Global embeddings of scalar-tensor theories in 2+1 dimensions

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We obtain (3+3)- or (3+2)-dimensional global flat embeddings of four uncharged and charged scalar-tensor theories with the parameters B or L in 2+1 dimensions, which are the nontrivially modified versions of the Bañados-Teitelboim-Zanelli (BTZ) black holes. The limiting cases $B=0$ or $L=0$ exactly are reduced to the global embedding Minkowski space solution of the BTZ black holes.

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I. INTRODUCTION

After Unruh's work [1], it has been known that a thermal Hawking effect on a curved manifold [2] can be looked at as an Unruh effect in a higher flat dimensional spacetime. According to the global embedding Minkowski space (GEMS) approach [3], several authors [4,5] recently have shown that this approach could yield a unified derivation of temperature for various curved manifolds such as the rotating Bañados-Teitelboim-Zanelli (BTZ) model, [6–8], the Schwarzschild [9] together with its anti–de Sitter (AdS) extension, the Reissner-Nordström (RN) [10,11], and the RN-AdS [12].

On the other hand, since the pioneering work in 1992, the (2+1)-dimensional BTZ black hole [6] has become a useful model for realistic black hole physics [7]. Moreover, significant interest in this model has recently increased with the novel discovery that the thermodynamics of higher dimensional black holes can often be interpreted in terms of the BTZ solution [13]. It is therefore interesting to study the geometry of (2+1)-dimensional black holes and their thermodynamics through further investigation. Very recently we have analyzed the Hawking and Unruh effects of the (2+1)-dimensional black holes in terms of the GEMS approach [14]. As a result, we have obtained the novel global higher dimensional flat embeddings of the (2+1)-dimensional static, rotating, and charged de Sitter black holes, which are the counterpart of the usual BTZ black holes as well as the charged static BTZ one.

In this paper we will futher analyze the (2+1)-dimensional scalar-tensor (ST) theories [15] as an alternative theory of gravity in three space-time dimensions in terms of the GEMS approach. As you may know three dimensional vacuum general relativity (GR) admits no black hole but rather a trivial locally flat (globally conical) solution. One has to either couple matter to GR, or consider alternative vacuum (or nonvacuum) gravitational theories in order to get black hole solutions. Motivated by this, we will consider the GEMS of the new black hole solutions in GR coupled to the vacuum ST theories [16], which are modifications of the BTZ black hole by an asymptotically constant scalar.

In Sec. II, we will consider the novel GEMS of the two uncharged (2+1)-dimensional ST theories, which have the usual BTZ black hole as a substructure. In Sec. III, we will also generalize these ST theories to the charged cases.

II. GEMS OF UNCHARGED SCALAR-TENSOR THEORIES

In three dimensions, the ST black holes have been obtained in Ref. [15]. The most general action coupled to a scalar can be written as [17]

$$S=\int d^3x\sqrt{-g}[C(\phi)R-\omega(\phi)(\nabla\phi)^2+V(\phi)], \qquad (1)$$

where R is the scalar curvature, and $V(\phi)$ is a potential function for ϕ. $C(\phi)$ and $\omega(\phi)$ are collectively known as the coupling functions.

On the other hand, the field equations for the action [Eq. (1)] with $C(\phi)=\phi$, which is a choice for the ST theories without loss of generality, can be obtained by varying Eq. (1) with respect to the metric and scalar fields, respectively, as follows:

$$\phi R_{\mu\nu}=\omega\nabla_\mu\phi\nabla_\nu\phi-g_{\mu\nu}V+g_{\mu\nu}\nabla^2\phi+\nabla_\mu\nabla_\nu\phi, \qquad (2)$$

$$2\omega\nabla^2\phi+\frac{dV}{d\phi}+\frac{d\omega}{d\phi}(\nabla\phi)^2+R=0. \qquad (3)$$

The special cases to Eq. (1) in three dimensions were previously considered by a number of authors. The first example is the static BTZ black hole solution of $C(\phi)=1$, $\omega(\phi)=0$, and $V(\phi)=2\Lambda$ [6]. The second example corresponds to the same $C(\phi)$ as above, but with a nontrivial ϕ, $\omega(\phi)=4$, and $V(\phi)=2\Lambda e^{b\phi}$, for which the static black hole solutions have been previously derived in Ref. [16]. These examples have the condition $C(\phi)=1$, for which the metric coupling to matter is the Einstein metric. In the ST theories, this is no longer true for the nontrivial case of $C(\phi)\neq 1$, and the gravitational force is governed by a mixture of the metric and scalar fields.

We now look for the GEMS of the ST gravity theories described by field equations (2) and (3), which have already been analyzed by Chan [18].

