Functional integral formulation of the Thirring model with two fermion species

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A two-dimensional Abelian model with quartic interaction among two species of Fermi fields is analyzed through a functional integral formulation. We consider the bosonization of the generating functional of the model in order to establish the fermion-boson mapping in the Hilbert space of states. The grand partition function of the statistical mechanical system associated with the effective bosonized theory is obtained and the corresponding exact equation of state exhibits a critical line characterizing a Kosterlitz-Thouless phase transition.

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I. INTRODUCTION

In the last few years, an impressive amount of effort has been made by many physicists to understand the underlying properties of quantum field theories in two dimensions [1], as well as to try to picture these models as theoretical laboratories to obtain insight into more realistic four-dimensional field theories and, more recently, to apply them to low-dimensional condensed matter systems [2], as well as to *N*-body problems in nuclear physics [3].

The structure of a relativistic quantum-mechanical two-body system in one space dimension has been discussed in Ref. [4], through a solvable two-body Dirac equation with an interaction in the form of a delta function. The Dirac equation is solved analytically for bound and scattering states. Exact solutions for a relativistic three-body bound state in one space dimension are found in Ref. [5]. In this generalization of the two-body model introduced in Ref. [4], the three Dirac particles interact with each other through pairwise delta function potentials of vector type. In a further publication [6], the exact solution for *N*-body bound state of Dirac particles interacting through a delta function potential in one space dimension is discussed. The boundary condition for the solution leads to a constraint for the coupling constant and the particle number.

The field theory generalization of the quantum-mechanical model introduced in Ref. [4] was discussed in a recent paper by Sakamoto and Heike [3]. The field theoretical model corresponds to a massive Thirring-like model [7] in which the quartic Fermi field interaction is a current-current interaction of two different Fermi field species. Using the functional integral bosonization the effective bosonized theory is obtained and corresponds to a model with two coupled sine-Gordon fields.

The functional integral bosonization scheme used in Ref. [3] requires the introduction of auxiliary vector fields in order to recast the theory in terms of a Lagrangian quadratic in the Fermi fields. As stressed in Refs. [8–11], the bosonization procedure introduces a redundant Bose field algebra which contains more degrees of freedom than those needed for the description of the physical content of the model. The main purposes of the present article are the following.

(i) A discussion of structural aspects related with the appearance of decoupled massless Bose fields in the functional integral bosonization scheme. To this end, and in order to obtain the fermion-boson mapping in the Hilbert space of states, we shall perform the bosonization of the generating functional of the theory without disregarding the role played by the ''decoupled'' massless Bose fields. In order to obtain some insight into the bosonization of the non-Abelian model, we review the presentation of Ref. [3] by using the Abelian reduction of the Wess-Zumino-Witten theory [12,13,11]. The redundant decoupled massless Bose fields are kept through the bosonization of the generating functional of the theory. We show that their only effect is to generate constant contributions to the Wightman functions in the Hilbert space of states. In the present approach close attention is paid to maintaining a complete control on the Hilbert space structure needed for the representation of the intrinsic field algebra generated by the set of fundamental fields whose Wightman functions define the model. We show that the factorization of the partition function of the effective bosonized theory can lead to incorrect conclusions concerning the physical content of the model, such as the existence of infinitely delocalized states and the violation of the asymptotic factorization property. The present approach clarifies some delicate aspects not evident in the presentation of Ref. [3] and also streamlines the discussion of the functional integral bosonization of two-dimensional models presented in Refs. [14,15]. This is done in Sec. II.

(ii) To provide a statistical-mechanical description of the effective bosonized theory corresponding to the two-body field theory model discussed in Ref. [3]. This is performed in Sec. III. We obtain the grand partition function of the statistical-mechanical system associated with the effective bosonized theory. The exact equation of state is obtained and exhibits a critical line characterizing a Kosterlitz-Thouless phase transition.

II. FUNCTIONAL INTEGRAL BOSONIZATION

The two-dimensional Thirring model with current-current interaction of two Fermi fields species is defined by the following Lagrangian density [3]:

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$$\mathcal{L}=\overline{\psi}_a(i\partial\!\!\!/-m_0)\psi_a+\overline{\psi}_b(i\partial\!\!\!/-m_0)\psi_b-\frac{g^2}{2}J_{\mu a}J^\mu_b, \quad (2.1)$$

where the vector current is[1] $J^\mu_k=\overline{\psi}_k\gamma^\mu\psi_k$, and $k=a,b$ denote the two fermion species.

Within the operator formulation, the Hilbert space $\mathcal{H}$ of the model is constructed as a representation of the intrinsic field algebra $\Im$, generated by the set of fundamental local field operators $\{\overline{\psi}_a,\psi_a,\overline{\psi}_b,\psi_b\}$, and whose Wightman functions define the theory: $\mathcal{H}\doteq\Im|0\rangle$. In the functional integral formalism, the Hilbert space of the model can be built from the generating functional with source terms for the basic fields that generate the polynomial field algebra $\Im$,

$$\mathcal{Z}[\theta_a,\overline{\theta}_a,\theta_b,\overline{\theta}_b]=\mathcal{N}^{-1}\int \mathcal{D}\overline{\psi}_a\mathcal{D}\psi_a\mathcal{D}\overline{\psi}_b\mathcal{D}\psi_b \times e^{iW[\overline{\psi}_a,\psi_a,\overline{\psi}_b,\psi_b,\theta_a,\overline{\theta}_a,\theta_b,\overline{\theta}_b]}, \quad (2.2)$$

where $W[\overline{\psi}_a,\psi_a,\overline{\psi}_b,\psi_b,\theta_a,\overline{\theta}_a,\theta_b,\overline{\theta}_b]$ is the action in the presence of external Grassmann-valued sources θ_k and $\overline{\theta}_k$:

$$W[\overline{\psi}_a,\psi_a,\overline{\psi}_b,\psi_b,\theta_a,\overline{\theta}_a,\theta_b,\overline{\theta}_b] =\int d^2x\{\mathcal{L}+\overline{\psi}_a\theta_a+\overline{\theta}_a\psi_a+\overline{\psi}_b\theta_b+\overline{\theta}_b\psi_b\}. \quad (2.3)$$

Following the procedure adopted in Refs. [3,15], we next modify the field algebra slightly and define an enlarged field algebra $\Im'$ by introducing the ''auxiliary'' vector fields a_μ and b_μ, in such a way that the new field algebra is given by $\Im'\equiv\Im'\{a_\mu,b_\mu,\overline{\psi}_a,\psi_a,\overline{\psi}_b,\psi_b\}$. The Hilbert space of this enlarged theory can be built from the ''interpolating'' generating functional

$$\begin{aligned}\mathcal{Z}'[\theta_a,\overline{\theta}_a,\theta_b,\overline{\theta}_b,\zeta^\mu_a,\zeta^\mu_b] &=\mathcal{N}^{-1}\int \mathcal{D}\overline{\psi}_a\mathcal{D}\psi_a\mathcal{D}\overline{\psi}_b\mathcal{D}\psi_b\int \mathcal{D}a_\mu\mathcal{D}b_\mu\\ &\times\exp\left(i\int d^2x\left\{\frac{1}{2}a^\mu b_\mu+a_\mu\zeta^\mu_a+b_\mu\zeta^\mu_b\right\}\right.\\ &\left.+iW[\overline{\psi}_a,\psi_a,\overline{\psi}_b,\psi_b,\theta_a,\overline{\theta}_a,\theta_b,\overline{\theta}_b]\right). \quad (2.4)\end{aligned}$$

The source terms for the auxiliary vector fields a_μ and b_μ are included in order to control the effects of the bosonization procedure on the construction of the Hilbert space $\mathcal{H}'\doteq\Im'|0\rangle$. As we shall see, the bosonized generating functional $\mathcal{Z}'$ defines an enlarged positive semidefinite Hilbert space $\mathcal{H}'$.

