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Remarks on topological brane theories

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We consider the theory of closed p-branes propagating on $(p+1)$-dimensional space-time manifolds. This theory has no local degrees of freedom. Here we study the canonical and BRST structures of the theory. In the case of locally flat backgrounds one can show that the p-brane theory is related to another known topological field theory. In the general situation some equivalent actions can also be written for the topological p-brane theory.

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I. INTRODUCTION

In the present paper we consider a p-brane propagating on a $(p+1)$-dimensional background manifold. In this model all degrees of freedom can be gauged away locally. However, there may still be nontrivial, nonlocal (topological) degrees of freedom. The motivation for the study of this model is twofold. First this theory is interesting by itself as a topological field theory [1], and we shall see that there is something one can learn about its canonical and Becchi-Rouet-Stora-Tyutin (BRST) structures. Second the present theory may serve as a toy model for the study of general extended objects which play an important role in modern string theory. Thus the present model can give some insight into the more general case of p-branes propagating on background manifolds of dimension higher than $(p+1)$.

The p-brane theory describes the embedding of a $(p+1)$-dimensional world volume into a d-dimensional space-time manifold $\mathcal{M}$. The action is given by the volume of the embedded $(p+1)$-dimensional manifold

$$S[X]=-T\int d^{p+1}\xi\sqrt{-\det(G_{\mu\nu}(X)\partial_\alpha X^\mu\partial_\beta X^\nu)}, \quad (1)$$

where $G_{\mu\nu}$ is the metric in the d-dimensional space-time manifold. Throughout the paper we shall look at the case of the Minkowskian signature, and we shall see that the results can be generalized to the Euclidean case in a straightforward way. It is assumed that the metric $G_{\mu\nu}$ is not degenerate at any point of the manifold $\mathcal{M}$, $\det(G_{\mu\nu})\neq 0$. The action (1) is called the Nambu-Goto action. The parameter T is called tension and it is a direct generalization of the concept of mass to p-branes. The p-brane action can be equivalently written in the Polyakov form

$$S[X,h]=-\frac{T}{2}\int d^{p+1}\xi\sqrt{-h}\times[h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X)-(p-1)], \quad (2)$$

where the $h_{\alpha\beta}$ transform as a world-volume metric $[h\equiv\det(h_{\alpha\beta})]$ and play the role of Lagrange multipliers. The equivalence can be checked by varying this action with respect to $h_{\alpha\beta}$, solving the resulting equations of motions and putting the resulting solution for $h_{\alpha\beta}$ into Eq. (2). The result is the action (1). One should notice that within the equivalence between actions (1) and (2) the induced world-volume metric must be nondegenerate.

Now let us discuss the symmetries of the theory. There is a local diffeomorphism invariance for the action (1)

$$\delta X^\mu=\pounds_\zeta X^\mu, \quad (3)$$

where $\pounds_\zeta$ stands for the Lie derivative along ζ^α. For the Polyakov action (2) the transformation (3) should be supplemented by the appropriate transformation for the auxiliary world-volume metric $\delta h^{\alpha\beta}=\pounds_\zeta h^{\alpha\beta}$. For the case $p=1$ there is an extra local symmetry $\delta h^{\alpha\beta}=\Lambda h^{\alpha\beta}$ (Weyl rescaling). In addition both actions (1) and (2) are invariant under arbitrary diffeomorphisms on $\mathcal{M}$, if $G_{\mu\nu}$ is transformed properly.

The local symmetry (3) allows one to choose locally the following gauge:

$$X^\mu=\xi^\mu, \quad \mu=0,1,\ldots,p, \quad (4)$$

which is usually called the static gauge. The existence of static gauge can be argued from the picture of embedding of a manifold into another one. Thus in the case of interest $d=p+1$ one can locally gauge away all degrees of freedom. However on a nontrivial background manifold $\mathcal{M}$ one can-

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not do this globally and therefore there are nontrivial global (topological) degrees of freedom. One can also see that degenerate situations (when $\det[G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu]=0$) do not appear because of Eq. (4). In the static gauge the determinant of the induced metric is equal to the determinant of the background metric which is assumed to be nondegenerate. Therefore as soon as we want to keep the local diffeomorphism symmetry (3) (i.e., the picture of embedding of one manifold into another) it is assumed everywhere that

$$\det[G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu]\neq 0. \tag{5}$$

Throughout the paper we use the following notation: μ, ν denote space-time indices α, β world-volume indices and a, b, c spatial world-volume indices.

