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Brane world cosmology: Gauge-invariant formalism for perturbation

Hideo Kodama, Akihiro Ishibashi, and Osamu Seto
Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
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In the present paper the gauge-invariant formalism is developed for perturbations of the brane world model in which our universe is realized as a boundary of a higher dimensional spacetime. For the background model in which the bulk spacetime is $(n+m)$ dimensional and has the spatial symmetry corresponding to the isometry group of an n-dimensional maximally symmetric space, gauge-invariant equations are derived for perturbations of the bulk space-time. Further, for the case corresponding to the brane world model in which $m=2$ and the brane is a boundary invariant under the spatial symmetry in the unperturbed background, relations between the gauge-invariant variables describing the bulk perturbations and those for brane perturbations are derived from Israel's junction condition under the assumption of $\mathbb{Z}_2$ symmetry. In particular, for the case in which the bulk spacetime is a constant-curvature spacetime, it is shown that the bulk perturbation equations reduce to a single hyperbolic master equation for a master variable, and that the physical condition on the gauge-invariant variable describing the intrinsic stress perturbation of the brane yields a boundary condition for the master equation through the junction condition. On the basis of this formalism, it is pointed out that it seems to be difficult to suppress brane perturbations corresponding to massive excitations for a brane motion giving a realistic expanding universe model.

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I. INTRODUCTION

Motivated by M theory [1,2], anti–de Sitter (AdS) conformal field theory (CFT) correspondence in string theories [3,4], and the hierarchy problem in particle theory [5–8], brane world models in which our universe is realized as a boundary of a higher-dimensional spacetime have been actively studied recently [9–38]. In particular, for the case in which the bulk spacetime is five dimensional, anti–de Sitter spacetime and the brane is realized as a flat four-dimensional spacetime, the gravitational interaction between matter in the brane is well described by the standard one on scales much larger than the scale corresponding to the brane tension [12–15].

Further, as an extension of the analysis to a dynamical situation, the embedding of Robertson-Walker universe models into five-dimensional anti–de Sitter and anti–de Sitter-Schwarzschild spacetimes has been discussed by many people [19–30]. In such high-symmetry cases, although the evolution equation for the cosmic scale factor is modified from the standard one, our universe is still a dynamically closed system, and the difference in the evolution equation can be neglected when the energy density of the universe becomes much smaller than the brane tension. Thus the brane world model gives a new world model consistent with present day observations. However, if one goes beyond this lowest-level approximation, it is not clear whether the brane world model is consistent with all available observations because our universe is not dynamically closed in this model [10].

One of the simplest ways to analyze this problem is to investigate the behavior of perturbations of the brane world model. Since perturbations of the brane are inevitably associated with perturbations in the geometry of the bulk spacetime, such investigation will make clear whether or not the open nature of the universe dynamics is controllable. It will also make possible an observational test of the model in terms of the anisotropy of the cosmic microwave background.

As the starting point of investigations in this line, in the present paper, we develop a gauge-invariant formalism for perturbations of the brane world model. The basic approach is the same as that originally developed for four-dimensional spacetime by Gerlach and Sengupta [39–41] and utilized by some people in analysis of the interaction between a domain wall and gravitational waves in four-dimensional spacetimes [42–44].

The formalism consists of two parts. The first is a gauge-invariant formalism for perturbations in the geometry of the bulk spacetime. This problem has already been investigated by some people for the standard case in which the bulk spacetime is vacuum and maximally symmetric [45]. In the present paper, taking account of the developing nature of the brane world model, we extend the formalism to the case in which the bulk spacetime is $(m+n)$ dimensional and its unperturbed geometry has only the isometry corresponding to the maximally symmetric space of dimension n $(n\geqslant 1)$. This symmetry is utilized to expand perturbations in terms of the harmonic functions on n-dimensional maximally symmetric space and define gauge-invariant variables.

The second part establishes relations between the gauge-invariant variables describing perturbations of the brane and those for the bulk perturbations. In this part we assume that $m=2$ and the $(n+1)$-dimensional brane is invariant under the isometry group of the bulk in the unperturbed model. Thus the brane represents an expanding Robertson-Walker universe in general.

The paper is organized as follows. In the next section we first classify perturbations into tensor, vector, and scalar types in terms of the tensorial behavior with respect to the maximally symmetric n-dimensional spacetime. Then for each type we define the gauge-invariant variables describing perturbations of the bulk geometry and express the Einstein equations in terms of them. In Sec. III, after introducing a gauge-invariant variable describing the motion of the brane, we express Israel's junction condition corresponding to the $\mathbb{Z}_2$ symmetry in terms of it and the bulk variables. We will show that this gives expressions for the intrinsic perturbation variables, for the brane in terms of the bulk variables, and a boundary condition on the latter in terms of the intrinsic stress perturbations of the brane. In Sec. IV we specialize the formalism to the standard brane world model in which the bulk spacetime is vacuum. We reduce the perturbation equations to a single hyperbolic equation for a master variable Ω in a two-dimensional spacetime and express the junction conditions in terms of the master variable. We will show that the condition that the anisotropic stress perturbation of the brane should vanish yields the Neumann and Dirichlet boundary conditions on the master variable for the tensor and vector perturbations, respectively, while the boundary condition for the scalar perturbation is obtained from the condition on the entropy perturbation of the brane. The last condition becomes nonlocal with respect to time except for the cases in which the brane is vacuum or $p=-\rho$. Section V is devoted to summary and discussion.

II. BULK PERTURBATION EQUATIONS

A. Background spacetime

In this section we consider perturbations of spacetime structure on $(m+n)$-dimensional spacetime $\mathcal{M}$, which is locally written as a product

$$\mathcal{M}^{m+n}=\mathcal{N}^m\times\mathcal{K}^n\ni(y^a,x^i)=(z^M). \tag{1}$$

Its unperturbed background geometry is given by the metric

$$d\overline{s^2}=\overline{g}_{MN}\,dz^M\,dz^N=g_{ab}(y)dy^a\,dy^b+r^2(y)d\sigma_n^2, \tag{2}$$

where the metric

$$d\sigma_n^2=\gamma_{ij}(x)dx^i\,dx^j \tag{3}$$

is that with a constant sectional curvature K on $\mathcal{K}^n$. We denote the covariant derivatives, the connection coefficients, and the curvature tensors for the three metrics $d\overline{s^2}$, $g_{ab}\,dy^a\,dy^b$, and $d\sigma_n^2$ as

$$d\overline{s}^2\Rightarrow\overline{\nabla}_M,\overline{\Gamma}^M_{NL},\overline{R}_{MNLS}, \tag{4}$$

$$g_{ab}(y)dy^a\,dy^b\Rightarrow D_a,\ {}^m\Gamma^a_{bc}(y),\ {}^mR_{abcd}(y), \tag{5}$$

$$d\sigma_n^2\Rightarrow\hat{D}_i,\hat{\Gamma}^i_{jk}(x),\hat{R}_{ijkl}(x)=K(\gamma_{ik}\gamma_{jl}-\gamma_{il}\gamma_{jk}). \tag{6}$$

The expressions for $\overline{\Gamma}^M_{NL}$ and $\overline{R}_{MNLS}$ in terms of the corresponding quantities for the metrics $g_{ab}(y)dy^a\,dy^b$ and $d\sigma_n^2$ are given in Appendix A.

From the symmetry structure of $\overline{G}_{MN}$ the energy-momentum tensor $\overline{T}_{MN}$ for the background bulk geometry has the structure

$$\overline{T}_{ai}=0,\quad \overline{T}^i_j=\overline{P}\delta^i_j. \tag{7}$$

Hence the Einstein equations for the bulk spacetime,

$$\overline{G}_{MN}+\Lambda\overline{g}_{MN}=\kappa^2\overline{T}_{MN}, \tag{8}$$

are reduced in the unperturbed background to

$$\overline{G}_{ab}+\Lambda g_{ab}=\kappa^2\overline{T}_{ab}, \tag{9}$$

$$\overline{G}^i_i=n(\kappa^2\overline{P}-\Lambda). \tag{10}$$

B. Gauge transformation of perturbations

For the infinitesimal gauge transformation represented in terms of the coordinates as $\bar{\delta}z^M=\xi^M$, the metric perturbation $h_{MN}=\delta\overline{g}_{MN}$ transforms as

$$\bar{\delta}h_{MN}=-\pounds_\xi g_{MN}=-\overline{\nabla}_M\xi_N-\overline{\nabla}_N\xi_M. \tag{11}$$

By decomposing the connection this yields

$$\bar{\delta}h_{ab}=-D_a\xi_b-D_b\xi_a, \tag{12}$$

$$\bar{\delta}h_{ai}=-r^2D_a\left(\frac{\xi_i}{r^2}\right)-\hat{D}_i\xi_a, \tag{13}$$

$$\bar{\delta}h_{ij}=-\hat{D}_i\xi_j-\hat{D}_j\xi_i-2rD^ar\xi_a\gamma_{ij}. \tag{14}$$

Similarly, the gauge transformation of the perturbation of the energy-momentum tensor $\bar{\delta}(\delta\overline{T})_{MN}$,

$$\bar{\delta}(\delta\overline{T})_{MN}=-\pounds_\xi\overline{T}_{MN}=-\xi^L\overline{\nabla}_L\overline{T}_{MN}-\overline{T}_{ML}\overline{\nabla}_N\xi^L-\overline{T}_{NL}\overline{\nabla}_M\xi^L, \tag{15}$$

is written as

$$\bar{\delta}(\delta\overline{T})_{ab}=-\xi^cD_c\overline{T}_{ab}-\overline{T}_{ac}D_b\xi^c-\overline{T}_{bc}D_a\xi^c, \tag{16}$$

$$\bar{\delta}(\delta\overline{T})_{ai}=-\overline{T}_{ab}\hat{D}_i\xi^b-r^2\overline{P}D_a(r^{-2}\xi_i), \tag{17}$$

$$\bar{\delta}(\delta\overline{T})_{ij}=-\xi^aD_a(r^2\overline{P})\gamma_{ij}-\overline{P}(\hat{D}_i\xi_j+\hat{D}_j\xi_i). \tag{18}$$

C. Gauge-invariant perturbation equations

In general, each tensor with rank at most 2 on the maximally symmetric space $\mathcal{K}^n$ is uniquely decomposed into components of the three types, scalar, vector, and tensor, and each component can be expanded in terms of harmonic functions of the same type [46].

1. Tensor perturbation

First we consider the tensor perturbation, which can be expanded in terms of the harmonic tensors $\mathbb{T}_{ij}$,

$$(\hat{\Delta}+k^2)\mathbb{T}_{ij}=0, \tag{19}$$

with the properties

$$\mathbb{T}^i_i=0, \quad \hat{D}_j\mathbb{T}^j_i=0. \tag{20}$$

In the present paper we omit the index labeling the harmonics as well as the summation symbol with respect to the index, because expansion coefficients corresponding to different eigenvalues decouple on the maximally symmetric space.

Here note that the eigenvalue k^2 is always non-negative under a boundary condition making the operator $\hat{\Delta}$ self-adjoint in the L^2 space. In particular, $k^2=0$ appears only for the flat space ($K=0$) since the corresponding eigentensors satisfy $\hat{D}_k\mathbb{T}_{ij}=0$, which yields $0=\hat{D}^i\hat{D}_k\mathbb{T}_{ij}=nK\mathbb{T}_{jk}$. Thus the eigentensors for $k^2=0$ are constant tensors. In the framework of the expansion in the L^2 sense, such eigentensors should be discarded. Thus we assume $k^2>0$ in the following unless otherwise stated.

For the tensor perturbation the metric perturbation is expanded as

$$h_{ab}=0, \quad h_{ai}=0, \quad h_{ij}=2r^2H_T\mathbb{T}_{ij}. \tag{21}$$

Since the infinitesimal gauge transformation $\xi=(\xi^a,\xi^i)$ has no tensor component, it follows that H_T is gauge invariant. Similarly, $\delta\overline{T}_{MN}$ is expanded as

$$\delta\overline{T}_{ab}=0, \quad \delta\overline{T}^a_i=0, \quad \delta\overline{T}^i_j=\tau_T\mathbb{T}^i_j, \tag{22}$$

where τ_T is the gauge-invariant variable representing the tensor-type anisotropic stress perturbation.

