Gravitational waves in open de Sitter space

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We compute the spectrum of primordial gravitational wave perturbations in open de Sitter spacetime. The background spacetime is taken to be the continuation of an O(5) symmetric instanton saddle point of the Euclidean no boundary path integral. The two-point tensor fluctuations are computed directly from the Euclidean path integral. The Euclidean correlator is then analytically continued into the Lorentzian region where it describes the quantum mechanical vacuum fluctuations of the graviton field. Unlike the results of earlier work, the correlator is shown to be unique and well behaved in the infrared. We show that the infrared divergence found in previous calculations is due to the contribution of a discrete gauge mode inadvertently included in the spectrum.

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I. INTRODUCTION

One appeal of inflationary cosmology is its mechanism for the origin of cosmological perturbations. The de Sitter phase of exponentially rapid expansion quickly redshifts away any local perturbations, leaving behind only the quantum mechanical vacuum fluctuations in the various fields. During inflation, these perturbations are stretched to macroscopic length scales and subsequently amplified, to later seed the growth of the large scale structures in the present-day universe. A particularly clean example of this effect are the gravitational wave perturbations of the spacetime itself. These tensor perturbations contribute to the cosmic microwave background anisotropy via the Sachs-Wolfe effect. They may potentially provide an observational discriminant between different theories of open (or closed) inflation because their long-wavelength modes strongly depend on the boundary conditions at the instanton that describes the beginning of the inflationary universe [1].

Although the tensor spectrum has been successfully computed in realistic $O(3,1)$ invariant models for an open inflationary universe [1], the problem of calculating the primordial gravitational waves in perfect open de Sitter spacetime has remained a paradox for some time. The previous literature claims that the spectrum of gravitational waves in perfect de Sitter space is infrared divergent for all physically well-motivated initial quantum states of an eternally inflating universe [2–4]. Breaking the $O(4,1)$ invariance of de Sitter space by going to a realistic inflationary model introduces a potential barrier for the tensor fluctuation modes, and it has been argued that the bubble wall acts to regularize the divergent spectrum in perfect de Sitter space [3].

Previous calculations of the gravitational wave spectrum [2,3] in open de Sitter space are based on a mode-by-mode analysis. One has a prescription for the vacuum state of the graviton that is imposed on every mode separately, on some Cauchy surface for the de Sitter spacetime. Then one propagates each mode into the open universe region. In this paper we instead compute the two-point tensor correlator in real space. In doing so, we have obtained an infrared finite tensor spectrum. The difference in the two approaches is related to the non-uniqueness of the mode decomposition in an open universe, as we shall explain.

As an aside, we mention in this context that also fluctuations of a massless minimally coupled scalar field in de Sitter space do not break $O(4,1)$. In some prior literature (see e.g., [5]) it is shown that there is no de Sitter invariant propagator for such a scalar field. However, the scalar field is not itself an observable since the action depends only on its derivative, and there is a symmetry $\phi \to \phi +$ constant. In fact, correlators of space or time derivatives of ϕ are de Sitter invariant, and since these are the only physical correlators in the theory, de Sitter invariance is unbroken.

We implement the Hartle-Hawking no boundary proposal [6] in our work by "rounding off" open de Sitter space on a compact Euclidean instanton, namely a round four sphere. The fluctuations are computed in the Euclidean region directly from the Euclidean path integral, to first order in $\hbar$ around the instanton saddle point. The Euclidean two-point correlator is analytically continued into the Lorentzian region where it describes the quantum mechanical vacuum fluctuations of the graviton field in the state described by the no boundary proposal initial conditions. There is no ambiguity in the choice of initial conditions because the Euclidean correlator is unique.

II. TENSOR FLUCTUATIONS ABOUT COSMOLOGICAL INSTANTONS

In quantum cosmology the basic object is the wavefunctional $\Psi[h_{ij},\phi]$, the amplitude for a three-geometry with metric h_{ij} and field configuration ϕ. It is formally given by a path integral

$$\Psi[h_{ij},\phi] \sim \int^{h_{ij},\phi} [\mathcal{D}g][\mathcal{D}\phi] e^{iS[g,\phi]}. \tag{1}$$

Following Hartle and Hawking [6] the lower limit of the path integral is defined by continuing to Euclidean time and

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integrating over all compact Riemannian metrics g and field configurations ϕ. If one can find a saddle point of Eq. (1), namely a classical solution satisfying the Euclidean no boundary condition, one can in principle at least compute the path integral as a perturbative expansion to any desired power in $\hbar$.

In this paper we wish to compute the two-point tensor fluctuation correlator in open de Sitter spacetime,

$$ds^2 = -dt^2 + \sinh^2(t)(d\chi^2 + \sinh^2(\chi) d\Omega_2^2). \tag{2}$$

Open de Sitter space may be obtained by analytic continuation of an O(5) invariant instanton, describing the beginning of a semi-eternally inflating universe. The analytic continuation is given by setting $t = -i\sigma$ and the radial coordinate $\chi = i\Omega$, where Ω is the polar angle on the three sphere (see [7,8]). The instanton obtained in this way is a solution of the Euclidean equations of motion with the maximal symmetry allowed in four dimensions. It takes the form of a round four sphere with line element $ds^2 = d\sigma^2 + \sin^2(\sigma) d\Omega_3^2$, where $d\Omega_3^2$ is the line element on S^3. It is useful to introduce a conformal spatial coordinate X defined by $\int_\sigma^{\pi/2}(d\sigma'/\sin\sigma')$, so that the line element takes the form

$$ds^2 = \cosh^{-2} X (dX^2 + d\Omega_3^2). \tag{3}$$

On the four sphere X then ranges from $-\infty$ to $+\infty$.

The principles of our method to calculate cosmological perturbations are described in detail in [1,8]. The instanton solution provides the classical background with respect to which the quantum fluctuations are defined. In the Euclidean region the exponent iS in the path integral becomes $-S_E = -(S_0 + S_2)$, where S_E is the Euclidean action, S_0 is the instanton action and S_2 the action for fluctuations. We keep the latter only to second order. The path integral for the two-point tensor fluctuation about a particular instanton background is then given by

$$\langle t_{ij}(x) t_{i'j'}(x') \rangle = \frac{\int [\mathcal{D}\delta g][\mathcal{D}\delta\phi] e^{-S_2} t_{ij}(x) t_{i'j'}(x')}{\int [\mathcal{D}\delta g][\mathcal{D}\delta\phi] e^{-S_2}}. \tag{4}$$

To first order in $\bar{h}$ the quantum fluctuations are specified by a Gaussian integral. The Euclidean action determines the allowed perturbation modes because divergent modes are suppressed in the path integral. The Euclidean two-point tensor correlator is then analytically continued into the Lorentzian region where it describes the quantum mechanical vacuum fluctuations of the graviton field in the state described by the no boundary proposal initial conditions.

