Magnetic fields, branes, and noncommutative geometry

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We construct a simple physical model of a particle moving on the infinite noncommutative 2-plane. The model consists of a pair of opposite charges moving in a strong magnetic field. In addition, the charges are connected by a spring. In the limit of large magnetic field, the charges are frozen into the lowest Landau levels. Interactions of such particles include Moyal-bracket phases characteristic of field theories on noncommutative space. The simple system arises in the light cone quantization of open strings attached to D-branes in antisymmetric tensor backgrounds. We use the model to work out the general form of light cone vertices from string splitting. We then consider the form of Feynman diagrams in (uncompactified) noncommutative Yang-Mills theories. We find that for all planar diagrams the commutative and noncommutative theories are exactly the same apart from trivial external line factors. This means that the large N theories are equivalent in the 't Hooft limit. Non-planar diagrams are generally more convergent than their commutative counterparts.

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I. MODEL

Gauge theories on noncommutative spaces [1,2] are believed to be relevant to the quantization of D-branes in background $B_{\mu\nu}$ fields [3]. The structure of such theories is similar to that of ordinary gauge theory except that the usual product of fields is replaced by a ''star product'' defined by

$$\phi * \chi = \phi(X)\exp\left\{-i\theta^{\mu\nu}\frac{\partial}{\partial X^{\mu}}\frac{\partial}{\partial Y^{\nu}}\right\}\chi(Y) \tag{1}$$

where $\theta^{\mu\nu}$ is an antisymmetric constant tensor. The effect of such a modification is reflected in the momentum space vertices of the theory by factors of the form

$$\exp[i\theta^{\mu\nu}p_{\mu}q_{\nu}] \equiv e^{ip\wedge q}. \tag{2}$$

The purpose of this paper is to show how these factors arise in an elementary way. We will begin by describing a simple quantum mechanical system which is fundamental to our construction. We then consider string theory in the presence of a D3-brane and a constant large $B_{\mu\nu}$ field. In the light cone frame the first quantized string is described by our elementary model. We use the model to compute the string splitting vertex and show how the factors in Eq. (2) emerge. We then turn to the structure of the perturbation series for the non-commutative theory in infinite flat space. We find that planar diagrams with any number of loops are identical to their commutative counterparts apart from trivial external line phase factors.

Compactification, which can lead to entirely new features, is not studied in this paper.

A. Model at classical level

Consider a pair of unit charges of opposite sign in a magnetic field B in the regime where the Coulomb and the radiation terms are negligible. The coordinates of the charges are $\vec{x}_1$ and $\vec{x}_2$ or in component form x_1^i and x_2^i. The Lagrangian is

$$\mathcal{L} = \frac{m}{2}[(\dot{x}_1)^2 + (\dot{x}_2)^2] + \frac{B}{2}\epsilon_{ij}(\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j) - \frac{K}{2}(x_1 - x_2)^2 \tag{3}$$

where the first term is the kinetic energy of the charges, the second term is their interaction with the magnetic field and the last term is a harmonic potential between the charges.

In what follows we will be interested in the limit in which the first term can be ignored. This is typically the case if B is so large that the available energy is insufficient to excite higher Landau levels [4]. Thus we will focus on the simplified Lagrangian

$$\mathcal{L} = \frac{B}{2}\epsilon_{ij}(\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j) - \frac{K}{2}(x_1 - x_2)^2. \tag{4}$$

Let us first discuss the classical system. The canonical momenta are given by

$$p_i^1 = \frac{\partial\mathcal{L}}{\partial\dot{x}_1^i} = B\epsilon_{ij}x_1^j$$

$$p_i^2 = -B\epsilon_{ij}x_2^j. \tag{5}$$

Let us define center of mass and relative coordinates X, Δ:

$$\vec{X} = (\vec{x}_1 + \vec{x}_2)/2$$

$$\vec{\Delta} = (\vec{x}_1 - \vec{x}_2)/2. \tag{6}$$

The Lagrangian is

$$\mathcal{L} = m[(\dot{X})^2 + (\dot{\Delta})^2] + 2B\epsilon_{ij}\dot{X}^i\Delta^j - 2K(\Delta)^2. \tag{7}$$

Dropping the kinetic terms gives

$$\mathcal{L}=2B\epsilon_{ij}\dot{X}^i\Delta^j-2K(\Delta)^2. \tag{8}$$

The momentum conjugate to X is

$$\frac{\partial\mathcal{L}}{\partial\dot{X}^i}=2B\epsilon_{ij}\Delta^j=P_i\,. \tag{9}$$

This is the center of mass momentum.