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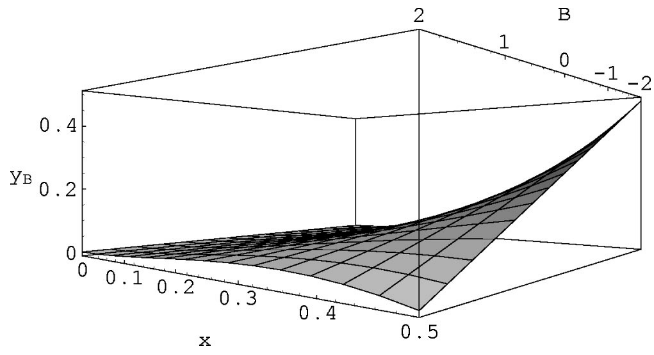

FIG. 1. Graph of $y_B(x) = -Bx^3 + x^2$: For a given constant $1/Ml^2$, it shows that there exist event horizons along the abscissa x on a constant B plane.

A. Case I: $\phi = r/(r-3B/2)$

Let us consider the action and the choice of a scalar field

$$\mathcal{L} = \phi R - \frac{2}{1-\phi}(\nabla\phi)^2 + 2(3-3\phi+\phi^2)\Lambda\phi + \frac{8M}{27B^2}(1-\phi)^3,$$

$$\phi = \frac{r}{r - \frac{3B}{2}}, \tag{4}$$

whose solution is given as

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 d\theta^2, \tag{5}$$

$$N(r) = -M + \frac{MB}{r} + \frac{r^2}{l^2}, \tag{6}$$

where $l^2 = \Lambda^{-1}$ and M is the positive mass parameter calculated using the quasilocal mass [6,11,19]. Here one notes that the metric looks like the Schwarzschild-AdS metric. If $\Lambda = 0$, the metric is exactly the same form as the four-dimensional Schwarzschild case.

To study the metric, Eq. (5) it is convenient to define the radial coordinate r as $r \equiv 1/x$. Then, the lapse function Eq. (6) can be rewritten as

$$N = \frac{M}{x^2}\left(\frac{1}{Ml^2} - y_B(x)\right),$$

$$y_B(x) = -Bx^3 + x^2. \tag{7}$$

Note that the parameter B may have either positive or negative values. The positions of event horizons obtained from $N=0$ can be now read off in Fig. 1 from cross sectional curve formed by the surface $y_B(x)$[1] and a (x,B) plane associated with a given value $1/Ml^2$. Moreover, the slope of the curve $y_B(x)$ at intersections along the abscissa on a constant B plane gives the surface gravity of the horizon, which is $k_H \equiv \frac{1}{2} dN/dr|_{N=0} = M(dy_B/dx)$. The positive B region of the graph contains a curve of maximum value $4/27B^2$ along the ordinate B. Thus, when satisfied with $(1/Ml^2) \leqslant (4/27B^2)$, there exist two intersections, the outer and inner event horizons, r_+ and r_-, respectively. An extremal black hole appears at the point $x = 2/3B$ coinciding with r_+ and r_- [16]. On the other hand, for negative B there is only one event horizon for any choice of $1/Ml^2$.

Now, let us consider the GEMS approach to embed this curved space time into a higher dimensional flat one. We restrict ourselves to the region of $r > r_+$ according to the usual GEMS embedding [3,4,14].

First, for the case of positive B the GEMS embedding is obtained by comparing the three metric in Eq. (6) with $ds^2 = \eta_{ab}\, dz^a\, dz^b$, where $(a,b=0,\cdots,5)$ and $\eta_{ab} = \mathrm{diag}(+,-,-,-,+,+)$. Now, let us find the $-r^2 d\theta^2$ term in the three metric by introducing two coordinates z^3, z^4 in Eq. (13) (see below), giving $-(dz^3)^2 + (dz^4)^2 = -r^2 d\theta^2 + (l^2/r_+^2) dr^2$. Then, in order to obtain the $N^2 dt^2$ term, we make an ansatz of two coordinates (z^0, z^1) in Eq. (13) which, together with the above (z^3, z^4), yields

$$(dz^0)^2 - (dz^1)^2 - (dz^3)^2 + (dz^4)^2 = N^2 dt^2 - \left(k_H^{-2} \frac{\left(-\frac{MB}{2r^2} + \frac{r}{l^2}\right)^2}{\left(-M + \frac{MB}{r} + \frac{r^2}{l^2}\right)} - \frac{l^2}{r_+^2} \right) dr^2 - r^2\, d\theta^2, \tag{8}$$

where the Hawking-Bekenstein horizon surface gravity is given by

$$k_H = \frac{r_+}{l^2} - \frac{B r_+}{2l^2(r_+ - B)}. \tag{9}$$

Since the combination of $N^{-2}\, dr^2$ and dr^2 terms in Eq. (8) can be separated into a positive definite part and a negative one as follows:

[1]In Fig. 1 the parameter B is regarded as a continuous variable and the limit of $x \to 0$ corresponds to $r \to \infty$. By choosing a plane with constant B, one can easily see that a curve is defined on the (x, y_B) plane. Note that for a fixed negative B there exists only one intersection of x associated with the value $1/Ml^2$.

