Oscillators with constrained energy difference: Coherent states and a quantum clock

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In order to study the ''problem of time in quantum gravity,'' Rovelli proposed a model of a two harmonic oscillator system where one of the oscillators can be thought of as a ''clock'' for the other oscillator. In this paper we examine a model where the Hamiltonian is a difference between two harmonic oscillators, and we consider one of them which has the minus sign as a ''clock,'' since the gravitational degree of freedom has a minus sign in the Hamiltonian of quantum cosmology. Klauder's projection operator approach to generalized coherent states is used to define physical states and operators. The resolution of unity is derived in terms of a gauge invariant coordinate. We investigate the ''quantum clock'' and show that the evolution described by it is identical to the classical motion when the energy becomes large.

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I. INTRODUCTION

One of the measure conceptual problems in quantum gravity is the ''problem of time'' [1]. In quantum gravity physical states and physical operators do not depend on the time coordinate (t) because of the constraints in Dirac quantization [2]. In order to study the ''problem of time in quantum gravity,'' Rovelli proposed an interesting model of a two harmonic oscillator system where one of the oscillators can be thought of as a ''clock'' for the other oscillator [3]. He showed that the ''clock'' can describe a natural time evolution, even though the system has a time reparametrization invariance. In a similar model Lawrie and Epp made one gauge invariant oscillator from the original two harmonic oscillators and studied an evolution which is governed by an exact Heisenberg equation [4]. They considered coherent states and introduced a window function to investigate an approximate analytical time dependence of the system. Recently, Ashworth utilized Klauder's projection operator approach to generalized coherent states [5] for the double harmonic oscillator system [6]. Using Marolf's gauge invariant statement [7], he introduced ''time'' by the phase of an oscillator, a ''clock,''[1] and he showed that the time evolution described by the ''clock'' agrees with the classical equation of motion when the energy becomes large.

On the other hand, it is well known that the gravitational degree of freedom has a minus sign in the Hamiltonian of quantum cosmology [9]. The Hamiltonian can be written as a difference between two harmonic oscillators in some cases, for example, the five-dimensional Kaluza-Klein cosmology by Wudka [10] and the minisuperspace model by Hartle-Hawking [11] if time variable is redefined and the cosmological constant is assumed to be zero. Other examples were also considered in Refs. [12,13]. Such a cosmological model and some aspects of its coherent states have been extensively discussed in Refs. [14–16].

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[1]The idea to use a phase variable of an oscillator as a quantum clock was given already in Ref. [8].

In this paper we examine a model where the Hamiltonian is a difference between two harmonic oscillators, and we consider one of them which has the minus sign in the Hamiltonian as a ''clock.'' The projection operator approach to generalized coherent states is used to define physical states. We deduce a resolution of unity with respect to gauge invariant states by virtue of a coordinate transformation. In the same way physical operators are expressed in terms of gauge invariant states and physical symbols. We investigate the ''quantum clock'' and show that the evolution described by it is identical to the classical motion when the energy becomes large.

In Sec. II we will consider a model where the Hamiltonian is a difference between two harmonic oscillators, and we will use the projection operator approach to generalized coherent states in order to obtain physical states. In Sec. III the resolution of unity will be derived in terms of a gauge invariant coordinate. In Sec. IV we will project operators to the physical space, and we will define a ''quantum clock'' and show that the evolution described by it is the same with the classical motion when the energy becomes large. We summarize in Sec. V. Appendix A is devoted to derive the gauge transformation of our system. In Appendix B it will be shown that our result of the resolution of unity agrees with that in Ref. [17].

II. A MODEL OF TWO HARMONIC OSCILLATORS

Let us consider the following action which is a difference between two harmonic oscillators:

$$S=\int dt\ \ L,$$

$$L=\frac{1}{2N}\left[\left(\frac{dq_1}{dt}\right)^2-\left(\frac{dq_2}{dt}\right)^2\right]-\frac{N}{2}[\omega^2(q_1^2-q_2^2)-2E], \tag{1}$$

where $q_1(t),q_2(t)$ are the harmonic oscillators with the same frequency ω, $N(t)$ is the lapse function, and E is the energy difference. The action (1) has the time reparametrization invariance, and the Hamiltonian reads

$$H=N(H_1-H_2-E), \tag{2}$$

where $H_i=\frac{1}{2}(p_i^2+\omega^2 q_i^2)$ $(i=1,2)$. If we define the proper time $\tau=\int_0^t dt' N(t')$, the classical equations of motion for q_1, q_2 are $\ddot{q}_i=-\omega^2 q_i$ $(i=1,2)$, $\ddot{q}_i=d^2q_i/d\tau^2$. Therefore, q_1 and q_2 are ordinary harmonic oscillators with only one exceptional point that q_2 has a minus sign in the Hamiltonian (2).

