s_2,s_3)&=\tilde{g}_6^{Z^*Z^*Z^*}(s_2,s_1,s_3)\\
&=\tilde{g}_4^{Z^*Z^*Z^*}(s_3,s_2,s_1)\\
&=\tilde{g}_4^{Z^*Z^*Z^*}(s_3,s_1,s_2)\\
&=\tilde{g}_5^{Z^*Z^*Z^*}(s_1,s_3,s_2)\\
&=\tilde{g}_5^{Z^*Z^*Z^*}(s_2,s_3,s_1),\\
\tilde{g}_7^{Z^*Z^*Z^*}(s_1,s_2,s_3)&=-\tilde{g}_7^{Z^*Z^*Z^*}(s_2,s_1,s_3)\\
&=\tilde{g}_8^{Z^*Z^*Z^*}(s_3,s_2,s_1)\\
&=-\tilde{g}_8^{Z^*Z^*Z^*}(s_3,s_1,s_2)\\
&=\tilde{g}_9^{Z^*Z^*Z^*}(s_1,s_3,s_2)\\
&=-\tilde{g}_9^{Z^*Z^*Z^*}(s_2,s_3,s_1),\\
\tilde{g}_{10}^{Z^*Z^*Z^*}(s_1,s_2,s_3)&=\tilde{g}_{10}^{Z^*Z^*Z^*}(s_2,s_1,s_3)\\
&=\tilde{g}_{10}^{Z^*Z^*Z^*}(s_3,s_2,s_1)\\
&=\tilde{g}_{10}^{Z^*Z^*Z^*}(s_1,s_3,s_2).
\end{aligned} \tag{B5}$$

The $Z^*Z^*\gamma^*$ case. Restarting from the initial list of CP-violating $V_1^*V_2^*V_3^*$ forms, with (q_3,μ) corresponding to the photon, and imposing the CVC constraint $q_3^{\mu}\ \Gamma^{Z^*Z^*\gamma^*}_{\alpha\beta\mu}(s_1,s_2,s_3)=0$ and Bose symmetry for Z^*Z^*, we get

$$\Gamma^{Z^*Z^*\gamma^*}_{\alpha\beta\mu}(q_1,q_2,q_3)=i\sum_{j=1}^{4}\tilde{I}^{Z^*Z^*\gamma^*,j}_{\alpha\beta\mu}\tilde{f}_j^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)+i\sum_{j=1}^{5}\tilde{J}^{Z^*Z^*\gamma^*,j}_{\alpha\beta\mu}\tilde{g}_j^{Z^*Z^*\gamma^*}(s_1,s_2,s_3), \tag{B6}$$

involving four transverse and five scalar forms. These are

$$\tilde{I}^{Z^*Z^*\gamma^*,1}_{\alpha\beta\mu}=g^{\alpha\beta}\left((q_1-q_2)^\mu-\frac{(s_2-s_1)}{s_3}q_3^\mu\right),$$

$$\tilde{I}^{Z^*Z^*\gamma^*,2}_{\alpha\beta\mu}=g^{\mu\beta}(q_3-q_2)^\alpha+g^{\mu\alpha}(q_3-q_1)^\beta-\frac{q_3^\mu(q_3-q_1)^\beta(q_3-q_2)^\alpha}{s_3}+\frac{q_3^\mu}{2s_3}[q_2^\beta(q_3-q_2)^\alpha+q_1^\alpha(q_3-q_1)^\beta],$$

$$\tilde{I}^{Z^*Z^*\gamma^*,3}_{\alpha\beta\mu}=g^{\mu\beta}(q_3-q_2)^\alpha-g^{\mu\alpha}(q_3-q_1)^\beta+\frac{q_3^\mu}{2s_3}[q_2^\beta(q_3-q_2)^\alpha-q_1^\alpha(q_3-q_1)^\beta],$$

$$\tilde{I}^{Z^*Z^*\gamma^*,4}_{\alpha\beta\mu}=(q_2-q_3)^\alpha(q_1-q_3)^\beta(q_1-q_2)^\mu-\frac{s_2-s_1}{s_3}q_3^\mu(q_3-q_1)^\beta(q_3-q_2)^\alpha,$$