$$\left(k_H^{-1}\frac{lN_1^{1/2}(B)}{2r_+^2 r^{3/2}(r_+-B)[r_+r(r+r_+)-B(r^2+r_+r+r_+^2)]^{1/2}}\right)^2$$
$$-\left(k_H^{-1}\frac{lN_2^{1/2}(B)}{2r_+^2 r^{3/2}(r_+-B)[r_+r(r+r_+)-B(r^2+r_+r+r_+^2)]^{1/2}}\right)^2 \equiv (dz^2)^2-(dz^5)^2, \tag{10}$$

where

$$N_1(B)=\frac{B^2r_+^5}{l^4}[r_+^3(r^2+r_+r+r_+^2)+9r^4(r+r_+)+21r_+^2r^3],$$

$$N_2(B)=\frac{Br_+^4}{l^4}[(8r_+^2+14B^2)r_+^2r^3+B^2r_+^3(r^2+r_+r+r_+^2)+(4r_+^2+5B^2)r^4(r+r_+)], \tag{11}$$

we can obtain the flat global embeddings of the corresponding curved three metric as

$$ds^2=(dz^0)^2-(dz^1)^2-(dz^2)^2-(dz^3)^2+(dz^4)^2+(dz^5)^2=N^2\,dt^2-N^{-2}\,dr^2-r^2\,d\theta^2. \tag{12}$$

As a result, the desired coordinate transformations to the (3+3)-dimensional AdS GEMS are obtained for $r \geqslant r_+$ as

$$z^0=k_H^{-1}\left(-M+\frac{MB}{r}+\frac{r^2}{l^2}\right)^{1/2}\sinh k_H t,$$

$$z^1=k_H^{-1}\left(-M+\frac{MB}{r}+\frac{r^2}{l^2}\right)^{1/2}\cosh k_H t,$$

$$z^2=k_H^{-1}\int dr\,\frac{lN_1^{1/2}(B)}{2r_+^2 r^{3/2}(r_+-B)[r_+r(r+r_+)-B(r^2+r_+r+r_+^2)]^{1/2}},$$

$$z^3=\frac{l}{r_+}r\sinh\frac{r_+}{l}\theta,$$

$$z^4=\frac{l}{r_+}r\cosh\frac{r_+}{l}\theta,$$

$$z^5=k_H^{-1}\int dr\,\frac{lN_2^{1/2}(B)}{2r_+^2 r^{3/2}(r_+-B)[r_+r(r+r_+)-B(r^2+r_+r+r_+^2)]^{1/2}}. \tag{13}$$

In static detectors (θ, $r=$const) described by a fixed point in the (z^2, z^3, z^4, z^5) hyperplane, one can have constant three acceleration

$$a=\frac{\frac{r}{l^2}-\frac{MB}{2r^2}}{\left(-M+\frac{MB}{r}+\frac{r^2}{l^2}\right)^{1/2}}, \tag{14}$$

and constant accelerated motion in the (z^0,z^1) plane with the Hawking temperature

$$2\pi T=a_6=\frac{\frac{r_+}{l^2}-\frac{MB}{2r_+^2}}{\left(-M+\frac{MB}{r}+\frac{r^2}{l^2}\right)^{1/2}}. \tag{15}$$

Here one notes that the above Hawking temperature is also given by the relation [2,20]

$$T=\frac{1}{2\pi}\frac{k_H}{g_{00}^{1/2}}. \tag{16}$$

One can easily check that, in the limit of $B=0$ where the spacelike z^2 and timelike z^5 coordinates in Eq. (13) vanish, the above (3+3)-dimensional coordinate transformations are exactly reduced to the (2+2)-dimensional GEMS of the usual BTZ case [4,14].

We now see how the scalar-tensor solution, which is a modified version of the BTZ, yields a finite Unruh area due to the periodic identification of θ mod 2π. The Rindler horizon condition $(z^1)^2-(z^0)^2=0$ implies $r=r_+$ and the embedding constraints yield $z^2=f_1(r)$, $z^5=f_2(r)$, and $(z^4)^2-(z^3)^2=l^2$, where $f_1(r)$ and $f_2(r)$ can be read from Eq. (13). The area of the Rindler horizon is now described as

$$\int dz^2\, dz^3\, dz^4\, dz^5\, \delta(z^2-f_1(r))\,\delta(z^5-f_2(r))\,\delta([(z^4)^2 -(z^3)^2]^{1/2}-l),$$

which, after performing trivial integrations over z^2 and z^5, yields the desired entropy of the scalar-tensor theory as

$$\int_{-l\sinh(\pi r_+/l)}^{l\sinh(\pi r_+/l)} dz^3 \int_0^{[(z^3)^2+l^2]^{1/2}} dz^4\, \delta([(z^4)^2-(z^3)^2]^{1/2}-l)$$
$$=\int_{-l\sinh(\pi r_+/l)}^{l\sinh(\pi r_+/l)} dz^3\, \frac{l}{[l^2+(z^3)^2]^{1/2}}=2\pi r_+(B), \tag{17}$$

which reproduces the entropy $2\pi r_H$ of the uncharged BTZ case in the limit $B=0$.

