Wave function of the radion in a brane world

Christos Charmousis and Ruth Gregory
Centre for Particle Theory, Durham University, South Road, Durham DH1 3LE, United Kingdom

Valery A. Rubakov
Institute for Nuclear Research of the Russian Academy of Sciences, 60th October Anniversary prospect, 7a, Moscow 117312, Russia and Isaac Newton Institute for Mathematical Sciences, University of Cambridge, 20 Clarkson Road, Cambridge CB3 0EH, United Kingdom

(Received 22 December 1999; published 28 August 2000)

We calculate the linearized metric perturbation corresponding to a massless four-dimensional scalar field, the radion, in a five-dimensional two-brane model of Randall and Sundrum. In this way we obtain the relative strengths of the radion couplings to matter residing on each of the branes. The results are in agreement with the analysis of Garriga and Tanaka of gravitational and Brans-Dicke forces between matter on the branes. We also introduce a model with an infinite fifth dimension and ''almost'' confined graviton, and calculate the radion properties in that model.

PACS number(s): 04.50.+h, 11.25.Mj

Recently, considerable interest has been raised by a five-dimensional model with an S^1/Z_2 orbifold extra dimension with two 3-branes residing at its boundaries [1]. This model and its noncompact analogues [2–5] (see [6] for an account of earlier works) provide a novel setting for discussing various conceptual and phenomenological issues related to compactification of extra dimensions in models motivated by M theory. In the two-brane Randall-Sundrum model [1], the branes have tensions $+\sigma$ and $-\sigma$, and the bulk cosmological constant is chosen in such a way that the classical solution describes five-dimensional space-time whose four-dimensional slices are flat:

$$ds^2 = a^2(z)\,\eta_{\mu\nu}dx^\mu dx^\nu - dz^2. \tag{1}$$

Here $a(z)=e^{-k|z|}$, the fifth coordinate z runs from $z_+=0$ to $z_-=r_c$ and $k=(4\pi/3)G_5\sigma$, where G_5 is Newton's constant in five dimensions. The orbifold symmetry, a local reflection symmetry at each brane, is assumed to hold for all fields in this space-time.

The excitations above the background metric Eq. (1), contain a massless four-dimensional graviton (whose wave function is peaked at the positive tension brane) and the corresponding Kaluza-Klein tower [3]. This is not the whole story, however. In general when one has a wall in space-time, one might expect a translational zero mode giving rise to free motion of the wall. In the case of anti–de Sitter (AdS) space-time, ∂_z is not a translational Killing vector but a conformal Killing vector; nonetheless we can identify solutions to the perturbation equations which correspond to the proper motion of the wall (although these will be singular on the AdS horizon). In the conventional application of the Israel equations, one identifies the extrinsic curvature $K_{\mu\nu}$ on each side of the wall, and then applies a Z_2 symmetry across the wall leading to $\tilde{K}_{\mu\nu}=(K^+_{\mu\nu}+K^-_{\mu\nu})/2=0$; geometrically this means that the wall is locally ''flat,'' i.e., totally geodesic. To describe proper dynamical motion of the wall, we require a nonzero $\tilde{K}_{\mu\nu}$, which is possible if the Z_2 symmetry is not imposed. The appropriate solution for such a motion then turns out to be

$$\delta g_{\mu\nu} = \frac{\sinh 2kz}{2k}\tilde{K}_{\mu\nu}, \tag{2}$$

where $\tilde{K}^\mu_\mu=0$, which is recognized as the ''Nambu'' equation for a brane. Since this solution blows up at large z, it does not correspond to a small perturbation of the space-time, and is indicative that in the presence of such free motion, the asymptotic structure of the space-time is altered, similar to the difference between the metrics of a straight cosmic string and a crinkly cosmic string [7]. As such, this perturbation is not considered in the general spectrum of localized perturbations of the Randall-Sundrum wall.

Once we have two branes, however, the situation is different: there are now two sorts of motion; a center of mass (which will still be divergent) and relative motion—the radion for which the second wall acts as a regulator on the divergence of Eq. (2). It is this second mode that we wish to identify, which will correspond to a massless four-dimensional scalar. One may or may not suspect that the radion is also accompanied by its own Kaluza-Klein tower.