Finally, the Hamiltonian is

$$\mathcal{H}=2K(\Delta)^2=2K\left(\frac{P}{2B}\right)^2=\frac{K}{2B^2}P^2. \tag{10}$$

This is the Hamiltonian of a nonrelativistic particle with mass

$$M=\frac{B^2}{K}. \tag{11}$$

Evidently the composite system of opposite charges moves like a Galileian particle of mass M. What is unusual is that the spatial extension Δ of the system is related to its momentum so that the size grows linearly with P according to Eq. (9). How does this growth with momentum effect the interactions of the composite? Let us suppose charge 1 interacts locally with an ''impurity'' centered at the origin. The interaction has the form

$$V(\vec{x}_1)=\lambda\,\delta(\vec{x}_1). \tag{12}$$

In terms of X and Δ this becomes

$$V=\lambda\,\delta(X+\Delta)=\lambda\,\delta\left(X^i-\frac{1}{2B}\epsilon^{ij}P_j\right). \tag{13}$$

Note that the interaction in terms of the center of mass coordinate is nonlocal in a particular way. The interaction point is shifted by a momentum dependent amount. This is the origin of the peculiar momentum dependent phases that appear in interaction vertices on the noncommutative plane. More generally, if particle 1 sees a potential $V(x_1)$, the interaction becomes

$$V\left(X-\frac{\epsilon P}{2B}\right). \tag{14}$$

B. Quantum level

The main problem in quantizing the system is to correctly define expressions like Eq. (14) which in general have factor ordering and other quantum ambiguities. In order to define them, let us assume that V can be expressed as a Fourier transform

$$V(x)=\int dq\tilde{V}(q)e^{iqx}. \tag{15}$$

We can then formally write

$$V\left(X-\frac{\epsilon P}{2B}\right)=\int dq\tilde{V}(q)e^{iq(X-\epsilon P/2B)}. \tag{16}$$

The factor ordering is not ambiguous because

$$[q_iX^i,q_l\epsilon^{lj}P_j]=q_iq_l\epsilon^{lj}[X^iP_j]=0. \tag{17}$$

Consider the matrix element

$$\left\langle k\left|\exp\left[iq\left(X-\frac{\epsilon p}{2B}\right)\right]\right|l\right\rangle \tag{18}$$

where $\langle k|$ and $|l\rangle$ are momentum eigenvectors. Using Eq. (17) we can write this as

$$a\left\langle k\left|\exp[iqX]\exp\left[-i\frac{q\epsilon P}{2B}\right]\right|l\right\rangle. \tag{19}$$

Since $|l\rangle$ is an eigenvector of P, this becomes

$$\langle k|\exp[iqX]|l\rangle\exp\left[-i\frac{q\epsilon l}{2B}\right]=\delta(k-q-l)\exp[-iq\epsilon l/2B]. \tag{20}$$

The phase factor is the usual Moyal bracket phase that is ubiquitous in noncommutative geometry.

II. STRING THEORY IN MAGNETIC FIELDS

Let us consider bosonic string theory in the presence of a D3-brane. The coordinates of the brane are x^0, x^1, x^2, x^3. The remaining coordinates will play no role. We will also assume a background antisymmetric tensor field $B_{\mu\nu}$ in the 1,2 direction. We will study the open string sector with string ends attached to the D3-brane in the light cone frame.