Next, for the case of $B<0$, since $N_2(B)$ is an odd function of B, the combination of $N^{-2}\,dr^2$ and dr^2 terms in Eq. (8) can be written by introducing only one extra space[2] dimension z'^2 as follows:

$$-\left(k_H^{-1}\frac{l(N_2-N_1)^{1/2}(B)}{2r_+^2 r^{3/2}(r_+-B)[r_+r(r+r_+)-B(r^2+r_+r+r_+^2)]^{1/2}}\right)^2 \equiv -(dz'^2)^2. \tag{18}$$

Then, we can obtain the following flat embedding of the corresponding curved three metric as

$$ds^2=(dz^0)^2-(dz^1)^2-(dz'^2)^2-(dz^3)^2+(dz^4)^2=N^2\,dt^2-N^{-2}\,dr^2-r^2\,d\theta^2. \tag{19}$$

As a result, the desired coordinate transformations to the (3+2)-dimensional GEMS are for $r>r_+$

$$z'^2=k_H^{-1}\int dr\,\frac{l(N_2-N_1)^{1/2}(B)}{2r_+^2 r^{3/2}(r_+-B)[r_+r(r+r_+)-B(r^2+r_+r+r_+^2)]^{1/2}}, \tag{20}$$

while (z^0,z^1,z^3,z^4) are of those forms in Eq. (13). Similar to the previous $B>0$ case, one can easily obtain the desired entropy of the ST theory as $2\pi r(B)$, where r is only one event horizon in this case.

It seems appropriate to comment on the minimal extra dimensions needed for a desired GEMS. As you may know, spaces of constant curvature can be embedded into flat space with only single extra dimension. This is seen in our previous work [14] for the static and rotating BTZ cases, which are embedded in the (2+2)-dimensional spaces. On the other hand, since the scalar-tensor solution is Schwarzschild-like [4,12], we have introduced (1+2) or (1+1) extra dimensions for the desired GEMS with the positive or negative B, respectively. In Sec. II B, we will also obtain similar results for the charged scalar-tensor theories.

B. Case II: $\phi=r^2/(r^2-2L)$

Next, another choice of an asymptotically constant scalar yields

$$\mathcal{L}=\phi R-\frac{4\phi-1}{2\phi(1-\phi)}(\nabla\phi)^2+\frac{M}{2L}+6\left(2\Lambda-\frac{M}{2L}\right)\phi +18\left(-\Lambda+\frac{M}{4L}\right)\phi^2+2\left(4\Lambda-\frac{M}{L}\right)\phi^3,$$

$$\phi=\frac{r^2}{r^2-2L}, \tag{21}$$

to yield the solution

$$ds^2=N^2\,dt^2-N^{-2}\,dr^2-r^2\,d\theta^2,$$

$$N(r)=-M+\frac{ML}{r^2}+\frac{r^2}{l^2}, \tag{22}$$

where $l^2=\Lambda^{-1}$. The metric has a curvature singularity at $r=0$, and the scalar and its potential both diverge at $r^2=2L$ with $L>0$. Note that only the case of $L>0$ is physically meaningful since we require it to have the positive r.

[2]By a simple test with $B<0$, we can show that Eq. (18) is really a monotonic decreasing function, and thus can be defined as a spacelike variable.

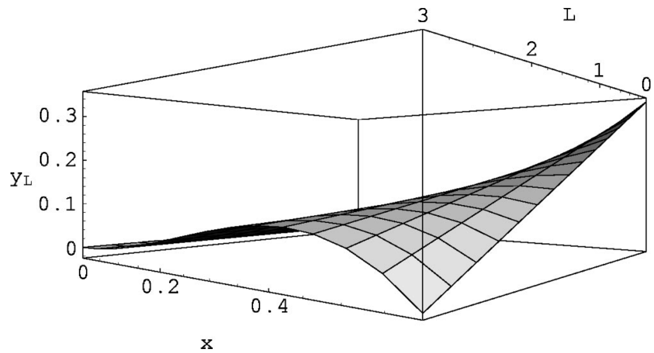

FIG. 2. Graph of $y_L(x)=-Lx^4+x^2$: For a given constant $1/Ml^2$, it shows that there exist event horizons along the abscissa x on a constant L plane.

Now, defining the radial coordinate r as $r=1/x$ as in Sec. II A, the lapse function can be rewritten as

$$N=\frac{M}{x^2}\left(\frac{1}{Ml^2}-y_L(x)\right), \tag{23}$$

$$y_L(x)=-Lx^4+x^2. \tag{24}$$

As shown in Fig. 2, for a specific value of $1/Ml^2$, there exist two event horizons, r_+ and r_-, on a (x,y_L) plane with a constant L, if satisfied $(1/Ml^2)\leqslant(1/4L)$. Here, the maximum value of $1/4L$ is obtained at $x=(1/\sqrt{2L})$ $(r=\sqrt{2L})$. Moreover, the extremal limit is $4L=Ml^2$ at $x=1/\sqrt{2L}$ $(r=\sqrt{2L})$ coinciding with r_+ and r_-. In this limit, the third term in the Lagrangian becomes 2Λ, while fourth, fifth, and sixth terms all vanish. Here note that, in the limit $L=(J^2/4M)$, the solution seems to be related to a rotationg BTZ black hole. However, since the three metric, Eq. (22), does not contain a shift function, our ST theory does not allow such a rotating BTZ solution.