The radion field $T(x)$ has been introduced by considering metrics of the form [1,3,8,9]

$$ds^2 = e^{-2k|z|T(x)}g_{\mu\nu}(x)dx^\mu dx^\nu - T^2(x)dz^2, \tag{3}$$

where $g_{\mu\nu}$ is the four-dimensional graviton. This form has been used for deriving the effective four-dimensional action describing the large-distance dynamics. The complete characterization of the radion excitation everywhere in the five-dimensional space-time, however, requires a solution to the field equations about the background Eq. (1). The ansatz (3) does not in fact solve these linearized field equations.

The purpose of this report is to calculate in linearized theory the five-dimensional metric perturbation corresponding to the propagating radion field. We will see that this perturbation does not vanish on either of the branes. We will point out also that there is no Kaluza-Klein tower above the radion, i.e., that all massive states have been accounted for in the analysis of [3].

To deal with the Z_2 symmetry as well as with junction conditions on the branes, it is convenient to choose Gaussian normal (GN) coordinates

$$g_{zz}=-1, \quad g_{z\mu}=0. \tag{4}$$

Such a system can always be chosen in the neighborhood of the brane by integrating out along its normal, in which case z will be the proper distance from the brane. However, note that this system is slightly more general, in that we can make coordinate transformations which shift the wall, but preserve the metric components [Eq. (4)].

Then the linearized theory is described by the metric

$$ds^2=a^2(z)\,\eta_{\mu\nu}dx^\mu dx^\nu+h_{\mu\nu}(x,z)dx^\mu dx^\nu-dz^2. \tag{5}$$

We will explicitly consider the region $r_c>z>0$; the orbifold symmetry giving $h_{\mu\nu}$ for other values of z. The four-dimensional indices will be raised and lowered using the Minkowski metric $\eta_{\mu\nu}$. The linearized Einstein equations are

$$\delta R_{zz}=8\pi G_5\left(\frac{2}{3}T_{zz}+\frac{1}{3a^2}T^\lambda_\lambda\right), \tag{6a}$$

$$\delta R_{z\mu}=8\pi G_5 T_{z\mu}, \tag{6b}$$

$$\delta R_{\mu\nu}-4k^2h_{\mu\nu}=8\pi G_5\left(T_{\mu\nu}-\frac{1}{3}\eta_{\mu\nu}T^\lambda_\lambda+\frac{a^2}{3}\eta_{\mu\nu}T_{zz}\right). \tag{6c}$$

Here T_{ab} is the energy-momentum tensor of additional matter, if present, and

$$\delta R_{zz}=-\left(\frac{h'}{2a^2}\right)'-2k\,\delta(z)h+\frac{2k}{a^2(r_c)}\,\delta(z-r_c)h, \tag{7a}$$

$$\delta R_{z\mu}=\left(\frac{1}{2a^2}(h^\nu_{\mu,\nu}-h_{,\mu})\right)',$$

$$\begin{aligned}\delta R_{\mu\nu}=&\frac{1}{2}h''_{\mu\nu}+2k^2h_{\mu\nu}-\left(k^2h+\frac{k}{2}h'\right)\eta_{\mu\nu}\\&+2k(\delta(z)-\delta(z-r_c))h_{\mu\nu}\\&+\frac{1}{2a^2}(2h^\lambda_{(\mu,\nu)\lambda}-h_{\mu\nu,\lambda}{}^{\lambda}-h_{,\mu\nu}),\end{aligned} \tag{7b}$$

where $h=h^\mu_\mu$. Equations (7b) are invariant under residual gauge transformations

$$h_{\mu\nu}\rightarrow h_{\mu\nu}+a^2(\epsilon_{\mu,\nu}+\epsilon_{\nu,\mu})+\frac{1}{k}\epsilon^z{}_{,\mu\nu}-2ka^2\,\eta_{\mu\nu}\epsilon^z, \tag{8}$$

where ϵ^z and ϵ^μ depend only on x. These transformations correspond to general coordinate transformations $z\rightarrow z+\xi^z$, $x^\mu\rightarrow x^\mu+\xi^\mu$; their consistency with the gauge conditions Eq. (4) requires $\xi^z=\epsilon^z(x)$, $\xi^\mu=(2k)^{-1}a^{-2}\epsilon^{z,\mu}(x)+\epsilon^\mu(x)$.