We write the classical solution of this system as

$$q_1^{cl}=A\cos(\omega\tau+\phi_1), \quad q_2^{cl}=B\cos(-\omega\tau+\phi_2), \tag{3}$$

where we have assumed that the two harmonic oscillators have opposite dependence on the proper time. The reason of this assumption is because under the gauge transformation, that is the time translation generated by the Hamiltonian, the phases of the two harmonic oscillators are transformed into opposite direction, which is discussed in Appendix A.[2] Then the classical motion of each harmonic oscillator can be also written by another harmonic oscillator as

$$q_1^{cl}=A\cos\left(-\cos^{-1}\frac{q_2^{cl}}{B}+\phi_1+\phi_2\right),$$

$$q_2^{cl}=B\cos\left(-\cos^{-1}\frac{q_1^{cl}}{A}+\phi_1+\phi_2\right), \tag{4}$$

where we have assumed that arccosine takes the principal value, namely, $0\leqslant\cos^{-1}x\leqslant\pi$. This expression shows that either q_1^{cl} or q_2^{cl} can be used for a classical clock for q_2^{cl} or q_1^{cl}, respectively.

According to the Dirac procedure [2], we obtain a primary constraint on the momentum p_N which is canonical conjugate to N:

$$p_N=0, \tag{5}$$

since the action (1) has no time derivative of N. As this constraint must hold throughout all time, we get a secondary constraint

$$H_1-H_2-E=0. \tag{6}$$

Equations (3), (6) imply that the classical amplitudes of the oscillators must satisfy

$$(A\omega)^2-(B\omega)^2=2E. \tag{7}$$

The two constraints (5), (6) are the first class constraints, and they come from the time reparametrization invariance. In the Dirac quantization, the physical states are defined by imposing the first class constraints on the full states. Because the constraint (5) means that the physical states do not contain N, hereafter we consider the dynamical variables are q_1, q_2.

[2]It was pointed out by Professor T. Kubota that the phases of two harmonic oscillators are transformed into opposite direction under the gauge transformation.

To quantize this model, we impose the canonical commutation relations for Heisenberg operators $\hat{Q}_j$, $\hat{P}_k$

$$[\hat{Q}_j,\hat{P}_k]=i\hbar\delta_{jk},$$

$$[\hat{Q}_j,\hat{Q}_k]=[\hat{P}_j,\hat{P}_k]=0. \tag{8}$$

Provided we define annihilation operators by

$$a=\sqrt{\frac{\omega}{2\hbar}}\hat{Q}_1+\frac{i}{\sqrt{2\hbar\omega}}\hat{P}_1, \quad b=\sqrt{\frac{\omega}{2\hbar}}\hat{Q}_2+\frac{i}{\sqrt{2\hbar\omega}}\hat{P}_2, \tag{9}$$

then we obtain $[a,a^\dagger]=[b,b^\dagger]=1$, $[a,b]=[a^\dagger,b^\dagger]=0$. Now Eq. (6) suggests that the Hamiltonian H_1-H_2-E becomes the generator of the gauge transformation associated with the time translation (see Appendix A). From Eqs. (6) and (9) the constraint operator can be written as

$$\hat{\Phi}=a^\dagger a-b^\dagger b-E', \tag{10}$$

with $E'=E/\hbar\omega$.

We start from the coherent states for the two harmonic oscillators:

$$|\alpha,\beta\rangle=e^{-(|\alpha|^2+|\beta|^2)/2}\sum_{n,m=0}^{\infty}\frac{\alpha^n\beta^m}{\sqrt{n!}\sqrt{m!}}|n,m\rangle, \tag{11}$$

where

$$|n,m\rangle=\frac{1}{\sqrt{n!}}(a^\dagger)^n\frac{1}{\sqrt{m!}}(b^\dagger)^m|0,0\rangle$$

and α,β are arbitrary complex numbers [18]. These coherent states satisfy the properties

$$a|\alpha,\beta\rangle=\alpha|\alpha,\beta\rangle, \quad b|\alpha,\beta\rangle=\beta|\alpha,\beta\rangle,$$