Inserting these expansions into the expression for $\delta\overline{R}_{ij}$, we obtain the following gauge-invariant perturbation equation:

$$-\Box H_T-\frac{n}{r}Dr\cdot DH_T+\frac{k^2+2K}{r^2}H_T=\kappa^2\tau_T, \tag{23}$$

where $\Box=D^aD_a$ is the d'Alembertian on the m-dimensional space $\mathcal{N}^m$.

2. Vector perturbation

Divergence-free vector fields can be expanded in terms of the vector harmonic $\mathbb{V}_i$ defined by

$$(\hat{\Delta}+k^2)\mathbb{V}_i=0, \tag{24}$$

$$\hat{D}_i\mathbb{V}^i=0. \tag{25}$$

From this we can define the vector-type harmonic tensor as

$$\mathbb{V}_{ij}=-\frac{1}{2k}(\hat{D}_i\mathbb{V}_j+\hat{D}_j\mathbb{V}_i), \tag{26}$$

which has the properties

$$[\hat{\Delta}+k^2-(n+1)K]\mathbb{V}_{ij}=0, \tag{27}$$

$$\mathbb{V}^i_i=0, \quad \hat{D}_j\mathbb{V}^j_i=\frac{k^2-(n-1)K}{2k}\mathbb{V}_i, \tag{28}$$

and expands a vector-type perturbation of a second-rank tensor.

As in the case of tensor harmonics, the eigenvalue k^2 is always non-negative and $k^2=0$ occurs only for $K=0$, for which the harmonic vectors become constant vectors. Thus, for the same reason as in the tensor harmonics, we assume $k^2>0$ in the following. One subtle point of the vector harmonics is that $k^2>0$ does not imply $k^2-(n+1)K>0$ for $K>0$. Hence, for $k^2<(n+1)K$ and $K>0$, the vector-type tensor harmonics defined by Eq. Eq. (26) should vanish, which implies that $\mathbb{V}^i$ is a Killing vector on S^n. In this case it follows from Eq. Eq. (28) that the eigenvalue should be given by $k^2=(n-1)K$.

The vector perturbation of the metric is expanded in terms of the vector harmonics as

$$h_{ab}=0, \quad h_{ai}=rf_a\mathbb{V}_i, \quad h_{ij}=2r^2H_T\mathbb{V}_{ij}, \tag{29}$$

and the vector perturbation of the energy-momentum tensor as

$$\delta\overline{T}_{ab}=0, \quad \delta\overline{T}^a_i=r\tau^a\mathbb{V}_i, \quad \delta\overline{T}^i_j=\tau_T\mathbb{V}^i_j. \tag{30}$$

For the reason stated above, H_T and τ_T are not defined for the mode $k^2=(n-1)K$ with $K>0$.

Since the infinitesimal gauge transformation ξ has only the vector component

$$\xi_a=0, \quad \xi_i=rL\mathbb{V}_i, \tag{31}$$

the expansion coefficients of the perturbation transform as

$$\bar{\delta}f_a=-rD_a\left(\frac{L}{r}\right), \quad \bar{\delta}H_T=\frac{k}{r}L, \quad \bar{\delta}\tau_a=0, \quad \bar{\delta}\tau_T=0. \tag{32}$$

Hence, except the mode $k^2=(n-1)K$ for $K>0$, the vector perturbation is described by the three gauge-invariant variables τ_a, τ_T, and

$$F_a=f_a+\frac{r}{k}D_aH_T. \tag{33}$$

On the other hand, for the mode $k^2=(n-1)K$ with $K>0$, only the combination

$$F^{(1)}_{ab}=rD_a\left(\frac{f_b}{r}\right)-rD_b\left(\frac{f_a}{r}\right) \tag{34}$$

is gauge invariant.

From the components $\delta\overline{G}^a_i$ and $\delta\overline{G}^i_j$ of the Einstein equations we obtain the following gauge-invariant perturbation equations except the mode $k^2=(n-1)K$ with $K>0$:

$$\frac{1}{r^{n+1}}D^b\left\{r^{n+2}\left[D_b\left(\frac{F_a}{r}\right)-D_a\left(\frac{F_b}{r}\right)\right]\right\} - \frac{k^2-(n-1)K}{r^2}F_a=-2\kappa^2\tau_a, \tag{35}$$

$$\frac{k}{r^n}D_a(r^{n-1}F^a)=-\kappa^2\tau_T. \tag{36}$$

On the other hand, for the mode $k^2=(n-1)K$ with $K>0$, the second equation does not appear and the first equation is written as

$$\frac{1}{r^{n+1}}D^b(r^{n+1}F^{(1)}_{ab})=-2\kappa^2\tau_a. \tag{37}$$

3. Scalar perturbation

From the scalar harmonic functions

$$(\hat{\Delta}+k^2)\mathbb{S}=0, \tag{38}$$

we can construct the scalar-type harmonic vectors $\mathbb{S}_i$ as

$$\mathbb{S}_i=-\frac{1}{k}\hat{D}_i\mathbb{S}, \tag{39}$$

$$[\hat{\Delta}+k^2-(n-1)K]\mathbb{S}_i=0, \tag{40}$$

$$\hat{D}_i\mathbb{S}^i=k\mathbb{S}, \tag{41}$$

and the scalar-type harmonic tensors $\mathbb{S}_{ij}$ as

$$\mathbb{S}_{ij}=\frac{1}{k^2}\hat{D}_i\hat{D}_j\mathbb{S}+\frac{1}{n}\gamma_{ij}\mathbb{S}, \tag{42}$$

$$\mathbb{S}^i_i=0, \quad \hat{D}_j\mathbb{S}^j_i=\frac{n-1}{n}\frac{k^2-nK}{k}\mathbb{S}_i, \tag{43}$$

$$(\hat{\Delta}+k^2-2nK)\mathbb{S}_{ij}=0. \tag{44}$$

In contrast to the vector and tensor harmonics, a constant function becomes the normalizable $k=0$ mode for $K>0$, for which $\mathbb{S}_i$ and $\mathbb{S}_{ij}$ vanish identically. Since $\mathbb{S}_i\equiv 0$ implies $\mathbb{S}=\text{const}$, no degeneracy occurs for the scalar-type harmonic vectors except for this constant mode, and $k^2>(n-1)K$ if $k^2>0$. On the other hand, $\mathbb{S}_{ij}$ vanishes identically for $k^2=nK$. For $k^2>0$ this occurs only for $K>0$. Since the spectrum of k^2 is given by $k^2=l(l+n-1)K$ with non-negative integer l, it corresponds to the $l=1$ harmonics. For other modes $k^2>2nK$.

A scalar perturbation of the metric is expanded in terms of the scalar harmonics as

$$h_{ab}=f_{ab}\mathbb{S}, \quad h_{ai}=rf_a\mathbb{S}_i, \quad h_{ij}=2r^2(H_L\gamma_{ij}\mathbb{S}+H_T\mathbb{S}_{ij}), \tag{45}$$

and a scalar perturbation of the energy-momentum tensor as

$$\delta\overline{T}_{ab}=\tau_{ab}\mathbb{S}, \quad \delta\overline{T}^a_i=r\tau^a\mathbb{S}_i, \quad \delta\overline{T}^i_j=\delta\overline{P}\delta^i_j\mathbb{S}+\tau_T\mathbb{S}^i_j. \tag{46}$$

In these expansions terms corresponding to H_T and τ_T for $k^2=nK>0$ and those corresponding to f_a, H_T, τ_a, and τ_T for $k^2=0$ do not exist.

For $k^2(k^2-nK)\neq 0$, under the infinitesimal gauge transformation

$$\xi_a=T_a\mathbb{S}, \quad \xi_i=rL\mathbb{S}_i, \tag{47}$$

these expansion coefficients transform as

$$\bar{\delta}f_{ab}=-D_aT_b-D_bT_a, \tag{48}$$

$$\bar{\delta}f_a=-rD_a\left(\frac{L}{r}\right)+\frac{k}{r}T_a, \tag{49}$$

$$\bar{\delta}X_a=T_a, \tag{50}$$

$$\bar{\delta}H_L=-\frac{k}{nr}L-\frac{D^ar}{r}T_a, \tag{51}$$

$$\bar{\delta}H_T=\frac{k}{r}L, \tag{52}$$

$$\bar{\delta}\tau_{ab}=-T^cD_c\overline{T}_{ab}-\overline{T}_{ac}D_bT^c-\overline{T}_{bc}D_aT^c, \tag{53}$$

$$\bar{\delta}\tau_a=\frac{k}{r}(\overline{T}_{ab}T^b-\overline{P}T_a), \tag{54}$$

$$\bar{\delta}(\delta\overline{P})=-T^aD_a\overline{P}, \tag{55}$$

$$\bar{\delta}\tau_T=0, \tag{56}$$

where X_a is defined as

$$X_a=\frac{r}{k}\left(f_a+\frac{r}{k}D_aH_T\right). \tag{57}$$

Hence, in addition to τ_T we can construct five independent gauge-invariant quantities as

$$F=H_L+\frac{1}{n}H_T+\frac{1}{r}D^arX_a, \tag{58}$$

$$F_{ab}=f_{ab}+D_aX_b+D_bX_a, \tag{59}$$

$$\Sigma_{ab}=\tau_{ab}+\overline{T}^c_bD_aX_c+\overline{T}^c_aD_bX_c+X^cD_c\overline{T}_{ab}, \tag{60}$$

$$\Sigma_a=\tau_a-\frac{k}{r}(\overline{T}^b_aX_b-\overline{P}X_a), \tag{61}$$

$$\Sigma=\delta\overline{P}+X^aD_a\overline{P}. \tag{62}$$

On the other hand, for the modes $k^2(k^2-nK)=0$, these become gauge dependent if we define them by setting undefined variables to zero.

From the components $\delta\overline{G}_{ab}$, $\delta\overline{G}^a_i$, $\delta\overline{G}^i_i$, and the traceless part of δG^i_j of the Einstein equations, we obtain the following four gauge-invariant perturbation equations for modes $k^2(k^2-nK)\neq 0$:

$$
\begin{aligned}
&-\Box F_{ab}+D_aD_cF^c_b+D_bD_cF^c_a+n\frac{D^c r}{r}(-D_cF_{ab}+D_aF_{cb}+D_bF_{ca})+{}^mR^c_aF_{cb}+{}^mR^c_bF_{ca}-2\,{}^mR_{acbd}F^{cd}\\
&+\left(\frac{k^2}{r^2}-\overline{R}+2\Lambda\right)F_{ab}-D_aD_bF^c_c-2n\left(D_aD_bF+\frac{1}{r}D_arD_bF+\frac{1}{r}D_brD_aF\right)-\Bigg[D_cD_dF^{cd}+\frac{2n}{r}D^crD^dF_{cd}\\
&+\left(-{}^mR^{cd}+\frac{2n}{r}D^cD^dr+\frac{n(n-1)}{r^2}D^crD^dr\right)F_{cd}-2n\Box F-\frac{2n(n+1)}{r}Dr\cdot DF\\
&+2(n-1)\frac{k^2-nK}{r^2}F-\Box F^c_c-\frac{n}{r}Dr\cdot DF^c_c+\frac{k^2}{r^2}F^c_c\Bigg]g_{ab}=2\kappa^2\Sigma_{ab},
\end{aligned}
\tag{63}
$$

$$
\frac{k}{r}\left[-\frac{1}{r^{n-2}}D_b(r^{n-2}F^b_a)+rD_a\left(\frac{1}{r}F^b_b\right)+2(n-1)D_aF\right]=2\kappa^2\Sigma_a,
\tag{64}
$$

$$
\begin{aligned}
&-\frac{1}{2}D_aD_bF^{ab}-\frac{n-1}{r}D^arD^bF_{ab}+\left(\frac{1}{2}\,{}^mR^{ab}-\frac{(n-1)(n-2)}{2r^2}D^arD^br-(n-1)\frac{D^aD^br}{r}\right)F_{ab}\\
&+\frac{1}{2}\Box F^c_c+\frac{n-1}{2r}Dr\cdot DF^c_c-\frac{n-1}{2n}\frac{k^2}{r^2}F^c_c+(n-1)\Box F+\frac{n(n-1)}{r}Dr\cdot DF\\
&-\frac{(n-1)(n-2)}{n}\frac{k^2-nK}{r^2}F=\kappa^2\Sigma,
\end{aligned}
\tag{65}
$$

$$
-\frac{k^2}{2r^2}[2(n-2)F+F^a_a]=\kappa^2\tau_T.
\tag{66}
$$

For the exceptional case $k^2=nK>0$ Eq. (66) does not exist, and for the case $k^2=0$ Eqs. (64) and (66) do not appear. The other equations still hold although each variable is gauge dependent.