The next step in the functional integral bosonization is to reduce the action of the Thirring model with two fermion species to quadratic actions in the Fermi fields ψ_k. This can be done by performing the ''change of variables'' [3,14,15]

$$a^\mu=\mathcal{A}^\mu-gJ^\mu_b, \quad b^\mu=\mathcal{B}^\mu-gJ^\mu_a, \quad (2.5)$$

such that

$$\begin{aligned}&\int \mathcal{D}a_\mu\mathcal{D}b_\mu\exp\left(i\int d^2x\frac{1}{2}\{a^\mu b_\mu-g^2J^\mu_aJ^b_\mu\}\right)\\ &=\int \mathcal{D}\mathcal{A}_\mu\mathcal{D}\mathcal{B}_\mu\exp\left(i\int d^2x\frac{1}{2}\{\mathcal{A}_\mu\mathcal{B}^\mu-gJ^\mu_a\mathcal{A}_\mu\right.\\ &\left.-gJ^\mu_b\mathcal{B}_\mu\}\right). \quad (2.6)\end{aligned}$$

In this way, the generating functional (2.4) can be written in terms of the effective Lagrangian density

$$\begin{aligned}\mathcal{L}'_{eff}&=\overline{\psi}_a\mathbb{D}(\mathcal{A})\psi_a+\overline{\psi}_b\mathbb{D}(\mathcal{B})\psi_b\\ &-m_0\overline{\psi}_a\psi_a-m_0\overline{\psi}_b\psi_b+\frac{1}{2}\mathcal{A}_\mu\mathcal{B}^\mu, \quad (2.7)\end{aligned}$$

where the covariant derivatives are defined by $\mathbb{D}(\mathcal{A})\doteq i\partial\!\!\!/-(g/2)\mathcal{A}\!\!\!/$. The local gauge noninvariance of the model is now explicited by the last term in Eq. (2.7). In order to decouple the Fermi and vector fields, we introduce the parametrization of the vector fields $(\mathcal{A}_\pm,\mathcal{B}_\pm)$, in terms of the U(1)-group-valued Bose fields (U_k,V_k) as follows [11]:

$$\mathcal{G}^k_+=\frac{2}{g}U_k^{-1}i\partial_+U_k, \quad \mathcal{G}^k_-=\frac{2}{g}V_ki\partial_-V_k^{-1}, \quad (2.8)$$

where $\mathcal{G}^k_\pm\equiv(\mathcal{A}_\pm,\mathcal{B}_\pm)$ for $k=a,b$. Performing the fermion chiral rotation [11]

$$\psi_k\equiv\begin{pmatrix}\psi_{(1)k}\\ \psi_{(2)k}\end{pmatrix}=\begin{pmatrix}V_k\chi_{(1)k}\\ U_k^{-1}\chi_{(2)k}\end{pmatrix}=\Omega_k\chi_k, \quad (2.9)$$

introducing in the functional integral the identities

$$1=\int \mathcal{D}U_k[\det\mathcal{D}_+(U_k)]\delta\left(\frac{g}{2}\mathcal{G}^k_+-U_k^{-1}i\partial_+U_k\right), \quad (2.10)$$

$$1=\int \mathcal{D}V_k[\det\mathcal{D}_-(V_k)]\delta\left(\frac{g}{2}\mathcal{G}^k_--V_ki\partial_-V_k^{-1}\right), \quad (2.11)$$

and taking into account the corresponding change in the fermionic integration measure [11], we get

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[1] Our conventions are

$$g^{00}=1=-g^{11}, \quad \epsilon^{01}=-\epsilon^{10}=1,$$

$$\gamma^0=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \quad \gamma^1\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \quad \gamma^5=\gamma^0\gamma^1,$$

$$\gamma^\mu\gamma^5=\epsilon^{\mu\nu}\gamma_\nu, \quad x^\pm=x^0\pm x^1, \quad \partial_\pm=\partial_0\pm\partial_1,$$

$$A^\pm=A^0\pm A^1.$$

$$\prod_{k=a}^{b} \mathcal{D}\bar{\psi}_k \mathcal{D}\psi_k \mathcal{D}\mathcal{A}_\pm \mathcal{D}\mathcal{B}_\pm = \prod_{k=a}^{b} \mathcal{D}\bar{\chi}_k \mathcal{D}\chi_k \mathcal{D}U_k \mathcal{D}V_k \mathcal{J}[U,V], \tag{2.12}$$

with

$$\begin{aligned}\mathcal{J}[U,V] &= \exp\left\{-i\sum_k (\Gamma[U_k]+\Gamma[V_k]) + ic\int d^2z(\mathcal{A}_\mu\mathcal{A}^\mu + \mathcal{B}_\mu\mathcal{B}^\mu)\right\}\\ &= \exp\left\{-i\sum_k \left[\Gamma[U_k]+\Gamma[V_k] + \frac{4c}{g^2}\int d^2z(U_k^{-1}\partial_+U_k)(V_k\partial_-V_k^{-1})\right]\right\},\end{aligned} \tag{2.13}$$

where $\Gamma[G]$ is the Wess-Zumino-Witten (WZW) functional [12], which enters in Eq. (2.13) with a negative level. In the Abelian case the WZW functional reduces to the free action

$$\Gamma[G] = \frac{1}{8\pi}\int d^2z\,\partial_\mu G^{-1}\partial^\mu G. \tag{2.14}$$

The last term in Eq. (2.13) has been added, exploiting the regularization freedom in the computation of the Jacobians, due to the local gauge noninvariance of the model. Using the Abelian version of the Polyakov-Wiegmann identity [13]

$$\Gamma[UV] = \Gamma[U]+\Gamma[V]+\frac{1}{4\pi}\int d^2z(U^{-1}\partial_+U)(V\partial_-V^{-1}), \tag{2.15}$$

and defining the ''regularization'' parameter $\boldsymbol{a}$ by

$$\frac{\boldsymbol{a}}{2\pi} \doteq \frac{1}{4\pi} - \frac{4c}{g^2}, \tag{2.16}$$

we can write Eq. (2.13) as

$$\mathcal{J} = \exp\left[\sum_k \left\{-i\Gamma[\Sigma_k] + i\frac{\boldsymbol{a}}{2\pi}\int d^2z \times (U_k^{-1}\partial_+U_k)(V_k\partial_-V_k^{-1})\right\}\right], \tag{2.17}$$

with $\Sigma_k \doteq U_k V_k$. The so-called ''gauge invariant regularization,''[2] which is adopted in Ref. [3], corresponds to set $\boldsymbol{a}=0$. In this case the Jacobian (2.13) depends only on the gauge invariant variables Σ_k. In what follows we shall consider the general case with $0 \leqslant \boldsymbol{a} < 1$. In this way, the generating functional (2.4) can be written in terms of the effective action

$$\boldsymbol{W}_{eff} = \boldsymbol{W}[U,V] + \int d^2z \sum_{k=a}^{b} \{\bar{\chi}_k i\partial\!\!\!/ \chi_k - m_0(\chi^*_{(1)k}\chi_{(2)k} \times \Sigma_k^{-1} + \chi^*_{(2)k}\chi_{(1)k}\Sigma_k)\}, \tag{2.18}$$

where

$$\begin{aligned}\boldsymbol{W}[U,V] = \sum_{k=a}^{b}\left\{-\Gamma[U_kV_k] + \frac{\boldsymbol{a}}{2\pi}\int d^2z(U_k^{-1}\partial_+U_k) \times (V_k\partial_-V_k^{-1})\right\} - \frac{1}{g^2}\int d^2z \sum_{\substack{k,k'=a\\ k\neq k'}}^{b} (U_k^{-1}\partial_+U_k) \times (V_{k'}\partial_-V_{k'}^{-1}).\end{aligned} \tag{2.19}$$

The vector fields in two dimensions can be decomposed as

$$\mathcal{G}^k_\mu = -\frac{1}{g}(\epsilon_{\mu\nu}\partial^\nu\tilde{\phi}_k + \partial_\mu\eta_k), \tag{2.20}$$

and this corresponds to parametrizing the Bose fields (U_k, V_k) as follows:

$$U_k = e^{i(\tilde{\phi}_k+\eta_k)/2}, \quad V_k = e^{i(\tilde{\phi}_k-\eta_k)/2}. \tag{2.21}$$

