Addenda and corrections to work done on the path-integral approach to classical mechanics

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We continue the study of the path-integral approach to classical mechanics and in particular we *correct* and better clarify, with respect to previous papers, the geometrical meaning of the variables entering this formulation. We show that the space spanned by the whole set of variables $(\varphi, c, \lambda, \bar{c})$ of our path integral is the cotangent bundle to the *reversed-parity* tangent bundle of the phase space $\mathcal{M}$ of our system and it is indicated as $T^\star(\Pi T\mathcal{M})$. We also show that it is possible to build a different path integral made only of *bosonic* variables. These turn out to be the coordinates of $T^\star(T^\star\mathcal{M})$ which is the double cotangent bundle to phase space.

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We continue the study of the *path-integral* formulation of Hamiltonian classical mechanics started in [1] and continued in [2]. Starting from a Hamiltonian H defined on a phase space $\mathcal{M}$ whose $2n$ coordinates we indicate as φ^a, our path integral [1] naturally generates a weight whose Lagrangian is

$$\tilde{\mathcal{L}} = \lambda_a[\dot{\varphi}^a - \omega^{ab}\partial_b H] + i\bar{c}_a[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H]c^b, \quad (1)$$

where ω^{ab} is the symplectic matrix of the system and $(\lambda_a, c^a, \bar{c}_a)$ are auxiliary variables needed in our formulation. In this paper we will try to better characterize geometrically the space spanned by the $8n$ variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$.

Let us start with λ_a. From the Lagrangian (1) one immediately finds how the λ_a transform under the Hamiltonian flow:

$$\lambda'_a = [\delta^b_a - \omega^{bj}\partial_j\partial_a H\Delta t]\lambda_b - i\bar{c}_i\omega^{ij}(\partial_j\partial_b\partial_a H\Delta t)c^b. \quad (2)$$

If we perform any Hamiltonian diffeomorphism [3] on φ, $\varphi'^a = \varphi^a - \varepsilon^a(\varphi)$, it is easy to understand, in analogy to Eq. (2), how λ_a will change under the above transformation:

$$\lambda'_a = \lambda_a + \lambda_b\partial_a\varepsilon^b + i\bar{c}_i(\partial_a\partial_b\varepsilon^i)c^b, \quad (3)$$

where $\varepsilon^a = \omega^{ab}\partial_b G(\varphi)$ and G is any function of φ multiplied by an infinitesimal parameter. The above transformation properties of λ seem to be at odds with the representation of λ_a that we found at the operatorial level [1]: $\lambda_a = -i\,\partial/\partial\varphi^a$. In fact in [1] we thought that this would imply that the λ_a, being proportional to $\partial/\partial\varphi^a$, transform as the basis of the vector fields [3], $\lambda'_a = (\partial\varphi^b/\partial\varphi'^a)\lambda_b = [\delta^b_a + \partial_a(\omega^{bc}\partial_c G)]\lambda_b$, and not as in Eq. (3). The way to reconcile these two facts is the following: $\partial/\partial\varphi^a$ would transform as vector fields if they were applied to functions only of φ, i.e., $F(\varphi)$, but they would transform differently if they were applied to functions of φ and c, i.e., $F(\varphi, c)$. We will now show that, if applied to these last functions, the $\partial/\partial\varphi^a$ transform as the λ_a do in Eq. (3). As explained in Appendix A of Ref. [2], under the same (symplectic) diffeomorphism which is applied to λ, the φ and c transform as $\varphi'^a = \varphi^a - \omega^{ab}\partial_b G$, $c'^a = [\delta^a_b - \omega^{ac}\partial_c\partial_b G]c^b$. Note that the transformed c' depend on φ via G. Let us now take a function of φ and c, i.e., $F(\varphi, c)$, and let us transform its arguments: $F(\varphi, c) = F[\varphi(\varphi'), c(\varphi', c')] \equiv \mathcal{S}(\varphi', c')$.