Let us assume that the space-time manifold $\mathcal{M}$ is compact and oriented. When we come to the Hamiltonian treatment we also assume that $\mathcal{M}=R\times\Sigma$ where Σ is a compact and oriented spatial manifold. p-branes can be closed or open. For closed p-branes periodicity conditions must be imposed along all spatial directions. The analysis of open p-branes is more involved since the theory should be supplemented by appropriate boundary conditions. In this paper we look only at closed p-branes. In the case of closed branes the Nambu-Goto action is a constant for all field configurations and it is equal to the volume of the background manifold $\mathcal{M}$. We hope to come back to the case of open p-branes elsewhere.

In this paper we study mainly the classical aspects of the theory. The paper is organized as follows. In Sec. II we go through the Hamiltonian treatment of the closed brane theory. Three equivalent sets of constraints are presented. In Sec. III we take a look at the construction of BRST generators for these three sets of constraints. Three different BRST generators are related to each other through canonical transformations in the extended phase space. In Sec. IV we look at equivalent forms of the action and specifically discuss the case of locally flat backgrounds. The degrees of freedom are briefly considered and the subtleties related to the degenerate solutions are pointed out. In the last section we discuss the results and outline possible generalizations of the model.

II. HAMILTONIAN TREATMENT

In this section we take a look at the Hamiltonian treatment of the system. The p-brane theory is a generally covariant system and therefore the naive Hamiltonian vanishes identically. In this theory the full Hamiltonian is given by a linear combination of the corresponding constraints which are first class. Our goal is to write down three different sets of constraints for the model.

In order to carry out the Hamiltonian formulation of the theory we choose one of the integration variables ξ^α as the evolution parameter (in the case of a relativistic metric with signature $(-1,1,\dots,1)$, the one-parameter group of diffeomorphisms defined by the translations in that variable should be generated by a timelike vector field), which we take to be ξ^0; the remaining integration variables, which parametrize the brane itself, are represented by ξ^a, with the small Latin letters taking values from 1 to p. The system is totally constrained since the theory is invariant under redefinitions of the evolution parameter.

Denoting by P_μ the momenta conjugate to the X^μ and starting from either the Polyakov action (2) or the Nambu-Goto action (1) the constraints can be worked out as [2]

$$\mathcal{H}_I=\begin{pmatrix}\mathcal{H}\\ \mathcal{H}_a\end{pmatrix}=\begin{pmatrix}G^{\mu\nu}(X)P_\mu P_\nu+T^2\det[q_{ab}]\\ P_\mu\partial_a X^\mu\end{pmatrix}, \tag{6}$$

where

$$q_{ab}=G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu \tag{7}$$

is the induced spatial metric on the brane. The constraints (6) are first class and obey the algebra

$$\{\mathcal{H}_a[M^a],\mathcal{H}_b[N^b]\}=\mathcal{H}_a[\pounds_M N^a], \tag{8}$$

$$\{\mathcal{H}_a[M^a],\mathcal{H}[N]\}=\mathcal{H}[\pounds_M N], \tag{9}$$

$$\{\mathcal{H}[M],\mathcal{H}[N]\}=\mathcal{H}_a[qq^{ab}(M\partial_b N-N\partial_b M)], \tag{10}$$

where $\pounds_N$ stands for the Lie derivative along the vector field N^a, and $q=\det(q_{ab})$. Since there are d pairs of canonical conjugate variables and $p+1$ constraints, the theory possesses $(d-p-1)$ degrees of freedom per brane point. Therefore in the case of a $(d-1)$-brane one has got no dynamical degrees of freedom and the theory is purely topological. The algebra (8)–(10) is called the algebra of many-fingered time (the name is due to Wheeler). The constraints (6) and their algebra (8)–(10) are true for a p-brane in any space-time dimension d. The algebra (8)–(10) is closed only for the case $p<2$. Now let us analyze the specific properties for a p-brane propagating on a $(p+1)$-dimensional space-time.