$$\tilde{J}^{Z^*Z^*\gamma^*,1}_{\alpha\beta\mu}=g^{\mu\beta}q_1^\alpha+g^{\mu\alpha}q_2^\beta-\frac{q_3^\mu}{2s_3}[q_1^\alpha(q_3-q_1)^\beta+q_2^\beta(q_3-q_2)^\beta]+\frac{q_1^\alpha q_2^\beta q_3^\mu}{s_3},$$

$$\tilde{J}^{Z^*Z^*\gamma^*,2}_{\alpha\beta\mu}=g^{\mu\beta}q_1^\alpha-g^{\mu\alpha}q_2^\beta-\frac{q_3^\mu}{2s_3}[q_1^\alpha(q_3-q_1)^\beta-q_2^\beta(q_3-q_2)^\alpha],$$

$$\tilde{J}^{Z^*Z^*\gamma^*,3}_{\alpha\beta\mu}=q_1^\alpha(q_1-q_3)^\beta(q_1-q_2)^\mu+q_2^\beta(q_2-q_3)^\alpha(q_2-q_1)^\mu+\frac{s_2-s_1}{s_3}q_3^\mu[q_1^\alpha(q_3-q_1)^\beta-q_2^\beta(q_3-q_2)^\alpha],$$

$$\tilde{J}^{Z^*Z^*\gamma^*,4}_{\alpha\beta\mu}=q_1^\alpha(q_1-q_3)^\beta(q_1-q_2)^\mu-q_2^\beta(q_2-q_3)^\alpha(q_2-q_1)^\mu+\frac{s_2-s_1}{s_3}q_3^\mu[q_1^\alpha(q_3-q_1)^\beta+q_2^\beta(q_3-q_2)^\alpha],$$

$$\tilde{J}^{Z^*Z^*\gamma^*,5}_{\alpha\beta\mu}=q_1^\alpha q_2^\beta(q_1-q_2)^\mu-\frac{s_2-s_1}{s_3}q_1^\alpha q_2^\beta q_3^\mu, \tag{B7}$$

implying the Bose relations

$$\tilde{f}_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\tilde{f}_1^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),\quad \tilde{f}_2^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=\tilde{f}_2^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),$$

$$\tilde{f}_3^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\tilde{f}_3^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),\quad \tilde{f}_4^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\tilde{f}_4^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),$$

$$\tilde{g}_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=\tilde{g}_1^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),\quad \tilde{g}_2^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\tilde{g}_2^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),$$

$$\tilde{g}_3^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=\tilde{g}_3^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),\quad \tilde{g}_4^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\tilde{g}_4^{Z^*Z^*\gamma^*}(s_2,s_1,s_3),$$

$$\tilde{g}_5^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\tilde{g}_5^{Z^*Z^*\gamma^*}(s_2,s_1,s_3). \tag{B8}$$

The $\gamma^*\gamma^*Z^*$ *case*. Imposing on the general $V_1^*V_2^*V_3^*$ vertex the two CVC constraints and Bose symmetry for the two photons leaves six invariant forms

$$\Gamma^{\gamma^*\gamma^*Z^*}_{\alpha\beta\mu}(q_1,q_2,q_3)=i\sum_{i=1}^{4}\tilde{I}^{\gamma^*\gamma^*Z^*,i}_{\alpha\beta\mu}\tilde{f}_i^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)+i\sum_{i=1,2}\tilde{J}^{\gamma^*\gamma^*Z^*,i}_{\alpha\beta\mu}\tilde{g}_i^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3), \tag{B9}$$