The Israel junction conditions on a brane are most easily formulated in the GN frame, in which the brane is located at fixed z. In the absence of matter on the brane, these junction conditions are $h'_{\mu\nu}+2kh_{\mu\nu}=0$. They are *not* invariant under the gauge transformations, Eq. (8), if $\epsilon^z\neq 0$. The importance of the gauge transformations Eq. (8) becomes clear from the observation that the coordinate system, which is GN with respect to one brane, need not be GN with respect to the other. Hence, one is led to consider two coordinate patches, the first (second) of which includes the positive (negative) tension brane. The coordinate systems in each of these patches are GN to the respective brane. A residual coordinate transformation is needed then to relate the metrics in the overlap of these patches.

In other words, to describe the propagating degrees of freedom, we introduce two sets of fields, $h^{(+)}_{\mu\nu}(x,z)$ and $h^{(-)}_{\mu\nu}(x,z)$. The first of them, $h^{(+)}_{\mu\nu}$, is defined in the interval in the fifth direction that includes $z_+=0$ but excludes $z_-=r_c$, and conversely for $h^{(-)}_{\mu\nu}$. Both $h^{(\pm)}_{\mu\nu}$ obey source-free equations (7). The boundary conditions are

$$h^{(\pm)\prime}_{\mu\nu}+2kh^{(\pm)}_{\mu\nu}=0 \quad \text{at} \quad z=z_\pm. \tag{9}$$

The relation between the two fields in the bulk is the gauge transformation of the form Eq. (8) with yet unknown gauge functions.

For $T_{ab}=0$, the linearized Einstein equations (7) with boundary conditions Eq. (9) are straightforward to solve by ''brute force.'' The outcome can be understood as follows. In each of the patches we write

$$h^{(\pm)}_{\mu\nu}=\tilde{h}^{(\pm)}_{\mu\nu}+\frac{1}{k}f^{(\pm)}_{,\mu\nu}-2ka^2\,\eta_{\mu\nu}f^{(\pm)}, \tag{10}$$

where $f^{(\pm)}(x)$ are yet to be determined and $\tilde{h}^{(\pm)}_{\mu\nu}$ is transverse-tracefree (TT) $\tilde{h}^{(\pm)\mu}_{\mu}=0$, $\tilde{h}^{(\pm)\nu}_{\mu,\nu}=0$. Then the field equations in the bulk become

$$\tilde{h}^{(\pm)\prime\prime}_{\mu\nu}-2k^2\tilde{h}^{(\pm)}_{\mu\nu}-\frac{1}{2a^2}\Box^{(4)}\tilde{h}^{(\pm)}_{\mu\nu}=0, \tag{11}$$

while the junction conditions on the respective branes read

$$\tilde{h}^{(\pm)\prime}_{\mu\nu}+2k\tilde{h}^{(\pm)}_{\mu\nu}=-2f^{(\pm)}_{,\mu\nu}. \tag{12}$$

The latter are consistent with TT property of $\tilde{h}_{\mu\nu}$ if $\Box^{(4)}f^{(\pm)}=0$. Hence, if the four-dimensional momenta are such that $p^2\neq 0$, one is left with Eq. (11) and homogeneous boundary conditions [i.e., Eq. (12) with $f^{(\pm)}=0$]. This is precisely the system of equations analyzed in [3], so we see that all massive propagating modes have been revealed by that analysis.