Here, note that from the Bianchi identities not all of these equations are independent, and some combinations of them yield the energy-momentum conservation law for the bulk matter perturbation. For example, if we eliminate $D_bF^b_a$ and F^a_a in Eq. (65) using Eqs. (64) and (66), we obtain

$$
\begin{aligned}
&\frac{1}{r^{n+1}}D_a(r^{n+1}\Sigma^a)-\frac{k}{r}\Sigma+\frac{n-1}{n}\frac{k^2-nK}{kr}\tau_T\\
&+\frac{k}{2r}(\overline{T}^{ab}F_{ab}-\overline{P}F^a_a)=0.
\end{aligned}
\tag{67}
$$

This is just the equation $\delta(\overline{\nabla}_M\overline{T}^M_i)=0$. Similarly, applying the same procedure to the divergence of Eq. (63), we obtain the equation $\delta(\overline{\nabla}_M\overline{T}^M_a)=0$, which is expressed as

$$
\begin{aligned}
&\frac{1}{r^n}D_b[r^n(\Sigma^b_a-\overline{T}^c_aF^b_c)]+\frac{k}{r}\Sigma_a-n\frac{D_ar}{r}\Sigma+\overline{T}^b_aD_bF\\
&-\overline{P}D_aF+\frac{1}{2}(\overline{T}^b_aD_bF^c_c-\overline{T}^{bc}D_aF_{bc})=0.
\end{aligned}
\tag{68}
$$

Thus, naively speaking, only $m(m-1)/2$ components of Eq. (63) are independent under Eqs. (64) and (66), provided that the bulk energy-momentum conservation laws (67) and (68) are satisfied. However, it is in general difficult to extract such a component explicitly.

III. JUNCTION CONDITION

In the brane world model the bulk spacetime $\mathcal{M}$ has one or two boundaries, and we live in a boundary Σ. Hence the intrinsic geometry of Σ is determined by the continuity of the bulk metric $\overline{g}_{MN}$ and is described by the induced metric $g_{\mu\nu}$. The intrinsic metric $g_{\mu\nu}$ determined in this way, however, is dependent on the location of the boundary Σ in the bulk spacetime even if the geometry of the bulk spacetime is given. Furthermore, in the spacetime with boundaries, the bulk geometry is not uniquely determined by an initial condition unless some appropriate boundary condition is imposed at Σ. Thus, in order for the brane world model to be well formulated, we must give some additional prescription to determine the motion of branes and the boundary condition at the branes for the bulk geometry.

In the brane world models proposed so far, this prescription is obtained by assuming that the bulk spacetime with boundaries is obtained from a spacetime $\tilde{\mathcal{M}}$ with $\mathbb{Z}_2$ symmetry by identifying points connected by the corresponding $\mathbb{Z}_2$ transformation. The boundaries correspond to fixed points of the transformation in the original covering spacetime $\tilde{\mathcal{M}}$. This implies that the hypersurface in $\tilde{\mathcal{M}}$ corresponding to a boundary Σ is in general a singular surface in the sense that the extrinsic curvatures $K_{\mu\nu}$ of Σ on its two sides have the same absolute value but their signs are different. Such a singular spacetime is obtained when the surface has an intrinsic energy-momentum with finite surface density $T_{\mu\nu}$.

As is shown by Israel [47], this energy-momentum surface density is related to the difference of the extrinsic curvature on the two sides of the singular surface Σ. If we define $K_{\mu\nu}$ in terms of the unit normal n_M to Σ as

$$K_{\mu\nu}=-\bar{\nabla}_\mu n_\nu, \tag{69}$$

and denote its value on the side in the direction of n^M as $K_{+\ \mu\nu}$ and that on the other side as $K_{-\ \mu\nu}$, this relation is written as

$$K_{+\ \nu}^{\ \mu}-K_{-\ \nu}^{\ \mu}=\kappa^2\left(T^\mu_\nu-\frac{1}{n}T\delta^\mu_\nu\right), \tag{70}$$

where the dimension of Σ is $n+1$. In the brane world model, if we choose the normal vector so that it is directed toward the inside of the bulk spacetime, $K_{+\ \nu}^{\ \mu}=-K_{-\ \nu}^{\ \mu}=K^\mu_\nu$. Hence the junction condition can be rewritten as

$$\kappa^2 T^\mu_\nu=2(K^\mu_\nu-K\delta^\mu_\nu). \tag{71}$$

Thus, when the intrinsic dynamics of matter in the brane is given, the motion of brane is constrained by this junction condition.

In this section we express the perturbation of the above junction condition in terms of gauge-invariant variables. We consider only the case in which the unperturbed geometry of the brane is spatially homogeneous and isotropic. This implies the case $m=2$ for the bulk spacetime, i.e., $\mathcal{M}=\mathcal{N}^2\times\mathcal{K}^n$ locally, and the brane is represented by a manifold

$$\Sigma=\mathbb{R}\times\mathcal{K}^n\ni(\tau,x^i)=(x^\mu), \tag{72}$$

where $\mathcal{K}^n$ corresponds to the maximally symmetric space in the unperturbed background.

A. Constraints

The junction condition (71) together with the Hamiltonian constraint and the momentum constraint for the bulk spacetime gives relations between quantities intrinsic to the brane and the bulk energy-momentum density. First, from the momentum constraint

$$\nabla_\nu(K^\nu_\mu-K\delta^\nu_\mu)=-\kappa^2\overline{T}_{\mu\perp}, \tag{73}$$

where ∇ is the covariant derivative with respect to the induced metric $g_{\mu\nu}$ on Σ, and $\perp$ denotes the component along n, we obtain

$$\nabla_\nu T^\nu_\mu=-2\overline{T}_{\mu\perp}. \tag{74}$$

Thus when the bulk spacetime is vacuum, the intrinsic energy-momentum tensor is conserved.

Secondly, from the Hamiltonian constraint

$$K^2-K^\mu_\nu K^\nu_\mu-R=2\kappa^2\overline{T}_{\perp\perp}-2\Lambda, \tag{75}$$

where R is the Ricci scalar of Σ, we obtain

$$-R-\frac{\kappa^4}{4}\left(T^\mu_\nu T^\nu_\mu-\frac{1}{n}T^2\right)=2\kappa^2\overline{T}_{\perp\perp}-2\Lambda. \tag{76}$$

This implies that the expansion law of the brane universe is different from the one without the extra dimension for which the relation

$$(n-1)R=-2\kappa^2 T \tag{77}$$

holds if the cosmological constant is included in $T_{\mu\nu}$.

B. Unperturbed brane motion

In the unperturbed background the brane motion is described by the dependence of the y^a coordinates on the proper time τ of Σ, i.e., the set of functions $y^a(\tau)$. We define the unit timelike vector u^a by $u^a=\dot{y}^a$. Here and from now on the overdot denotes differentiation with respect to the proper time τ. The unit normal to Σ in the unperturbed background is uniquely determined by u as

$$n_a=-\epsilon_{ab}u^b, \quad u_a=-\epsilon_{ab}n^b. \tag{78}$$

The extrinsic curvature is calculated as

$$K_{\tau\tau}=n_b u^a D_a u^b, \quad K_{\tau i}=0, \quad K^i_j=-\frac{D_\perp r}{r}\delta^i_j, \tag{79}$$

and the unperturbed energy-momentum tensor of the brane is written as

$$T_{\tau\tau}=\rho, \quad T_{\tau i}=0, \quad T^i_j=p\,\delta^i_j. \tag{80}$$

Hence the junction condition is expressed as

$$\frac{D_\perp r}{r}=-\frac{\kappa^2}{2n}\rho, \tag{81}$$

$$(n-1)\frac{D_\perp r}{r}-K^\tau_\tau=\frac{\kappa^2}{2}p. \tag{82}$$

The first of these equations implies that the energy density of our universe is determined by the brane motion. If the equation of state of the cosmic matter is given, these equations determine the brane motion because K^τ_τ represents the acceleration of the brane. Further, by differentiating the first equation by τ and eliminating K^τ_τ, we obtain

$$\dot{\rho}+n(\rho+p)\frac{\dot{a}}{a}=2u^a\overline{T}_{a\perp}. \tag{83}$$

This equation coincides with Eq. (74) obtained from the momentum constraint. Here a denotes the value of r at the brane and represents the cosmic scale factor of the Robertson-Walker universe on the brane whose metric is written as

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu=-d\tau^2+a^2(\tau)d\sigma_n^2. \tag{84}$$

C. Perturbation of the junction condition

The extrinsic curvature of the brane depends on the configuration of the brane as well as on the bulk geometry. If we denote the deviation of the brane configuration from the background one as

$$\delta z^M=Z^M(\tau,x)=Z^M_\parallel+Z_\perp n^M, \tag{85}$$

where $Z^M_\parallel$ is the component of Z^M parallel to the brane, the perturbation of the extrinsic curvature is in general expressed as

$$\delta K_{\mu\nu}=(\mathcal{L}_{Z_\parallel}K)_{\mu\nu}+\nabla_\mu\nabla_\nu Z_\perp+(\overline{R}_{\perp\mu\perp\nu}-K^\lambda_\mu K_{\lambda\nu})Z_\perp$$
$$+n_a\delta\overline{\Gamma}^a_{\mu\nu}+\frac{1}{2}h_{ab}n^a n^b K_{\mu\nu}. \tag{86}$$

The perturbation of the intrinsic metric of the brane also depends both on the perturbation of the bulk metric and on the brane configuration. To be explicit, these relations are expressed as

$$\delta g_{\tau\tau}=h_{ab}u^a u^b-2\dot{Z}^\tau+2K^\tau_\tau Z_\perp, \tag{87}$$

$$\delta g_{\tau i}=h_{ai}u^a-\hat{D}_i Z^\tau+a^2(Z_i/a^2)^{\cdot}, \tag{88}$$

$$\delta g_{ij}=h_{ij}+\hat{D}_i Z_j+\hat{D}_j Z_i+2a^2\gamma_{ij}\frac{D_a r}{r}Z^a. \tag{89}$$

To proceed further, we must treat the tensor, the vector, and the scalar perturbations separately.

1. Tensor perturbation

For the tensor perturbation the perturbation of the intrinsic metric of the brane is expanded in terms of the tensor harmonics as

$$\delta g_{\tau\tau}=0,\quad \delta g_{\tau i}=0,\quad \delta g_{ij}=2a^2 h_T\mathbb{T}_{ij}. \tag{90}$$

Since $Z^M=0$ for the tensor perturbation, h_T is simply related to the bulk perturbation as $h_T=H_T$.