Starting from the Nambu-Goto action (1) one can see that the constraints can be written in a form in which all of them are linear in the momenta. In order to do so, we observe that for p-branes the dimension of the world-volume in Eq. (1) is the same as the dimension of the embedding space-time, namely, $p+1$. Consequently $\partial_\alpha X^\mu$ is a square matrix, and one can write

$$\det(G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu)=G(X)\det{}^2(\partial_\alpha X^\mu), \tag{11}$$

with

$$G(X)=\det[G_{\mu\nu}(X)]. \tag{12}$$

The action (1) becomes then

$$S=\pm T\int d^{p+1}\xi\sqrt{-\mathcal{G}}\,\frac{1}{(p+1)!}\times\epsilon^{\alpha_0\cdots\alpha_p}\epsilon_{\mu_0\cdots\mu_p}\partial_{\alpha_0}X^{\mu_0}\cdots\partial_{\alpha_p}X^{\mu_p}, \tag{13}$$

where $\pm$ corresponds to the two possible solutions of the square root. Let us keep both signs in all calculations and eventually one can see that the sign ambiguity corresponds to the two possible orientations on the manifold. The equations of motion for Eq. (13) are somewhat trivial. They tell us that the exterior derivative of the volume form is zero. The $(p+1)$ decomposition of Eq. (1) is straightforward:

$$S=\pm T\int d\xi^0 d^p\xi n_\mu \dot{X}^\mu, \tag{14}$$

where an overdot represents a derivative with respect to the evolution parameter ξ^0, and the vector n_μ is a function of the configuration variables X^μ given by

$$n_\mu=\frac{1}{p!}\sqrt{-G}\epsilon_{\mu\nu_1\cdots\nu_p}\epsilon^{a_1\cdots a_p}\partial_{a_1}X^{\nu_1}\cdots\partial_{a_p}X^{\nu_p}. \tag{15}$$

The vector n_μ satisfies the following properties:

$$G^{\mu\nu}n_\mu n_\nu=-\det[q_{ab}], \tag{16}$$

$$n_\mu\partial_a X^\mu=0. \tag{17}$$

In this form it is clear that the momenta conjugate to the X^μ are

$$P_\mu=\pm T n_\mu. \tag{18}$$

The Hamiltonian vanishes and one must include the primary constraints given by Eq. (18) with the aid of some Lagrange multiplier functions λ^μ:

$$L^\pm=\int d^p\xi[P_\mu\dot{X}^\mu-\lambda^\mu\mathcal{C}^\pm_\mu], \tag{19}$$

$$\mathcal{C}^\pm_\mu=P_\mu\mp T n_\mu(X). \tag{20}$$

The obtained constraints are linear in the momenta and its Poisson bracket algebra vanishes strongly. The two sets of constraints $\mathcal{C}^+_\mu$ and $\mathcal{C}^-_\mu$ correspond to two different branches of the constraint surface. These two sets intersect only on the degenerate solutions

$$\mathcal{C}^+_\mu=\mathcal{C}^-_\mu=0\Rightarrow n_\mu=0\Rightarrow\det(q_{ab})=0. \tag{21}$$

Thus exluding degenerate solutions one has two independent branches of the theory with different Lagragians $L^\mp$. The constraints $\mathcal{C}^\pm_\mu$ generate the following transformation:

$$\delta X^\mu=\{X^\mu,\mathcal{C}^\pm_\mu[N^I]\}=N^\mu. \tag{22}$$

In fact one can see that this is the real symmetry of the action (13). The natural question might arise about the relation between Eq. (22) and the local diffeomorphism invariance (3). There is a one to one map between the two transformations:

$$\delta X^\mu=\pounds_\zeta X^\mu=(\partial_\alpha X^\mu)\zeta^\alpha=N^\mu, \tag{23}$$

when the quadratic matrix $(\partial_\alpha X^\mu)$ is assumed to be nondegenerate. Thus for every vector ζ there is unique vector N and vice versa. However it should be stressed that in general the gauge symmetries (3) and (22) have different properties. By using transformation (22) one can bring a nondegenerate solution to a degenerate one. One cannot do this by using the transformation (3).