If we now apply $\partial/\partial\varphi'$ on $\mathcal{S}$, we have

$$(\partial/\partial\varphi')\mathcal{S}(\varphi', c') = (\partial/\partial\varphi')\,F[\varphi(\varphi'), c(\varphi', c')] \quad (4)$$

$$= \frac{\partial\varphi}{\partial\varphi'}\frac{\partial F}{\partial\varphi} + \frac{\partial c}{\partial\varphi'}\frac{\partial F}{\partial c}. \quad (5)$$

Comparing the two right-hand sides (RHSs) of the above equations we can say that

$$\frac{\partial}{\partial\varphi'^a} = \frac{\partial\varphi^b}{\partial\varphi'^a}\frac{\partial}{\partial\varphi^b} + \frac{\partial c^b}{\partial\varphi'^a}\frac{\partial}{\partial c^b}. \quad (6)$$

Using the operatorial correspondence described in Ref. [1], $\partial/\partial\varphi'^a = i\lambda'_a$, $\partial/\partial\varphi^a = i\lambda_a$, and $\partial/\partial c^b = \bar{c}_b$, we can rewrite Eq. (6) as

$$i\lambda'_a = i\frac{\partial\varphi^b}{\partial\varphi'^a}\lambda_b + \frac{\partial c^b}{\partial\varphi'^a}\bar{c}_b. \quad (7)$$

One could say that in general $\partial c^b/\partial\varphi'^a = 0$ because c and φ' are independent coordinates. Actually it is not so because, as we saw before the transformed c depend on φ and, vice versa, the original c depend on the transformed φ. So $\partial c^b/\partial\varphi'^a \neq 0$. Let us now proceed to see if the transformations of the λ above are the same as Eq. (3). Using $\varphi^a = \varphi'^a + \omega^{ab}\partial_b G(\varphi')$, $c^a = [\delta^a_b + \omega^{ac}\partial_c\partial_b G]c'^b$, we get $\partial\varphi^b/\partial\varphi'^a = \delta^b_a + \omega^{bi}\partial_i\partial_a G$, $\partial c^b/\partial\varphi'^a = \omega^{bc}\partial_c\partial_i\partial_a G c'^i$. Inserting these expressions into Eq. (7) we obtain

$$\lambda'_a = [\delta^b_a + \omega^{bi}\partial_i\partial_a G]\lambda_b + i\bar{c}_b\omega^{bc}\partial_c\partial_i\partial_a G c^i, \quad (8)$$

where we have replaced c' with c because we keep only terms which are first order in the infinitesimal G and we have brought $\bar{c}$ in front in the last term of the RHS of the equation above. Equation (8) is exactly the transformation for λ we had in Eq. (3). So this proves that as an operator λ_a act as

$\partial/\partial\varphi^a$ but over the functions $F(\varphi,c)$ and this in turn implies that the *base space* we should consider is the one[1] made of (φ,c).

Then let us first find out which kind of space this is. φ^a are the $2n$ coordinates of the phase space $\mathcal{M}$. The c^a transform under a diffeomorphism as the forms $d\varphi^a$, so we stated in Refs. [1,2] that, *identifying* c with $d\varphi$, the space (φ^a,c^a) makes up the cotangent bundle [3] to phase space: $T^\star\mathcal{M}$. That is *wrong*. In fact c^a are at most *a basis* in the fiber $T^\star_\varphi\mathcal{M}$ and not a generic vector in $T^\star_\varphi\mathcal{M}$. Since the c are a basis, they belong to what is called [5] the *bundle of linear frames*[2] to $\mathcal{M}$. So the (φ^a,c^a) are nothing else than a *section* of the linear frame bundle. We say a ''section'' because there are other bases (or frames), besides c^a, which one could choose in the fibers of the linear frame bundle.

As we have stressed before, the structure above holds if one *identifies* c^a with $d\varphi^a$. We did that identification in Ref. [1] and used it to turn the whole Cartan calculus into operations which could be performed via our path integral and the structures present in it. Of course the fact that the c^a transform as the $d\varphi^a$ does not force us to identify them so explicitly with the $d\varphi^a$ as we have done in Ref. [1]. For example, if we build a generic vector field $V\equiv V^a\partial/\partial\varphi^a$, we would have that the components V^a transform as the c^a. So one could say that the c^a are the components or *coordinates* (and not the *basis*) of the vectors in the tangent fibers. Then they would make up, together with the φ, the tangent bundle to phase space $T\mathcal{M}$. The only difference is that the c^a have a Grassmannian nature and not a bosonic one like the V^a. In that case the bundle is known in the literature [4] as a *reversed-parity* tangent bundle and indicated as $\Pi T\mathcal{M}$.