Define

$$x^{\pm}=x^0\pm x^3 \tag{21}$$

and make the usual light cone choice of world sheet time

$$\tau=x^+. \tag{22}$$

The string action is

$$\mathcal{L}=\frac{1}{2}\int_{-L}^{L}d\tau d\sigma\left[\left(\frac{\partial x^i}{\partial\tau}\right)^2-\left(\frac{\partial x^i}{\partial\sigma}\right)^2+B_{ij}\left(\frac{\partial x^i}{\partial\tau}\right)\left(\frac{\partial x^j}{\partial\sigma}\right)\right]. \tag{23}$$

We have numerically fixed α' and the parameter L can be identified with P_-, the momentum conjugate to x_-.

In what follows we will be interested in the limit $B\to\infty$. Let us make the following rescalings:

$$x^i=\frac{y^i}{\sqrt{B}}$$

$$\tau=tB. \tag{24}$$

Then

$$\mathcal{L}=\frac{1}{2}\int_{-L}^{L} d\tau d\sigma\left[\frac{1}{B^2}\left(\frac{\partial y}{\partial t}\right)^2-\left(\frac{\partial y}{\partial\sigma}\right)^2+\epsilon_{ij}\left(\frac{\partial y^i}{\partial t}\right)\left(\frac{\partial y^j}{\partial\sigma}\right)\right]. \tag{25}$$

Now for $B\to\infty$ we can drop the first term. Furthermore, by an integration by parts and up to a total time derivative the last term can be written

$$\epsilon_{ij}\frac{\partial y_i}{\partial t}y_j\Big|_{-L}^{L}. \tag{26}$$

Thus

$$\mathcal{L}=\frac{1}{2}\int d\sigma d\tau\left(\frac{\partial y}{\partial\sigma}\right)^2+\epsilon_{ij}\dot{y}_i y_j\Big|_{-L}^{L}. \tag{27}$$

Since for $\sigma\neq\pm L$ the time derivatives of y do not appear in S, we may trivially integrate them out. The solution of the classical equation of motion is

$$y(\sigma)=y+\frac{\Delta\sigma}{L} \tag{28}$$

with Δ and y independent of σ. The resulting action is

$$\mathcal{L}=\left[-\frac{2\Delta^2}{L}+\dot{y}\epsilon\Delta\right]. \tag{29}$$

Evidently, the action is of the same form as the model in Sec. I with B and K rescaled.

III. INTERACTION VERTEX

Interactions in light cone string theory are represented by string splitting and joining. Consider two incoming strings with momenta p_1, p_2 and center of mass positions y_1, y_2. If their endpoints coincide, they can join to form a third string with momentum $-p_3$. The constraints on the endpoints are summarized by the overlap δ function:

$$\begin{aligned}\nu=&\delta((y_1-\Delta_1)-(y_2+\Delta_2))\delta((y_2-\Delta_2)\\&-(y_3+\Delta_3))\delta((y_3-\Delta_3)-(y_1+\Delta_1)).\end{aligned} \tag{30}$$

From Eq. (29) we see that the center of mass momentum is related to Δ by

$$P=\epsilon\Delta. \tag{31}$$

Inserting this in Eq. (30) gives the vertex

$$\begin{aligned}\nu=&\delta(y_1-y_2+(\epsilon p_1+\epsilon p_2))\\&\times\delta(y_2-y_3+(\epsilon p_2+\epsilon p_3))\\&\times\delta(y_3-y_1+(\epsilon p_3+\epsilon p_1)).\end{aligned} \tag{32}$$

To get the vertex in momentum space multiply by $e^{i(p_1y_1+p_2y_2+p_3y_3)}$ and integrate over y. This yields

$$\nu=e^{i(p_1\epsilon p_2)}\delta(p_1+p_2+p_3). \tag{33}$$

This is the usual form of the vertex in noncommutative field theory. We have scaled the "transverse" coordinates x^1, x^2 (but not x^0, x^3) and momenta so that the B field does not appear in the vertex. If we go back to the original units, the phases in Eq. (33) will be proportional to $1/B$.

Evidently a quantum of noncommutative Yang-Mills theory may be thought of as a straight string connecting two opposite charges. The separation vector Δ is perpendicular to the direction of motion P.