where (q_3,μ) correspond to Z^* and

$$
\begin{aligned}
\tilde{I}^{\gamma^*\gamma^*Z^*,1}_{\alpha\beta\mu} =& g^{\alpha\beta}(q_1-q_2)^\mu - \frac{q_1^\alpha}{2s_1}(q_1-q_3)^\beta(q_1-q_2)^\mu \\
&+ \frac{q_2^\beta}{2s_2}(q_2-q_3)^\alpha(q_2-q_1)^\mu + \frac{s_3}{2s_1s_2}q_1^\alpha q_2^\beta(q_1-q_2)^\mu, \\
\tilde{I}^{\gamma^*\gamma^*Z^*,2}_{\alpha\beta\mu} =& g^{\mu\beta}(q_3-q_2)^\alpha + g^{\mu\alpha}(q_3-q_1)^\beta - \frac{s_2-s_3}{s_1}q_1^\alpha g^{\mu\beta} - \frac{s_1-s_3}{s_2}q_2^\beta g^{\mu\alpha} \\
&+ \frac{q_2^\beta}{2s_2}(q_2-q_3)^\alpha(q_2-q_1)^\mu + \frac{q_1^\alpha}{2s_1}(q_1-q_3)^\beta(q_1-q_2)^\mu + \frac{s_1-s_2}{2s_1s_2}q_1^\alpha q_2^\beta(q_1-q_2)^\mu \\
&+ \frac{q_3^\mu}{2s_2}q_2^\beta(q_3-q_2)^\alpha + \frac{q_3^\mu}{2s_1}q_1^\alpha(q_3-q_1)^\beta + \frac{2s_3-s_1-s_2}{2s_1s_2}q_1^\alpha q_2^\beta q_3^\mu, \\
\tilde{I}^{\gamma^*\gamma^*Z^*,3}_{\alpha\beta\mu} =& g^{\mu\beta}(q_3-q_2)^\alpha - g^{\mu\alpha}(q_3-q_1)^\beta - \frac{s_2-s_3}{s_1}q_1^\alpha g^{\mu\beta} + \frac{s_1-s_3}{s_2}q_2^\beta g^{\mu\alpha} \\
&+ \frac{q_2^\beta}{2s_2}(q_2-q_3)^\alpha(q_2-q_1)^\mu - \frac{q_1^\alpha}{2s_1}(q_1-q_3)^\beta(q_1-q_2)^\mu + \frac{s_1-s_2}{2s_1s_2}q_1^\alpha q_2^\beta q_3^\mu \\
&+ \frac{q_3^\mu}{2s_2}q_2^\beta(q_3-q_2)^\alpha - \frac{q_3^\mu}{2s_1}q_1^\alpha(q_3-q_1)^\beta + \frac{2s_3-s_1-s_2}{2s_1s_2}q_1^\alpha q_2^\beta(q_1-q_2)^\mu, \\
\tilde{I}^{\gamma^*\gamma^*Z^*,4}_{\alpha\beta\mu} =& (q_2-q_3)^\alpha(q_1-q_3)^\beta(q_1-q_2)^\mu - \frac{s_3-s_2}{s_1}q_1^\alpha(q_1-q_3)^\beta(q_1-q_2)^\mu \\
&+ \frac{s_3-s_1}{s_2}q_2^\beta(q_2-q_3)^\alpha(q_2-q_1)^\mu + \frac{(s_3-s_2)(s_3-s_1)}{s_1s_2}q_1^\alpha q_2^\beta(q_1-q_2)^\mu, \\
\tilde{J}^{\gamma^*\gamma^*Z^*,1}_{\alpha\beta\mu} =& g^{\alpha\beta}q_3^\mu + \frac{q_3^\mu}{2s_2}q_2^\beta(q_3-q_2)^\alpha + \frac{q_3^\mu}{2s_1}q_1^\alpha(q_3-q_1)^\beta + \frac{s_3}{2s_1s_2}q_1^\alpha q_2^\beta q_3^\mu, \\
\tilde{J}^{\gamma^*\gamma^*Z^*,2}_{\alpha\beta\mu} =& q_3^\mu(q_3-q_1)^\beta(q_3-q_2)^\alpha - \frac{s_1-s_3}{s_2}q_3^\mu q_2^\beta(q_3-q_2)^\alpha \\
&- \frac{s_2-s_3}{s_1}q_3^\mu q_1^\alpha(q_3-q_1)^\beta + \frac{(s_2-s_3)(s_1-s_3)}{s_1s_2}q_1^\alpha q_2^\beta q_3^\mu,
\end{aligned}
\tag{B10}
$$