The effective Lagrangian density, corresponding to the action (2.18), can be written as

$$\begin{aligned}\mathcal{L}'_{eff} &= \sum_{k=a}^{b}\bar{\chi}_k i\partial\!\!\!/ \chi_k - m_0\sum_{k=a}^{b}\{\chi^*_{(1)k}\chi_{(2)k}e^{-i\tilde{\phi}_k} + \chi^*_{(2)k}\chi_{(1)k}e^{i\tilde{\phi}_k}\} - \frac{1}{8\pi}(1-\boldsymbol{a})\\ &\quad\times\sum_{k=a}^{b}(\partial_\mu\tilde{\phi}_k)^2 - \frac{1}{2g^2}\partial_\mu\tilde{\phi}_a\partial^\mu\tilde{\phi}_b - \frac{\boldsymbol{a}}{8\pi}\\ &\quad\times\sum_{k=a}^{b}(\partial_\mu\eta_k)^2 + \frac{1}{2g^2}\partial_\mu\eta_a\partial^\mu\eta_b.\end{aligned} \tag{2.22}$$

Introducing the fields $\{\tilde{\phi}_\pm, \chi_\pm\}$, defined by

$$\tilde{\phi}_\pm \doteq \frac{1}{\sqrt{2}}(\tilde{\phi}_a \pm \tilde{\phi}_b), \quad \eta_\pm \doteq \frac{1}{\sqrt{2}}(\eta_a \pm \eta_b), \tag{2.23}$$

we can write the effective Lagrangian density (2.22) as

[2]The gauge-invariant regularization can be used in the computation of the Jacobian taking into account the local gauge invariance of the fermionic piece of the effective theory.

$$\mathcal{L}'_{eff}=\overline{\chi}_a i\partial\!\!\!/\chi_a+\overline{\chi}_b i\partial\!\!\!/\chi_b-\frac{1}{2}\delta_+(\partial_\mu\tilde{\phi}_+)^2+\frac{1}{2}\delta_-(\partial_\mu\tilde{\phi}_-)^2+\frac{1}{2}\alpha_+(\partial_\mu\eta_+)^2-\frac{1}{2}\alpha_-(\partial_\mu\eta_-)^2-m_0\{\chi^*_{(1)a}\chi_{(2)a}e^{(-i/\sqrt{2})}(\tilde{\phi}_++\tilde{\phi}_-)+\chi^*_{(1)b}\chi_{(2)b}e^{(-i/\sqrt{2})}(\tilde{\phi}_+-\tilde{\phi}_-)+\text{H. c.}\}, \tag{2.24}$$

where the constants $\alpha_\pm$ and $\delta_\pm$ are defined by

$$\delta_\pm\doteq\frac{1}{2g^2}\left[1\pm\frac{g^2}{2\pi}(1-\boldsymbol{a})\right],\quad \alpha_\pm\doteq\frac{1}{2g^2}\left[1\mp\frac{g^2\boldsymbol{a}}{2\pi}\right]. \tag{2.25}$$

For $0\leqslant\boldsymbol{a}<1$, the model exhibits two distinct ranges for the coupling constant g:

$$\text{(i)}\quad 0<g^2<\frac{2\pi}{(1-\boldsymbol{a})},\quad \delta_+>0,\quad \delta_->0, \tag{2.26}$$

which from Eq. (2.24) implies that the field $\tilde{\phi}_+$ is quantized with negative metric and the field $\tilde{\phi}_-$ is quantized with positive metric; in this coupling constant range, the field η_+ is quantized with positive metric and the field η_- is quantized with negative metric ($0<g^2\boldsymbol{a}<2\pi$); and

$$\text{(ii)}\quad g^2>\frac{2\pi}{(1-\boldsymbol{a})},\quad \delta_+>0,\quad \delta_-<0. \tag{2.27}$$

In this case both fields $\tilde{\phi}_\pm$ are quantized with negative metric. In this range the model loses its physical meaning, since the unitarity is spoiled. We shall consider only the first range (2.26) in which the free theory limit $g\to 0$ exists and can be performed. For $\boldsymbol{a}=0$, Eq. (2.26) reduces to the boundary condition compatibility constraint for the coupling constant and the particle number ($n=2$) found in Ref. [6] for the two-body bound state solution for two-dimensional Dirac particles.

Introducing canonical fields by rescaling the fields $\tilde{\phi}_\pm\to\delta_\pm^{-1/2}\tilde{\phi}_\pm$, and defining the constants $\beta_\pm\doteq\delta_\pm^{-1/2}/\sqrt{2}$, the effective Lagrangian density can be written as

$$\mathcal{L}'_{eff}=\frac{1}{2}\alpha_+(\partial_\mu\eta_+)^2-\frac{1}{2}\alpha_-(\partial_\mu\eta_-)^2-\frac{1}{2}(\partial_\mu\tilde{\phi}_+)^2+\frac{1}{2}(\partial_\mu\tilde{\phi}_-)^2+\overline{\chi}_a i\partial\!\!\!/\chi_a+\overline{\chi}_b i\partial\!\!\!/\chi_b-m_0\{\chi^*_{(1)a}\chi_{(2)a}e^{i(\beta_+\tilde{\phi}_++\beta_-\tilde{\phi}_-)}+\chi^*_{(1)b}\chi_{(2)b}e^{i(\beta_+\tilde{\phi}_+-\beta_-\tilde{\phi}_-)}+\text{H. c.}\}. \tag{2.28}$$

The bozonized effective theory is obtained introducing the bosonized expression for the Fermi fields $\{\chi_a,\chi_b\}$ in terms of the Bose fields $\{\tilde{\varphi}_a,\tilde{\varphi}_b\}$, according to

$$\chi_k(x)=\left(\frac{\mu}{2\pi}\right)^{1/2}e^{-i\pi\gamma^5/4}{:}\times\exp\left(-i\beta\left\{\gamma^5\tilde{\varphi}_k(x)+\int_{x^1}^{+\infty}\dot{\tilde{\varphi}}_k(x^0,z^1)dz^1\right\}\right){:}, \tag{2.29}$$

$$\overline{\chi}_k(x)i\partial\!\!\!/\chi_k(x)=\frac{1}{2}{:}[\partial_\mu\tilde{\varphi}_k(x)]^2{:}, \tag{2.30}$$

$$\chi^*_{(1)k}(x)\chi_{(2)k}(x)=\frac{\mu}{2\pi}{:}e-i\beta\tilde{\varphi}_k(x){:}, \tag{2.31}$$

with $\beta=2\sqrt{\pi}$, and the double overdots indicate normal ordering with respect to the free propagator $(\Box+\mu^2)^{-1}$ in the limit $\mu\to 0$. Defining the fields $\tilde{\varphi}_\pm\doteq(1/\sqrt{2})\{\tilde{\varphi}_a\pm\tilde{\varphi}_b\}$, and the constants $\beta'=\beta/\sqrt{2}$, $m'_0=m_0\mu/\pi$, we can write the effective bosonized Lagrangian density as

$$\mathcal{L}'_{eff}=\frac{1}{2}\alpha_+(\partial_\mu\eta_+)^2-\frac{1}{2}\alpha_-(\partial_\mu\eta_-)^2-\frac{1}{2}(\partial_\mu\tilde{\phi}_+)^2+\frac{1}{2}(\partial_\mu\tilde{\phi}_-)^2+\frac{1}{2}(\partial_\mu\tilde{\varphi}_+)^2+\frac{1}{2}(\partial_\mu\tilde{\varphi}_-)^2-2m'_0{:}\cos\{\beta'\tilde{\varphi}_+(z)+\beta_+\tilde{\phi}_+(z)\}{:}{:}\cos\{\beta'\tilde{\varphi}_-(z)+\beta_-\tilde{\phi}_-(z)\}{:}. \tag{2.32}$$

For $g=0$ $(\alpha_\pm=0,\beta_\pm=0,\delta_\pm=0)$ we recover the bosonized action corresponding to the free theory of two independent massive Fermi fields, and which is given by

$$\mathcal{L}'_{eff}|_{g=0}=\frac{1}{2}{:}[\partial_\mu\tilde{\varphi}_a(z)]^2{:}+\frac{1}{2}{:}[\partial\tilde{\varphi}_b(z)]^2{:}-m'_0{:}\cos\{2\sqrt{\pi}\tilde{\varphi}_a(z)\}{:}-m_0{:}\cos\{2\sqrt{\pi}\tilde{\varphi}_b(z)\}{:}. \tag{2.33}$$