After similar algebraic manipulation for the region of $r>r_+$ by following the previous steps described in Sec. II A, we obtain the desired coordinate transformations to the (3+3)-dimensional AdS GEMS, $ds^2=(dz^0)^2-(dz^1)^2-(dz^2)^2-(dz^3)^2+(dz^4)^2+(dz^5)^2$, which are obtained for $r\geqslant r_+$:

$$z^0=k_H^{-1}\left(-M+\frac{ML}{r^2}+\frac{r^2}{l^2}\right)^{1/2}\sinh k_H t,$$

$$z^1=k_H^{-1}\left(-M+\frac{ML}{r^2}+\frac{r^2}{l^2}\right)^{1/2}\cosh k_H t,$$

$$z^2=k_H^{-1}\int dr\,\frac{lN_3^{1/2}(L)}{r_+^2r^2(r_+^2-L)[r_+^2r^2-L(r^2+r_+^2)]^{1/2}}, \tag{25}$$

$$z^3=\frac{l}{r_+}r\sinh\frac{r_+}{l}\theta,$$

$$z^4=\frac{l}{r_+}r\cosh\frac{r_+}{l}\theta,$$

$$z^5=k_H^{-1}\int dr\,\frac{lN_4^{1/2}(L)}{r_+^2r^2(r_+^2-L)[r_+^2r^2-L(r^2+r_+^2)]^{1/2}},$$

where the Hawking-Bekenstein horizon surface gravity is given by

$$k_H=\frac{r_+}{l^2}-\frac{Lr_+}{l^2(r_+^2-L)}, \tag{26}$$

and

$$N_3(L)=\frac{L^2r_+^6}{l^4}[r_+^2(r^4+r_+^2r^2+r_+^4)+5r^4(r^2+r_+^2)+3r_+^2r^4],$$

$$N_4(L)=\frac{Lr_+^4}{l^4}[(r_+^4+3L^2)r_+^2r^4+L^2r_+^2(r^4+r_+^2r^2+r_+^4)+(2r_+^4+3L^2)r^4(r^2+r_+^2)]. \tag{27}$$

In static detectors (θ, $r=$const) described by a fixed point in the $(z^2,\ z^3,\ z^4,\ z^5)$ hyperplane, one can have constant three acceleration

$$a=\frac{\dfrac{r}{l^2}-\dfrac{ML}{r^3}}{\left(-M+\dfrac{ML}{r^2}+\dfrac{r^2}{l^2}\right)^{1/2}}, \tag{28}$$

and constant accelerated motion in the (z^0,z^1) plane with the Hawking temperature

$$2\pi T=a_6=\frac{\dfrac{r_+}{l^2}-\dfrac{ML}{r_+^3}}{\left(-M+\dfrac{ML}{r^2}+\dfrac{r^2}{l^2}\right)^{1/2}}. \tag{29}$$

Here one notes that the above Hawking temperature is also given by the relation (16).

Similar to the previous case, in the limit of $L=0$, where the spacelike z^2 and timelike z^5 coordinates in Eq. (25) vanish, the (3+3)-dimensional coordinate transformations are exactly reduced to the (2+2)-dimensional GEMS of the usual BTZ case [4,14]. We also obtain the entropy $2\pi r_+(L)$ of the scalar-tensor theory with $\phi=r^2/(r^2-2L)$, which reproduces the uncharged static BTZ entropy $2\pi r_H$ [4,14] in the $L=0$ limit.

III. GEMS OF CHARGED SCALAR-TENSOR THEORIES

A. Case I: $\phi=r/(r-3B/2)$

Now consider the charged scalar-tensor theory for the modified BTZ black hole where the three metric, Eq. (5), is described by the charged lapse

$$N(r)=-M+\frac{MB}{r}+\frac{r^2}{l^2}-2Q^2\ln r. \tag{30}$$

Here we only consider the case in which the parameter B is positive because the analysis for the case of $B<0$ is highly nontrivial due to the addition of the charged term in contrast to the uncharged cases.

The coordinate transformations to the (3+3)-dimensional AdS GEMS $ds^2=(dz^0)^2-(dz^1)^2-(dz^2)^2-(dz^3)^2+(dz^4)^2+(dz^5)^2$ are obtained for $r\geqslant r_+$:

$$\begin{aligned}
z^0&=k_H^{-1}\left(-M+\frac{MB}{r}+\frac{r^2}{l^2}-2Q^2\ln\right)^{1/2}\sinh k_Ht,\\
z^1&=k_H^{-1}\left(-M+\frac{MB}{r}+\frac{r^2}{l^2}-2Q^2\ln r\right)^{1/2}\cosh k_Ht,\\
z^2&=k_H^{-1}\int dr\,\frac{lN_1^{1/2}(B,Q)}{2r_+^{3/2}r^{3/2}(r_+-B)D_1^{1/2}(B,Q)},\\
z^3&=\frac{l}{r_+}r\sinh\frac{r_+}{l}\theta,\\
z^4&=\frac{l}{r_+}r\cosh\frac{r_+}{l}\theta,\\
z^5&=k_H^{-1}\int dr\,\frac{lN_2^{1/2}(B,Q)}{2r_+^{3/2}r^{3/2}(r_+-B)D_1^{1/2}(B,Q)},
\end{aligned} \tag{31}$$