with the Bose relations

$$
\begin{aligned}
&\tilde{f}_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3) = -\tilde{f}_1^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3), \quad \tilde{f}_2^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3) = \tilde{f}_2^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3), \\
&\tilde{f}_3^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3) = -\tilde{f}_3^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3), \quad \tilde{f}_4^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3) = -\tilde{f}_4^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3), \\
&\tilde{g}_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3) = \tilde{g}_1^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3), \quad \tilde{g}_2^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3) = \tilde{g}_2^{\gamma^*\gamma^*Z^*}(s_2,s_1,s_3).
\end{aligned}
\tag{B11}
$$

APPENDIX C: FERMIONIC TRIANGLE ONE-LOOP CONTRIBUTIONS TO THE OFF-SHELL $V_1^*V_2^*V_3^*$ COUPLINGS

The basic triangle diagram is depicted in Fig. 2. For simplicity we only consider the case that a single fermion is running along the loop with the couplings defined in Eq. (42).[10] Only a restricted set of CP-conserving NAGC are generated by this

[10]In models such as SUSY we could also have fermion loops, where two different charginos mix through their Z couplings, while running along the loop. Such contributions were calculated in Ref. [9] in the case that only one of the neutral gauge bosons were off shell, and they were found to be rather small. Here they are neglected.

triangle loop (no CP-violating coupling appear). They are explicitly given below in terms of the Passarino-Veltman one-loop functions[11] [18].

Application to $Z^*Z^*Z^*$:

$$f_1^{Z^*Z^*Z^*}(s_1,s_2,s_3)=-\frac{e^2 g_{aF}}{32\pi^2 s_W^3 c_W^3}\{(3g_{vF}^2+g_{aF}^2)\mathcal{G}_1(s_1,s_2,s_3)-(g_{aF}^2-g_{vF}^2)\mathcal{G}_3(s_1,s_2,s_3)\},$$

$$f_2^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{e^2 g_{aF}}{32\pi^2 s_W^3 c_W^3}\{(3g_{vF}^2+g_{aF}^2)\mathcal{G}_2(s_1,s_2,s_3)-(g_{aF}^2-g_{vF}^2)\mathcal{G}_4(s_1,s_2,s_3)\},$$

$$f_3^{Z^*Z^*Z^*}(s_1,s_2,s_3)=0,$$

$$g_j^{Z^*Z^*Z^*}(s_1,s_2,s_3)=\frac{e^2}{8\pi^2 s_W^3 c_W^3}g_{aF}(3g_{vF}^2+g_{aF}^2)\mathcal{G}'_j(s_1,s_2,s_3), \tag{C1}$$

where

$$\begin{aligned}\mathcal{G}_1(s_1,s_2,s_3)=\frac{1}{\lambda^2}\Big\{&C_0(s_1,s_2,s_3)[s_3(2s_3-s_1-s_2)-(s_1-s_2)^2](\lambda M_F^2+2s_1s_2s_3)\\&-\frac{1}{2}[B_0(s_1)-B_0(s_2)](s_1-s_2)[\lambda(2M_F^2+s_3)+12s_1s_2s_3]\\&-\frac{s_3}{2}[B_0(s_1)+B_0(s_2)-2B_0(s_3)][2\lambda M_F^2+s_3^2(s_1+s_2)-2s_3(s_1^2+s_2^2-4s_1s_2)\\&+(s_1+s_2)(s_1-s_2)^2]\Big\}+\frac{2s_3^2-s_3(s_1+s_2)-(s_1-s_2)^2}{3\lambda},\end{aligned} \tag{C2}$$