Now consider the geometry of the 3-body vertex. The string endpoints u, v, w define a triangle with sides

$$\begin{aligned}\Delta_1&=(u-v)\\\Delta_2&=(v-w)\\\Delta_3&=(w-u)\end{aligned} \tag{34}$$

and the three momenta are perpendicular to the corresponding Δ. It is straightforward to see that the phase

$$\epsilon_{ij}p_iq_j/B\equiv p\wedge q \tag{35}$$

is just the area of the triangle times B. In other words, it is the magnetic flux through the triangle. Note that it can be of either sign.

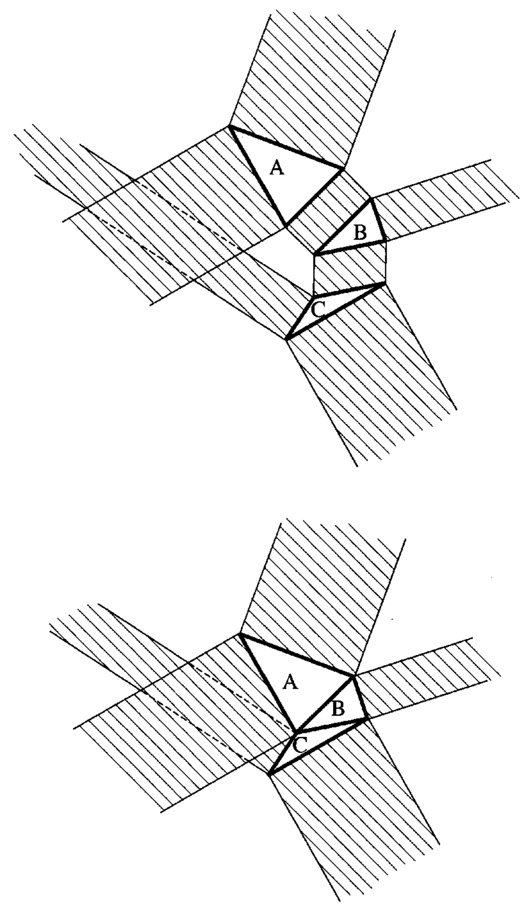

FIG. 1. "Feynman tree diagram" for the scattering of strings.

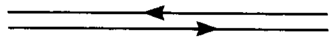

FIG. 2. Double line representation of the propagator.

More generally, we may consider a Feynman tree diagram constructed from such vertices. For example consider Fig. 1(a). The overall phase is the total flux through the triangles A, B and C. In fact we can simplify this by shrinking the internal propagators to get Fig. 1(b). Thus the phase is the flux through a polygon formed from the Δ's of the external lines. The phase depends only on the momenta of the external lines and their cyclic order.

IV. STRUCTURE OF PERTURBATION THEORY

In this section we will consider the effects of the Moyal phases on the structure of Feynman amplitudes in noncommutative Yang-Mills theory. Let us first review the diagram rules for ordinary Yang-Mills theory in 't Hooft double-line representation.

The gauge propagator can be represented as a double line as if the gauge boson were a quark-antiquark pair as in Fig. 2. Each gluon is equipped with a pair of gauge indices i,j, a momentum p and a polarization ε satisfying $\varepsilon\cdot p=\varepsilon^{\mu}p_{\mu}=0$.

The vertex describing the 3-gauge boson interaction is shown in Fig. 3. In addition to the Kronecker δ for the gauge indices and momentum δ functions the vertex contains the factor

$$(\varepsilon_1\cdot p_3+\varepsilon_3\cdot p_2+\varepsilon_2\cdot p_1). \tag{36}$$

The factor is antisymmetric under interchange of any pair and so it must be accompanied by an antisymmetric function of the gauge indices. For a purely Abelian theory the vertex vanishes when symmetrized.