To get another set of constraints one can contract $\mathcal{C}^\pm_\mu$ with the set of independent vectors $G^{\mu\nu}n_\nu$ and $\partial_a X^\mu$, resulting, respectively, in

$$\mathcal{H}^\pm_I=\begin{pmatrix}\mathcal{H}^\pm\\ \mathcal{H}_a\end{pmatrix}=\begin{pmatrix}G^{\mu\nu}n_\mu P_\nu\pm T\det[q_{ab}]\\ P_\mu\partial_a X^\mu\end{pmatrix}. \tag{24}$$

Comparing with Eqs. (6) we see that only the scalar constraint is modified, now being linear in the momenta, not quadratic. The constraints (24) obey the same Poisson bracket algebra as Eqs. (8)–(10). Again the two sets of constraints $\mathcal{H}^+_I$ and $\mathcal{H}^-_I$ describe the two separate branches if the degenerate solutions are excluded, and they obey two many-fingered algebras on the corresponding independent branches of the theory. The constraints (24) basically tell us that the system can be thought of as a parametrized field theory (parametrized cosmological constant term) [3]. The equation of motion for X^μ is given by

$$\dot{X}^\mu=\{X^\mu,\mathcal{H}_a[N^a]+\mathcal{H}^\pm[M]\}=N^a\partial_a X^\mu+MG^{\mu\nu}n_\nu, \tag{25}$$

which is nothing else but the geometrodynamical canonical decomposition [3] with respect to basic vectors $(G^{\mu\nu}n_\nu, \partial_a X^\mu)$.

Now one can check explicitly the equivalence of these three sets of constraints $\mathcal{C}^\pm_\mu$, $\mathcal{H}^\pm_I$, and $\mathcal{H}_I$. It is clear that $\mathcal{C}^\pm_\mu$ implies both $\mathcal{H}^\pm_I$ and $\mathcal{H}_I$. To check the converse we note that the second equations $\mathcal{H}_a=0$ in the sets of constraints (6) and (24), have the general solution

$$P_\mu=\alpha n_\mu, \tag{26}$$

where α is any function which does not carry indices. Plugging this result into the last equation of $\mathcal{H}$ one gets

$$(T^2-\alpha^2)\det(q_{ab})=0\Rightarrow T=\alpha \ \text{ or } \ T=-\alpha \ \text{ or } \ \det[q_{ab}]=0. \tag{27}$$

The first solution is just $\mathcal{C}^+_\mu$, the second is $\mathcal{C}^-_\mu$ and the last one corresponds to a degenerate solution. Applying the same procedure to $\mathcal{H}^\pm$ one finds that

$$\mathcal{C}^+_\mu=0\Leftrightarrow\mathcal{H}^+_I=0,\quad \mathcal{C}^-_\mu=0\Leftrightarrow\mathcal{H}^-_I=0, \tag{28}$$

if $\det(q_{ab})\neq 0$ is assumed.

Thus we have shown that all three sets of constraints are equivalent

$$(\mathcal{C}^+_\mu=0 \text{ and } \mathcal{C}^-_\mu=0)\Leftrightarrow\mathcal{H}_I=0\Leftrightarrow(\mathcal{H}^+_I=0 \text{ and } \mathcal{H}^-_I=0), \tag{29}$$

if the case of degenerate metric is excluded and therefore these three sets of constraints describe the same constrained surface. Since the manifold Σ is assumed oriented the two branches of the theory are dynamically independent.

III. BRST GENERATORS

In this section we construct the BRST generators which correspond to the different sets of constraints discussed in the previous section. We have a first class constrained system and its Hamiltonian is a linear combination of the constraints Ψ_I. Introducing the ghost variables η^I and the ghost momenta $\mathcal{P}_I$ one can define the classical Grassmann-odd BRST generator (charge) Q in the extended phase space [4]

$$Q=\int d^p\xi\,\eta^I(\xi)\Psi_I(\xi)+\sum_{n=0}^{r}\int d^p\xi_1\cdots\int d^p\xi_n Q^{I_1\cdots I_n}(\xi_1,\ldots,\xi_n)\times\mathcal{P}_{I_1}(\xi_1)\cdots\mathcal{P}_{I_n}(\xi_n), \tag{30}$$

such that Q is nilpotent and real. The ghost number of Ω should be equal to 1. The BRST construction is important because it reveals that the different representations of the constraint surface can be thought of as being obtained from each other by a canonical transformation in the extended phase space. In expression (30) the number r is called the rank of the BRST generator. The concept of rank is not intrinsic and can be made equal to zero by appropriate redefinitions of constraints [4].