$$\begin{aligned}\mathcal{G}_2(s_1,s_2,s_3)=&-\frac{(s_1-s_2)(s_3-s_1-s_2)}{\lambda}\\&+\frac{1}{\lambda^2}\Big\{-3(s_1-s_2)(s_3-s_1-s_2)(\lambda M_F^2+2s_1s_2s_3)C_0(s_1,s_2,s_3)\\&-\frac{1}{2}[B_0(s_1)-B_0(s_2)][2\lambda M_F^2(s_3-2s_1-2s_2)-s_3(s_1+s_2)(s_3^2+s_1^2+s_2^2+14s_1s_2)\\&+2s_3^2(s_1^2+s_2^2+6s_1s_2)-4s_1s_2(s_1-s_2)^2]\\&+\frac{1}{2}[B_0(s_1)+B_0(s_2)-2B_0(s_3)](s_1-s_2)(2\lambda M_F^2+\lambda s_3+12s_1s_2s_3)\Big\},\end{aligned} \tag{C3}$$

$$\begin{aligned}\mathcal{G}_3(s_1,s_2,s_3)=\frac{M_F^2}{\lambda}\{&-[2s_3^2-s_3(s_1+s_2)-(s_1-s_2)^2]C_0(s_1,s_2,s_3)+3(s_1-s_2)[B_0(s_1)-B_0(s_2)]\\&+3s_3[B_0(s_1)+B_0(s_2)-2B_0(s_3)]\},\end{aligned} \tag{C4}$$

[11]We follow the same notation as in the last paper in Ref. [18], but we omit the common fermion mass M_F from the arguments of the one-loop B_0 and C_0 functions. We also note that in this case $C_0(s_1,s_2,s_3)$ is a fully symmetric function of s_1,s_2,s_3 .

$$\mathcal{G}_4(s_1,s_2,s_3)=\frac{3M_F^2}{\lambda}\{(s_1-s_2)[(s_3-s_1-s_2)C_0(s_1,s_2,s_3)-B_0(s_1)-B_0(s_2)+2B_0(s_3)]$$
$$+(s_3-2s_1-2s_2)[B_0(s_1)-B_0(s_2)]\},\tag{C5}$$

while the scalar functions are determined through

$$\mathcal{G}_1'(s_1,s_2,s_3)=\frac{1}{\lambda^2}\Big\{-C_0(s_3,s_2,s_1)\{\lambda M_F^2(s_1-s_3-s_2)+s_3s_2[2s_1^2-s_1(s_3+s_2)-(s_3-s_2)^2]\}$$
$$+\frac{1}{2}[B_0(s_3)-B_0(s_1)]s_3[s_1^2+2s_1(2s_2-s_3)+s_3^2+4s_3s_2-5s_2^2]$$
$$+\frac{1}{2}[B_0(s_2)-B_0(s_1)]s_2[s_1^2+2s_1(2s_3-s_2)+s_2^2+4s_3s_2-5s_3^2]$$
$$-\frac{\lambda}{2}(s_1-s_3-s_2)\Big\}\tag{C6}$$

and the Bose result

$$\mathcal{G}_3'(s_1,s_2,s_3)=\mathcal{G}_3'(s_2,s_1,s_3)=\mathcal{G}_2'(s_1,s_3,s_2)=\mathcal{G}_2'(s_2,s_3,s_1)=\mathcal{G}_1'(s_3,s_2,s_1)=\mathcal{G}_3'(s_3,s_1,s_2),\tag{C7}$$

derived from Eq. (A5) and $f_3^{Z^*Z^*Z^*}=0$. In all cases we define

$$\lambda=s_3^2+s_1^2+s_2^2-2s_1s_2-2s_3(s_1+s_2).\tag{C8}$$

Application to $Z^*Z^*\gamma^*$:

$$f_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=-\frac{e^2Q_Fg_{aF}g_{vF}}{8\pi^2s_W^2c_W^2}[\mathcal{G}_1(s_1,s_2,s_3)+\mathcal{G}_5(s_1,s_2,s_3)],$$

$$f_2^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=\frac{e^2Q_Fg_{aF}g_{vF}}{8\pi^2s_W^2c_W^2}\left[\mathcal{G}_2(s_1,s_2,s_3)+\frac{1}{3}\mathcal{G}_4(s_1,s_2,s_3)\right],$$

$$g_1^{Z^*Z^*\gamma^*}(s_1,s_2,s_3)=g_2^{Z^*Z^*\gamma^*}(s_2,s_1,s_3)=\frac{e^2Q_Fg_{aF}g_{vF}}{2\pi^2s_W^2c_W^2}\mathcal{G}_1'(s_1,s_2,s_3),\tag{C9}$$