$$\langle\alpha,\beta|\alpha,\beta\rangle=1, \quad \mathcal{I}=\int\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}|\alpha,\beta\rangle\langle\alpha,\beta|, \tag{12}$$

with $d^2\alpha=d(\mathrm{Re}\,\alpha)d(\mathrm{Im}\,\alpha)$. Using Eqs. (9), (12), we obtain the diagonal element of $\hat{Q}_1$ and $\hat{Q}_2$ as

$$q_1(\alpha,\beta)=\langle\alpha,\beta|\hat{Q}_1|\alpha,\beta\rangle=\sqrt{\frac{\hbar}{2\omega}}(\alpha+\overline{\alpha}),$$

$$q_2(\alpha,\beta)=\langle\alpha,\beta|\hat{Q}_2|\alpha,\beta\rangle=\sqrt{\frac{\hbar}{2\omega}}(\beta+\overline{\beta}). \tag{13}$$

In the same way as Ref. [6], we utilize Klauder's projection operator approach to generalized coherent states [5]. Projecting $|\alpha,\beta\rangle$ on the physical states as

$$|\alpha,\beta\rangle_{\rm phys}=\mathcal{P}|\alpha,\beta\rangle,\quad \mathcal{P}=\int d\mu(\lambda)\ e^{-i\lambda\hat{\Phi}},$$

$$d\mu(\lambda)=d\lambda\frac{\sin(\epsilon\lambda)}{\pi\lambda}\quad (0<\epsilon\ll 1), \tag{14}$$

we have

$$|\alpha,\beta\rangle_{\rm phys}=e^{-(|\alpha|^2+|\beta|^2)/2}\sum_{n,m=0}^{\infty}\frac{\alpha^n\beta^m}{\sqrt{n!}\sqrt{m!}}\times\int d\mu(\lambda)\ e^{-i\lambda(n-m-E')}|n,m\rangle$$
$$=e^{-(|\alpha|^2+|\beta|^2)/2}\sum_{m=0}^{\infty}\frac{\alpha^{m+m'}\beta^m}{\sqrt{(m+m')!}\sqrt{m!}}|m+m',m\rangle. \tag{15}$$

Here we have set $E'=m'=n-m$. The norms of these states are

$${}_{\rm phys}\langle\alpha,\beta|\alpha,\beta\rangle_{\rm phys}=\langle\alpha,\beta|\mathcal{P}|\alpha,\beta\rangle$$
$$=e^{-(|\alpha|^2+|\beta|^2)}\sum_{m=0}^{\infty}\frac{|\alpha|^{2(m+m')}|\beta|^{2m}}{m!(m+m')!}$$
$$=e^{-(|\alpha|^2+|\beta|^2)}\left|\frac{\alpha}{\beta}\right|^{m'}I_{m'}(2|\alpha\beta|), \tag{16}$$

where we have used the formula [19]

$$\sum_{n=0}^{\infty}\frac{(x/2)^{2n}}{n!(k+n)!}=\left(\frac{2}{x}\right)^k I_k(x), \tag{17}$$

and $I_k(x)$ is a modified Bessel function. Normalized physical states, $|\alpha,\beta\rangle_{\rm n\ phys}$ can be written as

$$|\alpha,\beta\rangle_{\rm n\ phys}=|\alpha,\beta\rangle_{\rm phys}/\sqrt{{}_{\rm phys}\langle\alpha,\beta|\alpha,\beta\rangle_{\rm phys}}$$
$$=\frac{|\beta/\alpha|^{m'/2}\alpha^{m'}}{\sqrt{I_{m'}(2|\alpha\beta|)}}\sum_{m=0}^{\infty}\frac{(\alpha\beta)^m}{\sqrt{(m+m')!}\sqrt{m!}}\times|m+m',m\rangle. \tag{18}$$

These states are analogous to Eq. (14) in the first of Ref. [6]. Lawrie and Epp [4] showed that the reduced physical space contains only one oscillator owing to the Hamiltonian constraint. It seems that the coherent states for this gauge invariant oscillator are equivalent to the physical states in Ref. [6] and the difference is the way how to construct the physical states.