The perturbation of the energy-momentum tensor intrinsic to the brane is also expressed by a single expansion coefficient representing the anisotropic stress perturbation of the brane as

$$\delta T^\tau_\tau=0,\quad \delta T^\tau_i=0,\quad \delta T^i_j=\pi_T\mathbb{T}^i_j. \tag{91}$$

On the other hand, the harmonic expansion of Eq. (86) yields

$$\delta K^\tau_\tau=0,\quad \delta K^\tau_i=0,\quad \delta K^i_j=-D_\perp H_T\mathbb{T}^i_j. \tag{92}$$

Hence the junction condition (71) reduces to the single equation

$$D_\perp H_T=-\frac{\kappa^2}{2}\pi_T. \tag{93}$$

In general, the anisotropic stress perturbation is not an independent dynamical variable and is expressed by other dynamical variables when the model is specified. In particular, in the linear perturbation framework, it is natural to assume that $\pi_T=0$ for the tensor perturbation. In this case Eq. (93) gives a Neumann-type boundary condition for the wave equation of H_T obtained in Sec. II C 1. Thus we obtain a well-posed system describing the evolution of perturbations.

2. Vector perturbation

For the vector perturbation the perturbation of the brane configuration is expressed in the harmonic expansion as

$$Z^\tau=0,\quad Z_\perp=0,\quad Z_i=aZ\mathbb{V}_i. \tag{94}$$

On the other hand the intrinsic metric perturbation is expressed as

$$\delta g_{\tau\tau}=0,\quad \delta g_{\tau i}=-a\beta\mathbb{V}_i,\quad \delta g_{ij}=2a^2 h_T\mathbb{V}_{ij}. \tag{95}$$

Hence we obtain the relations

$$\beta=-f_\parallel-a\left(\frac{Z}{a}\right)^{\cdot}, \tag{96}$$

$$h_T=H_T-\frac{k}{a}Z. \tag{97}$$

If we construct the standard gauge-invariant variables for the intrinsic perturbation from these metric perturbation variables and the matter perturbation variables defined by

$$\delta T^\tau_\tau=0,\quad \delta T^\tau_i=a(\rho+p)(v-\beta)\mathbb{V}_i,\quad \delta T^i_j=\pi_T\mathbb{V}^i_j, \tag{98}$$

we obtain

$$\sigma_g=\frac{a}{k}\dot{h}_T-\beta=F_\parallel, \tag{99}$$

$$V=v-\beta=v-\frac{a}{k}\dot{h}_T+F_\parallel. \tag{100}$$

Note that Z disappears in these expressions because it corresponds to an intrinsic diffeomorphism of the brane. On the other hand, in the present case the perturbation of the extrinsic curvature is expressed as

$$\delta K^\tau_\tau = 0, \quad \delta K^\tau_i = \frac{a^2}{2}\epsilon^{ab}D_a\left(\frac{F_b}{r}\right)\mathbb{V}_i, \quad \delta K^i_j = -\frac{k}{a}F_\perp \mathbb{V}^i_j. \tag{101}$$

Inserting these equations into Eq. (71), we obtain the following two equations:

$$\kappa^2(\rho+p)V = r\epsilon^{ab}D_a\left(\frac{F_b}{r}\right), \tag{102}$$

$$\kappa^2\pi_T = -2\frac{k}{a}F_\perp. \tag{103}$$

The first of these gives the expression for the intrinsic perturbation variable in terms of the bulk perturbation variable. The second can be regarded as the boundary condition on the bulk perturbation equations in Sec. II C 2. It will be shown later that it gives a Dirichlet-type boundary condition when the bulk spacetime is vacuum.

3. Scalar perturbation

For the scalar perturbation for which

$$Z^\tau = Z^\tau \mathbb{S}, \quad Z_\perp = Z_\perp \mathbb{S}, \quad Z_i = aZ\mathbb{S}_i, \tag{104}$$

the harmonic expansion coefficients for the intrinsic metric perturbation defined by

$$\delta g_{\tau\tau} = -2\alpha\mathbb{S}, \quad \delta g_{\tau i} = -a\beta\mathbb{S}_i,$$
$$\delta g_{ij} = 2a^2(h_L\mathbb{S}\gamma_{ij} + h_T\mathbb{S}_{ij}) \tag{105}$$

are related to those for the bulk metric perturbation as

$$\alpha = -\frac{1}{2}F_{\parallel\parallel} + \dot{Y}^\tau - K^\tau_\tau Y_\perp, \tag{106}$$

$$\beta = -\frac{k}{a}Y^\tau + \frac{a}{k}\dot{h}_T, \tag{107}$$

$$h_L = H_L + \frac{k}{na}Z + \frac{\dot{a}}{a}Z^\tau + \frac{D_\perp r}{r}Z_\perp, \tag{108}$$

$$h_T = H_T - \frac{k}{a}Z, \tag{109}$$

where

$$Y^\tau = Z^\tau - X^\tau, \quad Y_\perp = Z_\perp - X_\perp. \tag{110}$$

Hence the intrinsic gauge-invariant variables constructed from these are related to the bulk gauge-invariant variables as

$$\Phi = h_L + \frac{1}{n}h_T - \frac{\dot{a}}{k}\sigma_g = F + \frac{D_\perp r}{r}Y_\perp, \tag{111}$$

$$\Psi = \alpha - \frac{1}{k}(a\sigma_g)^{\cdot} = -\frac{1}{2}F_{\parallel\parallel} - K^\tau_\tau Y_\perp, \tag{112}$$

where

$$\sigma_g = \frac{a}{k}\dot{h}_T - \beta. \tag{113}$$

In addition to these, we can construct gauge-invariant variables from the harmonic expansion of the intrinsic matter perturbation

$$\delta T^\tau_\tau = -\delta\rho\mathbb{S}, \quad \delta T^\tau_i = a(\rho+p)(v-\beta)\mathbb{S}_i,$$
$$\delta T^i_j = \delta p\mathbb{S}\delta^i_j + \pi_T\mathbb{S}^i_j \tag{114}$$

as

$$V = v - \frac{a}{k}\dot{h}_T, \tag{115}$$

$$\rho\Delta = \delta\rho - \frac{a}{k}\dot{\rho}(v-\beta), \tag{116}$$

$$\Gamma = \delta p - c_s^2\delta\rho. \tag{117}$$

Among these equations the last represents the amplitude of entropy perturbation of the matter.

The perturbation of the extrinsic curvature is now expressed in terms of the gauge-invariant variables as

$$\delta K^\tau_\tau = \Big[-\frac{1}{2}\dot{F}_{\perp\parallel} + \frac{1}{2}n_aD_bF^{ab} - \frac{1}{2}D_\perp F^a_a + \frac{1}{2}K^\tau_\tau F_{\parallel\parallel}$$
$$+ \dot{K}^\tau_\tau Y^\tau - \ddot{Y}_\perp + \left(\frac{1}{2}{}^2R + K^2_{\tau\tau}\right)Y_\perp\Big]\mathbb{S}, \tag{118}$$

$$\delta K^\tau_i = k\left[\frac{1}{2}F_{\perp\parallel} - \left(K^\tau_\tau + \frac{D_\perp r}{r}\right)Y^\tau + a\left(\frac{Y_\perp}{a}\right)^{\cdot}\right]\mathbb{S}_i, \tag{119}$$

$$\delta K^i_j = \Bigg\{-D_\perp F - \frac{\dot{a}}{a}F_{\perp\parallel} + \frac{D_\perp r}{2r}F_{\perp\perp} - \left(\frac{D_\perp r}{r}\right)^{\cdot}Y^\tau - \frac{\dot{a}}{a}\dot{Y}_\perp$$
$$-\left[\frac{k^2}{na^2} + \frac{n^an^bD_aD_br}{r} - \left(\frac{D_\perp r}{r}\right)^2\right]Y_\perp\Bigg\}\mathbb{S}\delta^i_j$$
$$+\frac{k^2}{a^2}Y_\perp\mathbb{S}^i_j. \tag{120}$$

Hence the junction condition (71) yields the following four relations among the gauge-invariant variables for the bulk and the brane:

$$D_\perp F+\frac{\dot a}{a}F_{\perp\parallel}-\frac{D_\perp r}{2r}F_{\perp\perp}+\frac{\dot a}{a}\dot Y_\perp +\left[\frac{k^2}{na^2}+\frac{n^a n^b D_a D_b r}{r}-\left(\frac{D_\perp r}{r}\right)^2\right]Y_\perp =-\frac{\kappa^2}{2n}\left(\rho\Delta+\frac{a}{k}\dot\rho V\right), \tag{121}$$

$$-\frac{1}{2}\dot F_{\perp\parallel}+\frac{1}{2}n_a D_b F^{ab}-\frac{1}{2}D_\perp F^a_a+\frac{1}{2}K^\tau_\tau F_{\parallel\parallel}-\ddot Y_\perp +\left(\frac{1}{2}{}^2R+K^2_{\tau\tau}\right)Y_\perp =-\frac{\kappa^2}{2}\left[\Gamma+\left(\frac{n-1}{n}+c_s^2\right)\left(\rho\Delta+\frac{a}{k}\dot\rho V\right)\right], \tag{122}$$

$$\frac{1}{2}F_{\perp\parallel}+a\left(\frac{Y_\perp}{a}\right)^{\cdot}=\frac{\kappa^2}{2}\frac{a}{k}(\rho+p)V, \tag{123}$$

$$2\frac{k^2}{a^2}Y_\perp=\kappa^2\pi_T. \tag{124}$$

These conditions have some features that are not shared by the vector and tensor perturbations. First, although the variables Z^τ and Z disappear as in the other types of perturbation, $Y_\perp=Z_\perp-X_\perp$ remains in the final expressions. This is because $Y_\perp$ defines the gauge-invariant amplitude of the perturbation of the brane motion, unlike Z^τ and Z, which correspond to intrinsic diffeomorphism of the brane. Secondly, from the last equation one finds that a condition on the anisotropic stress perturbation does not give any boundary condition on the bulk perturbation. Instead, it constrains the perturbation, which cannot be simply attributed either to the intrinsic structure of the brane or to the bulk.

Then where does the boundary condition come from? We can find an answer to this question by closely inspecting the structure of the above equations. First, note that the gauge invariants representing the perturbation of the intrinsic geometry of the brane are determined by the bulk variables through Eqs. (111) and (112). Meanwhile, Eqs. (121) and (123) yield the expressions of the gauge-invariants Δ and V for the intrinsic matter in terms of the bulk variables. Inserting these expressions into Eq. (122), we obtain an expression for the amplitude of the entropy perturbation Γ in terms of the bulk variables. Like π_T, Γ is not a dynamical variable and should be expressed in terms of Δ, V, and other intrinsic dynamical perturbation variables whose dynamics is determined when a model of the intrinsic matter is given. Hence we should regard Eq. (122) or an equation derived from it by eliminating the independent dynamical variable as the boundary condition on the bulk perturbation. This means that the boundary condition is dependent on the type of intrinsic matter perturbation, e.g., adiabatic or isocurvature. In the next section we will show that this boundary condition becomes nonlocal with respect to the time coordinate of the brane.

IV. MASTER VARIABLE

As was shown in Sec. II, the metric perturbation in the bulk spacetime for the tensor perturbation is described by the single gauge-invariant variable H_T, and it obeys a simple wave equation. Further, the junction condition gives a simple boundary condition on it. In contrast, for the vector and the scalar perturbations, the bulk perturbation is described by multicomponent variables and their equations have structures too complicated to be solved. Fortunately, in the case in which the unperturbed background of the bulk spacetime is vacuum (and the two-dimensional orbit space $\mathcal{N}^2$ is maximally symmetric for the scalar perturbation), we can find a single master variable for the bulk perturbation and reduce the perturbation equation to a single wave equation. In this section we analyze the structure of the junction condition in terms of that master variable.

A. Vacuum background

We consider the case in which $m=2$ in the notation of Sec. II A and $\overline{T}_{MN}=0$. Hence the bulk spacetime is $(n+2)$ dimensional and has the isometry group corresponding to the n-dimensional maximally symmetric space in the unperturbed background. In this case, from the generalized Birkhoff theorem, the geometry of the background spacetime is given by either of the following two families of solutions.