where the Hawking-Bekenstein horizon surface gravity is given by

$$k_H=\frac{r_+}{l^2}-\frac{Br_+^2-2BQ^2l^2\ln r_+}{2l^2r_+(r_+-B)}-\frac{Q^2}{r_+}, \tag{32}$$

and

$$\begin{aligned}
N_1(B,Q)&=4Q^4r_+^3r^2(r+r_+)[r_+^2+r^2(2f+1)]\\
&\quad+\frac{B^2r_+^5}{l^4}[r_+^3(r^2+r_+r+r_+^2)+9r^4(r+r_+)\\
&\quad+21r_+^2r^3]+c_{12}BQ^2+c_{16}BQ^6+c_{24}B^2Q^4\\
&\quad+c_{32}B^3Q^2+c_{36}B^3Q^6,
\end{aligned}$$

$$\begin{aligned}
N_2(B,Q)&=\frac{4Q^2r_+^3r^4}{l^2}(r+r_+)\left(2r_+^2+\frac{r_+^4+Q^4l^4}{r_+^2}f\right)\\
&\quad+\frac{Br_+^4}{l^4}[(8r_+^2+14B^2)r_+^2r^3+B^2r_+^3(r^2+r_+r\\
&\quad+r_+^2)+(4r_+^2+5B^2)r^4(r+r_+)]+c_{14}BQ^4\\
&\quad+c_{22}B^2Q^2+c_{26}B^2Q^6+c_{34}B^3Q^4,
\end{aligned} \tag{33}$$

$$\begin{aligned}
D_1(B,Q)&=r_+^2r(r+r_+)-Br_+(r^2+r_+r+r_+^2)\\
&\quad-Q^2l^2(r+r_+)[rf-B(\ln r_++1)g],
\end{aligned}$$

and the coefficients are given by

$$\begin{aligned}
c_{12}&=\frac{4r_+r}{l^2}[r_+^3(2r^2+r_+^2+2r^2\ln r_+)(r^2+r_+r+r_+^2)\\
&\quad+r_+^3r^3(r+r_+)(3f+5)+r_+^4r^2(r+r_+)\\
&\quad\times(\ln r_++1)g],\\
c_{14}&=4r_+^2r[r^2(r^2+r_+r+r_+^2)(2\ln r_++3)+2r_+^3(r\\
&\quad+r_+)\ln r_++r_+^2r(r+3r_+)+r^3(r+r_+)\\
&\quad\times(2\ln r_++5)f+2r_+r^2(r+r_+)(\ln r_++1)g],\\
c_{16}&=4l^2r^3(r+r_+)(\ln r_++1)(2rf+r_+g),\\
c_{22}&=\frac{r_+^3}{l^2}[20r^3(r^2+r_+r+r_+^2)(\ln r_++1)+8r_+^2r^3\ln r_+\\
&\quad+4r_+^2(r^2+r_+r+r_+^2)(2r+r_+\ln r_+)+3r^4(r\\
&\quad+r_+)(3f+4)+12r_+r^3(r+r_+)(\ln r_++1)g],\\
c_{24}&=4r_+[r^3(r^2+r_+r+r_+^2)(\ln r_++1)^2+3r_+^4r\ln r_+\\
&\quad+r_+^2(r^2+r_+r+r_+^2)(r+r_+\ln r_+)\ln r_+(r+r_+)\\
&\quad\times(\ln r_++1)(3r^4f+2r^4+3r_+^2r^2)+r_+r^3(r+r_+)\\
&\quad\times(\ln r_++1)(2\ln r_++5)g],\\
c_{26}&=4l^2r^3\frac{r+r_+}{r_+}(\ln r_++1)^2(rf+2r_+g),\\
c_{32}&=\frac{r_+^2}{l^2}[4r(r_+^2+3r^2)(r^2+r_+r+r_+^2)(\ln r_++1)\\
&\quad+4r_+^2(2r^3+r_+^3)\ln r_++9r_+r^3(r+r_+)\\
&\quad\times(\ln r_++1)g],\\
c_{34}&=4r^3(r^2+r_+r+r_+^2)(\ln r_++1)^2+4r_+^4(r+r_+)\\
&\quad\times(\ln r_+)^2+8r_+^4r\ln r_++4r_+r^2(r+r_+)\\
&\quad\times(\ln r_++1)^2(r_++3rg),
\end{aligned}$$

$$c_{36}=4l^2r^3\frac{r+r_+}{r_+}(\ln r_+ +1)^3 g,$$

$$f(r,r_+)=\frac{2r_+^2\ln(r/r_+)}{r^2-r_+^2}, \tag{34}$$

$$g(r,r_+)=\frac{2r_+(r\ln r-r_+\ln r_+)}{(r^2-r_+^2)(\ln r_+ +1)}.$$

Here both $f(r,r_+)$ and $g(r,r_+)$ approach unities as r goes to r_+, due to L'Hospital's rule.