Now we add the new factor coming from the Moyal bracket. This factor is

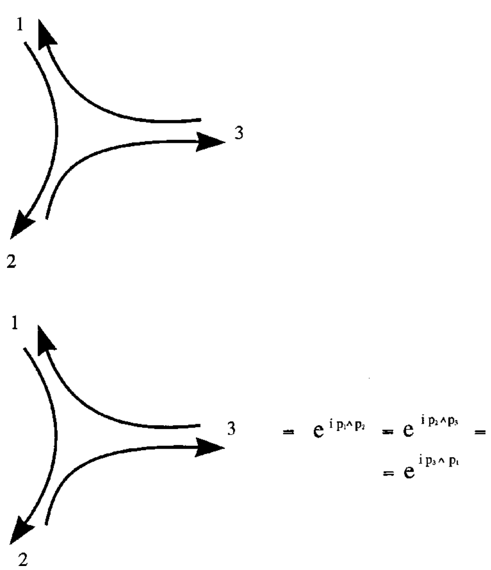

FIG. 3. The three-boson interaction and its associated phase.

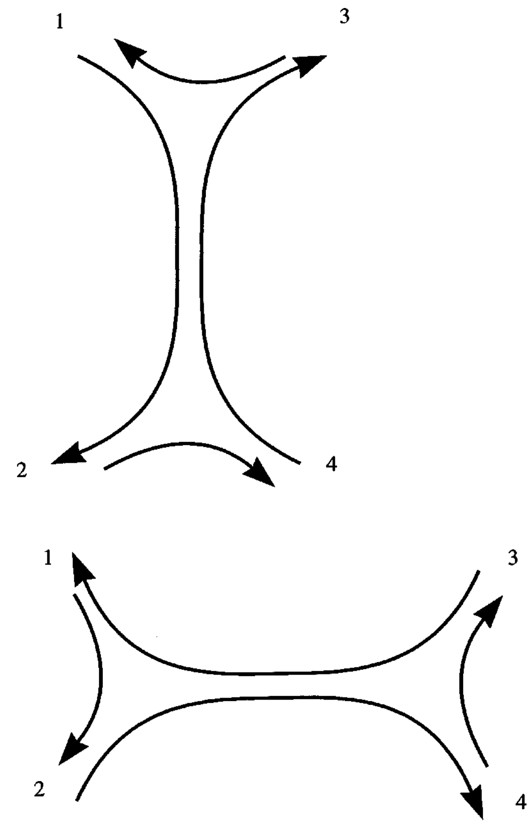

FIG. 4. The elementary ''duality exchange'' move.

$$e^{ip_1\wedge p_2}=e^{ip_2\wedge p_3}=e^{ip_3\wedge p_1} \tag{37}$$

where $p_a\wedge p_b$ indicates an antisymmetric product:

$$p\wedge q=p_{\mu}q_{\nu}\theta^{\mu\nu}$$

$$\theta^{\mu\nu}=-\theta^{\nu\mu}. \tag{38}$$

Because these factors are not symmetric under interchange of particles, the vertex no longer vanishes when Bose symmetrized even for the Abelian theory.

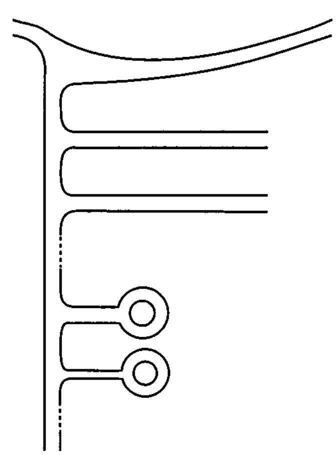

FIG. 5. Basic form of any planar diagram after an appropriate sequence of duality moves.

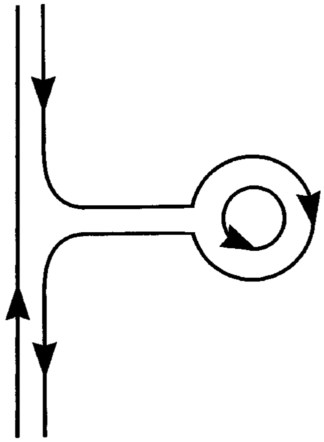

FIG. 6. The tadpole diagram.