In order to reduce the algebra of the interacting fields in the effective theory given by Eq. (2.32), and recover the two dynamical degrees of freedom of the theory in the original fermionic formulation of the model, consider the following canonical transformations:

$$\text{ß}_+\tilde{\Phi}_+\doteq\beta'\tilde{\varphi}_++\beta_+\tilde{\phi}_+,\quad \text{ß}_+\tilde{\xi}_+\doteq\beta_+\tilde{\varphi}_++\beta'\tilde{\phi}_+, \tag{2.34}$$

$$\text{ß}_-\tilde{\Phi}_-\doteq\beta'\tilde{\varphi}_-+\beta_-\tilde{\phi}_-,\quad \text{ß}_-\tilde{\xi}_-\doteq\beta_-\tilde{\varphi}_--\beta'\tilde{\phi}_-. \tag{2.35}$$

For the range considered for the coupling constant, we get

$$\text{ß}^2_\pm=\beta'^2\mp\beta^2_+=\frac{2\pi\mp\boldsymbol{a}g^2}{1\pm\frac{g^2}{2\pi}(1-\boldsymbol{a})}>0, \tag{2.36}$$

and the field ξ_+ is quantized with negative metric. In this way, the bosonized effective Lagrangian density is given by

$$\mathcal{L}'_{eff} = -\frac{1}{2}(\partial_\mu \tilde{\xi}_+)^2 + \frac{1}{2}(\partial_\mu \tilde{\xi}_-)^2 + \frac{1}{2}\alpha_+(\partial_\mu \eta_+)^2 - \frac{1}{2}\alpha_-(\partial_\mu \eta_-)^2 + \frac{1}{2}(\partial_\mu \tilde{\Phi}_+)^2 + \frac{1}{2}(\partial_\mu \tilde{\Phi}_-)^2 - 2m'_0 :\cos\{\text{ß}_+ \tilde{\Phi}_+(z)\}::\cos\{\text{ß}_- \tilde{\Phi}_-(z)\}:. \quad (2.37)$$

The effective bosonized theory is described by two coupled sine-Gordon theories and four decoupled massless fields quantized with opposite metric. As stressed in Ref. [15] the extraction of these decoupled Bose fields relies on a structural problem which is related to the fact that the fields $\eta_\pm$ and $\xi_\pm$ do not belong to the field algebra $\Im'$ and cannot be defined in the Hilbert space of states $\mathcal{H}'$. We shall discuss this question in the next section.

Setting $a \equiv b$, then $\eta_- = 0$, $\xi_- = 0$, $\Phi_- = 0$, and the effective Lagrangian density (2.37) reduces to the bosonized Lagrangian of the massive Thirring model with

$$\text{ß}^2_{Th} = \frac{[4\pi - 2\boldsymbol{a}g^2]}{\left[1 + \frac{g^2}{2\pi}(1-\boldsymbol{a})\right]}. \quad (2.38)$$

Setting $\boldsymbol{a} = 0$ we recover the standard massive Thirring model.

III. INTRINSIC FIELD ALGEBRA AND HILBERT SPACE HIERARCHY

It seems to be very instructive to express the set of fundamental fields $\{\mathcal{A}_\mu, \mathcal{B}_\mu, \psi_a, \psi_b\}$ that define the field algebra $\Im'$ in terms of the sine-Gordon-like fields $\tilde{\Phi}_\pm$. Performing the canonical transformations (2.34) on the auxiliary vector fields (2.20), we obtain $(k = a,b)$

$$\mathcal{G}^\mu_k = \frac{1}{g}\epsilon^{\mu\nu}\partial_\nu\left(\frac{\beta^2_+}{\text{ß}_+}\tilde{\Phi}_+ \mp \frac{\beta^2_-}{\text{ß}_-}\tilde{\Phi}_-\right) - l^\mu_k, \quad (3.1)$$

where $l^\mu_k = l^\mu_+ \pm l^\mu_-$ $(k = a,b)$ are longitudinal currents and

$$l^\pm_\mu = \frac{1}{g\sqrt{2}}\partial_\mu(\eta_\pm \pm \alpha_\pm^{-1/2}\xi_\pm). \quad (3.2)$$

Performing the chiral fermion rotation (2.9), together with the canonical transformations (2.34), and using the bosonized expression (2.29) for the free massive Fermi field, we get

$$\psi_k = \Omega_k \chi_k = \Psi_k \sigma_k, \quad (3.3)$$

where the Fermi fields Ψ_k can be factorized in terms of the fields $\Psi_\pm$ with Lorentz spin $s = \frac{1}{4}$ as follows:

$$\psi_a = \Psi_+ \Psi_-, \quad \psi_b = \Psi_+ \Psi^*_-, \quad (3.4)$$

with

$$\Psi_\pm(x) = \left(\frac{\mu}{2\pi}\right)^{1/4} e^{-i\pi\gamma^5/4} :\exp - i\left\{\gamma^5 \text{ß}_\pm \tilde{\Phi}_\pm(x) + \frac{\beta'}{\text{ß}_\pm}\int_{x^1}^{+\infty} \dot{\tilde{\Phi}}_\pm(x^0,z^1)dz^1\right\}:. \quad (3.5)$$

The fields σ_k are given by

$$\sigma_a = e^{-igl_a} = \sigma_+ \sigma_-, \quad (3.6)$$

$$\sigma_b = e^{-igl_b} = \sigma_+ \sigma^*_-, \quad (3.7)$$

with

$$\sigma_\pm = e^{-igl_\pm}, \quad (3.8)$$

where $l_\pm$ are the potentials for the longitudinal currents, $l^\mu_\pm = \partial^\mu l_\pm$.

Taking into account the invariance of the fermionic piece of the effective theory under local gauge transformations, the fermion currents are computed using a gauge-invariant regularization, and we obtain $(k = a,b.)$

$$J^k_\mu = \frac{1}{g^2}\epsilon_{\mu\nu}\partial^\nu\left(\frac{\beta^2_+}{\text{ß}_+}\tilde{\Phi}_+ \pm \frac{\beta^2_-}{\text{ß}_-}\tilde{\Phi}_-\right). \quad (3.9)$$

In this way, we can rewrite Eq. (3.1) as

$$\mathcal{A}^\mu = gJ^\mu_b - l_a, \quad \mathcal{B}^\mu = gJ^\mu_a - l_b. \quad (3.10)$$

The bosonized interpolating generating functional (2.4) can be written as

$$\boldsymbol{\mathcal{Z}}[\theta_a, \overline{\theta_a}, \theta_b, \overline{\theta_b}, \zeta^a_\mu, \zeta^b_\mu] = \mathcal{N}^{-1}\int \mathcal{D}\eta_+ e^{iW_0[\eta_+]}\int \mathcal{D}\eta_- e^{-iW_o[\eta_-]} \times \int \mathcal{D}\xi_+ \mathcal{D}\xi_- e^{-iW_0[\xi_+]} e^{iW_0[\xi_-]} \times \int \mathcal{D}\tilde{\Phi}_+ \mathcal{D}\tilde{\Phi}_- e^{iW[\tilde{\Phi}_+, \tilde{\Phi}_-]} \times \exp\left(i\int d^2x \sum_k \{(\overline{\Psi}_k \sigma^*_k)\theta_k + \overline{\theta}_k(\Psi_k \sigma_k) - \zeta^k_\mu l^\mu_k\}\right), \quad (3.11)$$

where $W_0[\eta_\pm]$ are the free actions for the noncanonical massless fields $\eta_\pm$, quantized with opposite metric, $W_0[\xi_\pm]$ are the free actions for the canonical massless fields $\xi_\pm$, quantized with opposite metric, and $W[\tilde{\Phi}_+, \tilde{\Phi}_-]$ is the action for the coupled sine-Gordon fields $\tilde{\Phi}_\pm$.