III. RESOLUTION OF UNITY

As indicated in Appendix A the gauge transformation generated by the constraint transforms the complex coordinates as $\alpha\to\alpha e^{i\varphi}$ and $\beta\to\beta e^{-i\varphi}$. If we define a complex coordinate $\xi=\alpha\beta$, then ξ is gauge independent. Now $|\xi|$ is a product of the amplitudes of the two harmonic oscillators, and $\arg\xi$ is the sum of their phases. We will see that ξ is sufficient to describe the resolution of unity in the physical space. Let us define the minus of the phase of the second harmonic oscillator q_2 that has the minus sign in the Hamiltonian (2) as θ, that is $\beta=|\beta|e^{-i\theta}(0\leqslant\theta<2\pi)$. In principle, any of the two oscillators could be used as a ''clock.'' However, the gravitational degree of freedom has a minus sign in the Hamiltonian of quantum cosmology. And we will consider the classical limit of our model as when the energy becomes large, namely, the first oscillator q_1 becomes large. So we will later regard q_2 as a ''clock'' and θ as ''time'' in this system. We can factor out the dependence on θ from $|\alpha,\beta\rangle_{\rm n\ phys}$,

$$|\alpha,\beta\rangle_{\rm n\ phys}=e^{im'\theta}|\xi\rangle,$$

$$|\xi\rangle=\frac{\xi^{m'}}{|\xi|^{m'/2}\sqrt{I_{m'}(|2\xi|)}}\sum_{m=0}^{\infty}\frac{\xi^m}{\sqrt{(m+m')!}\sqrt{m!}}\times|m+m',m\rangle. \tag{19}$$

The unity operator in the full phase space can be projected in the physical phase space

$$\mathcal{I}'=\mathcal{P}\mathcal{I}\mathcal{P}=\int\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}\mathcal{P}|\alpha,\beta\rangle\langle\alpha,\beta|\mathcal{P}. \tag{20}$$

Suppose we change the coordinates

$$\tilde{r}=|\alpha|^2-|\beta|^2,$$
$$e^{-i\theta}=\frac{\beta}{|\beta|},$$
$$\xi=\alpha\beta. \tag{21}$$

We have

$$\alpha=\sqrt{\frac{\tilde{r}_+}{2}}\frac{\xi}{|\xi|}e^{i\theta},\quad \beta=\sqrt{\frac{-\tilde{r}_-}{2}}\ e^{-i\theta},$$

where $\tilde{r}_\pm=\tilde{r}\pm\sqrt{\tilde{r}^2+4|\xi|^2}$, $\tilde{r}_+\tilde{r}_-=-4|\xi|^2$. The absolute value of the Jacobian $|J|$ associated with this change of coordinates is calculated as[3]

$$|J|=\frac{1}{2\sqrt{\tilde{r}^2+4|\xi|^2}}=\frac{1}{2r}, \tag{22}$$

where we have defined $r=|\alpha|^2+|\beta|^2=\sqrt{\tilde{r}^2+4|\xi|^2}$, $(r\geqslant 2|\xi|)$. Using Eqs. (16), (19)–(22), we deduce the resolution of unity

[3]It is easier to consider the inverse change of coordinates and to derive $|J^{-1}|=2r$ than to calculate $|J|$ directly, which was suggested by Professor T. Kubota.

$$\mathcal{I}'=\int\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}|\langle\alpha,\beta|\mathcal{P}|\alpha,\beta\rangle||\xi\rangle\langle\xi|$$
$$=\int d\tilde{r}d\theta d^2\xi\,\frac{1}{2\pi^2 r}e^{-r}\left(\frac{\tilde{r}_+}{2|\xi|}\right)^{m'}I_{m'}(2|\xi|)|\xi\rangle\langle\xi|.$$

Because $\tilde{r}=\pm\sqrt{r^2-4|\xi|^2}$ ($+$ for $|\alpha|\geqslant|\beta|$, $-$ for $|\alpha|<|\beta|$), we are led to

$$\mathcal{I}'=\int\frac{d^2\xi}{\pi}\frac{I_{m'}(2|\xi|)}{(2|\xi|)^{m'}}f_{m'}(|\xi|)|\xi\rangle\langle\xi|,$$

$$f_{m'}(|\xi|)=\int_{-\infty}^{\infty}d\tilde{r}\,\frac{e^{-r}}{r}(\tilde{r}_+)^{m'}$$
$$=\int_{2|\xi|}^{\infty}dr\,\frac{e^{-r}}{\sqrt{r^2-4|\xi|^2}}[r_+^{m'}+r_-^{m'}],\qquad(23)$$

with $r_\pm=r\pm\sqrt{r^2-4|\xi|^2}$. Owing to the formula [20]