To find the perturbed action S_2 that enters in the path integral (4), we write the perturbed line element in open de Sitter space as

$$\begin{aligned} ds^2 = \sinh^{-2}(\tau)(-(1+2A)d\tau^2 + S_i dx^i d\tau \\ + (\gamma_{ij} + h_{ij}) dx^i dx^j), \end{aligned} \tag{5}$$

where the fields A, S_i and h_{ij} are small perturbations. Because we are interested in the gravitational wave spectrum in the open slicing of de Sitter space, we will only retain $O(3,1)$ invariance in our calculation.

The quantities S_i and h_{ij} may be uniquely decomposed as follows [9]:

$$\begin{aligned} h_{ij} &= \frac{1}{3} h \gamma_{ij} + 2\left(\nabla_i \nabla_j - \frac{\gamma_{ij}}{3}\Delta_3\right) E + 2F_{(i|j)} + t_{ij}, \\ S_i &= B_{|i} + V_i. \end{aligned} \tag{6}$$

Here Δ_3 is the Laplacian on S^3 and $|j$ the covariant derivative on the three-sphere. With respect to reparametrizations of the three-sphere, h, B and E are scalars, V_i and F_i are divergenceless vectors and t_{ij} is a transverse traceless symmetric tensor, describing the gravitational waves. Because gauge transformations are scalar or vector, the perturbations t_{ij} are automatically gauge invariant.

It is important to note that the gauge invariance of t_{ij} follows from the uniqueness of the above decomposition. This is only true however for bounded (asymptotically decaying) perturbations [9]. If one does not impose suitable asymptotic conditions on the fields, a degeneracy appears between scalar and tensor perturbations that introduces a discrete gauge mode in the tensor spectrum, which plays a crucial role in the divergent behavior of the correlator. We come back to this point in Sec. V.

We now substitute the decomposition (6) into the Lorentzian action for gravity plus a cosmological constant,

$$S = \frac{1}{2\kappa}\int d^4x \sqrt{-g}(R - 2\Lambda) - \frac{1}{\kappa}\int d^3x \sqrt{\gamma} K. \tag{7}$$

The scalar, vector and tensor quantities decouple. Keeping all terms to second order, we continue the perturbed Lorentzian action to the Euclidean region. The scalar and vector fluctuations are pure gauge in perfect de Sitter space. The tensor perturbations t_{ij} yield the following well-known positive Euclidean action [10]:

$$S_2 = \frac{1}{8\kappa}\int d^4x \frac{\sqrt{\gamma}}{\cosh^2 X}(t'^{ij} t'_{ij} + t^{ij|k} t_{ij|k} + 2t^{ij} t_{ij}). \tag{8}$$

Here prime denotes differentiation with respect to the conformal coordinate X. After performing the rescaling $\tilde{t}_{ij} = t_{ij}/\cosh X$ and integrating by parts we obtain

$$\begin{aligned} S_2 = \frac{1}{8\kappa}\int d^4x \sqrt{\gamma} \tilde{t}_{ij}(\hat{K} + 3 - \Delta_3)\tilde{t}^{ij} \\ + \frac{1}{8\kappa}\left[\int d^3x \sqrt{\gamma} \tilde{t}_{ij} \tilde{t}^{ij} \tanh(X)\right], \end{aligned} \tag{9}$$

where the Schrödinger operator

$$\hat{K} = -\frac{d^2}{dX^2} - \frac{2}{\cosh^2(X)} \equiv -\frac{d^2}{dX^2} + U(X). \tag{10}$$

Because the fluctuations are specified by a Gaussian integral, we can solve the path integral (4) by looking for the Green function of the operator in its exponent. The potential $U(X)$ for the fluctuation modes is well known to be perfectly reflectionless. However, changing its shape slightly would introduce some reflection which becomes increasingly significant at small momenta. Such a change corresponds to breaking the $O(5)$ invariance of Euclidean de Sitter space and is exactly what happens in the O(4) invariant Hawking-Turok [11] and Coleman–De Luccia [12] instantons that describe the beginning of realistic open inflationary universes. This difference between both classes of instantons has profound implications for the tensor perturbations about them, especially for their long-wavelength regime [1]. The operator $\hat{K}$ has in all three cases a positive continuum starting at eigenvalue $p^2=0$, as well as a single bound state $\tilde{t}_{ij}=b(X)q_{ij}(\Omega)$ at $p=i$ which turns out to be a trivial gauge mode.

III. THE EUCLIDEAN GREEN FUNCTION

To evaluate the path integral (4), we first look for the Green function $G_E^{iji'j'}(X,X',\Omega,\Omega')$ of the operator in Eq. (9). The Euclidean fluctuation correlator (4) will then be given by $\cosh(X)\cosh(X')G_E^{iji'j'}$. The Euclidean Green function satisfies

$$\frac{1}{4\kappa}(\hat{K}+3-\Delta_3)G^{ij}_{Ei'j'}(X,X',\Omega,\Omega') = \delta(X-X')\gamma^{-(1/2)}\delta^{ij}{}_{i'j'}(\Omega-\Omega'). \tag{11}$$

If we think of the scalar product as defined by integration over S^3 and summation over tensor indices, then the right hand side is the normalized projection operator onto transverse traceless tensors on S^3.

The Green function $G^{ij}_{Ei'j'}$ can only be a function of the geodesic distance $\mu(\Omega,\Omega')$ if it is to be invariant under isometries of the three-sphere. This suggests that

$$G^{ij}_{Ei'j'}(\mu,X,X')=4\kappa\sum_{p=3i}^{+i\infty}G_p(X,X')W^{ij}_{(p)i'j'}(\mu), \tag{12}$$

where $W^{ij}_{(p)i'j'}(\mu)$ is a bitensor that is invariant under the isometry group O(4). It equals the sum (A2) of the normalized rank-two tensor eigenmodes with eigenvalue $\lambda_p=p^2+3$ of the Laplacian on S^3. Note that the indices i,j lie in the tangent space over the point Ω while the indices i',j' lie in the tangent space over the point Ω'. On S^3 we have

$$\Delta_3 W^{ij}_{(p)i'j'}(\mu)=\lambda_p W^{ij}_{(p)i'j'}(\mu). \tag{13}$$

The motivation for the unusual labeling of the eigenvalues of the Laplacian is that, as demonstrated in the Appendix, in terms of the label p the bitensor on S^3 has precisely the same formal expression as the corresponding bitensor on H^3. It is precisely this property that will enable us in Sec. IV to continue the Green function from the Euclidean instanton into open de Sitter space without decomposing it in Fourier modes. The relation between the bitensors on S^3 and H^3 together with some useful formulas and properties of maximally symmetric bitensors are given in the Appendix.