At $p^2=0$, however, there are two types of solutions. One of them is $f^{(\pm)}(x)=0$, $\tilde{h}_{\mu\nu}(x,z)=a^2\chi_{\mu\nu}(x)$, and does not require the gauge transformation in the overlap of the two patches. These solutions have been considered in [3], and describe massless four-dimensional gravitons. The other type of solution is

$$\tilde{h}^{(\pm)}_{\mu\nu}=-\frac{a_\pm^2}{2ka^2}f^{(\pm)}_{,\mu\nu}, \tag{13}$$

where $f^{(\pm)}(x)$ are yet arbitrary and $a_\pm=a(z_\pm)$. The relation between $f^{(+)}$ and $f^{(-)}$ is found using Eq. (8). Since $\tilde{h}_{\mu\nu}$ are proportional to a^{-2}, they should coincide in the two patches, so one requires $a_+f^{(+)}(x)=a_-f^{(-)}(x)\equiv f(x)$. One finally obtains

$$h^{(+)}_{\mu\nu}=-\frac{1}{2ka^2}f_{,\mu\nu}+\frac{1}{k}f_{,\mu\nu}-2ka^2\eta_{\mu\nu}f, \tag{14a}$$

$$h^{(-)}_{\mu\nu}=-\frac{1}{2ka^2}f_{,\mu\nu}+\frac{e^{2kr_c}}{k}f_{,\mu\nu}-2ka^2e^{2kr_c}\eta_{\mu\nu}f, \tag{14b}$$

where $f(x)$ is a massless four-dimensional scalar mode. The first term on the right hand side (RHS) is clearly identifiable as the growing part of the mode in Eq. (2), and hence corresponds to motion of the wall; the coincidence of this term in Eqs. (14a) and (14b) identifies this as a relative motion. One can quantify this by noting that the transition function between the two patches is

$$\epsilon^z(x)=(e^{2kr_c}-1)f(x), \quad \epsilon_\mu=0. \tag{15}$$

This transition function then determines the physical distance between the branes, $r(x)-r_c=\epsilon^z(x)$ [recall that Eq. (8) is the coordinate transformation between the coordinate systems in which the branes are located exactly at $z=0$ and $z=r_c$, respectively]. These properties show that $f(x)$ is indeed the (unnormalized) radion field in the linearized theory.

Equations (14a) and (14b) determine the induced metrics on each of the branes in the presence of the radion field. The first two terms on the RHS's of these equations can be gauged away *on the branes*. With the graviton field $\chi_{\mu\nu}$ included, the induced metrics on each brane are

$$\overline{h}^{(\pm)}_{\mu\nu}(x)=a_\pm^2\left(\eta_{\mu\nu}+\chi_{\mu\nu}(x)-\frac{2k}{a_\pm^2}f(x)\,\eta_{\mu\nu}\right). \tag{16}$$

If matter is present on the branes, it couples to the induced metrics through $L_{\rm int}\propto\overline{h}_{\mu\nu}T^{\mu\nu}$. Clearly, the radion field couples to the trace of the energy-momentum tensor. The corresponding effective coupling constants at each brane are proportional to $g_\pm\propto a_\pm^{-1}$. Indeed, the elementary vertex of a graviton to matter at each brane is proportional to $\sqrt{G_{N\pm}}$, where $G_{N\pm}\propto a_\pm^2$ are effective four-dimensional Newton's constants at each brane [1]; from Eq. (16) it follows that the radion vertex contains an extra factor $a_\pm^{-2}$. Hence, the radion field couples to matter on the negative tension brane exponentially stronger than to matter on the positive tension one, $g_-^2/g_+^2=e^{2kr_c}$. This relation is just the opposite to the case of graviton, and it is in accord with the results of Garriga and Tanaka [10]. The overall strength of these interactions can be also read off from [10]: the interaction Lagrangian of the normalized radion field $\hat{f}(x)$ with matter on each brane is $L^{(\pm)}_{\rm int}=g_\pm\hat{f}T^{(\pm)\mu}_\mu$ with

$$g_\pm^2=\frac{16\pi}{3}G_5k\,\frac{e^{\mp kr_c}}{\sinh kr_c}. \tag{17}$$

The fact that the radion couples to matter on the negative tension brane much stronger than the graviton does has been observed also in [8,9].