$$\int_a^\infty dx\,\frac{(x+\sqrt{x^2-a^2})^\nu+(x-\sqrt{x^2-a^2})^\nu}{\sqrt{x^2-a^2}}e^{-px}$$
$$=2a^\nu K_\nu(ap)\qquad(a>0,\ \mathrm{Re}\,p>0),$$

where K_ν is a modified Bessel function, we can obtain the explicit expression of $f_{m'}(|\xi|)$ as

$$f_{m'}(|\xi|)=2(2|\xi|)^{m'}K_{m'}(2|\xi|).\qquad(24)$$

Finally we can derive the resolution of unity

$$\mathcal{I}'=\frac{2}{\pi}\int d^2\xi\ \ I_{m'}(2|\xi|)K_{m'}(2|\xi|)|\xi\rangle\langle\xi|,\qquad(25)$$

from Eqs. (23), (24).

Now the constraint equations (7), (10) suggest that the underlying symmetry of our model is SU(1,1). As shown in Appendix B, it is possible to prove that our result (25) agrees with Eq. (3.22) in Ref. [17] which is the resolution of unity for generalized coherent states associated with the Lie algebra of SU(1,1). Our coherent states (19) and the resolution of unity (25) are also closely analogous to Eqs. (3.6), (3.7) of conserved charge coherent states in Ref. [21].

IV. PROJECTION OF OPERATORS AND QUANTUM CLOCK

According to Ref. [6], let us define a symbol for an arbitrary operator $\tilde{O}(\hat{Q},\hat{P})$ on the physical space as

$$o(q,p)|_{\rm phys}=\frac{\langle q,p|\mathcal{P}\tilde{O}(\hat{Q},\hat{P})\mathcal{P}|q,p\rangle}{|\langle q,p|\mathcal{P}|q,p\rangle|},\qquad(26)$$

and let us project $\tilde{O}(\hat{Q},\hat{P})$ to a well-defined operator on the physical states as

$$\tilde{O}(\hat{Q},\hat{P})|_{\rm phys}=\int d\mu(q,p)o(q,p)\mathcal{P}|q,p\rangle\langle q,p|\mathcal{P}.\qquad(27)$$

In the same way as the resolution of unity, we can rewrite this equation into the form

$$\tilde{O}(\hat{Q},\hat{P})|_{\rm phys}=\int\frac{d^2\alpha}{\pi}\frac{d^2\beta}{\pi}o(\alpha,\beta)\mathcal{P}|\alpha,\beta\rangle\langle\alpha,\beta|\mathcal{P}$$
$$=\frac{2}{\pi}\int d^2\xi\ I_{m'}(2|\xi|)K_{m'}(2|\xi|)\ o'(\xi)|\xi\rangle\langle\xi|,$$

$$o'(\xi)=\frac{1}{2(2|\xi|)^{m'}K_{m'}(2|\xi|)}$$
$$\times\int_{-\infty}^{\infty}d\tilde{r}\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{e^{-r}}{r}\tilde{r}_+^{m'}o(\xi,\tilde{r},\theta),\qquad(28)$$

where r and $\tilde{r}_+$ were defined in Eqs. (21), (22). Note that $o'(\xi)$ is the projected symbol and $o'(\xi)=1$ when $o(\alpha,\beta)=1$.

Unless the symbol $o(\xi,\tilde{r},\theta)$ changes very much with respect to $\tilde{r}$, the integrand of $o'(\xi)$ (X below) approaches a Gaussian function around $\tilde{r}\approx m'$, when the energy of the system $E'=m'$ becomes large

$$X=\frac{e^{-\sqrt{\tilde{r}^2+4|\xi|^2}}}{2K_{m'}(2|\xi|)\sqrt{\tilde{r}^2+4|\xi|^2}}\left(\frac{\tilde{r}+\sqrt{\tilde{r}^2+4|\xi|^2}}{2|\xi|}\right)^{m'}$$
$$\to\frac{1}{\sqrt{2\pi m'}}\exp\left[-\frac{(\tilde{r}-m')^2}{2m'}\right].\qquad(29)$$