Since the tensor eigenmodes of the Laplacian on S^3 form a complete basis, we can also write

$$\gamma^{-1/2}\delta^{ij}_{i'j'}(\Omega-\Omega')=\sum_{p=3i}^{+i\infty}W^{ij}_{(p)i'j'}(\mu(\Omega,\Omega')). \tag{14}$$

Hence by substituting our ansatz (12) for the Green function into Eq. (11) we obtain an equation for the X-dependent part of the Green function,

$$(\hat{K}-p^2)G_p(X,X')=\delta(X-X'). \tag{15}$$

The solution to Eq. (15) is

$$G_p(X,X')=\frac{1}{\Delta_p}[\Psi^r_p(X)\Psi^l_p(X')\Theta(X-X') + \Psi^l_p(X)\Psi^r_p(X')\Theta(X'-X)]. \tag{16}$$

$\Psi^l_p(X)$ is the solution to the Schrödinger equation that tends to e^{-ipX} as $X\to-\infty$, and $\Psi^r_p(X)$ is the solution going as e^{ipX} as $X\to+\infty$. The factor Δ_p is the Wronskian of the two solutions. Since the potential is reflectionless on the round four sphere the left- and right-moving waves do not mix and they equal the Jost functions $g_{\pm p}(X)$ with nice analytic properties. The solutions may be found explicitly and are given by

$$\Psi^r_p(X)=(\tanh X-ip)e^{ipX}$$
$$\Psi^l_p(X)=(\tanh X+ip)e^{-ipX} \tag{17}$$

and their Wronskian $\Delta_p=-2ip(1+p^2)$, independent of X. The zero of the Wronskian at $p=i$ corresponds to the bound state mentioned above. Taking $X>X'$, we obtain the Euclidean Green function as a discrete sum

$$G_E^{iji'j'}(\mu,X,X')=4\kappa\sum_{p=3i}^{i\infty}\frac{i}{2p}\frac{\Psi^r_p(X)\Psi^l_p(X')}{(1+p^2)}W^{iji'j'}_{(p)}(\mu). \tag{18}$$

Before proceeding, let us demonstrate that the Euclidean Green function is regular at the poles of the four sphere. This is a nontrivial check because the coordinates σ and X are singular there, and the rescaling becomes divergent too. In the large X, X' limit, Eq. (18) becomes

$$G_E^{iji'j'}(\mu,X,X')=2\kappa\sum_{n=3}^{\infty}\frac{1}{n}e^{-n(X-X')}W^{iji'j'}_{(in)}(\mu). \tag{19}$$

For $n\geqslant 3$ the Gaussian hypergeometric functions $F(3+n,3-n,7/2,z)$ that constitute the bitensor $W^{iji'j'}_{(n)}$ have a series expansion that terminates, and they essentially reduce to Gegenbauer polynomials $C^{(3)}_{n-3}(1-2z)$. Using then the identity [13]

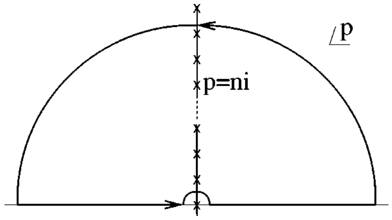

FIG. 1. Contour for the Euclidean correlator.

$$\sum_{l=0}^{\infty} C_l^{\nu}(x) q^l = (1-2xq+q^2)^{-\nu} \tag{20}$$

with $q=e^{-(X-X')}$, one easily sees that the sum (19) indeed converges.

We have the Euclidean Green function defined as an infinite sum (18). However, the eigenspace of the Laplacian on H^3 suggests that the Lorentzian Green function is most naturally expressed as an integral over real p. To do so we must extend the summand into the upper half p-plane. We have already defined the wave functions $\Psi_p(X)$ as analytic functions for all complex p but we need to extend the bitensor as well. When the Green function is expressed as a discrete sum, it involves the bitensor $W^{iji'j'}_{(p)}(\mu)$ evaluated at $p=ni$ with n integral. At these values of p, the bitensor is regular at both coincident and opposite points on S^3, that is at $\mu=0$ and $\mu=\pi$. However, if we extend p into the complex plane we lose regularity at $\mu=0$, essentially because the bitensor obeys the differential equation (11) with a delta function source at $\mu=0$. Similarly we must maintain regularity at $\mu=\pi$, since there is no delta function source there. The condition of regularity at π imposed by the differential equation for the Green function is sufficient to uniquely specify the analytic continuation of $W^{iji'j'}_{(in)}(\mu)$ into the complex p-plane. The continuation is described in the Appendix, and the extended bitensor $W^{iji'j'}_{(p)}(\mu)$ is defined by Eqs. (A4) and (A7).

Now we are able to write the sum in Eq. (18) as an integral along a contour $\mathcal{C}_1$ encircling the points $p=3i,4i,\ldots Ni$, where N tends to infinity. For $X>X'$ we have

$$G_E^{iji'j'}(\mu,X,X') = \kappa \int_{\mathcal{C}_1} \frac{dp}{p \sinh p\pi} \times \frac{\Psi^r_p(X)\Psi^l_p(X')}{(1+p^2)} W^{iji'j'}_{(p)}(\mu). \tag{21}$$

To see that Eq. (21) is equivalent to the sum (18) introduce $1=\cosh p\pi/\cosh p\pi$ into the integral. Then note that $\coth p\pi$ has residue π^{-1} at every integer multiple of i. Finally, use Eq. (A10) to rewrite $W^{iji'j'}_{(p)}(\mu)$ in the form regular at $\mu=0$ used in Eq. (18). The factor of $\cosh p\pi$ from Eq. (A10) cancels that in the integrand.