In static detectors (θ, $r=$const) described by a fixed point in the (z^2, z^3, z^4, z^5) hyperplane, one can have constant three acceleration

$$a=\frac{\dfrac{r}{l^2}-\dfrac{MB}{2r^2}-\dfrac{Q^2}{r}}{\left(-M+\dfrac{MB}{r}+\dfrac{r^2}{l^2}-2Q^2\ln r\right)^{1/2}}, \tag{35}$$

and constant accelerated motion in the (z^0,z^1) plane with the Hawking temperature

$$2\pi T=a_6=\frac{\dfrac{r_+}{l^2}-\dfrac{MB}{2r_+^2}-\dfrac{Q^2}{r_+}}{\left(-M+\dfrac{MB}{r}+\dfrac{r^2}{l^2}-2Q^2\ln r\right)^{1/2}}. \tag{36}$$

On the other hand, the above Hawking temperature is also given by relation (16). Note that one can easily check that, since in the uncharged limit $Q=0$, $N_1(B,Q)$, and $N_2(B,Q)$ in Eq. (33) are exactly reduced to the $N_1(B)$ and $N_2(B)$ in Eq. (11), respectively, the (3+3)-dimensional coordinate transformations Eq. (31) are also exactly reduced to the uncharged case, Eq. (13), having the same (3+3)-dimensional GEMS structure in contrast to the usual BTZ case [14]. Since in this case the metric is Schwarzschild-like, the GEMS structure coinsides with that of the (3+1)-dimensional Schwarzschild black hole, which needs (1+1) additional extra dimensions to yield the (4+2) GEMS structure [4,12]. Furthermore, in the $B=0$ limit, the transformations, Eq. (31), are exactly reduced to the charged BTZ case [14], which still has the (3+3)-dimensional GEMS structure.

B. Case II: $\phi=r^2/(r^2-2L)$

Now consider the charged scalar-tensor theory with $L>0$ for the modified BTZ black hole where the three metric, Eq. (22), is described by the charged lapse function:

$$N(r)=-M+\frac{ML}{r^2}+\frac{r^2}{l^2}-2Q^2\ln r. \tag{37}$$

The coordinate transformations to the (3+3)-dimensional AdS GEMS, $ds^2=(dz^0)^2-(dz^1)^2-(dz^2)^2-(dz^3)^2+(dz^4)^2+(dz^5)^2$ are obtained for $r\geqslant r_+$:

$$\begin{aligned}
z^0&=k_H^{-1}\left(-M+\frac{ML}{r^2}+\frac{r^2}{l^2}-2Q^2\ln r\right)^{1/2}\sinh k_H t,\\
z^1&=k_H^{-1}\left(-M+\frac{ML}{r^2}+\frac{r^2}{l^2}-2Q^2\ln r\right)^{1/2}\cosh k_H t,\\
z^2&=k_H^{-1}\int dr\,\frac{lN_3^{1/2}(L,Q)}{r_+^2r^2(r_+^2-L)D_2^{1/2}(L,Q)},\\
z^3&=\frac{l}{r_+}r\sinh\frac{r_+}{l}\theta,\\
z^4&=\frac{l}{r_+}r\cosh\frac{r_+}{l}\theta,\\
z^5&=k_H^{-1}\int dr\,\frac{lN_4^{1/2}(L,Q)}{r_+^2r^2(r_+^2-L)D_2^{1/2}(L,Q)},
\end{aligned} \tag{38}$$

where the Hawking-Bekenstein horizon surface gravity is given by

$$k_H=\frac{r_+}{l^2}-\frac{Lr_+^2-2LQ^2l^2\ln r_+}{l^2r_+(r_+^2-L)}-\frac{Q^2}{r_+}, \tag{39}$$

and

$$\begin{aligned}
N_3(L,Q)&=Q^4r_+^6r^4[r_+^2+r^2(2f+1)]+\frac{L^2r_+^6}{l^4}[r_+^2(r^4+r_+^2r^2\\
&\quad+r_+^4)+5r^4(r^2+r_+^2)+3r_+^2r^4]+d_{12}LQ^2\\
&\quad+d_{16}LQ^6+d_{24}L^2Q^4+d_{32}L^3Q^2+d_{36}L^3Q^6,\\
N_4(L,Q)&=\frac{Q^2r_+^6r^6}{l^2}\left(2r_+^2+\frac{r_+^4+Q^4l^4}{r_+^2}f\right)+\frac{Lr_+^4}{l^4}[(r_+^4\\
&\quad+3L^2)r_+^2r^4+L^2r_+^2(r^4+r^2r_+^2+r_+^4)+(2r_+^4\\
&\quad+3L^2)r^4(r^2+r_+^2)]+d_{14}LQ^4+d_{22}L^2Q^2\\
&\quad+d_{26}L^2Q^6+d_{34}L^3Q^4,\\
D_2(L,Q)&=r_+^2r^2-L(r^2+r_+^2)-Q^2l^2[r^2f-L(2\ln r_+\\
&\quad+1)g],
\end{aligned} \tag{40}$$