For the general case of p-branes with constraints $\mathcal{H}_I$ given by Eq. (6) the classical BRST generator Q has been constructed by Henneaux [5]. The rank of Q is equal p. Now we can construct the BRST generator for the other two sets of constraints $\mathcal{C}^{\pm}_{\mu}$ and $\mathcal{H}^{\pm}_I$. Since the Poisson brackets algebra for $\mathcal{C}^{\pm}_{\mu}$ vanishes strongly (thus the algebra is commutative) the BRST operator $Q^{\pm}$ has rank 0:

$$Q^{\pm}=\int d^p\xi\,\eta^{\mu}(\xi)\mathcal{C}^{\pm}_{\mu}(\xi), \tag{31}$$

where Q^+ and Q^- are defined for the two different sectors. In this case the BRST transformation ($\delta_{\pm}A=\{A,Q_{\pm}\}$) in the extended phase space has a simple form

$$\delta_{\pm}X^{\mu}=\eta^{\mu},\delta_{\pm}\eta^{\mu}=0,\delta_{\pm}\mathcal{P}_{\mu}=-\Phi_{\mu},$$

$$\delta_{\pm}P_{\mu}=\mp T\frac{1}{(p-1)!}\sqrt{-G}\epsilon_{\mu\nu_1\nu_2\cdots\nu_p}\times\epsilon^{a_1\cdots a_p}\partial_{a_1}\eta^{\nu_1}\partial_{a_2}X^{\nu_2}\cdots\partial_{a_p}X^{\nu_p}. \tag{32}$$

The BRST generator $\mathcal{Q}^{\pm}$ for the constraints $\mathcal{H}^{\pm}_I$ can also be worked out, being given by

$$\mathcal{Q}^{\pm}=\int d^p\xi[\eta H^{\pm}+\eta^a H_a+(\eta^a\partial_a\eta+\eta\partial_a\eta^a)\mathcal{P}+(q\circ q^{ab}\eta\partial_a\eta+\eta^a\partial_a\eta^b)\mathcal{P}_b], \tag{33}$$

where $(\eta,\mathcal{P})$ and $(\eta^a,\mathcal{P}_b)$ are the ghost pairs associated with $\mathcal{H}^{\pm}$ and $\mathcal{H}_a$, respectively. Its rank is 1. Thus we have constructed three different BRST generators for the same theory. They should relate to each other by canonical transformations in the extended phase space. We were unable to construct these canonical transformations in any simple closed form. However, one can certainly construct them perturbatively in same fashion as in Refs. [4] and [6].

As we saw the three sets of constraints $\mathcal{H}_I$, $\mathcal{H}^{\pm}_I$, and $\mathcal{C}^{\pm}_{\mu}$ describe the same constraint surface if we exclude degenerate solutions. There should be the following relation among the sets of constraints which describe the same constrain surface:

$$\mathcal{H}_I=(S^{\pm})^{\mu}_I\mathcal{C}^{\pm}_{\mu},\quad \mathcal{H}^{\pm}_I=(S)^{\mu}_I\mathcal{C}^{\pm}_{\mu}, \tag{34}$$

where $(S^{\pm})^{\mu}_I$ and $(S)^{\mu}_I$ must be nondegenerate. It is not difficult to construct these matrices explicitly. Thus for $S^{\pm}$ we have the following expression:

$$S^{\pm}=\begin{pmatrix}\partial_1X^0 & \ldots & \partial_1X^p\\ \partial_2X^0 & \ldots & \partial_2X^p\\ \ldots & \ldots & \ldots\\ G^{0\nu}(P_{\nu}\pm Tn_{\nu}) & \ldots & G^{p\nu}(P_{\nu}\pm Tn_{\nu})\end{pmatrix}, \tag{35}$$

and, for S,

$$S=\begin{pmatrix}\partial_1X^0 & \partial_1X^1 & \ldots & \partial_1X^p\\ \partial_2X^0 & \partial_2X^1 & \ldots & \partial_2X^p\\ \ldots & \ldots & \ldots & \ldots\\ G^{0\nu}n_{\nu} & G^{1\nu}n_{\nu} & \ldots & G^{p\nu}n_{\nu}\end{pmatrix}. \tag{36}$$

Let us calculate the determinants of these matrices

$$\det(S^{\pm})=(-1)^p n_{\mu}G^{\mu\nu}(P_{\nu}\pm Tn_{\nu})=(-1)^p\tilde{\mathcal{H}}^{\pm}, \tag{37}$$

$$\det(S)=(-1)^{p+1}\det[q_{ab}]. \tag{38}$$

We see that S is non degenerate if degenerate solutions ($\det[q_{ab}]=0$) are excluded. The matrix S^+ is not degenerate either as long as we stay at the branch defined by $\mathcal{C}^+_{\mu}=0$ (or, equivalently, by $\mathcal{H}^+_I$) and S^- is not degenerate at the branch defined by $\mathcal{C}^-_{\mu}$ (or, equivalently, by $\mathcal{H}^-_I$). Therefore using these matrices one can construct perturbatively the relevant canonical transformations in the extended phase space.