We now distort the contour for the p integral to run along the real p axis (Fig. 1). At large imaginary p the integrand decays exponentially and the contribution vanishes in the limit of large N. However as we deform the contour towards the real axis we encounter two poles in the $\sinh p\pi$ factor, the latter at $p=i$ becoming a double pole due to the simple zero of the Wronskian. For the $p=2i$ pole, it follows from the normalization of the tensor harmonics that $W^{iji'j'}_{(2i)}=0$. Indirectly, this is a consequence of the fact that spin-2 perturbations do not have a monopole or dipole component. At $p=i$ we have a double pole, but although the relevant Schrödinger operator possesses a bound state, it does not generate a ''super-curvature mode.'' Instead the relevant mode is a time-independent shift in the metric perturbation which may be gauged away [1,3]. We conclude that up to a term involving a pure gauge mode, we can deform the contour $\mathcal{C}_1$ into the contour shown in Fig. 1. For the moment, since the integrand involves a factor $p \sinh p\pi$ which has a double pole at $p=0$, we leave the contour avoiding the origin on a small semicircle in the upper half p-plane.

Finally, in order to deal with the pole at $p=0$, we reexpress the integrand in Eq. (21) as a sum of its p-symmetric and p-antisymmetric parts. Denoting the integrand by I_p we then have

$$G_E^{iji'j'} = \frac{1}{2}\int dp(I_p+I_{-p}) + \frac{1}{2}\int dp(I_p-I_{-p}), \tag{22}$$

where the integral is taken from $p=-\infty$ to ∞ along a path avoiding the origin above. But $\int dp I_{-p}$ along this contour is equal to the integral of I_p taken along a contour avoiding the origin below. The second term is therefore equal to the integral of I_p along a contour around the origin. Hence we have

$$\frac{1}{2}\int dp(I_p-I_{-p}) = -\pi i \,\mathrm{Res}\,(I_p;p=0). \tag{23}$$

We defer a detailed discussion of this term to Sec. V, because its interpretation is clearer in the Lorentzian region. Hence for the time being we just keep it, but it will turn out that it represents a non-physical contribution to the graviton propagator.

In the p-symmetric part of the correlator, we can leave the integrand as a sum of I_p and I_{-p}. We henceforth denote the path from $-\infty$ to $+\infty$ avoiding the origin above by $\mathcal{R}$. This shall turn out to be a regularised version of the integral over the real axis. Our final result for the Euclidean Green function then reads

$$G^E_{iji'j'}(\mu,X,X') = \frac{\kappa}{2}\int_{\mathcal{R}} \frac{dp}{p\sinh p\pi} \times \frac{W^{(p)}_{iji'j'}(\mu)}{(1+p^2)} (\Psi_p(X)\Psi_{-p}(X') + \Psi_{-p}(X)\Psi_p(X')) - \pi i \,\mathrm{Res}\,(I_p;p=0). \tag{24}$$

IV. TWO-POINT TENSOR CORRELATOR IN OPEN de SITTER SPACE

The analytic continuation into open de Sitter space is given by setting $\sigma = it$ and the polar angle $\Omega = -i\chi$. Without loss of generality we may take one of the two points, say Ω' to be at the north pole of the three-sphere. Then $\mu = \Omega$, and μ continues to $-i\chi$. We then obtain the correlator in open de Sitter space where one point has been chosen as the origin of the radial coordinate χ. The conformal coordinate X continues to conformal time τ as $X = -\tau - i\pi/2$ (see [8]).

Hence the analytic continuation of the Euclidean mode functions is given by

$$\Psi_p^r(X) \rightarrow -e^{p\pi/2}\Psi_p^L(\tau) \quad \text{and} \quad \Psi_p^l(X) \rightarrow -e^{-p\pi/2}\Psi_{-p}^L(\tau) \tag{25}$$

where the Lorentzian mode functions are

$$\Psi_p^L(\tau) = (\coth\tau + ip)e^{-ip\tau}. \tag{26}$$

They are solutions to the Lorentzian perturbation equation $\hat{K}\Psi_p^L(\tau) = p^2\Psi_p^L(\tau)$.

In order to perform the substitution $\mu = -i\chi$, where χ is the comoving separation on H^3, we use the explicit formula given in the Appendix for the bitensor regular at $\mu = \pi$. The continued bitensor $W_{iji'j'}^{(p)}(\chi)$ is defined by Eqs. (A7), (A11) and (A12). It can be seen from Eq. (A12) that it involves terms which behave as $e^{\pm p(i\chi+\pi)}$. One must extract the $e^{p\pi}$-factors in order for the bitensor to correspond to the usual sum of rank-two tensor harmonics on the real p-axis. To do so we use the following general identity. For $\tau' - \tau > 0$, we have (up to the $p = i$ gauge mode)

$$\int_{\mathcal{C}} \frac{dp}{p} \frac{\Psi_p^L(\tau)\Psi_{-p}^L(\tau')}{(1+p^2)} e^{ip\chi}F(p) = 0, \tag{27}$$

where $F(p)$ are the p-dependent coefficients occurring in the final (Lorentzian) form of the bitensor given in Eq. (A13). This identity follows from the analyticity of the integrand. By inserting $1 = \sinh p\pi/\sinh p\pi$ under the integral, it is clear that the integral (27) with a factor $e^{p\pi}/\sinh p\pi$ inserted equals that with a factor $e^{-p\pi}/\sinh p\pi$ inserted. The resulting identity allows us to replace the factors $e^{+p(i\chi+\pi)}$ in the bitensor by $e^{p(i\chi-\pi)}$, and vice versa in the analog integral of I_{-p} closed in the lower half p-plane.

For the tensor correlator we also need to restore the factor $ia^{-1}(\tau)$ to t_{ij}. It is convenient to define the eigenmodes $\Phi_p^L(\tau) = \Psi_p^L(\tau)/a(\tau)$. The extra minus sign hereby introduced is cancelled by a change in sign of the normalization factor Q_p of the bitensor, which then becomes $+(p^2+4)/(30\pi^2)$. This corresponds to requiring the spacelike metric to have postive signature. We finally obtain the Lorentzian tensor Feynman (time-ordered) correlator, for $\tau' - \tau > 0$,

$$\begin{aligned}\langle t_{ij}(x), t_{i'j'}(x')\rangle = {}& \frac{\kappa}{2}\int_R \frac{dp}{p\sinh p\pi} \frac{W_{iji'j'}^{L(p)}(\chi)}{(1+p^2)} \\ & \times \left(e^{-p\pi}\Phi_p^L(\tau)\Phi_{-p}^L(\tau') + e^{p\pi}\Phi_{-p}^L(\tau)\Phi_p^L(\tau')\right) \\ & - \pi i \,\mathrm{Res}\,(I_p^L; p=0),\end{aligned} \tag{28}$$

where the Lorentzian bitensor $W_{iji'j'}^{L(p)}$ is defined in the Appendix, Eqs. (A4) and (A13).