Next we have to consider the role of the $4n$ remaining variables $(\lambda_a,\bar{c}_a)$. Looking at the Lagrangian in Eq. (1) we see that they play the role of momenta to the variables (φ^a,c^a), so they will make cotangent fibers to the previous space. We can summarize all this in the following scheme:

$$(\varphi^a)\Rightarrow\mathcal{M},\quad (\varphi^a,c^a)\Rightarrow\Pi T\mathcal{M}, \tag{9}$$

$$(\varphi^a,c^a,\lambda_a,\bar{c}_a)\Rightarrow T^\star(\Pi T\mathcal{M}). \tag{10}$$

Anyhow this is not the only picture we can have of our space. In fact in Ref. [1] we proved that the $\bar{c}_a$ act, at the operatorial level, as $\bar{c}_a=\partial/\partial c^a$; moreover, in the previous section of this paper we proved that the λ_a, despite their strange transformation properties (3), still maintain their operatorial meaning of being $\lambda_a=-i\partial/\partial\varphi^a$. From these representations of λ and $\bar{c}$, we can say that $(\lambda_a,\bar{c}_a)$ form *a basis* in the tangent fibers to the *base space* (φ^a,c^a). As this base space is $\Pi T\mathcal{M}$, the overall $8n$ coordinates $(\varphi^a,c^a,\lambda_a,\bar{c}_a)$ can then be considered as a section of the bundle of linear frames over $\Pi T\mathcal{M}$. This is an *alternative* interpretation of our $8n$ variables with respect to the interpretation contained in Eq. (10). This sort of ''duality'' in treating each of our variables $(\lambda,c,\bar{c})$ either as a *basis* or *coordinate* could be considered at each of the levels of Eqs. (9),(10) and would give rise to all possible combinations. Anyhow we will stick here to the *coordinate* picture which will lead to the ''reversed-parity'' bundle of Eq. (10).

In addition to this sort of ''duality'' which would allow us to see in two different ways the spaces labeled by our variables, there is a further freedom. This is related to the scheme of Eqs. (9),(10). Let us first perform a partial integration in the kinetic piece of the Grassmannian variables present in the Lagrangian (1). The new Lagrangian, modulo surface terms, would be $\mathcal{L}'\equiv\lambda_a[\dot{\varphi}^a-\omega^{ab}\partial_b H]-i\dot{\bar{c}}_a c^a-i\bar{c}_a\omega^{ac}\partial_c\partial_b H c^b$. As this Lagrangian is different from $\tilde{\mathcal{L}}$ only by a surface term, the equations of motion for c^a and $\bar{c}_a$ are the same, but now $\bar{c}_a$ play the role of ''configurational'' variables while c^a are their relative momenta. Then it would seem natural to choose in Eq. (9) the $(\varphi^a,\bar{c}_a)$ as new fundamental coordinates. Let us now see the geometrical interpretation of all this: the $\bar{c}_a$ transform [2] as $\bar{c}'_a=\bar{c}_a+\bar{c}_b\partial_a\varepsilon^b$ and so, interpreting the $\bar{c}_a$ as *coordinates* and not *basis*, they transform as components of forms but with Grassmannian character, i.e., with the reversed-parity character. This means that the $(\varphi^a,\bar{c}_a)$ are the coordinates of the reversed-parity cotangent bundle: $(\varphi^a,\bar{c}_a)\Rightarrow\Pi T^\star M$. From the Lagrangian $\mathcal{L}'$ above we see that λ_a and c^a play the role of momenta to the previous variables, so they belong to the cotangent fibers of the previous space. Putting all this together we can then write down the following picture:

$$(\varphi^a)\Rightarrow\mathcal{M},\quad (\varphi^a,\bar{c}_a)\Rightarrow\Pi T^\star\mathcal{M}, \tag{11}$$

$$(\varphi^a,\bar{c}_a,\lambda_a,c^a)\Rightarrow T^\star(\Pi T^\star\mathcal{M}).$$

As the physics contained in the Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ are the same and the coordinates are the same, we can then say that our variables label both spaces: $T^\star(\Pi T\mathcal{M})$ and $T^\star(\Pi T^\star\mathcal{M})$. A more mathematically precise proof of this is contained in Ref. [10].