The phase factors satisfy an important identity. Let us consider the phase factors that accompany a given diagram. In fact from now on a diagram will indicate *only the phase factor* from the product of vertices. Now consider the diagram in Fig. 4(a). It is given by

$$e^{i(p_1\wedge p_2)}e^{i(p_1+p_2)\wedge p_3}=e^{i(p_1\wedge p_2+p_2\wedge p_3+p_1\wedge p_3)}. \quad (39)$$

On the other hand the dual diagram, Fig. 4(b), is given by $e^{i[p_1\wedge(p_2+p_3)+p_2\wedge p_3]}$. It is identical to the previous diagram. Thus the Moyal phases satisfy old fashioned ''channel duality.'' This conclusion is also obvious from the ''flux through polygon'' construction of the previous section.

In what follows, a ''duality move'' will refer to a replacement of a diagram such as in Fig. 4(a) by the dual diagram in Fig. 4(b).

Now consider any planar diagram with L loops. By a series of ''duality moves'' it can be brought to the form indicated in Fig. 5 consisting of a tree with L simple one-loop tadpoles.

Let us consider the tadpole, Fig. 6. The phase factor is just $e^{iq\wedge q}=1$. Thus the loop contributes nothing to the phase and the net effect of the Moyal factors is exactly that of the tree diagram. In fact all tree diagrams contributing to a given topology have the same phase, which is a function only of the external momentum. The result is that for planar diagrams the Moyal phases do not affect the Feynman integrations at all. In particular the planar diagrams have exactly the same divergences as in the commutative theory. Evidently in the large N limit noncommutative Yang-Mills theory $=$ordinary Yang-Mills theory.

On the other hand, divergences that occur in nonplanar diagrams can be regulated by the phase factors. For example consider the nonplanar diagram in Fig. 7. The Moyal phase for the diagram is

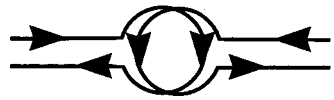

FIG. 7. Non-planar insertion for the self-energy.

$$e^{ip\wedge q}e^{ip\wedge q}=e^{2ip\wedge q} \quad (40)$$

and does not cancel. It is not difficult to see that such oscillating phases will regulate divergent diagrams and make them finite, unless the diagram contains divergent planar subdiagrams. Thus it seems that the leading high momentum behavior of the theory is controlled by the planar diagrams. Among other things it means that in this region the $1/N$ corrections to the 't Hooft limit must vanish.

An interesting question arises if we study the theory on a torus of finite size [5]. For an ordinary local field theory high momentum behavior basically corresponds to small distance behavior. For this reason we expect the high momentum behavior on a torus to be identical to that in infinite space once the momentum becomes much larger than the inverse size of the torus. However, in the noncommutative case the story is more interesting. We have seen that high momentum in the 1,2 plane is associated with *large* distances in the perpendicular direction. Most likely this means that the finite torus generically behaves very differently at high momentum than the infinite plane.

Indeed there is an exception to the rule that nonplanar diagrams are finite. If a line with a nonplanar self-energy insertion such as in Fig. 7 happens to have vanishing momentum in the 1,2 plane, then according to Eq. (40) its phase will vanish. Thus, for a set of measure zero, the nonplanar self-energy diagram can diverge. This presumably leads to no divergences in infinite space when the line in question is integrated over. The situation could be different for compact noncommutative geometries since integrals over momenta are replaced by sums [6].

The fact that the large N limit is essentially the same for noncommutative and ordinary Yang-Mills theories implies that in the AdS conformal field theory (CFT) correspondence the introduction of noncommutative geometry does not change the thermodynamics of the theory [7]. It may also be connected to the fact that in the matrix theory construction of Connes, Douglas and Schwartz [1] and Douglas and Hull [2], the large N limit effectively decompactifies X^{11} and should therefore eliminate the dependence on the 3-form potential. However, the argument is not straightforward since in matrix theory we are not usually in the 't Hooft limit.

ACKNOWLEDGMENTS

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