In this section, we concentrate on the first term in Eq. (24), the integral over p, and ignore for the moment the second, discrete term. We first extract the symmetrized part, $\langle\{t_{ij}(x), t_{i'j'}(x')\}\rangle$, which is just the real part of the Feynman correlator. The imaginary part involves an integrand which is analytic for $p \rightarrow 0$:

$$\begin{aligned}&\langle t_{ij}(x), t_{i'j'}(x')\rangle \\ &= \frac{\kappa}{2}\int_{\mathcal{R}} \frac{dp}{p(1+p^2)} W_{iji'j'}^{L(p)}(\chi) \\ &\quad \times \coth p\pi[\Phi_p^L(\tau)\Phi_{-p}^L(\tau') + \Phi_{-p}^L(\tau)\Phi_p^L(\tau')] \\ &\quad - 2i\int_0^\infty dp \frac{W_{iji'j'}^{L(p)}(\chi)}{(1+p^2)} \mathcal{I}\left[\frac{1}{p}\Phi_p^L(\tau)\Phi_{-p}^L(\tau')\right].\end{aligned} \tag{29}$$

It is straightforward to see that if we apply the Lorentzian version of the perturbation operator $\hat{K}$ to Eq. (29) with an appropriate heaviside function of $\tau - \tau'$, the imaginary term will produce the Wronskian of $\Phi_{-p}^L(\tau)$ and $\Phi_p^L(\tau)$, which is proportional to ip, times $\delta(\tau - \tau')$. Then the integral over p produces a spatial delta function. From this one sees that our Feynman correlator obeys the correct second order partial differential equation, with a delta function source. The delta function source term in Eq. (11) goes from being real in the Euclidean region to imaginary in the Lorentzian region because the factor $\sqrt{g}$ continues to $i\sqrt{-g}$.

The integral in Eq. (28) diverges as p^{-2} for $p \rightarrow 0$, in contrast with realistic models for inflationary universes where a reflection term in Eq. (29) regularizes the spectrum [1]. However, as we immediately show, even in perfect de Sitter space the integral over p is perfectly finite. We rewrite the symmetrized correlator as an integral over real $0 \leqslant p \leqslant \infty$ as follows. Because the integrand in Eq. (29) is even in p, we have

$$\begin{aligned}\langle\{t_{ij}(x), t_{i'j'}(x')\}\rangle = {}& 2\kappa\int_\epsilon^\infty \frac{dp}{\pi p^2} \frac{p\pi\coth p\pi}{(1+p^2)} \\ & \times \Re[\Phi_p^L(\tau)\Phi_{-p}^L(\tau')] W_{iji'j'}^{L(p)}(\chi) \\ & - \frac{2\kappa}{\pi\epsilon}\Phi_0^L(\tau)\Phi_0^L(\tau')W_{iji'j'}^{L(0)}(\chi) + O(\epsilon),\end{aligned} \tag{30}$$

the second term being the contribution from the small semicircle around $p=0$. Both terms may be combined under one integral. The resulting integrand is *analytic* as $p\rightarrow 0$ and one can safely take the limit $\epsilon\rightarrow 0$. The symmetrized correlator is then given by

$$\langle\{t_{ij}(x),t_{i'j'}(x')\}\rangle$$
$$=2\kappa\int_0^\infty \frac{dp}{\pi p^2}\left(\frac{p\pi\coth p\pi}{(1+p^2)}\Re[\Phi_p^L(\tau)\Phi_{-p}^L(\tau')]\right.$$
$$\left.\times W_{iji'j'}^{L(p)}(\chi)-\Phi_0^L(\tau)\Phi_0^L(\tau')W_{iji'j'}^{L(0)}(\chi)\right), \tag{31}$$

where the Lorentzian bitensor $W_{iji'j'}^{L(p)}$ is defined in the Appendix, Eqs. (A4) and (A13). In this integral it may be written as

$$W_{iji'j'}^{L(p)}(\chi)=\sum_{\mathcal{P}lm} q_{ij}^{(p)\mathcal{P}lm}(\Omega)q_{i'j'}^{(p)\mathcal{P}lm}(\Omega')^*. \tag{32}$$

The functions $q_{ij}^{(p)\mathcal{P}lm}(\Omega)$ are the rank-two tensor eigenmodes with eigenvalues $\lambda_p=-(p^2+3)$ of the Laplacian on H^3. Here $\mathcal{P}=e,o$ labels the parity, and l and m are the usual quantum numbers on the two-sphere. At large p, the coefficient functions $w_j^{(p)}$ of the bitensor (see the Appendix) behave like $p\sin p\chi$. Hence the above integral converges at large p, for both timelike and spacelike separations. Furthermore, the correlations asymptotically decay for large separation of the two points.

Equation (28), with the first term given by Eq. (31) is our final result for the two-point tensor correlator in open de Sitter space, with Euclidean no boundary initial conditions. Contracting the propagator with the harmonics $q_{(p)elm}^{i'j'}$ and integrating over the three sphere reveals that the second term leaves the spectrum completely unchanged apart from cancelling the (divergent) contribution from the $p^2=0$ divergence in the first term. We defer a detailed discussion of this result to the next section, in which we will also clarify the difficulties of the previous work on the graviton propagator in open de Sitter spacetime [2–4].

As an illustration let us compute the Sachs-Wolfe integral [14] and show that all the multipole moments are finite. The contribution of gravitational waves to the CMB anisotropy in perfect de Sitter space is given by

$$\frac{\delta T_{SW}}{T}(\theta,\phi)=-\frac{1}{2}\int_0^{\tau_0}d\tau t_{\chi\chi,\tau}(\tau,\chi,\theta,\phi)|_{\chi=\tau_0-\tau}, \tag{33}$$

where τ_0 is the observing time. The temperature anisotropy on the sky is characterized by the two-point angular correlation function $C(\gamma)$, where γ is the angle between two points located on the celestial sphere. It is customary to expand the correlation function in terms of Legendre polynomials as

$$C(\gamma)=\left\langle\frac{\delta T}{T}(0)\frac{\delta T}{T}(\gamma)\right\rangle=\sum_{l=2}^{\infty}\frac{2l+1}{4\pi}C_lP_l(\cos\gamma). \tag{34}$$