(1) Pure product type $(Dr=0)$:

$$dS^2(\sqrt{n/2\Lambda})\times S^n[\sqrt{n(n-1)/2\Lambda}],\quad \Lambda>0, \tag{125}$$

$$AdS^2(\sqrt{n/2|\Lambda|})\times H^n[\sqrt{n(n-1)/2|\Lambda|}],\quad \Lambda<0, \tag{126}$$

$$E^{n+1,1},\quad \Lambda=0. \tag{127}$$

(2) Schwarzschild type $(Dr\neq 0)$:

$$d\overline{s^2}=-U(r)dt^2+\frac{dr^2}{U(r)}+r^2d\sigma_n^2, \tag{128}$$

$$U(r)=K-\frac{2M}{r^{n-1}}-\lambda r^2, \tag{129}$$

$$\lambda=\frac{2\Lambda}{n(n+1)}. \tag{130}$$

The derivation of the solutions of the first family and their physical meaning were given by Nariai [48,49]. For the second family the following simple formulas hold:

$${}^2R=2\lambda+\frac{2n(n-1)M}{r^{n+1}}, \tag{131}$$

$$\frac{\Box r}{r}=-2\lambda+\frac{2(n-1)M}{r^{n+1}}, \tag{132}$$

$$\frac{K-(Dr)^2}{r^2}=\lambda+\frac{2M}{r^{n+1}}. \tag{133}$$

In particular, when the mass parameter M vanishes, the quantities on the left-hand side of these equations become constant, and the spacetime coincides with dS^{n+2}, AdS^{n+2}, and $E^{n+1,1}$ for $\Lambda>0$, <0, and $=0$, respectively.

The background configuration of the brane in the Schwarzschild-type background geometry is determined by solving Eqs. (81) and (82) with Eq. (133). In particular, from Eq. (83), the same energy equation as in the no-extra-dimension case holds for the energy density of the brane,

$$\dot{\rho}=-n(\rho+p)\frac{\dot{a}}{a}. \tag{134}$$

In contrast, from Eqs. (81) and (133), the decomposition $(Dr)^2=-\dot{a}^2+(D_\perp r)^2$ yields

$$\left(\frac{\dot{a}}{a}\right)^2=\left(\frac{\kappa^2}{2n}\rho\right)^2-\frac{K}{a^2}+\lambda+\frac{2M}{a^{n+1}}, \tag{135}$$

which is different from the standard expansion equation even in the case $M=0$ in the point that ρ is replaced by ρ^2. These equations form a closed system and determine ρ and a as functions of the intrinsic proper time τ. When these functions are given, the embedding of the brane $[t(\tau),r(\tau)]$ is determined by $r(\tau)=a(\tau)$ and a solution of the equation

$$\left(\frac{dt}{d\tau}\right)^2=\frac{\dot{a}^2+U(a)}{U(a)^2}. \tag{136}$$

In contrast to the Schwarzschild case, the background brane configuration becomes quite special for the pure product type background spacetime. In fact, since $r=\mathrm{const}$ in this case, it follows from Eqs. (81) and (82) that ρ should vanish and K^τ_τ is proportional to p. Since it is natural to assume $p=0$ for $\rho=0$, the latter condition implies that the background brane motion is represented by a geodesic in the two-dimensional constant-curvature space $\mathcal{N}$.

B. Expression in terms of a master variable

1. Tensor perturbation

For the tensor perturbation the system is already described by a single variable. For completeness we recapitulate the equations for the tensor perturbation in the vacuum case. We need no further symmetry assumption on the unperturbed bulk geometry.

The perturbation equation for the bulk is given by the homogeneous wave equation

$$-\Box H_T-\frac{n}{r}Dr\cdot DH_T+\frac{k^2+2K}{r^2}H_T=0. \tag{137}$$

The junction condition gives the boundary condition

$$D_\perp H_T=-\frac{\kappa^2}{2}\pi_T. \tag{138}$$

2. Vector perturbation

For the vector perturbation on the vacuum bulk spacetime τ_T vanishes. Hence for $k^2>(n-1)K$, taking account of the fact that the orbit space $\mathcal{N}$ is two dimensional, Eq. (36) implies that F_a is written in terms of a function Ω as

$$F^a=\frac{1}{r^{n-1}}\epsilon^{ab}D_b\Omega. \tag{139}$$

Hence the perturbation Eq. (35) for F_a is expressed in terms of Ω as

$$D^a\left[r^{n+2}D_b\left(\frac{D^b\Omega}{r^n}\right)-[k^2-(n-1)K]\Omega\right]=0. \tag{140}$$

The bulk perturbation equation is thus reduced to the single equation for the master variable Ω given by

$$\Box\Omega-\frac{n}{r}Dr\cdot D\Omega-\frac{k^2-(n-1)K}{r^2}\Omega=\frac{C}{r^2}, \tag{141}$$

where C is an integration constant, which can be set to zero by redefinition of Ω.

On the other hand, for the mode $k^2=(n-1)K>0$, the gauge-invariant $F^{(1)}_{ab}$ has a single independent component and is expressed as

$$F^{(1)}_{ab}=\epsilon_{ab}\Omega^{(1)}. \tag{142}$$

In terms of $\Omega^{(1)}$ Eq. (37) is expressed as

$$\epsilon_{ab}D^b(r^{n+1}\Omega^{(1)})=0. \tag{143}$$

This equation is easily solved to yield

$$F^{(1)}_{ab}=\epsilon_{ab}\frac{C}{r^{n+1}}, \tag{144}$$

where C is an integration constant.

For $k^2>(n-1)K$, the junction conditions (102) and (103) are expressed in terms of Ω as

$$\kappa^2\pi_T=-2\frac{k}{a^n}\dot{\Omega}, \tag{145}$$

$$\kappa^2(\rho+p)V=\frac{1}{a^{n+1}}[k^2-(n-1)K]\Omega, \tag{146}$$

$$\sigma_g=\frac{1}{a^{n-1}}D_\perp\Omega. \tag{147}$$

The first equation gives a Dirichlet-type boundary condition on Ω. The other two equations give expressions for the intrinsic gauge-invariant variables V and σ_g in terms of Ω. Thus the initial value problem is well posed for this system.

The situation for the exceptional mode $k^2=(n-1)K>0$ is slightly different. For this mode we do not have the equation for π_T. However, this does not cause trouble because $F_{ab}^{(1)}$ is explicitly given. The junction condition determines the only nontrivial gauge invariant intrinsic to the brane, V, as

$$\kappa^2 a^{n+1}(\rho+p)V=C. \tag{148}$$

Here note that the momentum constraint (74) reduces to the conservation of $T_{\mu\nu}$ in the present case and its perturbation gives

$$\frac{1}{a^{n+1}}[a^{n+1}(\rho+p)V]^{\cdot}=\frac{k^2-(n-1)K}{2ak}\pi_T. \tag{149}$$

It is easily checked that this equation is consistent with the above junction conditions. Thus the evolution of V is intrinsically determined and coincides with the no-extra-dimension case. In contrast, the evolution of σ_g is determined only by solving the master equation, in contrast to the no-extra-dimension case in which σ_g is related to V as [46]

$$2\kappa'^2 a^2(\rho+p)V=-[k^2-(n-1)K]\sigma_g, \tag{150}$$

where κ'^2 denotes the gravitational constant on the brane.

3. Scalar perturbation

As shown in Sec. II C 3, for the scalar perturbation on the vacuum background, Eq. (65) is automatically satisfied if Eqs. (63), (64), and (66) hold. Among the latter, Eqs. (64) and (66) are written as

$$F_a^a=-2(n-2)F, \tag{151}$$

$$D_b(r^{n-2}F_a^b)=2D_a(r^{n-2}F). \tag{152}$$

Here note that for the exceptional modes $k^2=0$ and $k^2=nK>0$ we do not have one or both of them. However, we can still assume that these equations hold by regarding missing equations as gauge conditions to fix the residual gauge freedom.

As was shown by Mukohyama, in the case that the two-dimensional orbit space $\mathcal{N}$ is a constant-curvature space, the general solutions to these equations are written in terms of a master variable Ω as

$$\tilde{F}=r^{n-2}F=\frac{1}{2n}(\Box+2\lambda)\Omega, \tag{153}$$

$$\tilde{F}_{ab}=r^{n-2}F_{ab}=D_aD_b\Omega-\left(\frac{n-1}{n}\Box+\frac{n-2}{n}\lambda\right)\Omega g_{ab}. \tag{154}$$

(See Appendix C for a simpler proof.)

On the other hand, for the background geometry (128) with $M=0$, Eq. (63) is reduced to the equation

$$\begin{aligned}-\Box F_{ab}-\frac{n}{r}Dr\cdot DF_{ab}+\left(\frac{k^2}{r^2}-2\lambda\right)F_{ab}&\\
+\frac{D^cr}{r}\Big[2(D_aF_{cb}+D_bF_{ca})&\\
+(n-2)\left(\frac{D_ar}{r}F_{cb}+\frac{D_br}{r}F_{ca}\right)\Big]&\\
=4\left(\frac{D_br}{r}D_aF+\frac{D_ar}{r}D_bF+(n-2)\frac{D_arD_br}{r^2}F\right)&.\end{aligned} \tag{155}$$

In terms of the master variable Ω, this equation is written as

$$(D_aD_b+\lambda g_{ab})E(\Omega)=0, \tag{156}$$

where

$$E(\Omega)\equiv r^2\left[\Box\Omega-\frac{n}{r}Dr\cdot D\Omega-\left(\frac{k^2-nK}{r^2}+(n-2)\lambda\right)\Omega\right]. \tag{157}$$

As is shown in Appendix D, the general solution of Eq. (156) is written as

$$E(\Omega)=C_0g_0(t,r)+C_1g_1(t,r)+C_2r, \tag{158}$$

where C_0, C_1, and C_2 are arbitrary constants. On the other hand, it is easy to see that the freedom in the definition of Ω is expressed in terms of a solution to $(D_aD_b+\lambda g_{ab})\omega=0$ as $\Omega\to\Omega+\omega$. Since ω is again written as $\omega=C_0'g_0(t,r)+C_1'g_1(t,r)+C_2'r$ with arbitrary constants $C_0'\sim C_2'$, the value of $E(\Omega)$ changes by the redefinition as

$$E(\omega)=\begin{cases}-(k^2-nK)(C_0'g_0+C_1'g_1)-k^2C_2'r, & K\neq 0,\\ -k^2(C_0'g_0+C_1'g_1)-(k^2C_2'-2n\lambda C_0')r, & K=0.\end{cases} \tag{159}$$

From this we immediately see that $C_0\sim C_2$ can be put to zero by an appropriate redefinition of Ω for $k^2(k^2-nK)\neq 0$. On the other hand, only C_0 and C_1 can be set to zero for $k^2=0$ and $K\neq 0$, while only C_2 can be put to zero for $k^2=nK>0$. In these cases, however, there still remains a residual gauge freedom in F and F_{ab}. As is shown in Appendix E, any solution Ω to the homogeneous equation $E(\Omega)=0$ can be set to zero by this residual gauge transformation, while the constants above that cannot be removed by the redefinition are just the gauge-invariants for the exceptional modes. Thus the gauge-equivalent classes of the solutions to the perturbed solutions form a one-dimensional space parametrized by C_2 for the mode $k^2=0$ and $K>0$ and a two-dimensional space parametrized by C_0 and C_1 for the mode $k^2=nK>0$.