Here we have used the asymptotic form of K_ν in Ref. [22],

$$K_\nu(\nu z)\sim\sqrt{\frac{\pi}{2\nu}}\frac{e^{-\nu\eta}}{(1+z^2)^{1/4}}\qquad(\nu\to\infty),$$

$$\eta=\sqrt{1+z^2}+\log\frac{z}{1+\sqrt{1+z^2}},\qquad(30)$$

and we have assumed $\tilde{r}\gg|\xi|,m'\gg1$. Figure 1 demonstrates the relation between X and $\tilde{r}$, when $m'=10,\ 100,\ 1000$ and $|\xi|=1$. The limit (29) of X means that $\lim_{m'\to\infty}\int_{-\infty}^{\infty}d\tilde{r}X=1$. Thus X becomes a delta function $\delta(\tilde{r}-m')$ in the classical limit. This means that, when m' is large, the projection of the symbol satisfies $o'(\xi)\approx\int_0^{2\pi}(d\theta/2\pi)o(\xi,m',\theta)$, and, if the symbol o is gauge independent, namely, o does not depend on θ, we have $o'(\xi)\approx o(\xi,m',\theta_0)$, where θ_0 is an arbitrary constant ($0\leqslant\theta_0<2\pi$).

For example, let us take $\hat{Q}_1$ and $\hat{Q}_2$ for $\tilde{O}(\hat{Q},\hat{P})$, then we have

$$q_1'(\xi)\propto\int_0^{2\pi}d\theta\cos(\phi_+ +\theta)=0,$$

$$q_2'(\xi)\propto\int_0^{2\pi}d\theta\cos\theta=0,$$

where we have used Eqs. (13), (28) and have defined $\xi=|\xi|e^{i\phi_+}$. This result is rather natural, since the average positions of operators over one period of the oscillator are zero [6]. Note that the gauge transformation is ''time translation'' in this system, so we must choose a specific time to avoid this result.

Following Ashworth, we use Marolf's gauge invariant statement [7]

$$o|_{q=s}=\int dt\frac{dq}{dt}\delta[q(t)-s]o(t). \tag{31}$$

Let us consider the second oscillator q_2 which has the minus sign in the Hamiltonian as a ''clock,'' and let us regard the minus of its phase θ as ''time'' in our system. So we take $q=q_2(\theta)$, $s=B\cos(\omega\tau-\phi_2)=q_2^{cl}$, and we obtain

$$\begin{aligned}o|_{q_2=s}&=\int d\theta\frac{dq_2}{d\theta}\delta[q_2(\theta)-B\cos(\omega\tau-\phi_2)]o(\theta)\\&=\int d\theta\delta[\theta-(\omega\tau-\phi_2)]o(\theta)=o(\omega\tau-\phi_2).\end{aligned}$$

This means that we can replace $o(\xi,\tilde{r},\theta)|_{q_2=s}$ by $o(\xi,\tilde{r},\omega\tau-\phi_2)$, so Eq. (28) gives

$$\begin{aligned}o'(\xi,s)&=o'(\xi)\Big|_{q_2=s}\\&=\frac{1}{2(2|\xi|)^{m'}K_{m'}(2|\xi|)}\\&\quad\times\int_{-\infty}^{\infty}d\tilde{r}\int_0^{2\pi}\frac{d\theta}{2\pi}\frac{e^{-r}}{r}\tilde{r}_+^{m'}o(\xi,\tilde{r},\theta)|_{q_2=s}\\&=\frac{1}{2(2|\xi|)^{m'}K_{m'}(2|\xi|)}\int_{-\infty}^{\infty}d\tilde{r}\frac{e^{-r}}{r}\tilde{r}_+^{m'}\\&\quad\times o(\xi,\tilde{r},\omega\tau-\phi_2).\end{aligned}\tag{32}$$

Choosing q_1 as o, we obtain

$$\begin{aligned}q_1'(\xi)|_{q_2=s}&=\frac{1}{2(2|\xi|)^{m'}K_{m'}(2|\xi|)}\\&\quad\times\int_{-\infty}^{\infty}d\tilde{r}\frac{e^{-r}}{r}\tilde{r}_+^{m'}\sqrt{\frac{\hbar}{\omega}}\sqrt{\tilde{r}_+}\\&\quad\times\cos[\phi_+ +(\omega\tau-\phi_2)]\\&=\sqrt{\frac{\hbar}{\omega}}\frac{\cos[\phi_+ +(\omega\tau-\phi_2)]}{2(2|\xi|)^{m'}K_{m'}(2|\xi|)}f_{m'+1/2}(|\xi|),\end{aligned}$$