IV. *p*-BRANE THEORY IN LOCALLY FLAT BACKGROUND

To understand the theory better we would like to study alternative representations of this model. In many cases alternative representations of a theory may help to analyze their degrees of freedom. In this section we study some classically equivalent actions and analyze the degrees of freedom corresponding to the topological p-brane theory. It is hard to say anything explicit about the degrees of freedom when the theory is formulated in the form (1) or (2). Intuitively we understand that the number of degrees of freedom is related to the number of patches needed to cover the manifold $\mathcal{M}$. However, it is hard to count them explicitly. Therefore we can try to reformulate the theory in a more transparent way. One can reach this goal by using new variables. Since the task is difficult for generic curved background manifolds, we look first at the case of locally flat manifolds $\mathcal{M}$:

$$G_{\mu\nu}=\eta_{\mu\nu}. \tag{39}$$

At the end of this section we will take a brief look on equivalent actions for the generic case. However, it is still problematic to analyze the degrees of freedom in all generality.

Now we are assuming that Eq. (39) holds. Let us enlarge the gauge symmetry of the system defining the tetrad fields

$$e_a{}^{\mu}=\partial_a X^{\mu} \tag{40}$$

as the new configuration variables. They are subject to the constraints

$$\partial_{[a}e_{b]}{}^{\mu}=0. \tag{41}$$

One can easily see that there is a one to one correspondence between new and old variables in the locally flat space-time. The static gauge (4) in new variables corresponds to $e_a{}^{\mu}=\delta_a{}^{\mu}$.

The action in these new variables and their canonical conjugate momenta $\pi^a{}_{\mu}$ can be obtained from the generating functional depending on the old coordinates and the new momenta

$$S_{X\pi}=-\int d^p\xi\,\partial_a X^{\mu}\pi^a{}_{\mu}. \tag{42}$$

One has

$$e_a{}^{\mu}=-\frac{\delta S_{X\pi}}{\delta\pi^a{}_{\mu}}=\partial_a X^{\mu}, \tag{43}$$

$$P_{\mu}=-\frac{\delta S_{X\pi}}{\delta X^{\mu}}=-\partial_a\pi^a{}_{\mu}. \tag{44}$$

Plugging this result into Eq. (20) one gets

$$S=\int d^{p+1}\xi\,\pi^a{}_{\mu}\dot{e}_a{}^{\mu}+\phi^{ab}{}_{\mu}\partial_{[a}e_{b]}{}^{\mu}+\lambda^{\mu}\bigg\{\partial_a\pi^a{}_{\mu} \pm T\frac{1}{p!}\epsilon^{a_1\cdots a_d}\epsilon_{\mu\nu_1\cdots\nu_p}e_{a_1}{}^{\nu_1}\cdots e_{a_p}{}^{\nu_p}\bigg\}, \tag{45}$$

where $\phi^{ab}{}_{\mu}$ are the Lagrange multiplier functions for the constraints (41). In the case of a nonflat metric the Lagrangian (45) would be nonlocal in the new variables since it involves the original coordinates X^{μ} present in the determinant of the metric $G_{\mu\nu}$. But this problem does not arise in the case of a flat metric. We have then the following action:

$$S=\int d^{p+1}\xi\bigg[\pi^a{}_{\mu}\partial_0 e_a{}^{\mu}+\phi^{ab}{}_{\mu}\partial_{[a}e_{b]}{}^{\mu}+\lambda^{\mu}\partial_a\pi^a{}_{\mu} \pm T\lambda^{\mu}\frac{1}{p!}\epsilon^{a_1\cdots a_p}\epsilon_{\mu\nu_1\cdots\nu_p}e_{a_1}{}^{\nu_1}\cdots e_{a_p}{}^{\nu_p}\bigg], \tag{46}$$

which can be given in a covariant form if one identifies the Lagrange multipliers λ^{μ} with the time components of the tetrad fields

$$\lambda^{\mu}=e_0{}^{\mu}, \tag{47}$$

and writes the momenta $\pi^a{}_{\mu}$ and the Lagrange multipliers $\phi^{ab}{}_{\mu}$ as the components of a $(p-1)$-form F_{μ},

$$\pi^a{}_{\mu}=\epsilon^{ab_1\cdots b_{p-1}}F_{b_1\cdots b_{p-1}\mu}, \tag{48}$$

$$\phi^{ab}{}_{\mu}=(p-1)\epsilon^{abc_1\cdots c_{p-2}}F_{0c_1\cdots c_{p-2}\mu}. \tag{49}$$