The reader may feel a little bit uneasy with the Grassmannian double bundles we have provided in the previous section and even with the alternative interpretation as sections of the frame bundle. For this reason in this section we will show that, at least for the *Hamiltonian* flow, it is possible to provide a path integral of classical mechanics made entirely of bosonic variables. Moreover, we will prove that these variables are just the coordinates of a standard double bundle such as $T^\star(T^\star\mathcal{M})$. The procedure to achieve what we said above is explained below.

The path integral [1] for *classical* mechanics (CM) was basically the following:

[1]If we had used the above transformation properties for λ in Ref. [2] we would have found that the Lagrangian $\tilde{\mathcal{L}}$ is a scalar under a diffeomorphism even off shell.

[2]Actually the bundle of linear frames is made out of the basis of the tangent fibers while ours is made of the basis of the cotangent fibers, but the two are isomorphic.

$$Z[0]=\int \mathcal{D}\varphi\tilde{\delta}[\varphi(t)-\varphi_{cl}(t)]$$
$$=\int \mathcal{D}\varphi\tilde{\delta}\left[\dot{\varphi}^a-\omega^{ab}\frac{\partial H}{\partial\varphi^b}\right]\det\left[\delta_l^a\partial_t-\omega^{ab}\frac{\partial^2 H}{\partial\varphi^b\partial\varphi^l}\right] \quad (12)$$

where the $\det[\cdot]$ appearing in Eq. (12) is a functional determinant [1]. The Lagrangian of Eq. (1) is obtained by doing the Fourier transform (via the variables λ_a) of the Dirac delta in the second term of Eq. (12) and exponentiating the $\det[\cdot]$ with Grassmannian variables c^a and $\bar{c}_a$. In order to avoid using the Grassmannian variables the trick we adopt now is to substitute the $\det[\cdot]$ in Eq. (12) with an inverse determinant:

$$\det\left[\delta_l^a\partial_t-\omega^{ab}\frac{\partial^2 H}{\partial\varphi^b\partial\varphi^l}\right]=\left\{\det\left[\delta_l^a\partial_t+\omega^{ab}\frac{\partial^2 H}{\partial\varphi^b\partial\varphi^l}\right]\right\}^{-1}. \quad (13)$$

The proof of this relation goes as follows: the determinants in Eq. (13) are functional determinants, which means

$$\det[\delta_l^a\partial_t-\omega^{ab}\,\partial^2H/\partial\varphi^b\partial\varphi^l]$$
$$=\{\det\partial_t\}\left\{\det\left[\delta_l^a\delta(t-t')-\theta(t-t')\omega^{ab}\frac{\partial^2 H}{\partial\varphi^b\partial\varphi^l}\right]\right\}. \quad (14)$$

If Eq. (13) holds, then the determinant of the product of the two matrices appearing, respectively, on the LHS and RHS of Eq. (13) is 1. The proof goes as follows:

$$\det\left\{\int dt'\left[\delta_b^a\delta(t-t')-\theta(t-t')\omega^{al}\frac{\partial^2 H}{\partial\varphi^l\partial\varphi^b}\right]\times\left[\delta_c^b\delta(t'-t'')+\theta(t'-t'')\omega^{bk}\frac{\partial^2 H}{\partial\varphi^k\partial\varphi^c}\right]\right\}$$
$$=\det\left\{\delta_c^a\delta(t-t'')-\int dt'\,\theta(t-t')\theta(t'-t'')\times\omega^{al}\frac{\partial^2 H}{\partial\varphi^l\partial\varphi^b}\omega^{bk}\frac{\partial^2 H}{\partial\varphi^k\partial\varphi^c}\right\} \quad (15)$$