It is instructive to return to Eq. (3) with the benefit of our perturbative calculation to see what the linearized metric with the walls fixed at some coordinate values 0 and r_c should look like. To derive this form, we take the two GN patches of Eqs. (14a) and (14b), and perform a gauge transformation to make the two identical. We now have a single coordinate chart between the walls, but the walls are no longer at $z=0,r_c$. We then perform another coordinate transformation which is determined by the dual requirements that the walls sit at the (new) $\tilde{z}$ coordinates 0 and r_c, and that there are no cross terms $\tilde{g}_{\tilde{z}\mu}$ in the metric. The price of having the walls at a rigid value of $\tilde{z}$ is that the system is no longer GN—a nontrivial $\tilde{g}_{\tilde{z}\tilde{z}}$ is introduced. After performing these transformations we find that the new metric can be written in the form

$$d\tilde{s}^2=e^{-2k(\tilde{z}+f(\tilde{x})e^{2k\tilde{z}})}g_{\mu\nu}(\tilde{x})d\tilde{x}^\mu d\tilde{x}^\nu -(1+2kf(\tilde{x})e^{2k\tilde{z}})^2d\tilde{z}^2, \tag{18}$$

where we have included the possibility of graviton perturbations in $g_{\mu\nu}(\tilde{x})$. This form of the metric correctly describes the linearized dynamics of the massless metric excitations (and also reduces to the appropriate expression for $\tilde{x}$ independent, although not necessarily small, displacements of the wall).

Clearly, the properties of the radion are quite different from the graviton. To stress this point, let us introduce a model in which gravitons are not confined, but the radion is. This model may be of interest by itself, as in an appropriate limit gravity on a brane is expected to be almost, but not exactly, Einsteinian.

Let us consider five-dimensional space-time with infinite fifth dimension. Let there be two branes, one with positive tension σ and another with negative tension $-\sigma/2$ (note the factor 1/2). The latter brane is placed to the right of the former in the fifth direction. The bulk cosmological constant between the two branes and to the left of the positive-tension one is the same as in the Randall-Sundrum model, and is zero to the right of the negative-tension brane. Then there exists a solution to the Einstein equations for which both branes are at rest, the coordinates of the positive and negative tension branes being $z=0$ and $z=r_c$, respectively, where r_c is again an arbitrary constant. This solution has the form of Eq. (1) but now with

$$a^2(z)=\begin{cases} e^{-2k|z|} & \text{for } z<r_c, \\ e^{-2kr_c}=\text{const} & \text{for } z>r_c. \end{cases} \tag{19}$$

The four-dimensional hypersurfaces $z=\text{const}$ are flat; the five-dimensional space-time is flat to the right of the negative tension brane, and AdS in the rest of the bulk.

An interesting feature of this model is that gravitons are almost but not exactly confined: the wave functions of gravitons, $h_{\mu\nu}=a^2(z)\chi_{\mu\nu}(x)$, are peaked at $z=0$ but are not normalizable. At large r_c, gravity experienced by matter residing on the positive tension brane should be almost, but not exactly, Einsteinian (the limit $r_c\to\infty$ corresponds to the noncompact Gogberashvili-Randall-Sundrum model [2,3], with gravitons confined to the positive tension brane). The background Eq. (19) is of interest for exploring possible deviations from the Einstein gravity in the brane world and, in particular, for analyzing the issue of (non)conservation of energy measured by a four-dimensional observer.

We leave the discussion of gravitational perturbations in our model for the future, and here we consider a simpler mode, the radion. For the confined radion, the metric perturbation analogous to Eqs. (14a) and (14b) has to be a solution to linearized Einstein equations [still in the gauge Eq. (4)] which tends to pure the gauge as $z\to+\infty$ and $z\to-\infty$. We again have to consider two coordinate patches, overlapping in a region between the branes. In the overlap, the perturbations $h^{(+)}_{\mu\nu}$ and $h^{(-)}_{\mu\nu}$ are to be related by a gauge transformation Eq. (8).