From the generating functional $\boldsymbol{\mathcal{Z}}'$ we obtain the general $2n$-point functions for the Thirring model with two fermion species, as, for instance,

$$\langle 0|\overline{\psi}_k(x_1)\cdots\overline{\psi}_k(x_n)\psi_k(y_1)\cdots\psi_k(y_n)|0\rangle' = \langle 0|\overline{\Psi}_k(x_1)\cdots\overline{\Psi}_k(x_n)\Psi_k(y_1)\cdots\Psi_k(y_n)|0\rangle \times\langle 0|\sigma_k^*(x_1)\cdots\sigma_k^*(x_n)\sigma_k(y_1)\cdots\sigma_k(y_n)|0\rangle_0, \tag{3.12}$$

where the notation $\langle 0|\bullet|0\rangle$ means average with respect to the coupled sine-Gordon theories and $\langle 0|\bullet|0\rangle_0$ means average with respect to the free theories of the massless Bose fields $\eta_\pm$ and $\xi_\pm$. As a result of the opposite metric quantization for the fields $\eta_\pm$ and $\xi_\pm$, the functional integration over the field $\xi_\pm$ cancels those arising from the integration over the field $\eta_\pm$ in such a way that the fields $\sigma_\pm$ generate constant contributions to the Wightman functions:

$$\langle 0|\sigma_\pm^*(x_1)\cdots\sigma_\pm^*(x_n)\sigma_\pm(y_1)\cdots\sigma_\pm(y_n)|0\rangle_o = 1. \tag{3.13}$$

This implies the following isomorphism:

$$\langle 0|\overline{\psi}_k(x_1)\cdots\overline{\psi}_k(x_n)\psi_k(y_1)\cdots\psi_k(y_n)|0\rangle' \equiv\langle 0|\overline{\Psi}_k(x_1)\cdots\overline{\Psi}_k(x_n)\;\;\Psi_k(y_1)\cdots\Psi_k(y_n)|0\rangle. \tag{3.14}$$

In this way, for any global gauge-invariant functional $\mathcal{F}\{\overline{\psi}_k,\psi_k\}\in\Im$, we obtain the general one-to-one fermion-boson mapping

$$\langle 0|\mathcal{F}\{\overline{\psi}_k,\psi_k\}|0\rangle' \equiv \langle 0|\mathcal{F}\{\overline{\Psi}_k,\Psi_k\}|0\rangle. \tag{3.15}$$

From the generating functional (3.11) we see that the longitudinal currents (3.2) create from the vacuum zero norm states in $\mathcal{H}'$,

$$\langle 0|l_k^\mu(x)l_k^\nu(y)|0\rangle = 0, \tag{3.16}$$

implying that the Hilbert space $\mathcal{H}'$ is positive semidefinite.

Although the partition function obtained from Eq. (3.11) factorizes in the form

$$\boldsymbol{\mathcal{Z}}'[0] = \boldsymbol{\mathcal{Z}}^0_{\eta_+}[0]\times\boldsymbol{\mathcal{Z}}^0_{\eta_-}[0]\times\boldsymbol{\mathcal{Z}}^o_{\xi_+}[0]\times\boldsymbol{\mathcal{Z}}^0_{\xi_-}[0] \times\boldsymbol{\mathcal{Z}}_{\{\Phi_+,\Phi_-\}}[0], \tag{3.17}$$

the fact that the spurious fields σ_k appears attached to the bosonized Fermi fields in the source terms implies that the generating functional (3.11) cannot be factorized and the massless scalar fields cannot be removed in a naive way, contrary to what is usually done [16,3]. As a matter of fact, the bosonization procedure leads to the appearance of spurious fields $\sigma_k(x)$ with zero scale dimension, implying a structural problem that refers to the existence of infinitely delocalized states $\sigma_k^n|0\rangle$ in $\mathcal{H}'$, and that would imply the violation of the asymptotic factorization property. Although the fields $\sigma_\pm$ generate constant contributions to the Wightman functions, these fields cannot be defined by itself in $\mathcal{H}'$ and the cluster decomposition property is not violated. This question can be clarified on the basis of general principles by examining the intrinsic algebraic structure of the model.

The set of fields $\{\overline{\psi}_k,\psi_k\}$ constitute the intrinsic mathematical structure of the model and generate the local polynomial field algebra $\Im=\Im\{\overline{\psi}_k,\psi_k\}$. The Wightman functions generated from the field algebra $\Im$ define the model and identify the Hilbert space $\mathcal{H}$ of the theory, $\mathcal{H}\doteq\Im|0\rangle$. The introduction of the auxiliary vector fields a_μ and b_μ enlarges the field algebra $\Im\to\Im'=\Im'\{a_\mu,b_\mu,\overline{\psi}_k,\psi_k\}$, and the change of variables (2.5) leads to a field algebra $\Im' = \Im'\{\mathcal{A}_\mu,\mathcal{B}_\mu,\overline{\psi}_k,\psi_k\}$. This field algebra is represented in the enlarged Hilbert space $\mathcal{H}'\doteq\Im'|0\rangle$. The field algebra $\Im$ is a proper subalgebra of $\Im'$, $\Im\subset\Im'$, such that $\mathcal{H}\subset\mathcal{H}'$. Within the bosonization procedure, the fundamental fields defining the field algebra $\Im'$ are written in terms of the Bose fields $\{\tilde{\Phi}_\pm,\eta_\pm,\xi_\pm\}$. This set of Bose fields defines an enlarged redundant field algebra $\Im^B$, which is represented in the indefinite metric Hilbert space $\mathcal{H}^B\doteq\Im^B|0\rangle$. These Bose fields are the building blocks in terms of which the bosonized solution is constructed and, as stressed in Refs. [8–11], should not be considered as elements of the intrinsic field algebra $\Im'$. Only some particular combinations of them belong to the field algebra $\Im'$, in such a way that, $\Im'\subset\Im^B$ and thus $\mathcal{H}'\subset\mathcal{H}^B$. The auxiliary vector fields $\mathcal{A}^\mu = gJ_b^\mu - l_a^\mu$, $\mathcal{B}^\mu = gJ_a^\mu - l_a^\mu$ belong to the field algebra $\Im'$, and since $J_k^\mu\in\Im'$, then $l_k^\mu\in\Im'$. In this way, the positive semidefinite Hilbert space $\mathcal{H}'$ is generated from the field algebra $\Im'\{\mathcal{A}_\mu,\mathcal{B}_\mu,\overline{\psi}_k,\psi_k\}=\Im'\{\Im_0,\overline{\Psi}_k\sigma_k^*,\Psi_k\sigma_k\}$, where $\Im_0\subset\Im'$ is the field subalgebra generated by the longitudinal currents l_k^μ, $\Im_0=\Im_0\{l_k^\mu\}$, and that generates zero norm states: $\mathcal{H}_0\doteq\Im_0|0\rangle\subset\mathcal{H}'$. The fields l_k, which act as potentials for the longitudinal currents l_k^μ, do not belong to the field algebra $\Im'$ and only their space-time derivatives occur in $\Im'$. In this way, the exponential fields σ_k also do not belong to $\Im'$. Since these fields cannot be defined by itself in $\mathcal{H}'$, the Hilbert space cannot be factorized, as, for instance, $\mathcal{H}'\neq\mathcal{H}_{\sigma_k}\otimes\mathcal{H}_{\Phi_+,\Phi_-}$. This implies that the asymptotic factorization property is not violated in $\mathcal{H}'$.