Hence, inserting the Sachs-Wolfe integral into Eq. (34) and substituting Eq. (31) for the two-point fluctuation correlator yields the multipole moments

$$C_l=\frac{\kappa}{2}\int_0^{+\infty}dp\int_0^{\tau_0}d\tau\int_0^{\tau_0}d\tau'$$
$$\times\left(\frac{\coth p\pi}{p(1+p^2)}\Re[\dot\Phi_p^L(\tau)\dot\Phi_p^L(\tau')]Q_{\chi\chi}^{pl}Q_{\chi'\chi'}^{pl}\right.$$
$$\left.-\dot\Phi_0^L(\tau)\dot\Phi_0^L(\tau')Q_{\chi\chi}^{0l}Q_{\chi'\chi'}^{0l}\right). \tag{35}$$

In this expression we have written the normalized tensor harmonics $q_{\chi\chi}^{(p)elm}(\chi,\theta,\phi)$ as $Q_{\chi\chi}^{pl}(\chi)Y_{lm}(\theta,\phi)$, where

$$Q_{\chi\chi}^{pl}(\chi)=\frac{N_l(p)}{p^2(p^2+1)}(\sinh\chi)^{l-2}\left(\frac{-1}{\sinh\chi}\frac{d}{d\chi}\right)^{l+1}(\cos p\chi) \tag{36}$$

and

$$N_l(p)=\left[\frac{(l-1)l(l+1)(l+2)}{\pi\prod_{j=2}^{l}(j^2+p^2)}\right]^{1/2}. \tag{37}$$

It can readily be seen that the multipole moments are finite. With the aid of the explicit expressions and the wave functions (26) they can be numerically computed.

V. CONCLUSIONS

We have computed the spectrum of primordial gravitational waves predicted in open de Sitter space, according to Euclidean no boundary initial conditions. The Euclidean path integral unambiguously specifies the tensor fluctuations with no additional assumptions. The real space Euclidean correlator has been analytically continued into the Lorentzian region without Fourier decomposing it, and we obtained an infrared finite two-point tensor correlator in open de Sitter space, contrary to previous results in the literature [2–4].

Let us now elaborate on the second, regularising term in the symmetrised correlator (31) and the discrete $p=0$ contribution to the Feynman correlator given from the last term in Eq. (24). Not surprisingly, they have a similar interpretation. Their angular part $W_{iji'j'}^{L(0)}(\chi)$ is equal to the sum of the tensor harmonics with eigenvalue $\lambda_p(p=0)=-3$ of the Laplacian on H^3. It has been known that a degeneracy appears between $p^2=0$ tensor modes and $p_s^2=-4$ scalar harmonics [3]. More specifically, one has $q_{ij}^{e(0)lm}=(\nabla_i\nabla_j-\frac{1}{3}\gamma_{ij}\nabla^2)q^{(2i)lm}$ where $q^{(2i)lm}=P_{(2i)lm}Y_{lm}$. The discrete $p^2=0$ tensor harmonics are the only transverse traceless tensor perturbations that can be constructed from a scalar quantity. But as a consequence of this, they are sensitive to scalar gauge transformations. Consider now the coordinate transformation $\xi^\alpha=(0,\epsilon\Phi_0^L(\tau)\nabla^i q^{(2i)lm})$. Under this transformation the transverse traceless part of the metric perturbation h_{ij} in the perturbed line element (5) changes exactly by $\epsilon t_{ij}^{(0)lm}=\epsilon\Phi_0^L(\tau)q_{ij}^{(0)lm}$. Using the transverse-traceless properties of t_{ij} it is easily seen that the action for tensor fluctuations is invariant under such transformations. Hence this tensor eigenmode is non-physical and can be gauged away. Note that since the functional form of ξ is completely fixed this corresponds to a global transformation, analogous to the transformation $\phi\rightarrow\phi+$ constant for a massless field. To compute the Green function for a massless field one has to project out this homogeneous mode, and it is necessary to do the same here. One should therefore disregard the contribution from the discrete term in Eq. (24) to the Lorentzian correlator. This was actually also done in our computation of the tensor fluctuation spectrum about $O(4)$ instantons [1], although in that case not because the mode was pure gauge, but because it couples to the inflaton field, and is not represented by a simple action of the form (8). If a scalar field is present, the mode is most simply treated as a part of the scalar perturbations, as was done in [8].

In our result (31) for the symmetrised correlator, the discrete gauge mode is set to zero because the second term cancels exactly the contribution from the $p^2=0$ mode implicitly contained in the continuous spectrum. This automatic cancellation does not happen in the conventional mode-by-mode analysis where, if one chooses the most degenerate continuous representation of the isometry group $O(3,1)$ of the hyperboloid H^3, corresponding to the range $p\in[0,\infty)$, one obtains a divergent correlator.

It is clear that the underlying reason for these subtleties has to do with the different nature of tensor harmonics on compact and non-compact spaces. Hence, we could have expected the generation of the two discrete gauge modes simply from the analytic continuation of the completeness relation (14) of the harmonics on S^3. Apart from the sum of the complete set of modes that constitute the delta function on H^3, one obtains also three extra terms $W_{(2i)}^{iji'j'}(\mu)$, $W_{(i)}^{iji'j'}(\mu)$ and $W_{(0)}^{iji'j'}(\mu)$. The first term is zero, and the remaining two terms should respectively be viewed as sums of vector—and scalar—harmonics. On the other hand, the fact that the scalar-tensor degeneracy appears precisely at the lower bound of the continuous spectrum is a peculiar feature of three dimensions. In the analogous computation in four dimensions for instance [15], this degeneracy happens at $p^2=-1/4$ and consequently, there is no regularizing term in the correlator.

There is yet another way in which the exclusion of the degenerate modes from the perturbation spectrum can be interpreted. Remember that in non-compact spacetimes the decomposition (6) is only uniquely defined for bounded perturbations. Hence, the only way there can appear a degeneracy between the different types of fluctuations is for the degenerate modes to be unbounded. Indeed, on the three-hyperboloid the scalar $p^2=-4$ modes describe divergent fluctuations because the scalar spherical harmonics $q^{(2i)lm}$ grow exponentially with distance. The action of the above tensor operator renders only the $q_{\chi j}^{(0)lm}$ components of $q_{ij}^{(0)lm}$ finite at infinity. The remaining components still diverge as $\sim e^{\chi}$ and correspond to exponentially growing fluctuations at large distances.[1] Since in cosmological perturbation theory one assumes the perturbation h_{ij} to be small, one must expand correlators in bounded harmonics.