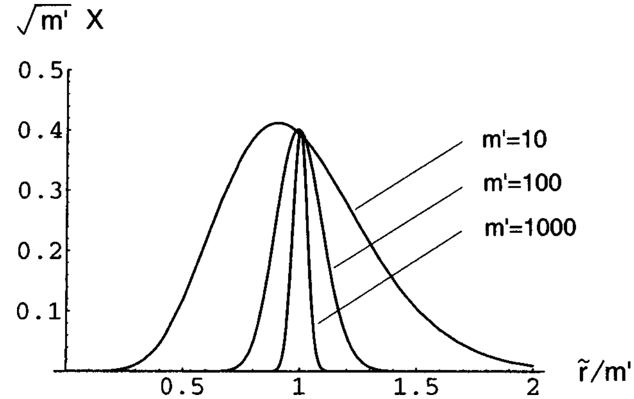

FIG. 1. The relation between X and $\tilde{r}$, when $m'=10,100,1000$ and $|\xi|=1$.

where ϕ_+ is the phase of ξ, and $f_{m'+1/2}(|\xi|)$ is defined in Eqs. (23) with the replacement $m'\to m'+\frac{1}{2}$. Since Eq. (24) means that $f_{m'+1/2}(|\xi|)=2(2|\xi|)^{m'+1/2}K_{m'+1/2}(2|\xi|)$, we arrive at

$$q_1'(\xi)|_{q_2=s}=\sqrt{\frac{\hbar}{\omega}}\sqrt{2|\xi|}\frac{K_{m'+1/2}(2|\xi|)}{K_{m'}(2|\xi|)}\cos(\omega\tau-\phi_2+\phi_+). \tag{33}$$

In the classical limit $E'=m'\to\infty$, the asymptotic form of the modified Bessel function (30) gives $K_{m'+1/2}(2|\xi|)/K_{m'}(2|\xi|)\approx\sqrt{m'/|\xi|}$ and

$$\begin{aligned}q_1'(\xi)|_{q_2=s}&\approx\sqrt{\frac{2\hbar}{\omega}}\sqrt{m'}\cos(\omega\tau-\phi_2+\phi_+)\\&\approx A\cos(\omega\tau-\phi_2+\phi_+).\end{aligned}\tag{34}$$

Here A is the amplitude of the first oscillator, and we have used $\tilde{r}\approx m'$, $\tilde{r}\gg|\xi|$. Note that ξ is gauge invariant and its phase ϕ_+ is the same as the initial phase sum $\phi_1+\phi_2$. Hence the right-hand side of Eq. (34) is identical to the classical solution q_1^{cl} in Eqs. (3). Namely, the evolution of the first operator q_1 described by the ''quantum clock'' q_2 is identical to the classical motion when the energy becomes large.

V. SUMMARY

We examined a model where the Hamiltonian is a difference between two harmonic oscillators, and we considered one of them which has the minus sign in the Hamiltonian as a ''clock,'' since the gravitational degree of freedom has a minus sign in quantum gravity. The projection operator approach to generalized coherent states was used to define physical states. We deduced a resolution of unity with respect to gauge invariant states . In the same way, physical operators were expressed in terms of gauge invariant states and physical symbols. We investigated the ''quantum clock'' and showed that the evolution described by it is identical to the classical motion when the energy becomes large.

As a future work, it will be interesting to apply the projection operator approach to coherent states in order to study the time evolution of the five-dimensional Kaluza-Klein cosmology by Wudka [10] and the minisuperspace model by Hartle-Hawking [11] when the cosmological constant is zero. It would be also interesting to extend our results from minisuperspace to full superspace and to examine the relation between our consideration and some recent papers on the "problem of time" [23–25].

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APPENDIX A

First we begin by a pair of creation and annihilation operators of a harmonic oscillator $a^\dagger, a$ which satisfy $[a,a^\dagger]=1$. Let us formally define the polar decomposition of a as

$$a=e^{i\theta_a}|a|, \tag{A1}$$

where $|a|$ and θ_a are the absolute value operator and the phase operator of a, respectively [26]. Then the number operator $N_a=a^\dagger a=|a|^2$ satisfies

$$[N_a, e^{i\theta_a}]=-e^{i\theta_a}. \tag{A2}$$

In the following expansion:

$$e^{i\theta_a}N_a e^{-i\theta_a}=N_a+i[\theta_a,N_a]+\frac{i^2}{2!}[\theta_a,[\theta_a,N_a]]+\cdots,$$

the left-hand side is equal to N_a+1 by Eq. (A2), and the right-hand side becomes $N_a+i[\theta_a,N_a]$, because $[\theta_a,N_a]$ is a c number. Therefore we obtain

$$[N_a,\theta_a]=i. \tag{A3}$$

Next let us consider another pair of creation and annihilation operators $b^\dagger, b$ with $[b,b^\dagger]=1$, then similar equations as Eqs. (A1)–(A3) hold with respect to b. We examine two cases where the Hamiltonian is the sum or the difference of two harmonic oscillators.