From the algebraic point of view, the fact that the fields σ_k do not belong to the field algebra $\Im'$ and thus are not defined as operator in $\mathcal{H}'$ follows from the charge content of $\mathcal{H}_B$ and $\mathcal{H}'$, since some *topological* charges[3] get trivialized in going from $\mathcal{H}^B$ to $\mathcal{H}'$ and $\mathcal{H}$ [8]. To begin with, consider the following currents that belong to the Bose field algebra $\Im^B$:

$$j_{k_1}^\mu \doteq \mathcal{G}_k^\mu + \partial^\mu\xi_k,\;\; j_{k_2}^\mu \doteq \mathcal{G}_k^\mu + \frac{1}{g}\partial^\mu\eta_k. \tag{3.18}$$

Although the vector fields $\mathcal{G}_\mu^k$ belong to the field algebra $\Im'$, and the field derivatives $\partial_\mu\eta_k$ and $\partial_\mu\xi_k$ belong to the Bose

[3]These charges are called *topological* in the sense that the corresponding conservation laws are totally unrelated to any Noether symmetry exhibited by the Lagrangian defining the model.

field algebra $\Im^B$, they only occur in $\Im'$ as the combination $l_k^\mu = -\partial_\mu(\xi_k + \alpha^{-1/2}\eta_k)$. This ensures that $j_{k_i}^\mu \in \Im^B$. The corresponding charges are given by

$$\mathcal{Q}_{k_i} \doteq \int_{-\infty}^{+\infty} dz^1 j_{k_i}^0(z), \tag{3.19}$$

such that $[\mathcal{Q}_{k_i}, \Im^B] \neq 0$. This implies that the charges $\mathcal{Q}_{k_i}$ do not vanish on $\mathcal{H}^B$: $\mathcal{Q}_{k_i}\mathcal{H}^B \neq 0$. The charges $\mathcal{Q}_{k_i}$ commute with ψ_k, J_k^μ, and l_k^μ, that is,

$$[\mathcal{Q}_{k_i}, \Im_0] = 0, \quad [\mathcal{Q}_{k_i}, \Im] = 0 \rightarrow [\mathcal{Q}_{k_i}, \Im'] = 0. \tag{3.20}$$

This means that the charges $\mathcal{Q}_{k_i}$ are trivialized in the restriction from $\mathcal{H}^B$ to $\mathcal{H}'$ and $\mathcal{H}$ [8–11]:

$$\mathcal{Q}_{k_i}\mathcal{H}^B \neq 0, \quad \mathcal{Q}_{k_i}\mathcal{H}' = 0, \quad \mathcal{Q}_{k_i}\mathcal{H} = 0. \tag{3.21}$$

Since $[\mathcal{Q}_{k_i}, \sigma_k] \neq 0$, the state $|\sigma_k\rangle = \sigma_k|0\rangle$ cannot belong to $\mathcal{H}'$ and the field σ_k cannot be defined as an operator in the Hilbert space $\mathcal{H}'$ [8–11].

The states in the positive semidefinite Hilbert space $\mathcal{H}'$ can be accommodated as equivalence classes modulo $l_k^\mu|0\rangle$ in such a way that the Hilbert space $\mathcal{H}$ of the model is a proper subspace of $\mathcal{H}'$, and is given by the quotient space $\mathcal{H} \sim \mathcal{H}'/\mathcal{H}_0$.

From the operator point of view, the equivalence established by Eq. (3.14) implies the algebraic isomorphism

$$\begin{aligned} \Im\{\overline{\psi}_a, \psi_a, \overline{\psi}_b, \psi_b\} &\sim \Im''\{\overline{\Psi}_a\sigma_a^*, \Psi_a\sigma_a, \overline{\Psi}_b\sigma_b^*, \Psi_b\sigma_b\} \\ &\sim \Im\{\overline{\Psi}_a, \Psi_a, \overline{\Psi}_b, \Psi_b\}, \end{aligned} \tag{3.22}$$

where $\Im'' = \Im' - \Im_0$. In this sense we obtain the fermion-boson mapping in the Hilbert space of states,

$$\begin{aligned} \mathcal{Z}'[\theta_a, \overline{\theta}_a, \theta_b, \overline{\theta}_b, 0, 0] &\sim \mathcal{Z}[\theta_a, \overline{\theta}_a, \theta_b, \overline{\theta}_b] \\ &\sim \mathcal{Z}^{\{\Phi_+, \Phi_+\}}[\theta_a, \overline{\theta}_a, \theta_b, \overline{\theta}_b], \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} \mathcal{Z}^{\{\Phi_+, \Phi_+\}}[\theta_a, \overline{\theta}_a, \theta_b, \overline{\theta}_b] = \mathcal{N}^{-1} \int & \mathcal{D}\tilde{\Phi}_+ \mathcal{D}\tilde{\Phi}_- e^{iW[\tilde{\Phi}_+, \tilde{\Phi}_-]} \\ & \times \exp\Bigg(i \int d^2x \{\overline{\Psi}_a\theta_a + \overline{\theta}_a\Psi_a \\ & + \overline{\Psi}_b\theta_b + \overline{\theta}_b\Psi_b\} \Bigg), \end{aligned} \tag{3.24}$$

and $W[\tilde{\Phi}_+, \tilde{\Phi}_-]$ is the action corresponding to the Lagrangian density of two coupled sine-Gordon theories:

$$\begin{aligned} \mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu\tilde{\Phi}_+)^2 + \frac{1}{2}(\partial_\mu\tilde{\Phi}_-)^2 \\ - 2m_0' : \cos\{ß_+\tilde{\Phi}_+(z)\} : : \cos\{ß_-\tilde{\Phi}_-(z)\} :. \end{aligned} \tag{3.25}$$

IV. STATISTICAL-MECHANICAL DESCRIPTION

In this section we shall consider in the two-dimensional Euclidean space the statistical-mechanical system associated with the effective bosonized theory in the quotient space $\mathcal{H}$, which is built from the generating functional (3.24).

Denoting by $\{\tilde{\Phi}_\alpha\}$ the set of Bose fields $\{\tilde{\Phi}_+, \tilde{\Phi}_-\}$, and by $W_0(\tilde{\Phi}_\alpha)$ the corresponding free field Euclidean actions, the vacuum functional of the effective theory is given by the Euclidean region functional integral (Euclidean Gell-Mann and Low formula)

$$\mathcal{Z} = \frac{1}{\mathcal{Z}_0} \int \prod_\alpha d\mu_0(\tilde{\Phi}_\alpha) \exp,\left(2m_0' \int d^2z \mathcal{F}(z) \right), \tag{4.1}$$

where $\tilde{\Phi}_\alpha \in \{\tilde{\Phi}_\alpha\}$ are random classical fields whose distributions are given by the free field (Gaussian) probability measures

$$d\mu_o(\tilde{\Phi}_\alpha) = e^{-W_0(\tilde{\Phi}_\alpha)}[\mathcal{D}\tilde{\Phi}_\alpha], \tag{4.2}$$

with $[\mathcal{D}\tilde{\Phi}_\alpha]$ the formal Lebesgue measure, and

$$\mathcal{Z}_0 = \int \prod_\alpha d\mu_0(\tilde{\Phi}_\alpha). \tag{4.3}$$

By expanding the exponential of the interaction action in the Gell'Mann-Low formula (4.1) in a power series of the bare mass m_0', the interaction term of the effective theory can be treated as a perturbation in the corresponding free field theories defined by the actions $W_0(\tilde{\Phi}_\alpha)$. This procedure, when applied to a sine-Gordon-like systems, corresponds to a *gas expansion* [17] and leads to a two-dimensional neutral-Coulomb-like gas description.

A. Grand partition function

In order to obtain the partition function of the statistical mechanical system associated with the effective theory described by action $W[\tilde{\Phi}_+, \tilde{\Phi}_-]$, we perform the gas expansion by expanding the exponential of the interaction term of the action in powers of m_0' [17]:

$$\exp\left(2m_0' \int d^2z \mathcal{F}(z) \right) = \sum_{n=0}^\infty \frac{(2m_0')^n}{n!} \int \prod_{i=1}^n d^2z_i \prod_{j=1}^n \mathcal{F}(z_j), \tag{4.4}$$

with

$$\mathcal{F}(z) = : \cos\{ß_+\tilde{\Phi}_+(z)\} : : \cos\{ß_-\tilde{\Phi}_-(z)\} :. \tag{4.5}$$