We want to emphasize that the regularity of the two-point tensor correlator does not depend on the Euclidean methods used in our work. One could have equally well computed the correlator on closed Cauchy surfaces for the de Sitter space where the subtleties encountered here do not arise, assuming the standard conformal vacuum for that slicing. One would then analytically continue the result to the open slicing. On the other hand, the Euclidean no boundary principle is an appealing prescription which avoids the arbitrary choice of vacuum otherwise needed. The path integral effectively defines its own initial conditions, yielding a unique and infrared finite Green function in the Lorentzian region. The initial quantum state of the perturbation modes, defined by the no boundary path integral, corresponds to the conformal vacuum in the Lorentzian spacetime. This is in many ways the most natural state in de Sitter space, but the regularity of the graviton propagator is independent of this choice. The most important technical advantage of our method is that we deal throughout directly with the real space correlator, which makes the derivation independent of the gauge ambiguities involved in the mode decomposition.

Finally, let us conclude by comparing the gravitational wave spectrum in perfect open de Sitter spacetime with the spectrum in realistic open inflationary universes. In both the Hawking-Turok and the Coleman–De Luccia model for open inflation there is an extra reflection term in the correlator because O(5) symmetry is broken on the instanton [1]. This term gives rise to long-wavelength bubble wall fluctuations in the Lorentzian region. At first sight, the wall fluctuations seem to regularize the spectrum. However, adding and subtracting the second term in Eq. (31) to the two-point tensor correlator in the $O(4)$ models [Eq. (34) in [1]] and comparing that with our result (31) reveals that the wall fluctuations actually appear as an extra long-wavelength continuum contribution *on top of* the spectrum in perfect de Sitter space. Hence in both the Hawking-Turok and Coleman–De Luccia model there is an enhancement of the fluctuations compared to the perturbations in perfect de Sitter space. But the singularity in Hawking-Turok instantons suppresses the wall fluctuations because it enforces Dirichlet boundary conditions on the perturbation modes [1]. Hence we expect the spectrum in perfect de Sitter space to be quite similar to the spectrum predicted by singular instantons. On the other hand, Coleman–De Luccia models typically predict large wall fluctuations, yielding a very different CMB anisotropy spec-

[1]The confusion arises because, due to the form of the metric inverse, scalar invariants are finite at infinity, e.g., $q_{ij}q^{ij}\sim e^{-2\chi}$. This also explains why the coefficient functions $w_j^{(0)}(\chi)$ in the bitensor $W_{iji'j'}^{L(0)}$, asymptotically decay.

trum on large angular scales. The tensor fluctuation spectrum therefore potentially provides an observational discriminant between different theories of open inflation [16].

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APPENDIX: MAXIMALLY SYMMETRIC BITENSORS

A maximally symmetric bitensor T is one for which $\sigma^*T=0$ for any isometry σ of the maximally symmetric manifold. Any maximally symmetric bitensor may be expanded in terms of a complete set of ''fundamental'' maximally symmetric bitensors with the correct index symmetries. For instance

$$\begin{aligned}T_{iji'j'}=&t_1(\mu)g_{ij}g_{i'j'}+t_2(\mu)[n_ig_{ji'}n_{j'}+n_jg_{ii'}n_{j'}\\&+n_ig_{jj'}n_{i'}+n_jg_{ij'}n_{i'}]+t_3(\mu)[g_{ii'}g_{jj'}+g_{ji'}g_{ij'}]\\&+t_4(\mu)n_in_jn_{i'}n_{j'}+t_5(\mu)[g_{ij}n_{i'}n_{j'}+n_in_jg_{i'j'}]\end{aligned}\tag{A1}$$

where the coefficient functions $t_j(\mu)$ depend only on the distance $\mu(\Omega,\Omega')$ along the shortest geodesic from Ω to Ω'. $n_{i'}(\Omega,\Omega')$ and $n_i(\Omega,\Omega')$ are unit tangent vectors to the geodesics joining Ω and Ω' and $g_{ij'}(\Omega,\Omega')$ is the parallel propagator along the geodesic; $V^ig_i^{j'}$ is the vector at Ω' obtained by parallel transport of V^i along the geodesic from Ω to Ω' [17].

The set of tensor eigenmodes on S^3 or H^3 forms a representation of the symmetry group of the manifold. It follows in particular that their sum over the parity states $\mathcal{P}=\{e,o\}$ and the quantum numbers l and m on the two-sphere defines a maximally symmetric bitensor on S^3 (or H^3) [17]

$$W^{ij}_{(p)i'j'}(\mu)=\sum_{\mathcal{P}lm}q^{(p)ij}_{\mathcal{P}lm}(\Omega)q^{(p)\mathcal{P}lm}_{i'j'}(\Omega')^*.\tag{A2}$$

On S^3 the label $p=3i,4i,\ldots$. It is related to the usual angular momentum k by $p=i(k+1)$. The ranges of the other labels is then $0\leqslant l\leqslant k$ and $-l\leqslant m\leqslant l$. On H^3 there is a continuum of eigenvalues $p\in[0,\infty)$. We will assume from now that the eigenmodes on are normalized by the condition

$$\int\sqrt{\gamma}d^3xq^{(p)ij}_{\mathcal{P}lm}q^{(p')*}_{\mathcal{P}'l'm'ij}=\delta^{pp'}\delta_{\mathcal{P}\mathcal{P}'}\delta_{ll'}\delta_{mm'}.\tag{A3}$$

The bitensor $W^{ij}_{(p)i'j'}(\mu)$ appearing in our Green function has some additional properties arising from its construction in terms of the transverse and traceless tensor harmonics $q^{(p)\mathcal{P}lm}_{ij}$. The tracelessness of $W^{(p)}_{iji'j'}$ allows one to eliminate two of the coefficient functions in Eq. (A1). It may then be written as

$$\begin{aligned}W^{(p)}_{iji'j'}(\mu)=&w^{(p)}_1[g_{ij}-3ninj][g_{i'j'}-n_{i'}n_{j'}]\\&+w^{(p)}_2[4n_{(i}g_{j)(i'}n_{j')}+4n_in_jn_{i'}n_{j'}]\\&+w^{(p)}_3[g_{ii'}g_{jj'}+g_{ji'}g_{ij'}-2n_ig_{i'j'}n_j\\&-2n_{i'}g_{ij}n_{j'}+6n_in_jn_{i'}n_{j'}].\end{aligned}\tag{A4}$$