where the only new function not already appearing in the $Z^*Z^*Z^*$ case is

$$\mathcal{G}_5(s_1,s_2,s_3)=\frac{M_F^2}{\lambda}\{-[s_3(s_1+s_2)-(s_1-s_2)^2]C_0(s_1,s_2,s_3)+[B_0(s_1)-B_0(s_2)]$$
$$\times(s_1-s_2)+s_3[B_0(s_1)+B_0(s_2)-2B_0(s_3)]\}+\frac{1}{3}.\tag{C10}$$

Application to $\gamma^*\gamma^*Z^*$:

$$f_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=-\frac{e^2Q_F^2g_{aF}}{8\pi^2s_Wc_W}[\mathcal{G}_6(s_1,s_2,s_3)+\mathcal{G}_7(s_1,s_2,s_3)],$$

$$f_2^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=-\frac{e^2Q_F^2g_{aF}}{8\pi^2s_Wc_W}[\mathcal{G}_6(s_1,s_2,s_3)-\mathcal{G}_7(s_1,s_2,s_3)],$$

$$g_1^{\gamma^*\gamma^*Z^*}(s_1,s_2,s_3)=\frac{e^2Q_F^2g_{aF}}{2\pi^2s_Wc_W}\mathcal{G}_1'(s_3,s_2,s_1), \tag{C11}$$

$$\begin{aligned}\mathcal{G}_6(s_1,s_2,s_3)=\frac{1}{\lambda^2}\Big\{&-2s_1C_0(s_3,s_2,s_1)[\lambda M_F^2(s_1-s_3-s_2)-s_3^3s_2+s_3^2s_2(2s_2-s_1)+s_3s_2(2s_1^2-s_1s_2-s_2^2)]\\&-\frac{s_1}{2}[B_0(s_1)-B_0(s_2)][s_3^3-2s_3^2(s_1+3s_2)+s_3(s_1^2+12s_1s_2+3s_2^2)\\&+2s_2(s_1-s_2)^2]-\frac{s_3s_1}{2}[B_0(s_1)+B_0(s_2)-2B_0(s_3)][s_3^2+2s_3(2s_2-s_1)\\&+s_1^2+4s_1s_2-5s_2^2]\Big\}-\frac{s_1(s_1-s_3-s_2)}{\lambda},\end{aligned} \tag{C12}$$

$$\begin{aligned}\mathcal{G}_7(s_1,s_2,s_3)=\frac{1}{\lambda^2}\Big\{&-2s_2C_0(s_3,s_2,s_1)[\lambda M_F^2(s_2-s_3-s_1)-s_3^3s_1+s_3^2s_1(2s_1-s_2)-s_3s_1(s_1^2+s_1s_2-2s_2^2)]\\&+\frac{s_2}{2}[B_0(s_1)-B_0(s_2)][s_3^3-2s_3^2(s_2+3s_1)+s_3(s_2^2+12s_1s_2+3s_1^2)+2s_1(s_1-s_2)^2]\\&-\frac{s_2s_3}{2}[B_0(s_1)+B_0(s_2)-2B_0(s_3)][s_3^2+2s_3(2s_1-s_2)+s_2^2+4s_1s_2-5s_1^2]\Big\}\\&-\frac{s_2(s_2-s_3-s_1)}{\lambda}.\end{aligned} \tag{C13}$$

In principle the triangular graph in Fig. 2 (with a single fermion of mass M_F running along it), could also include ambiguous axial anomaly contributions. Such contributions do not have the structure of a self-interaction among three neutral gauge bosons, and they are presumably cancelled by other (possibly extremely heavy) fermions. The cancellation of these anomalous contributions is easily imposed by requiring that all $\mathcal{G}_j$ and $\mathcal{G}_j'$ functions defined above vanish in the limit ($M_F^2 \gg |s_1|,\ |s_2|,\ |s_3|$). Thus, for cancelling the anomaly in the actual calculation of the functions above, we occasionally needed to subtract an appropriate M_F-independent term.

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