The vacuum functional (4.1) can be written in terms of mixed correlation functions of order (disorder) variables of the fields $\{\tilde{\Phi}_\alpha\}$, in which the averages are taken with respect to the free field probability measures $d\mu_0(\tilde{\Phi}_\alpha)$. The Euclidean vacuum functional can be written as

$$\mathcal{Z}=\sum_{n=0}^{\infty}\frac{(m_0')^n}{2^n n!}\mathcal{Z}^{(n)}, \tag{4.6}$$

where

$$\mathcal{Z}^{(n)}=\frac{1}{\mathcal{Z}_o}\sum_{\substack{\{\lambda_j^+\}_n\\ \{\lambda_k^-\}_n}}\int\prod_{j=1}^{n}d^2z_j \times\int\prod_{\alpha}d\mu_0(\tilde{\Phi}_\alpha)\prod_{j=1}^{n}e^{i\lambda_j^{+}\beta_+\tilde{\Phi}_+(z_j)}\prod_{k=1}^{n}e^{i\lambda_k^{-}\beta_-\tilde{\Phi}_-(z_k)}, \tag{4.7}$$

where $\lambda_j^+=\pm1$, $\lambda_k^-=\pm1$, and $\Sigma_{\{\lambda_j^+\}_n,\{\lambda_k^-\}_n}$ runs over all possibilities in the sets $\{\lambda_1^+,\ldots,\lambda_n^+\}$, $\{\lambda_1^-,\ldots,\lambda_n^-\}$. In this way the partition function (4.7) can be factorized in terms of statistical averages of order variables taken with respect to the free theories of the massless fields $\{\tilde{\Phi}_\alpha\}$:

$$\mathcal{Z}^{(n)}=\sum_{\substack{\{\lambda_j^+\}_n\\ \{\lambda_k^-\}_n}}\int\prod_{j=1}^{n}d^2z_j\left\langle\prod_{j=1}^{n}e^{i\lambda_j^{+}\beta_+\tilde{\Phi}_+(z_j)}\right\rangle_0 \times\left\langle\prod_{k=1}^{n}e^{i\lambda_k^{-}\beta_-\tilde{\Phi}_-(z_k)}\right\rangle_0, \tag{4.8}$$

where the averages of any functional $F[\tilde{\Phi}]$ of a field $\tilde{\Phi}\in\{\tilde{\Phi}_\alpha\}$ are given by

$$\langle F[\tilde{\Phi}]\rangle_0\equiv\frac{\int d\mu_0(\tilde{\Phi})F[\tilde{\Phi}]}{\int d\mu_o(\tilde{\Phi})}. \tag{4.9}$$

Defining the charge densities of finite support $j_\pm^n(x;z_1,\ldots,z_n)$ as

$$j_\pm^n(x;z_1,\ldots,z_n)=i\beta_\pm\sum_{j=1}^{n}\lambda_j^{\pm}\delta^{(2)}(x-z_j), \tag{4.10}$$

the n-point correlation function appearing in Eq. (4.8) is given by

$$\left\langle\prod_{k=1}^{n}e^{i\lambda_k^{\pm}\beta_\pm\tilde{\Phi}_\pm(z_k)}\right\rangle_0=\langle e^{-\tilde{\Phi}_\pm^n(j)}\rangle_0\equiv e^{-\langle j_\pm^n,-\Box^{-1}j_\pm^n\rangle/2}, \tag{4.11}$$

where we have defined

$$\tilde{\Phi}_\pm^n(j)\doteq\int d^2x\,j_\pm^n(x;z_1,\ldots,z_n)\tilde{\Phi}_\pm(x), \tag{4.12}$$

such that

$$e^{\langle j,-\Box^{-1}j\rangle/2}\equiv e\left(\frac{1}{2}\int d^2x\int d^2x'\,j(x;z_1,\ldots,z_n)D_0(x-x')\times j(x';z_1,\ldots,z_n)\right), \tag{4.13}$$

in which

$$D_o(x)=\lim_{\mu^2\to0}\Delta(x;\mu)=-\frac{1}{4\pi}\ln\{-\mu^2(|x|^2+\varepsilon^2)\} \tag{4.14}$$

is the infrared and ultraviolet regularized massless Green function of the two-dimensional Laplacian operator. We carry out the calculations in the presence of μ^2, and set $\mu^2\to0$ at the end. Thus, we get

$$\exp\left\{\frac{1}{2}\langle j_+^n,\Box^{-1}j_+^n\rangle+\frac{1}{2}\langle j_-^n,\Box^{-1}j_-^n\rangle\right\}=\exp\left\{\frac{1}{2}\beta_+^2\sum_{i,j=1}^{n}\lambda_i^+\lambda_j^+D_0(z_i-z_j)+\frac{1}{2}\beta_-^2\sum_{k,l=1}^{n}\lambda_k^-\lambda_l^-D_0(z_k-z_l)\right\}. \tag{4.15}$$

Using Eq. (4.15), we can write Eq. (4.8) as

$$\mathcal{Z}^{(n)}=\int\prod_{j=1}^{n}d^2z_j\times\sum_{\{\lambda_j^+\}_n}\exp\left(\frac{1}{2}\beta_+^2\sum_{i,j=1}^{n}\lambda_i^+\lambda_j^+D_0(z_i-z_j)\right)\times\sum_{\{\lambda_k^-\}_n}\exp\left(\frac{1}{2}\beta_-^2\sum_{k,l=1}^{n}\lambda_k^-\lambda_l^-D_0(z_k-z_l)\right). \tag{4.16}$$

The contributions of the infrared cutoff μ^2 in Eq. (4.16) can be factorized and are given by

$$f(\mu^2)=(\mu^2)^{(1/8\pi)\beta_+^2\left(\sum_{j=1}^{n}\lambda_j^+\right)^2}(\mu^2)^{(1/8\pi)\beta_-^2\left(\sum_{j=1}^{n}\lambda_j^-\right)^2}. \tag{4.17}$$

The correlation functions will be (infrared) instability free if both superselection rules are satisfied:

$$\sum_{j=1}^{n}\lambda_j^+=0,\quad\sum_{k=1}^{n}\lambda_k^-=0. \tag{4.18}$$

This means that only neutral configurations contribute, i.e., configurations with zero total charge:

$$\mathcal{Q}^n_{\pm}=\int_{-\infty}^{+\infty}dz^1 j^n_{\pm}(x;z_1,\dots,z_n)=0. \tag{4.19}$$

The only nonzero contributions for the correlation function are those with n even. In this way, we get

$$\mathcal{Z}^{(2n)}=\lim_{\varepsilon\to 0}f^{2n}(\varepsilon)\int\prod_{j=1}^{2n}d^2z_j \times\sum_{\{\lambda_j^+\}_n}\exp\left\{\frac{\beta_+^2}{8\pi}\sum_{i\neq j}^{2n}\lambda_i^+\lambda_j^+\ln(|z_i-z_j|^2+\varepsilon^2)\right\} \times\sum_{\{\lambda_k^-\}_n}\exp\left\{\frac{\beta_-^2}{8\pi}\sum_{k\neq l}^{2n}\lambda_k^-\lambda_l^-\ln(|z_k-z_l|^2+\varepsilon^2)\right\}, \tag{4.20}$$

where $f(\varepsilon)=(\varepsilon^2)^{\beta^2/8\pi}$, and $\beta^2=\beta_+^2+\beta_-^2$. The contributions of $f^{(2n)}(\varepsilon)$ are eliminated by a redefinition of the fugacity $\mathbf{z}\doteq m_0' f(\varepsilon)/2$. The grand-partition function (4.6) can be written as

$$\mathcal{Z}=\sum_{n=0}^{\infty}\frac{\mathbf{z}^{2n}}{(2n)!}\mathcal{Z}^{(2n)}, \tag{4.21}$$

where

$$\mathcal{Z}^{(2n)}=\lim_{\varepsilon\to 0}\int\prod_{j=1}^{2n}d^2z_j \times\sum_{\{\lambda_i^+\}_n}\exp\left\{\frac{\beta_+^2}{8\pi}\sum_{\substack{i,j=1\\ i\neq j}}^{2n}\lambda_i^+\lambda_j^+\ln(|z_i-z_j|+|\varepsilon|)\right\} \times\sum_{\{\lambda_k^-\}_n}\exp\left\{\frac{\beta_-^2}{8\pi}\sum_{\substack{k,l=1\\ k\neq l}}^{2n}\lambda_k^-\lambda_l^-\ln(|z_k-z_l|+|\varepsilon|)\right\}. \tag{4.22}$$