and the coefficients are given by

$$\begin{aligned}
d_{12}&=\frac{r_+^4r^2}{l^2}[2r_+^2r^2(r^2+r_+^2)(2\ln r_+ +1)+2r_+^2(r^4+r_+^2r^2\\
&\quad+r_+^4)+4r_+^2r^4(f+1)+r_+^4r^2(2\ln r_+ +1)g],
\end{aligned}$$

$$d_{14}=r_+^4 r^2[4(r^4+r_+^2 r^2+r_+^4)\ln r_+ + r^2(3r^2+4r_+^2) + 2r^4(2\ln r_+ +3)f+2r_+^2 r^2(2\ln r_+ +1)g],$$

$$d_{16}=l^2 r_+^2 r^4[2r^2 f+r_+^2(2\ln r_+ +1)g],$$

$$d_{22}=\frac{2r_+^4}{l^2}[2(r^4+r_+^2 r^2+r_+^4)(r^2+r_+^2\ln r_+)+3r^4(r^2+r_+^2)\times(2\ln r_+ +1)],$$

$$d_{24}=r_+^2[r^2(r^4+r_+^2 r^2+r_+^4)(2\ln r_+ +1)^2+4r_+^4(r^2+r_+^2\ln r_+)\ln r_+ +2r^6(2\ln r_+ +1)(2f+1)+2r_+^2 r^4(2\ln r_+ +1)(2\ln r_+ +3)g], \tag{41}$$

$$d_{26}=l^2 r^4(2\ln r_+ +1)^2(r^2 f+2r_+^2 g),$$

$$d_{32}=\frac{r_+^2 r^2}{l^2}[(r_+^4+r_+^2 r^2+r^4)+8r^2(r^2+r_+^2)\ln r_+ +2r^2(2r^2+r_+^2)],$$

$$d_{34}=4(r^4+r_+^2 r^2+r_+^4)(r^2+r_+^2\ln r_+)\ln r_+ + r_+^2 r^4(2\ln r_+ +1)^2+r^6[4(\ln r_+)^2+1]+4r_+^2 r^4(2\ln r_+ +1)^2 g,$$

$$d_{36}=l^2 r^4(2\ln r_+ +1)^3 g.$$

In static detectors (θ, $r=$const) described by a fixed point in the (z^2, z^3, z^4, z^5) hyperplane, one can have constant three acceleration

$$a=\frac{\dfrac{r}{l^2}-\dfrac{ML}{r^3}-\dfrac{Q^2}{r}}{\left(-M+\dfrac{ML}{r^2}+\dfrac{r^2}{l^2}-2Q^2\ln r\right)^{1/2}}, \tag{42}$$

and constant accelerated motion in the (z^0,z^1) plane with the Hawking temperature

$$2\pi T=a_6=\frac{\dfrac{r_+}{l^2}-\dfrac{ML}{r_+^3}-\dfrac{Q^2}{r_+}}{\left(-M+\dfrac{ML}{r^2}+\dfrac{r^2}{l^2}-2Q^2\ln r\right)^{1/2}}. \tag{43}$$

Similar to the previous case, one can also check that, since in the uncharged limit of $Q=0$, $N_3(L,Q)$, and $N_4(L,Q)$ are exactly reduced to the $N_1(L)$ and $N_2(L)$, respectively, the coordinate transformations Eq. (31) are also exactly reduced to the uncharged case Eq. (13) having the same (3+3)-dimensional GEMS structure in contrast to the usual BTZ case [14]. Furthermore, in the $L=0$ limit, the transformations Eq. (31) are exactly reduced to the charged BTZ case [14], which still has the (3+3)-dimensional GEMS structure.

IV. CONCLUSIONS

In conclusion, we have newly analyzed the (2+1)-dimensional four uncharged and two charged ST theories with the parameters B or L through the GEMS approach, which are the modified versions of the usual BTZ black holes. First, we have obtained the (3+3)- or (3+2)-dimensional GEMS of the uncharged ST theories in the (2+1) dimensions depending on the positive or negative signs of B, respectively. Second, we have generalized these embeddings to the charged ST theories with the definitely positive B. Third, we have also obtained the (3+3)-dimensional GEMS of the uncharged and charged ST theories with the definitely positive parameter L. Since in the uncharged limit $Q=0$, the (3+3)-dimensional coordinate transformations of the charged ST theories are exactly reduced to the uncharged case having the same (3+3)-dimensional GEMS structure in contrast to the usual BTZ case [14]. Since in the case with $\phi=r/(r-3B/2)$ the metric is Schwarzschild-like, the GEMS structure coinsides with that of the (3+1)-dimensional Schwarzschild black hole, which needs (1+1) additional extra dimensions to yield the (4+2)-dimensional GEMS structure. Furthermore, in the $B=0$ or $L=0$ limit, the coordinate transformations are exactly reduced to the charged BTZ case, which still has the (3+3)-dimensional GEMS structure.

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