From now on we consider only modes with $k^2(k^2-nK)\neq 0$. From the above argument, the master equation for these modes is always written as

$$\Box\Omega-\frac{n}{r}Dr\cdot D\Omega-\left(\frac{k^2-nK}{r^2}+(n-2)\lambda\right)\Omega=0. \tag{160}$$

In terms of the master variable, the junction conditions (121)–(124) are written as

$$rD_\perp\left(\frac{\Omega}{r}\right)=-\frac{\kappa^2}{k^2-nK}a^n\rho\Delta-\frac{\kappa^2}{k^2}a^n\pi_T, \tag{161}$$

$$(D_\perp\Omega)^\cdot+K_\tau^\tau\dot{\Omega}=\kappa^2\frac{a^{n-1}}{k}(\rho+p)V-\frac{\kappa^2}{k^2}a^{n-1}(a\pi_T)^\cdot, \tag{162}$$

$$\frac{1}{a}(aV)^\cdot=\frac{k}{a}\Psi+\frac{k}{a}\frac{\Gamma+c_s^2\rho\Delta}{\rho+p}-\frac{n-1}{n}\frac{k^2-nK}{ak}\frac{\pi_T}{\rho+p}, \tag{163}$$

$$2\frac{k^2}{a^2}Y_\perp=\kappa^2\pi_T, \tag{164}$$

where

$$\Phi=\frac{1}{2a^{n-2}}\left[-\frac{\dot{a}}{a}\dot{\Omega}+\frac{D_\perp r}{r}D_\perp\Omega+\left(\frac{k^2-nK}{na^2}+\lambda\right)\Omega\right]+\frac{D_\perp r}{r}Y_\perp, \tag{165}$$

$$\Psi=-\frac{1}{2a^{n-2}}\left[\ddot{\Omega}+\left(K_\tau^\tau+(n-1)\frac{D_\perp r}{r}\right)D_\perp\Omega-(n-1)\frac{\dot{a}}{a}\dot{\Omega}+\left(\frac{n-1}{n}\frac{k^2-nK}{a^2}+(n-2)\lambda\right)\Omega\right]-K_\tau^\tau Y_\perp. \tag{166}$$

Here, note that Eq. (163) is identical to the space component of the perturbation of the intrinsic conservation law of the energy-momentum tensor $\nabla_\mu T^\mu_\nu=0$. Further, the corresponding time component, which is written as

$$\frac{1}{a^n}(a^n\rho\Delta)^\cdot=-\frac{k}{a}(\rho+p)\left[1-n\frac{a^2}{k^2}\left(\frac{\dot{a}}{a}\right)^\cdot\right]V-n(\rho+p)\left(\dot{\Phi}-\frac{\dot{a}}{a}\Psi\right)-(n-1)\frac{k^2-nK}{k^2}\frac{\dot{a}}{a}\pi_T, \tag{167}$$

is obtained from the above junction conditions, as it should be.

As was discussed in Sec. III C 3, Eqs. (161) and (162) are the equations determining the intrinsic gauge invariants Δ and V. Hence Eq. (163), or the equation for the intrinsic entropy perturbation Γ, should be regarded as a boundary condition on the master variable. For $\pi_T=0$, this expression is given by

$$\left[rD_\perp\left(\frac{\Omega}{r}\right)\right]^{\cdot\cdot}+(2+nc_s^2)\frac{\dot{a}}{a}\left[rD_\perp\left(\frac{\Omega}{r}\right)\right]^\cdot+\left[-n(1+w)(2n-2+nw)\left(\frac{D_\perp r}{r}\right)^2+c_s^2\frac{k^2-nK}{a^2}\right]\times\left[rD_\perp\left(\frac{\Omega}{r}\right)\right]-(n-1)(1+w)\frac{k^2}{a^2}\frac{D_\perp r}{r}\Omega=\kappa^2a^{n-2}\Gamma, \tag{168}$$

where $w=p/\rho$. From this equation we immediately see that, except for the special case in which $p=-\rho$, the junction condition yields a boundary condition that is nonlocal in time.

In contrast, for the case $p=-\rho$, the junction condition yields a closed evolution equation for $rD_\perp(\Omega/r)$ or $\rho\Delta$. To be precise, $\delta\rho$ and δp becomes gauge invariant. Further, although V is ill-defined, the combination $(\rho+p)V=(\rho+p)\times(v-\beta)=\delta T_i^\tau/(a\mathbb{S}_i)$ is well-defined and can have a nonvanishing value. If we take these facts into account, the boundary condition for $\rho+p=0$ is given by the equation obtained from (168) by the replacements $c_s^2=0$, $w=-1$, and $\Gamma=\delta p$.

Even in this case, the gauge invariants Φ and Ψ representing the intrinsic perturbations of the spatial curvature and the gravitational potential of the brane are determined only by solving the wave equation for Ω under given initial data and a boundary condition. This is because we lack the relations that make the equations for intrinsic quantities closed in the no-extra-dimension case [46],

$$\kappa'^2\rho\Delta=(n-1)a^{-2}(k^2-nK)\Phi, \tag{169}$$

$$(n-2)\Phi+\Psi=-\kappa'^2a^2k^{-2}\pi_T. \tag{170}$$

Thus it may be difficult to find a natural initial condition for which the evolution law for the intrinsic perturbation becomes similar to the standard one.

V. DISCUSSION

In the present paper we have developed a gauge-invariant formalism for the perturbation of the brane world model for which the background configuration has a spatial symmetry corresponding to a maximally symmetric space with a dimension n lower than the dimension $n+m$ of the bulk spacetime. The formalism consisted of two parts. The first part gave a system of gauge-invariant equations for the perturbation of the bulk spacetime geometry. With applications to wider situations in mind, we derived the equations for generic values of n and m and for generic bulk matter. They give an extension of the formalism developed for the $n=2$ and $m=2$ case by Gerlach and Sengupta [39].

The second part was concerned with a situation specific to the brane world model in which $m=2$ and gave gauge-invariant equations for the junction condition corresponding to the $\mathbb{Z}_2$ symmetry along a brane with codimension 1. As an immediate consequence, we have shown that, when the stress perturbation intrinsic to the brane is specified or expressed in terms of other intrinsic quantities, the junction condition yields a boundary condition at the brane(s) on the evolution equation for the bulk perturbation.

In order to investigate the structure of the equations in more detail, we have introduced a master variable Ω for the bulk perturbation and reduced the bulk perturbation equations to a single wave equation for Ω in the case in which the bulk spacetime is vacuum. This reduction was already done by Mukohyama [45] in the case in which the background geometry of the bulk spacetime is maximally symmetric. Since we were able to introduce the master potential for the scalar perturbation only in the case in which the two-dimensional orbit space has a constant curvature, the master equation we obtained is the same as that derived by Mukohyama. However, the master equation for the vector and tensor perturbations is more general and holds also in the case in which the background geometry is of the Schwarzschild black hole type. We have also given a proof different from that given by Mukohyama for the existence of the master potential for the scalar perturbation.

We have also investigated the structure of the junction condition in terms of the master variable. In particular, we have shown that the boundary condition on the master potential obtained from the junction condition has a different structure depending on the type of perturbation: for the tensor and vector perturbations, the condition that the anisotropic stress perturbation vanishes yields a Neumann-type and a Dirichlet-type boundary condition, respectively, while the boundary condition for the scalar perturbation is given by a condition on the intrinsic entropy perturbation and is nonlocal in time in general.

Here, note that, although the master variable is used in an essential way in the analysis of the scalar perturbation, the introduction of the master variable is not the only way to make the problem tractable. For example, Fourier expansion of the original gauge-invariant variables in terms of time may also be used to make the equations simpler. If it works well, we can also treat the scalar perturbation in the Schwarzschild black hole type background.

Although the main purpose of the present paper is to develop a formalism, we briefly discuss here a possible consequence of the formalism for the brane world scenario. In the original Randall-Sundrum model, in which the brane is realized as a flat subspace in a five-dimensional anti–de Sitter spacetime, the bulk graviton modes which behave as massive particles inside the brane decouple from the massless mode. In our formalism this phenomenon is understood in the following way.

Since $n=3$, $K=0$, and $M=0$ in this case, in the units $\lambda=-1$, the gravitational wave in the bulk spacetime is described by H_T satisfying the wave equation

$$-\partial_t^2 H_T = -\frac{1}{r}\partial_r(r\partial_r H_T) + k^2 H_T. \tag{171}$$

Since the brane is static and located at $r=1$, the boundary condition is given by $\partial_r H_T=0$. Under the Fourier expansion with respect to the time t, the mode $H_T \propto e^{-i\omega t}$ is a solution to the equation

$$y^3 \frac{d}{dy}\left(\frac{1}{y^3}\frac{dH_T}{dy}\right) + \mu^2 H_T = 0, \tag{172}$$

where $y=1/r (1 \leq y < \infty)$ and $\mu^2=\omega^2-k^2$. If we require that the mode is normalizable in the generalized sense with respect to the natural metric $dr\, r \propto dy/y^3$, which makes the right-hand side of the above wave equation self-adjoint, the spectrum of μ^2 consists of two parts. One is the point spectrum $\mu^2=0$ for which H_T is constant. The other is the continuous spectrum $\mu^2>0$ for which H_T is proportional to $y^2 Z_2(\mu y)$ where Z_2 is a Bessel function of degree 2. Thus the general solution is written as

$$H_T = \Re\left(C e^{-ikt} + \int_0^\infty d\mu^2\, y^2 [A(\mu) J_2(\mu y) + B(\mu) N_2(\mu y)] e^{-i\omega t}\right). \tag{173}$$

The important point here is that the boundary condition is simply written as a relation between A and B. Hence the massless mode for which $A=B=0$ decouples from massive modes. If we apply the same argument to a dynamical case in which the brane is nonstatic and represents an expanding universe, the situation changes significantly. In this case the boundary condition $D_\perp H_T=0$ is expressed as a relation among A, B, and C. Hence all modes contain massive components.

Of course, since the expansion rate of the present universe is small, one might expect that there is a mode in which the amplitude of the massive component is negligible. However, such a mode contains massive components with large amplitudes in the early phase of the universe due to rapid cosmic expansion. Hence, if the initial condition of the universe is imposed in the early universe as in the argument of quantum generation of perturbations, it is in general expected that the present day universe contains a non-negligible amount of massive gravitons. The situation is quite similar to quantum particle creation due to cosmic expansion. Whether this problem is a crucial defect of the brane world model or rather provides a new model of dark matter is a very interesting problem.

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APPENDIX A: BACKGROUND QUANTITIES

1. Connection coefficients

$$\overline{\Gamma}^a_{bc} = {}^m\Gamma^a_{bc}(y), \quad \overline{\Gamma}^a_{ij} = -rD^a r\gamma_{ij},$$
$$\overline{\Gamma}^i_{aj} = \frac{D_a r}{r}\delta^i_j, \quad \overline{\Gamma}^i_{jk} = \hat{\Gamma}^i_{jk}(x). \tag{A1}$$

2. Curvature tensors

$$\overline{R}^a{}_{bcd} = {}^m R^a{}_{bcd}, \tag{A2}$$

$$\overline{R}^i{}_{ajb} = -\frac{D_a D_b r}{r}\delta^i_j, \tag{A3}$$

$$\overline{R}^i{}_{jkl} = [K-(Dr)^2](\delta^i_k\gamma_{jl} - \delta^i_l\gamma_{jk}). \tag{A4}$$

3. Ricci tensors

$$\overline{R}_{ab} = {}^m R_{ab} - \frac{n}{r}D_a D_b r, \tag{A5}$$

$$\overline{R}_{ai} = 0, \tag{A6}$$

$$\overline{R}^i_j = \left(-\frac{\Box r}{r} + (n-1)\frac{K-(Dr)^2}{r^2}\right)\delta^i_j, \tag{A7}$$

$$\overline{R} = {}^m R - 2n\frac{\Box r}{r} + n(n-1)\frac{K-(Dr)^2}{r^2}. \tag{A8}$$