$$\approx\exp\left(-\int dt'\,\theta(t-t')\theta(t'-t)\times\omega^{al}\frac{\partial^2 H}{\partial\varphi^l\partial\varphi^b}\omega^{bk}\frac{\partial^2 H}{\partial\varphi^k\partial\varphi^a}\right)=1, \quad (16)$$

where we have dropped the $\det\partial_t$ part, and in Eq. (16) we have used the ''exp tr log'' form for the determinant plus the fact that the product of the two $\theta(\cdot)$ is zero. So this proves the relation (13). Before proceeding we should better qualify the steps done in Eqs. (15) and (16). There we used the $\theta(t-t')$ as ''inverse'' (or Green function) of ∂_t. The $\theta(t-t')$ is the retarted or *causal* Green function. If we had used other Green functions such as the $\epsilon(t-t')$, we would not have obtained Eq. (16), which means also relation (13) would not have been valid. To make relation (13) valid we should better specify what we mean by a functional determinant.

It is actually well known [6,7] that all functional determinants of the form $\det[\partial_t\delta(t-t')-\delta(t-t')G'(\varphi)]$ depend on the *boundary conditions* under which we solve the associated differential equation, $[\partial_t-G'(\varphi)]c_n(t)=\sigma_n c_n(t)$, whose eigenvalues σ_n are needed to calculate the determinant in some *regularized* form: $\det[(\cdot)]=\{\Pi_{n=-\infty}^{n=\infty}\sigma_n\}_{regul}$. Solving the differential equation above with *causal* boundary conditions one obtains [7]

$$\det[\partial_t\delta(t-t')-\delta(t-t')G'(\varphi)]_{causal}=\exp\left(-\frac{1}{2}\int dt'\,G'(\varphi(t'))\right). \quad (17)$$

So by reversing the sign of $G'(\varphi)$ we get

$$\det[\partial_t\delta(t-t')+\delta(t-t')G'(\varphi)]_{causal}=\exp\left(+\frac{1}{2}\int dt'\,G'(\varphi(t'))\right). \quad (18)$$

By comparing the RHS of the last two equations above we see that the two determinants are the inverse of each other. This proves relation (13) provided we specify that the functional determinant has to be evaluated with *causal* boundary conditions. The reason we choose these boundary conditions is because, after all, we are just doing classical mechanics, which means just solving ordinary Hamiltonian equations of motion. These are usually solved by giving a value of φ at the initial time $t=0$ and looking for the evolution at *later* times using a *causal* propagator. The use of *periodic boundary conditions* and of a time-symmetric Green function for our path integral has been analyzed in full detail in Ref. [8]. The result is a path integral whose only *nonzero* expectation values are those associated with observables which are independent from deformations of the Hamiltonian H and of its symplectic form ω_{ab}. This means a path integral which is not affected by the form of H anymore. Something that does not feel the dynamics at all is *not* what we want to use here.

Having clarified the boundary conditions we use in evaluating the determinants in Eq. (13), the next step is to use relation (13) in Eq. (12) and then ''exponentiate'' the inverse of the matrix using bosonic variables by making use of the well-known formula[3] $\int dx^i dy_j\exp(ix^iA_i^jy_j)\propto\{\det[A_i^j]\}^{-1}$. Doing all that we get

[3]This formula requires that the determinant be positive and this is our case because the LHS of Eq. (13) is positive [1].

$$Z[J]=\int \mathcal{D}\varphi\tilde{\delta}\left[\dot{\varphi}^a-\omega^{ab}\frac{\partial H}{\partial \varphi^b}\right]\left\{\det\left[\delta^a_l\partial_t+\omega^{ab}\frac{\partial^2 H}{\partial\varphi^b\partial\varphi^l}\right]\right\}^{-1} \tag{19}$$

$$=\int \mathcal{D}\varphi^a\mathcal{D}\lambda_a\mathcal{D}\pi^a\mathcal{D}\xi_a \exp\left(i\int \mathcal{L}dt\right), \tag{20}$$

where

$$\mathcal{L}=\lambda_a\left[\dot{\varphi}^a-\omega^{ab}\frac{\partial H}{\partial\varphi^b}\right]+\pi^l\left[\delta^a_l\partial_t+\omega^{ab}\frac{\partial^2 H}{\partial\varphi^b\partial\varphi^l}\right]\xi_a. \tag{21}$$

The variables π^l and ξ_a are the *bosonic* variables we have used to exponentiate the inverse matrix and they replace the Grassmannian variables c^a and $\bar{c}_a$ present in $\tilde{\mathcal{L}}$ of Eq. (1).