B. Equation of state

Following the standard procedure [17], in order to obtain the equation of state of the statistical-mechanical system described by the partition function (4.21), we shall consider the system confined in a finite volume $V=\pi R^2$. The themodynamical limit is performed in the end of all calculations. Making the change of variables $z\to\hat{z}=z/\mathcal{R}$, we can write the partition function (4.21) as

$$\mathcal{Z}=\sum_{n=0}^{\infty}\frac{\mathbf{z}^{2n}}{(2n)!}V^{2n(1-\beta^2/8\pi)}\hat{\mathcal{Z}}^{(2n)}, \tag{4.23}$$

with

$$\hat{\mathcal{Z}}^{(2n)}=\lim_{\hat{\varepsilon}\to 0}\int_{|\hat{z}_i|<1}\prod_{j=1}^{2n}d^2\hat{z}_j \times\sum_{\{\lambda_i^+\}_n}\exp\left\{\frac{\beta_+^2}{8\pi}\sum_{\substack{i,j=1\\ i\neq j}}^{2n}\lambda_i^+\lambda_j^+\ln(|\hat{z}_i-\hat{z}_j|+|\hat{\varepsilon}|)\right\} \times\sum_{\{\lambda_k^-\}_n}\exp\left\{\frac{\beta_-^2}{8\pi}\sum_{\substack{k,l=1\\ k\neq l}}^{2n}\lambda_k^-\lambda_l^-\ln(|\hat{z}_k-\hat{z}_l|+|\hat{\varepsilon}|)\right\}. \tag{4.24}$$

Introducing the potential $\mathbf{\Omega}=-\gamma\ln\mathcal{Z}$, with $\gamma=kT$, the pression is given by

$$\mathcal{P}=-\left(\frac{\partial\mathbf{\Omega}}{\partial V}\right)=\gamma\frac{1}{\mathcal{Z}}\left(\frac{\partial\mathcal{Z}}{\partial V}\right). \tag{4.25}$$

The variation of Eq. (4.25) with respect to the volume leads to the following equation of state:

$$\mathcal{P}V=\left(1-\frac{\beta^2}{8\pi}\right)\langle\mathcal{N}\rangle kT, \tag{4.26}$$

where $\langle\mathcal{N}\rangle$ is the expected number of particles defined by

$$\langle\mathcal{N}\rangle=\frac{1}{\mathcal{Z}}\sum_{n=0}^{\infty}\frac{\mathbf{z}^{2n}}{(2n)!}V^{2n(1-\beta^2/8\pi)}(2n)\hat{\mathcal{Z}}^{(2n)}. \tag{4.27}$$

The equation of state (4.26) exhibits a Kosterlitz-Thouless (KT) phase transition at the critical temperature:

$$(\beta_+^2+\beta_-^2)_c=8\pi. \tag{4.28}$$

For atractive and repulsive Thirring couplings, the critical line characterizing the KT phase transition starts at the critical value of the coupling constant

$$g_c^2=\frac{2\pi}{\sqrt{(1-\boldsymbol{a})(2-\boldsymbol{a})}}. \tag{4.29}$$

For $g=0$, the equation of state (4.26) reduces to

$$\mathcal{P}V=\frac{1}{2}\langle\mathcal{N}\rangle kT. \tag{4.30}$$

In the free case, the equation of state can be writen in terms of the equation of state describing a Coulomb gas. Since in the free case the action is writen as a sum of two decoupled sine-Gordon actions, the partition function (4.1) factorizes as terms of two Coulomb gas partition functions $\mathcal{Z}=\mathcal{Z}_1\times\mathcal{Z}_2$, where $\mathcal{Z}_a$ are the partition functions of two noninteracting Coulomb gas in a volume V:

$$\mathcal{Z}_a=\sum_{n_a=0}^{\infty}\frac{\mathbf{z}^{2n_a}}{(2n_a!)^2}V^{n_a}\hat{\mathcal{Z}}^{(2n_a)}, \tag{4.31}$$

with

$$\hat{\mathcal{Z}}^{(2n_a)} = \lim_{\varepsilon \to 0} \int \prod_{j=1}^{2n_a} d^2 \hat{z}_j \times \exp\left\{ \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{2n_a} \lambda_i \lambda_j \ln(|\hat{z}_i - \hat{z}_j| + |\hat{\varepsilon}|) \right\}. \tag{4.32}$$

The total pressure is given by

$$\mathcal{P} = \gamma \frac{1}{\mathcal{Z}} \left(\frac{\partial \mathcal{Z}}{\partial V} \right) = \mathcal{P}_1 + \mathcal{P}_2 = \frac{kT}{v} \left(\frac{1}{2} \langle \mathcal{N}_1 \rangle_c + \frac{1}{2} \langle \mathcal{N}_2 \rangle_c \right), \tag{4.33}$$

where $\langle \mathcal{N}_a \rangle_c$ is the expected number of particles of a Coulomb gas in a volume V [17]:

$$\langle \mathcal{N}_a \rangle_c = \frac{1}{\mathcal{Z}_a} \sum_{n_a=0}^{\infty} \frac{\mathbf{z}^{2n_a}}{(2n_a!)^2} V^{n_a} (2n_a) \hat{\mathcal{Z}}^{(2n_a)}. \tag{4.34}$$

V. CONCLUSIONS

Using the Abelian reduction of the WZW theory we have considered the functional integral bosonization of a model with two fermionic fields with Thirring-like coupling. We showed that the use of auxiliary vector fields in the bosonization procedure enlarges the Hilbert space by the introduction of an external field algebra and that should not be considered as an element of the intrinsic algebraic structure defining the model. The correct identification of the original positive metric Hilbert space as a proper subspace of the enlarged indefinite metric Bose Hilbert space has been done without *a priori* disregarding the decoupled scalar fields. We have seen that a null metric combination of derivatives of scalar fields persists as an element of the algebra of observables after the dequartizing procedure. Contrary to naive expectations, in spite of the zero norm fields the Hilbert space $\mathcal{H}'$ does not contains infinitely delocalized states and the asymptotic factorization property is not violated. The factorization of the partition function will generally leads to incorrect conclusions concerning the physical content of the model. The algebraic structure of the model was identified according to general structural properties of quantum field theory, and which are related to the trivialization of some topological charges in going from the enlarged bosonic Hilbert space $\mathcal{H}^B$ to the Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}$. This procedure allows for the identification of the original Hilbert space as a coset space. The disregarding of the zero norm fields, as usually done in the treatment of functional integral bosonization, is rigorously justified through this result.

The statistical-mechanical description of the bosonized theory has been performed. As a result of the coupling of the sine-Gordon fields, the grand partition function of the associated statistical system does not factorizes as a product of partition functions of two Coulomb gas systems. Indeed the partition function describes a gas of two types of point particles carrying independent Abelian charges which can assume plus or minus signals independently. The Coulomb interaction only occurs between charges of the same type. The selection rules, which ensure the infrared stability of the composite statistical-mechanical system, are independent and lead to two conservation laws for each type of charge. In the free theory limit ($g=0$, $\beta_+^2 = \beta_-^2 = 2\pi$) the grand partition function factorizes as the product of two Coulomb gas partition functions. The associated equation of state is, however, very simple even in the interacting case. It exhibits the Kosterlitz-Thouless phase transition characteristics of some two-dimensional models.

The peculiar nature of the cosine interaction for this two Fermi fields model could be forecasted by a Fierz-like transformation of the fields. By interchanging the role of the left Lorentz components of both fields we obtain two Thirring models with the interaction in the mass term $\Sigma_i \overline{\psi}_i \psi_i \to \overline{\psi}_a \psi_b + \overline{\psi}_b \psi_a$. This redefinition of the fields calls attention to the close relation with Gross-Neveu O(2) model.

The two-fermion system with contact interaction that motivates the present study does not incorporate the interaction with the electromagnetic field. It would be instructive to study the generalization of the model here dealt with by introducing the coupling with a physical electromagnetic field, besides the vector auxiliary ones. Notice further that the subtetlies associated with the proper definition of the Hilbert space will come about in the analogous to the model here dealt with in higher dimensions using the bosonization of the currents.

After over a quarter of a century of investigations of two-dimensional field theories we have learned that, besides their peculiar formal aspects, two-dimensional models have also the value of providing a better conceptual and structural understanding of general properties of quantum field theory [18–20].

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