4. Einstein tensors

$$\overline{G}_{ab} = {}^m G_{ab} - \frac{n}{r}D_a D_b r - \left(\frac{n(n-1)}{2}\frac{K-(Dr)^2}{r^2} - \frac{n}{r}\Box r\right)g_{ab}, \tag{A9}$$

$$\overline{G}^i_j = \left(-\frac{1}{2}{}^m R - \frac{(n-1)(n-2)}{2}\frac{K-(Dr)^2}{r^2} + \frac{n-1}{r}\Box r\right)\delta^i_j, \tag{A10}$$

$$\overline{G}_{ai} = 0. \tag{A11}$$

APPENDIX B: PERTURBATIONS OF THE RICCI TENSORS OF THE BULK

In general the perturbation of the Ricci tensor is expressed in terms of $h_{MN} = \delta\overline{g}_{MN}$ as

$$2\,\delta\overline{R}_{MN} = -\overline{\nabla}^L\overline{\nabla}_L h_{MN} - \overline{\nabla}_M\overline{\nabla}_N h + \overline{\nabla}_M\overline{\nabla}_L h^L_N + \overline{\nabla}_N\overline{\nabla}_L h^L_M + \overline{R}_{ML}h^L_N + \overline{R}_{NL}h^L_M - 2\overline{R}_{MLNS}h^{LS}, \tag{B1}$$

$$\delta\overline{R} = -h_{MN}\overline{R}^{MN} + \overline{\nabla}^M\overline{\nabla}^N h_{MN} - \overline{\nabla}^M\overline{\nabla}_M h. \tag{B2}$$

By decomposing the connection ∇ into D and $\hat{D}$, we obtain

$$2\,\delta\overline{R}_{ab} = -\Box h_{ab} + D_a D_c h^c_b + D_b D_c h^c_a + n\frac{D^c r}{r}(-D_c h_{ab} + D_a h_{cb} + D_b h_{ca}) + {}^m R^c_a h_{cb} + {}^m R^c_b h_{ca} - 2\,{}^m R_{acbd}h^{cd} - \frac{1}{r^2}\hat{\Delta}h_{ab} + \frac{1}{r^2}(D_a\hat{D}^i h_{bi} + D_b\hat{D}^i h_{ai}) - \frac{D_b r}{r^3}D_a h_{ij}\gamma^{ij} - \frac{D_a r}{r^3}D_b h_{ij}\gamma^{ij} + \frac{4}{r^4}D_a r D_b r h_{ij}\gamma^{ij} - D_a D_b h, \tag{B3}$$

$$2\,\delta\overline{R}_{ai} = \hat{D}_i D_b h^b_a + \frac{n-2}{r}D^b r\hat{D}_i h_{ab} - r\Box\left(\frac{1}{r}h_{ai}\right) - \frac{n}{r}D^b r D_b h_{ai} - D_a r D_b\left(\frac{1}{r}h^b_i\right) + \frac{n+1}{r}D^b r D_a h_{bi} + rD_a D_b\left(\frac{1}{r}h^b_i\right) + \left((n+1)\frac{(Dr)^2}{r^2} + (n-1)\frac{K-(Dr)^2}{r^2} - \frac{\Box r}{r}\right)h_{ia} + \frac{1}{r^2}D^b r D_a r h_{bi} + (n+1)rD_a\left(\frac{1}{r^2}D^b r\right)h_{bi} - \frac{n+2}{r}D_a D^b r h_{ib} + {}^m R^b_a h_{bi} - \frac{1}{r^2}\hat{\Delta}h_{ai} + \frac{1}{r^2}\hat{D}_i\hat{D}^j h_{aj} + rD_a\left(\frac{1}{r^3}\hat{D}^j h_{ji}\right) + \frac{1}{r^3}D_a r\hat{D}^j h_{ji} - \frac{1}{r^3}D_a r\hat{D}_i h_{jk}\gamma^{jk} - rD_a\left(\frac{1}{r}\hat{D}_i h\right), \tag{B4}$$

$$2\,\delta\overline{R}_{ij}=[2rD^arD_bh^b_a+2(n-1)D^arD^brh_{ab}+2rD^aD^brh_{ab}]\gamma_{ij}+r\hat{D}_iD_a\left(\frac{1}{r}h^a_j\right)+r\hat{D}_jD_a\left(\frac{1}{r}h^a_i\right)$$
$$+(n-1)\frac{D^ar}{r}(\hat{D}_ih_{aj}+\hat{D}_jh_{ai})+2\frac{D^ar}{r}\hat{D}^kh_{ka}\gamma_{ij}-r^2\Box\left(\frac{1}{r^2}h_{ij}\right)-n\frac{D^ar}{r}D_ah_{ij}$$
$$+\frac{1}{r^2}(\hat{D}_i\hat{D}^kh_{kj}+\hat{D}_j\hat{D}^kh_{ki})-\frac{1}{r^2}\hat{\Delta}h_{ij}+2\left((n-1)\frac{K}{r^2}+2\frac{(Dr)^2}{r^2}-\frac{\Box r}{r}\right)h_{ij}$$
$$-2(\gamma^{kl}h_{kl}\gamma_{ij}-h_{ij})\frac{K-(Dr)^2}{r^2}-2\frac{(Dr)^2}{r^2}\gamma_{ij}\gamma^{kl}h_{kl}-\hat{D}_i\hat{D}_jh-rD^arD_ah\gamma_{ij}, \tag{B5}$$

$$\delta\overline{R}=D_aD_bh^{ab}+\frac{2n}{r}D^arD^bh_{ab}+\left(-{}^mR^{ab}+\frac{2n}{r}D^aD^br+\frac{n(n-1)}{r^2}D^arD^br\right)h_{ab}+\frac{2}{r^2}D_a\hat{D}^ih^a_i$$
$$+2(n-1)\frac{D^ar}{r^3}\hat{D}^ih_{ai}+\frac{1}{r^4}\hat{D}^i\hat{D}^jh_{ij}-\frac{D^ar}{r^3}D_ah_{ij}\gamma^{ij}-\frac{1}{r^2}\left((n-1)\frac{K}{r^2}-2\frac{(Dr)^2}{r^2}\right)h_{ij}\gamma^{ij}$$
$$-\Box h-n\frac{D^ar}{r}D_ah-\frac{1}{r^2}\hat{\Delta}h. \tag{B6}$$

APPENDIX C: SCALAR MASTER VARIABLE

In this Appendix we show by a method different from the proof given in [45] that F_{ab} and F satisfying Eqs. (151) and (152) are written in terms of the master variable Ω as in Eqs. (153) and (154) if the two-dimensional orbit space with the metric g_{ab} is a space $\mathcal{N}$ with a constant sectional curvature λ.

First, let W_{ab} be a symmetric, traceless, and divergenceless tensor field on $\mathcal{N}$. Let ξ^a be a (timelike) Killing vector, which exists because $\mathcal{N}$ is maximally symmetric. If we put $W_a=W_{ab}\xi^b$, from the divergenceless condition and the Killing equation, we obtain

$$D_aW^a=W_{ab}D^a\xi^b=0. \tag{C1}$$

In the same way, we obtain the conservation law for the combination $W_{ab}\epsilon^{bc}\xi_c$ as

$$\begin{aligned}D^a(W_{ab}\epsilon^{bc}\xi_c)&=W^a_b\epsilon^{bc}D_a\xi_c\\&=-W_{ab}\epsilon^{bc}\epsilon^a{}_c\left(\frac{1}{2}\epsilon^{ef}D_e\xi_f\right)\\&=W^c_c\left(\frac{1}{2}\epsilon^{ef}D_e\xi_f\right)=0.\end{aligned} \tag{C2}$$

Here, from the traceless condition, this vector is related to W_a as

$$W_{ab}\epsilon^{bc}\xi_c=-\epsilon^{ab}W_{bc}\epsilon^{ce}\epsilon_{ef}\xi^f=-\epsilon_{ab}W^b. \tag{C3}$$

Hence Eq. (C2) is written as

$$\epsilon^{ab}D_aW_b=0, \tag{C4}$$

which implies that W_a is written as a gradient of a function W:

$$W_a=D_aW. \tag{C5}$$

Equation (C1) yields the Laplace equation

$$\Box W=0. \tag{C6}$$

Since the vector defined by $\eta_a=\epsilon_{ab}\xi^b$ is orthogonal to ξ^a and has the norm $\eta_a\eta^a=-\xi_a\xi^a$, the metric g_{ab} is written as

$$g_{ab}=\frac{1}{U}(-\xi_a\xi_b+\eta_a\eta_b), \tag{C7}$$

where $U=-\xi_a\xi^a$. Utilizing this and the traceless condition, we obtain

$$\begin{aligned}W_{ab}&=W_{ac}\delta^c_a\\&=-\frac{1}{U}(W_a\xi_b+\epsilon_{ac}W^c\eta_b)\\&=-\frac{1}{U}(W_a\xi_b+W_b\xi_a-g_{ab}W_c\xi^c).\end{aligned} \tag{C8}$$

It is easily checked that the right-hand side of this equation is a symmetric, traceless, and divergenceless tensor if Eqs. (C5) and (C6) are satisfied.

In order to apply this formula to our problem, let us introduce the traceless tensor Z_{ab} as

$$r^{n-2}F_{ab}=Z_{ab}-(n-2)r^{n-2}Fg_{ab}. \tag{C9}$$

This tensor is not divergenceless:

$$D_b Z^b_a = n D_a(r^{n-2}F). \tag{C10}$$

In order to define a divergenceless tensor, let us introduce a variable Ω as

$$2nr^{n-2}F = (\Box + 2\lambda)\Omega, \tag{C11}$$

and define W_{ab} as

$$W_{ab} = Z_{ab} - \left(D_a D_b \Omega - \frac{1}{2}\Box\Omega g_{ab} \right). \tag{C12}$$

It is easy to check that W_{ab} is a symmetric, traceless, and divergenceless tensor if λ is constant; hence it is written in terms of a potential W as in Eq. (C8).

Here, note that in the definition of Ω there exists a freedom of replacement $\Omega \to \Omega + \phi$ where ϕ is a solution of the hyperbolic equation

$$(\Box + 2\lambda)\phi = 0. \tag{C13}$$

By this replacement W_{ab} changes as

$$W_{ab} \to W'_{ab} = W_{ab} - (D_a D_b \phi + \lambda \phi g_{ab}). \tag{C14}$$

Since ϕ is constrained by the hyperbolic equation, we can choose the initial condition of ϕ and $\partial_t \phi$ on an initial surface $t=$const so that $W'_{rr} = W'_{tr} = 0$, where t and r are the coordinates used in Eq. (128). This condition is written in terms of the potential W' for W'_{ab} as $\partial_t W' = \partial_r W' = 0$. For any boundary condition on W' that is linear and gives a well-posed initial value problem, the solution satisfying this initial condition is $W'=$const, which implies that $W'_{ab}=0$. Thus F_{ab} and F are expressed as in Eqs. (153) and (154).

APPENDIX D: GENERAL SOLUTION OF EQUATION (156)

In this Appendix we give the general solution to Eq. (156) on a two-dimensional maximally symmetric space. We work in the coordinates (t,r) used in Eq. (128). Since the general solution for the case $\lambda=0$ is obviously given by $E=C_0 + C_1 t + C_2 r$ with arbitrary constants $C_0 \sim C_2$, we assume $\lambda \neq 0$ below.