Case 1: $\hat{H}_+=\hbar\omega(N_a+N_b-E')$. Since Eq. (A3) means $[\hat{H}_+,\theta_a]=i\hbar\omega$ and $[\hat{H}_+,\theta_b]=i\hbar\omega$, we have

$$[\hat{H}_+,\theta_a+\theta_b]=2i\hbar\omega,\quad [\hat{H}_+,\theta_a-\theta_b]=0. \tag{A4}$$

Therefore the phase difference $\theta_a-\theta_b$ is gauge invariant, and the phase sum $\theta_a+\theta_b$ is not invariant in this case. This case was investigated in Ref. [6].

Case 2: $\hat{H}_-=\hbar\omega(N_a-N_b-E')$. Since Eq. (A3) means $[\hat{H}_-,\theta_a]=i\hbar\omega$ and $[\hat{H}_-,\theta_b]=-i\hbar\omega$, we have

$$[\hat{H}_-,\theta_a+\theta_b]=0,\quad [\hat{H}_-,\theta_a-\theta_b]=2i\hbar\omega. \tag{A5}$$

Hence the phase sum $\theta_a+\theta_b$ is gauge invariant, and the phase difference $\theta_a-\theta_b$ is not invariant in this case.

From Eq. (10), $\hat{H}_-=\hbar\omega\hat{\Phi}$, and $\hat{H}_-$ is the generator of the gauge transformation (time translation) of our system. So Eqs. (A5) suggest that the phase of each harmonic oscillator transforms into opposite direction under the gauge transformation.

APPENDIX B

Barut and Girardello [17] derived the resolution of unity for generalized coherent states associated with the Lie algebra of SU(1,1) as

$$\mathcal{I}'=\int d\sigma(z)\ |z\rangle\langle z|,$$

$$\sigma(z)=\sigma(\rho)\rho d\rho d\varphi,$$

$$\sigma(\rho)=\frac{4}{\pi\Gamma(-2\Phi)}(\sqrt{2}\rho)^{-2\Phi-1}K_{1+2\Phi}(2\sqrt{2}\rho), \tag{B1}$$

where $z=\rho e^{i\varphi}, |z|=\rho$ and $-2\Phi-1=0,1,2,\ldots$. Here we have corrected an erratum of $\sigma(\rho)$ in Ref. [17], namely, $K_{1/2+\Phi}\to K_{1+2\Phi}$. The reason for this is because the formula in Ref. [27] was wrong, and should be replaced by

$$\int_0^\infty dx\ \ 2x^{\alpha+\beta}K_{2(\alpha-\beta)}(2\sqrt{x})x^{s-1}=\Gamma(2\alpha+s)\Gamma(2\beta+s). \tag{B2}$$

This formula follows immediately from Eq. (6.561.16) in Ref. [28]:

$$\int_0^\infty dx\ x^\mu K_\nu(ax)=2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right),$$

where $\mathrm{Re}(\mu+1\pm\nu)>0$, $\mathrm{Re}\,a>0$. We can also easily assure Eq. (B2) in a special case that $\alpha=1/4$, $\beta=0$, $s=1$, using $K_{1/2}(z)=\sqrt{\pi/2z}\exp(-z)$ [29].

Let us write the normalized state of $|z\rangle$ as $|z\rangle_{\rm n}$, then we have

$$|z\rangle\langle z|=\langle z|z\rangle|z\rangle_{\rm n}\,{}_{\rm n}\langle z|,$$

$$\langle z|z\rangle=\Gamma(-2\Phi)\sum_{n=0}^{\infty}\frac{(\sqrt{2}|z|)^{2n}}{n!\Gamma(-2\Phi+n)}$$

$$=\frac{m'!}{(\sqrt{2}|z|)^{m'}}I_{m'}(2\sqrt{2}|z|), \tag{B3}$$

where we have used Eq. (17) and have identified $m'=-2\Phi-1$. If we write $\xi=\sqrt{2}z$ and $|z\rangle_{\rm n}\,{}_{\rm n}\langle z|=|\xi\rangle\langle\xi|$, then Eqs. (B1) is identical to Eq. (25). Therefore, we have established that our resolution of unity agrees with that in Ref. [17].

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