Equation (46) then becomes

$$S=\int d^{p+1}\xi\bigg[\epsilon^{\alpha_0\cdots\alpha_p}\partial_{\alpha_0}e_{\alpha_1}{}^{\mu}F_{\alpha_2\cdots\alpha_p\mu} \pm T\frac{1}{(p+1)!}\epsilon^{\alpha_0\cdots\alpha_p}\epsilon_{\nu_0\cdots\nu_p}e_{\alpha_0}{}^{\mu_0}\cdots e_{\alpha_p}{}^{\mu_p}\bigg], \tag{50}$$

which can be compactly written in the differential form language as

$$S=\int F_{\mu}\wedge de^{\mu}\pm T\frac{1}{(p+1)!}\epsilon_{\nu_0\cdots\nu_p}e^{\nu_0}\wedge\cdots\wedge e^{\nu_p}, \tag{51}$$

where e^{ν} is a one-form and F_{μ} is a $(p-1)$-form. The action (51) is explicitly topological since it does not involve the metric. After all one can see just at level of actions that the actions (13), (51) are equivalent to each other. This equivalence can be established by integrating out the field F_{μ}.

Now let us take a look at the symmetries and equations of motions of the action (51). The action has the following obvious symmetry

$$\delta F_{\mu}=dw_{\mu}, \tag{52}$$

which is the shift of F_{μ} by any exact $(p-1)$-form. There is one extra symmetry which is less obvious

$$\delta e^{\mu}=df^{\mu}, \tag{53}$$

$$\delta F_{\mu}=\pm\frac{T}{(p-1)!}\epsilon_{\mu\nu_1\nu_2\cdots\nu_p}f^{\nu_1}e^{\nu_2}\wedge\cdots\wedge e^{\nu_p}, \tag{54}$$

where f^{μ} is an arbitrary zero-form (function). The equations of motion are the following:

$$de^{\mu}=0, \tag{55}$$

$$dF_{\mu}=\mp\frac{T}{p!}\epsilon_{\mu\nu_1\ldots\nu_p}e^{\nu_1}\wedge\cdots\wedge e^{\nu_p}. \tag{56}$$

The classical moduli space is given by gauge non-equivalent solutions of equations (55) and (56). Thus we were able to reformulate the topological p-brane theory in a locally flat background as an Abelian BF-like model [7] with the action given by Eq. (51). The model has a bunch of U(1) fields e^{μ} and the nontriviality comes from the last ''mass'' term which mixes different gauge fields. For the case $p\geqslant 2$ the action (51) can be thought as the zero gravitational constant limit for the general relativity with cosmological constant in

$(p+1)$-dimensional space-time. This limit should be taken in the first order formalism [8].

The degrees of freedom (the classical moduli space) for the action (51) can be analyzed in a straightforward fashion through the cohomology groups. In general the situation depends on the details of the topology of the background manifold or more precisely, on the structure of the first cohomology group $H^1(\mathcal{M},R)$. Since the one-forms e^μ are closed and any two solutions that differ by an exact one-form are gauge equivalent, e^μ is a element of $H^1(\mathcal{M},R)$. The equations for F_μ are more difficult to analyze since the right hand side involves e^μ. If $\dim[H^1(\mathcal{M},R)]<p$ then the last equation of motion in Eq. (55) reduces to $dF_\mu=0$. We have not enough elements of the first cohomology group to construct a nonzero right-hand side. Thus in this case the model coincides with $(p+1)$ copies of an Abelian BF system [7]. Therefore the space of solutions for e^μ and F_μ is given by $p+1$ copies of $H^1(\mathcal{M},R)\oplus H^{p-1}(\mathcal{M},R)$. In the case $\mathcal{M}=R\times\Sigma$ we have

$$H^1(\mathcal{M},R)\approx H^1(\Sigma,R)\approx H^{p-1}(\Sigma,R)\approx H^{p-1}(\mathcal{M},R), \tag{57}$$