First, note that in the (t,r) coordinates the nonvanishing Christoffel symbols are given by

$$\Gamma^t_{tr} = \frac{U'}{2U}, \quad \Gamma^r_{tt} = \frac{1}{2}UU', \quad \Gamma^r_{rr} = -\frac{U'}{2U}. \tag{D1}$$

From this equation the (tr) component of Eq. (156) is written as

$$0 = D_t D_r E = U^{1/2}\partial_r(U^{-1/2}\partial_t E), \tag{D2}$$

which yields

$$E = f_1(t)U^{1/2} + f_2(r). \tag{D3}$$

Inserting this expression into the (tt) component of Eq. (156), we obtain

$$0 = (D_t D_t + \lambda g_{tt})E = U^{1/2}\left[\ddot{f}_1 - \left(\lambda U + \frac{(U')^2}{4} \right) f_1 \right] - U\left(\frac{1}{2}U' f'_2 + \lambda f_2 \right), \tag{D4}$$

where the overdot and the prime denote differentiation with respect to t and r, respectively. Since $\lambda U + (U')^2/4 = \lambda K$ is constant, this equation is equivalent to the following two ordinary differential equations:

$$\ddot{f}_1 - \lambda K f_1 = c, \tag{D5}$$

$$-r f'_2 + f_2 = \frac{c}{\lambda U^{1/2}}, \tag{D6}$$

where c is a separation constant. The general solution of the first equation is given by

$$f_1(t) = \begin{cases} \frac{1}{2}ct^2 + c_1 t + c_0, & K=0 \\ -\frac{c}{\lambda K} + c_0 e^{\sqrt{\lambda K}t} + c_1 e^{-\sqrt{\lambda K}t}, & K \neq 0. \end{cases} \tag{D7}$$

On the other hand, the general solution for the equation for f_2 is given by

$$f_2(r) = \begin{cases} c_2 r + \frac{c}{2(-\lambda)^{3/2} r}, & K=0 \\ c_2 r + \frac{c}{\lambda K}U^{1/2}, & K \neq 0. \end{cases} \tag{D8}$$

Hence, after redefinitions of constants, the general solution including the case $\lambda=0$ is expressed as

$$E = C_0 g_0(t,r) + C_1 g_1(t,r) + C_2 r, \tag{D9}$$

where

$$g_0(r) = \begin{cases} 1, & \lambda=0, K\neq 0 \\ \lambda^2 t^2 r + \frac{1}{r}, & \lambda \neq 0, K=0 \\ e^{\sqrt{\lambda K}t}U^{1/2}, & \lambda K \neq 0, \end{cases} \tag{D10}$$

$$g_1(r) = \begin{cases} t, & \lambda K = 0 \\ e^{-\sqrt{\lambda K}t}U^{1/2}, & \lambda K \neq 0. \end{cases} \tag{D11}$$

It is easy to check that this satisfies the remaining (rr) component of Eq. (156):

$$0 = (D_r D_r + \lambda g_{rr})E = \left(\partial_r^2 + \frac{U'}{2U}\partial_r + \frac{\lambda}{U} \right)E. \tag{D12}$$

APPENDIX E: EXCEPTIONAL MODES FOR SCALAR PERTURBATION WITH $K>0$

In this Appendix we show that the gauge-equivalent classes of the solutions to the perturbed Einstein equations are parametrized by a finite number of parameters for the exceptional modes $k^2(k^2-nK)=0$ $(K>0)$ of the bulk scalar perturbation on a maximally symmetric background.

First, let us consider the mode $k^2=0$. For this mode $\mathbb{S}_i$ and $\mathbb{S}_{ij}$ vanish, and f_a and H_T are undefined. Further, the gauge transformation is parametrized only by T_a. Hence, setting the undefined variables to zero, F and F_{ab} are written as $F=H_L$ and $F_{ab}=f_{ab}$, which transform under the gauge transformation as

$$\bar{\delta}F=-\frac{D_a r}{r}T^a,\quad \bar{\delta}F_{ab}=-D_aT_a-D_bT_a. \tag{E1}$$

For the same reason, Eqs. (64) and (66), or equivalently, Eqs. (151) and (152), do not exist for the mode $k^2=0$. However, we can recover these equations by regarding them as the gauge-fixing conditions. Then the residual gauge freedom is represented by T_a satisfying the following two conditions:

$$0=\bar{\delta}[\tilde{F}^a_a+2(n-2)\tilde{F}]=-2D_a\tilde{T}^a, \tag{E2}$$

$$\begin{aligned}0&=\bar{\delta}(D_b\tilde{F}^b_a-2D_a\tilde{F})\\ &=-\Box\tilde{T}_a+\frac{n-2}{r}Dr\cdot D\tilde{T}_a+\frac{2}{r}D^brD_a\tilde{T}_b\\ &\quad-\left((n-2)\frac{K}{r^2}+(2n-1)\lambda\right)\tilde{T}_a\\ &\quad-\frac{n}{r^2}D_arD_br\tilde{T}^b+D_aD_b\tilde{T}^b\\ &\quad+\frac{n-2}{r}D^arD_b\tilde{T}^b,\end{aligned} \tag{E3}$$

where

$$\tilde{F}=r^{n-2}F,\quad \tilde{F}_{ab}=r^{n-2}F_{ab},\quad \tilde{T}_a=r^{n-2}T_a. \tag{E4}$$

Equation (E2) implies that $\tilde{T}^a$ is represented by a scalar function T as

$$\tilde{T}^a=\epsilon^{ab}D_bT, \tag{E5}$$

because the orbit space $\mathcal{N}$ is two dimensional. Inserting this expression into Eq. (E3), we obtain

$$\epsilon^{ab}D_b[r^2\Box T-nrDr\cdot DT+2(n-1)KT]=0. \tag{E6}$$

Hence, by replacing T by $T+\text{const}$, we obtain

$$r^2\Box T-nrDr\cdot DT+2(n-1)KT=0. \tag{E7}$$

Since Eqs. (151) and (152) hold under the above gauge conditions, any solution of the perturbed Einstein equations is parametrized by Ω satisfying $(D_aD_b+\lambda g_{ab})E(\Omega)=0$ as for the generic mode. Let the set of solutions Ω to this equation be $\mathcal{S}_\Omega$. Then we have an onto map Φ_1 from $\mathcal{S}_\Omega$ to the space of solutions to the perturbed Einstein equations. The kernel of this map is spanned by the solutions of $(D_aD_b+\lambda g_{ab})\Omega=0$. On the other hand, F and F_{ab} obtained by setting $F=\bar{\delta}F$ and $F_{ab}=\bar{\delta}F_{ab}$ in Eq. (E1) with T_a satisfying the above gauge-fixing condition is also a solution to the perturbed Einstein equations belonging to the trivial gauge-equivalent class. This correspondence defines a map Φ_2 from the space $\mathcal{S}_G$ of solutions T to Eq. (E7). Then the set $\mathcal{S}_{\rm inv}$ of gauge-equivalent classes to the perturbed Einstein equations is represented as $\mathcal{S}_\Sigma/\Phi_1^{-1}\Phi_2\mathcal{S}_G$.

Here, note that $\mathcal{S}_\Sigma/\ker\Phi_1$ is parametrized by the solution to the equation $E(\Omega)=C_2r$, and hence by the initial data $(\Omega,\dot{\Omega})$ on the initial surface and the constant C_2. Similarly, $\mathcal{S}_G$ is parametrized by the initial data $(T,\dot{T})$ for Eq. (E7). Therefore, by representing the condition $\Phi_1(\Omega)=\Phi_2(T)$ as a relation between these initial data (and C_2), we can determine $\mathcal{S}_{\rm inv}$.

Now let us undertake this program. First, by redefining $-T$ as T, the condition $\Phi_1(\Omega)=\Phi_2(T)$ is expressed as

$$\begin{aligned}(D_aD_b+\lambda g_{ab})\Omega&=\epsilon_{ac}D_bD^cT+\epsilon_{bc}D_aD^cT\\ &\quad-\frac{n-2}{r}(D_ar\epsilon_{bc}+D_br\epsilon_{ac})D^cT\\ &\quad+\frac{2(n-1)}{r}\epsilon_{cd}D^crD^dTg_{ab}.\end{aligned} \tag{E8}$$

In the (t,r) coordinates used in Eq. (128), with the help of the equations for Ω and T, the trace and (t,r)-component of this equation are written as

$$U\left(\frac{\Omega}{r}\right)'+\frac{C_2}{nr}=\frac{2}{r}\dot{T}, \tag{E9}$$

$$\begin{aligned}U^{1/2}(U^{-1/2}\dot{\Omega})'&=2UT''+\left(U'-2(n-1)\frac{U}{r}\right)T'\\ &\quad+2(n-1)\frac{K}{r^2}T.\end{aligned} \tag{E10}$$

These equations have a solution for $(T,\dot{T})$ when data $(\Omega,\dot{\Omega})$ are given.

On the other hand, the (r,r) component is expressed as

$$U\Omega''+\frac{1}{2}U\Omega'+\lambda\Omega=2U^{1/2}(U^{-1/2}\dot{T})'+\frac{2}{r}\dot{T}, \tag{E11}$$

and gives a constraint on C_2. In fact, inserting the expression for $\dot{T}$ obtained from the trace, we obtain the condition $C_2=0$. This implies that the set $\Phi_1^{-1}\Phi_2\mathcal{S}_G$ coincides with the set of solutions to the homogeneous equation $E(\Omega)=0$. Thus C_2 is a gauge invariant and parametrizes the space $\mathcal{S}_{\rm inv}$.

Next we examine the mode $k^2=nK$. The argument is almost the same as in the above case. Now the harmonic scalar and the harmonic vector are nontrivial but the harmonic tensor $\mathbb{S}_{ij}$ vanishes. Hence only H_T is undefined, and X_a is defined as $X_a=rf_a/k$. The gauge transformations of F and F_{ab} are given by

$$\bar{\delta}F=-\frac{r}{k}\left[Dr\cdot D\left(\frac{L}{r}\right)+\frac{K}{r^2}L\right], \tag{E12}$$

$$\bar{\delta}F_{ab}=-\frac{1}{k}\left\{D_a\left[r^2D_b\left(\frac{L}{r}\right)\right]+D_b\left[r^2D_a\left(\frac{L}{r}\right)\right]\right\}. \tag{E13}$$

In the present case only Eq. (151) is lacking. Hence we regard this as the gauge-fixing condition. Then the residual gauge freedom is parametrized by L satisfying the wave equation

$$\Box\tilde{L}-\frac{n}{r}Dr\cdot\tilde{L}+\left(n\lambda+2(n-1)\frac{K}{r^2}\right)\tilde{L}=0, \tag{E14}$$

where $\tilde{L}=r^{n-1}L$. After the redefinition $-2\tilde{L}/k\rightarrow\tilde{L}$, the condition $\Phi_1(\Omega)=\Phi_3(L)$ is represented as

$$\begin{aligned}(D_aD_b+\lambda g_{ab})\Omega=&D_aD_b\tilde{L}-\frac{n-1}{r}(D_arD_b\tilde{L}+D_brD_a\tilde{L})\\&+\frac{n(n-1)}{r^2}D^arD^br\tilde{L}+\left[\frac{n-1}{r}Dr\cdot D\tilde{L}\right.\\&\left.+\left(n^2\lambda-(n-1)^2\frac{K}{r^2}\right)\tilde{L}\right]g_{ab}.\end{aligned} \tag{E15}$$

Here Φ_3 represents the map from the space $\mathcal{S}_L$ of solutions L to the set of solutions to the perturbed Einstein equations.

The trace and the (t,r) component of this equation are written as

$$\begin{aligned}U\Omega'+\lambda r\Omega+\frac{1}{nr}(C_0g_0+C_1g_1)\\=U\tilde{L}'+\left(n\lambda r-(n-1)\frac{K}{r}\right)\tilde{L},\end{aligned} \tag{E16}$$

$$U^{1/2}(U^{1/2}\dot{\Omega})'=U^{1/2}r^{n-1}\left(\frac{\dot{\tilde{L}}}{U^{1/2}r^{n-1}}\right)', \tag{E17}$$

which have a solution for $(L,\dot{L})$ for any data $(\Omega,\dot{\Omega})$. On the other hand, the (r,r) component

$$\begin{aligned}U\Omega''-\lambda r\Omega'+\lambda\Omega=U\tilde{L}''-\left(\lambda r+\frac{n-1}{r}U\right)\tilde{L}'\\+\left((n-1)\frac{K}{r^2}+n\lambda\right)\tilde{L}\end{aligned} \tag{E18}$$

gives the constraint $C_0=C_1=0$. Thus $\Phi_1^{-1}\Phi_3\mathcal{S}_T$ coincides with the space of solutions to the homogeneous equation $E(\Omega)=0$, and the space $\mathcal{S}_{\rm inv}$ of the gauge-equivalent classes of solutions is parametrized by the two gauge-invariant constants C_0 and C_1.

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