Proceeding as above, we find in the left patch

$$h^{(+)}_{\mu\nu}=\frac{1}{k}(1-e^{2kz})f_{,\mu\nu}(x)-2ka^2f(x)\,\eta_{\mu\nu}, \tag{20}$$

where $f(x)$ is the massless radion field. The forms of metric perturbation in the right patch are different in AdS and flat parts

$$h^{(-)}_{\mu\nu}=\begin{cases} \dfrac{2}{k}e^{2kr_c}\sinh[2k(r_c-z)]f_{,\mu\nu} & \text{for } z<r_c, \\ -4(z-r_c)e^{2kr_c}f_{,\mu\nu} & \text{for } z>r_c. \end{cases} \tag{21}$$

It is straightforward to see that these perturbations indeed obey the Einstein equations everywhere in the bulk and the Israel junction conditions on the branes. The gauge transformation relating $h^{(+)}_{\mu\nu}$ and $h^{(-)}_{\mu\nu}$ in the bulk between the two branes is

$$h^{(+)}_{\mu\nu}-h^{(-)}_{\mu\nu}=\frac{1}{k}f_{,\mu\nu}-2ka^2f\eta_{\mu\nu}-\frac{1}{k}e^{4kr_c}a^2f_{,\mu\nu}, \tag{22}$$

i.e., in the notation of Eq. (8)

$$\epsilon^z=f; \quad \epsilon_\mu=\frac{f_{,\mu}}{2k}(1-e^{4kr_c}a^2). \tag{23}$$

It is easy to see that this corresponds to proper relative motion of the wall, since computing the extrinsic curvature of the first wall gives $\tilde{K}^{(+)}_{\mu\nu}=-f_{,\mu\nu}$. Meanwhile, at the second wall $\tilde{K}^{(-)}_{\mu\nu}=-2e^{2kr_c}f_{,\mu\nu}$. Alternatively, the perturbation Eq. (21) is pure gauge for $z>r_c$, and changing coordinates to the right of the second wall so that the metric there is Minkowskian, we find that the wall is located at $\hat{z}^{(-)}=r_c-2e^{2kr_c}f$. Similarly, for $z<0$, the perturbation Eq. (20) is pure gauge, and changing coordinates for the first wall gives $\hat{z}^{(+)}=-f$, therefore we see how f does indeed encode a relative motion of the walls. Note how the radion field is nontrivial only inbetween the two branes and on the positive tension brane itself. In other words, there is not even short-ranged radion hair outside the two-brane system. It is likely that the absence of the radion hair outside a stack of branes is a general property of models with infinite extra dimensions.

Finally, we note that in our model the radion does not induce metric perturbations on the negative tension brane, $h^{(-)}_{\mu\nu}(r_c)=0$. Hence, the radion does not interact with matter residing on the negative tension brane, in sharp contrast to the Randall–Sundrum model discussed above. This seems to be a peculiarity of our model, which is related to the flatness of the five-dimensional space-time for $z>r_c$, since a perturbed wall in flat space-time written in GN coordinates can be shown to have four-dimensional metric $g_{\mu\nu}=\eta_{\mu\nu}+2zf_{,\mu\nu}+O(f^2)$, and so any perturbation always vanishes to leading order on the wall.

We would like to thank Jaume Garriga, Takahiro Tanaka, David Langlois, and Clifford Johnson for useful discussions. This work was done while participating in the workshop on *'Structure Formation in the Universe'* at the Isaac Newton Institute. C.C. was supported by PPARC, R.G. by the Royal Society, and V.R. by the Russian Foundation for Basic Research, Grant No. 990218410.

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).
[2] M. Gogberashvili, hep-ph/9908347.
[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).
[4] A. G. Cohen and D. B. Kaplan, Phys. Lett. B **470**, 52 (1999).
[5] R. Gregory, Phys. Rev. Lett. **84**, 2564 (2000).
[6] M. Visser, Phys. Lett. **159B**, 22 (1985).
[7] T. Vachaspati and A. Vilenkin, Phys. Rev. Lett. **67**, 1057 (1991).
[8] C. Csaki, M. Graesser, L. Randall, and J. Terning, Phys. Rev. D **62**, 045015 (2000).
[9] W. D. Goldberger and M. B. Wise, Phys. Lett. B **475**, 275 (2000).
[10] J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000).