where we used Poincaré duality on the p-dimensional manifold Σ . Thus the space of gauge inequivalent solutions is even dimensional and it is given by the product of $2(p+1)$ copies of the first cohomology group: $H^1(\Sigma,R)$. The situation with $\dim[H^1(\mathcal{M},R)]\geqslant p$ is more involved. One should analyze what kind of right hand side in the last equation (55) can be constructed from e^μ. For instance, in the case $\mathcal{M}=R\times\Sigma$ it might be possible to construct out of e^μ the volume form for Σ: $e^1\wedge\cdots\wedge e^p$. Since the volume form cannot be exact the corresponding equation has no solution. We will not analyze this situation in all generality. However, the task might be solved straightforwardly as soon as we know explicitly the content of $H^1(\mathcal{M},R)$. Above analysis of degrees of freedom is appropriate for the actions (13) and (51) where the degenerate solutions are included. However, to incorporate into the analysis the restriction of nondegeneracy can be hard since the removal of degennerate solutions from the phase space might destroy the gauge orbits. The similar problem appears in the relation between $2+1$ gravity and Chern-Simons theory [9].

On a curved space-time manifold there is no such simple BF-like action as in locally flat case. However, one can write the following action:

$$S=\int (dX^\mu-\eta^\mu)\wedge B_\mu \pm T\frac{1}{(p+1)!}\sqrt{-G}\,\epsilon_{\mu_0\cdots\mu_p}\eta^{\mu_0}\wedge\cdots\wedge\eta^{\mu_p}, \tag{58}$$

which is classically equivalent to the Nambu-Goto action (13). In the action (58) η^μ and B_μ are one-forms and p-forms, respectively. The action is nonlinear in X^μ and therefore it is difficult to analyze it in the same fashion as before. The case $p=1$ is definitly special. By itself the Nambu-Goto action (13) can be interpreted as topological sigma model [10] in two dimensions since $\sqrt{-G}\epsilon_{\mu\nu}$ might serve as closed symplectic form on $\mathcal{M}$. Also the Nambu-Goto action is equivalent to the following action:

$$S=\int dX^\mu\wedge\eta_\mu\pm\frac{T}{2}(-G)^{-1/2}\epsilon^{\mu\nu}\eta_\mu\wedge\eta_\nu \tag{59}$$

which is the Poisson sigma model on two-dimensional $\mathcal{M}$ [11]. Therefore we see that two-dimensional topological string theory is classically equivalent to other known theories up to some subtleties related to degenerate configurations.

V. DISCUSSION AND OUTLINE

In the present work we considered the classical aspects of closed p-brane theory defined on $(p+1)$-dimensional background manifolds $\mathcal{M}$. We analyzed the Hamiltonian and BRST structure of the theory. We saw that model has different equivalent realizations. However, the classical equivalence between the constraints and the actions might fail at the quantum level due to normal ordering problem (different regularizations). One can look at the most familiar example $p=1$. For the case of quadratic constraints there is an anomaly in the Virasoro algebra and therefore the system is not first class anymore. In the case of linear constraints (20) there is no anomaly possible since the constraints are completely linear. This discussion gives us an example that at the quantum level the Nambu-Goto action (natural source for linear constraints) and the Polyakov action (the natural source for quadratic constraints) are not equivalent to each other. As well at the classical level different status of degenerate solutions can bring extra problems into identification of two theories.

The actions (13) and (58) have a straightforward generalization to the following topological models:

$$S=T\frac{1}{(p+1)!}\int d^{p+1}\xi C_{\mu_0\cdots\mu_p}(X)\times\epsilon^{\alpha_0\cdots\alpha_p}\partial_{\alpha_0}X^{\mu_0}\ldots\partial_{\alpha_p}X^{\mu_p} \tag{60}$$

and

$$S=\int (dX^\mu-\eta^\mu)\wedge B_\mu+T\frac{1}{(p+1)!}\times C_{\mu_0\cdots\mu_p}(X)\,\eta^{\mu_0}\wedge\cdots\wedge\eta^{\mu_p}, \tag{61}$$

where C is a $(p+1)$-form defined on the d-dimensional background manifold $\mathcal{M}$ [d might be any value equal or greater than $(p+1)$]. If the form C is closed the model has many similarities with the topological p-brane studied in the present work. We shall consider the classical and quantum aspects of these theories in coming work.

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