Let us now see if we can give a geometrical understanding of the new variables π^a, ξ_a present here. Let us show how they change under the Hamiltonian flow, which means under their equation of motion, $\partial_t\xi_l+\xi_a\omega^{ab}\,\partial^2 H/\partial\varphi^b\partial\varphi^l=0$, which can easily be derived from the Lagrangian $\mathcal{L}$ above. This equation should be compared with the equations of motion of c^a derived [1] from $\tilde{\mathcal{L}}$ of Eq. (1) which are $\partial_t c^a-\omega^{ab}(\partial^2 H/\partial\varphi^b\partial\varphi^l)\,c^l=0$. From the above equations it is now easy to see that the quantity $\Xi\equiv\xi_a c^a$ is invariant under the Hamiltonian flow. This quantity would behave in the same way under any diffeomorphism of the phase space $\mathcal{M}$ and not just under the Hamiltonian flow.[4] The invariance of Ξ is the same thing that would happen to a form[5] $\tilde{\Xi}\equiv\tilde{\xi}_a d\varphi^a$ and by identifying the $d\varphi^a$ above with the c^a we see that we can identify the ξ_a with the $\tilde{\xi}_a$. So while the c^a are the basis of the fibers on $T^\star\mathcal{M}$, the ξ_a are the coordinates of the same space. Looking at the Lagrangian (21) we see that π^a and λ_a are the momenta associated with ξ_a and φ^a, which means they will make up the cotangent fibers to the previous space. So the overall set of variables $(\varphi^a,\xi_a,\lambda_a,\pi^a)$ makes up the coordinates of $T^\star(T^\star\mathcal{M})$. This is a double bundle but it may please the reader more than the reversed-parity one $T^\star(\Pi T\mathcal{M})$ associated with the Lagrangian $\tilde{\mathcal{L}}$ of Eq. (1). It might also be a space easier to handle for the study of various *physical* issues like the study of ergodicity and Lyapunov exponents [9] that we performed previously using the old Lagrangian (1). It may also be worthwhile to see if the universal symmetries (BRS and supersymmetry) we found in [1,9] are present (in a different form) also in the purely bosonic case presented here.

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[4]This is so because we would have to choose the transformations on π and ξ induced by the diffeomorphism in φ in such a way so as to keep invariant the Hamiltonian associated with $\mathcal{L}$.

[5]This is so because forms are objects totally coordinate free.

[1] E. Gozzi, M. Reuter, and W. D. Thacker, Phys. Rev. D **40**, 3363 (1989).
[2] E. Gozzi, M. Reuter, and W. D. Thacker, Phys. Rev. D **46**, 757 (1992).
[3] R. Abraham and J. Marsden, *Foundations of Mechanics* (Benjamin, New York, 1978).
[4] A. Schwarz, in *Topics in Statistical and Theoretical Physics*, edited by R. L. Dobrushin, R. A. Minlos, M. A. Shubin, and A. M. Vershik (AMS, Providence, RI, 1996).
[5] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Wiley, New York, 1963), Vol. I.
[6] H. Nakazato *et al.*, Nucl. Phys. **B346**, 611 (1990).
[7] S. Marculescu *et al.*, Nucl. Phys. **B349**, 463 (1991).
[8] E. Gozzi and M. Reuter, Phys. Lett. B **240**, 137 (1990).
[9] E. Gozzi and M. Reuter, Phys. Lett. B **233**, 383 (1989); Chaos, Solitons and Fractals **4**, 1117 (1994).
[10] Marco Regini, Tesi di Laurea, Universitá di Trieste, 1998.