This expression is traceless on either index pair ij or $i'j'$. The requirement that the bitensor be transverse $\nabla^iW^{(p)}_{iji'j'}=0$ and the eigenvalue condition $(\Delta_3-\lambda_p)W^{iji'j'}_{(p)}=0$ impose additional constraints on the remaining coefficient functions $w^{(p)}_j(\mu)$. To solve these constraint equations it is convenient to introduce the new variables [18] on S^3 (on H^3, μ is replaced by $-i\tilde{\mu}$)

$$\alpha(\mu)=w^{(p)}_1(\mu)+w^{(p)}_3(\mu)$$

$$\beta(\mu)=\frac{7}{(p^2+9)\sin\mu}\frac{d\alpha(\mu)}{d\mu}.\tag{A5}$$

In terms of a new argument $z=\cos^2(\mu/2)$ (or its continuation on H^3) the transversality and eigenvalue conditions imply for $\alpha(z)$

$$z(1-z)\frac{d^2\alpha(z)}{d^2z}+\left[\frac{7}{2}-7z\right]\frac{d\alpha(z)}{dz}=(p^2+9)\alpha(z)\tag{A6}$$

and then for the coefficient functions

$$\begin{aligned}w_1=&Q_p([2(\lambda_p-6)z(z-1)-2]\alpha(z)\\&+\tfrac{4}{7}[(\lambda_p+6)z(z-\tfrac{1}{2})(z-1)]\beta(z))\\w_2=&Q_p(2(1-z)[(\lambda_p-6)z+3]\alpha(z)\\&-\tfrac{4}{7}[(\lambda_p+6)z(z-1)(z-\tfrac{3}{2})]\beta(z))\\w_3=&Q_p([-2(\lambda_p-6)z(z-1)+3]\alpha(z)\\&-\tfrac{4}{7}[(\lambda_p+6)z(z-\tfrac{1}{2})(z-1)]\beta(z))\end{aligned}\tag{A7}$$

with $\lambda_p=(p^2+3)$.

The above conditions leave the overall normalization of the bitensor undetermined. To fix the normalization constant Q_p we contract the indices in the coincident limit $z\to1$. This yields [18]

$$W^{(p)ij}_{ij}(\Omega,\Omega)=\sum_{\mathcal{P}lm}q^{(p)\mathcal{P}lm}_{ij}(\Omega)q^{(p)\mathcal{P}lmij}(\Omega)^*=30Q_p\alpha(1).\tag{A8}$$

By integrating over the three-sphere and using the normalization condition (A3) on the tensor harmonics one obtains $Q_p=-(p^2+4)/30\pi^2\alpha(1)$.

Notice that Eq. (A6) is precisely the hypergeometric differential equation, which has a pair of independent solutions $\alpha(z)={}_2F_1(3+ip,3-ip,7/2,z)$ and $\alpha(1-z)={}_2F_1(3+ip,3-ip,7/2,1-z)$. The former of these solutions is singular at

$z=1$, i.e., for coincident points on the three-sphere, and the latter is singular for opposite points. The solution for $\beta(z)$ follows from Eq. (A5) and is given by

$$\beta(z)={}_2F_1(4-ip,4+ip,9/2,z). \tag{A9}$$

The hypergeometric functions are related by the transformation formula [Eq. [15.3.6] in [19])

$$\begin{aligned}{}_2F_1(a,b,c,z)=&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b,a+b-c,1-z)\\&+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}_2\\&\times F_1(c-a,c-b,c-a-b,1-z).\end{aligned} \tag{A10}$$

Only for the eigenvalues of the Laplacian on S^3, i.e., $p=in$ $(n\geqslant 3)$, the term on the second line vanishes for ${}_2F_1(3+ip,3-ip,7/2,z)$. For these special values, $\alpha(z)$ and $\alpha(1-z)$ are no longer linearly independent but related by a factor of $(-1)^{n+1}$, and they are both regular for any angle on the three-sphere. In fact, the hypergeometric series terminates for these parameter values and the hypergeometric functions reduce to Gegenbauer polynomials $C^{(3)}_{n-3}(1-2z)$. We have a choice between using $\alpha(z)$ and $\alpha(1-z)$ in the bitensor for these values of p. Since $F(1-z)\rightarrow 1$ for coincident points, it is more natural to choose $\alpha(1-z)$ in the bitensor appearing in the Euclidean Green function (18). However, to obtain the Lorentzian correlator, we had to express the discrete sum (18) as a contour integral. Since the Euclidean correlator obeys a differential equation with a delta function source at $\mu=0$, we must maintain regularity of the integrand at $\mu=\pi$ when extending the bitensor in the complex p-plane. In other words, for generic p, we need to work with the solution $\alpha(z)$, rather than $\alpha(1-z)$. Therefore, in order to write the Euclidean correlator as a contour integral, we first have replaced $F(1-z)$ by $F(z)(-1)^{n+1}$, by applying (A10) to Eq. (18), and we then have continued the latter term to $-(\cosh p\pi)^{-1}{}_2F_1(3+ip,3-ip,\frac{7}{2},z)$.

We conclude that the properties of the bitensor appearing in the tensor correlator completely determine its form. Notice that in terms of the label p we have obtained a ''unified'' functional description of the bitensor $W^{iji'j'}_{(p)}$ on S^3 and H^3. Its explicit form is very different in both cases however, because the label p takes on different values. But it is precisely this description that has enabled us in Sec. IV to analytically continue the correlator from the Euclidean instanton into open de Sitter space without Fourier decomposing it. We shall conclude this Appendix by giving the explicit formulas for the coefficient functions of the bitensor $W^{L(p)}_{iji'j'}$ appearing in our final result (31). With this description, they can be obtained by analytic continuation from S^3.

To perform the continuation to H^3 we note that the geodesic separation μ on S^3 continues to $-i\chi$ where χ is the comoving separation on H^3. Hence the hypergeometric functions on H^3 are defined by analytic continuation (Eq. 15.3.7 in [19]) and may be expressed in terms of associated Legendre functions as

$$\begin{aligned}\alpha(z)&=15\sqrt{\frac{\pi}{2}}(-\sinh\chi)^{-5/2}P^{-5/2}_{-1/2+ip}(-\cosh\chi),\\ \beta(z)&=15\sqrt{\frac{\pi}{2}}(-\sinh\chi)^{-7/2}P^{-7/2}_{-1/2+ip}(-\cosh\chi).\end{aligned} \tag{A11}$$

Using the relation $-\cosh(\chi)=\cosh(\chi-i\pi)$, the Legendre functions on H^3 may be expressed as

$$\begin{aligned}P^{-5/2}_{-1/2+ip}(-\cosh\chi)=&\sqrt{\frac{2}{-\pi\sinh\chi}}(1+p^2)^{-1}(4+p^2)^{-1}\Big[-3\coth\chi\cosh p(\pi+i\chi)\\&-\frac{i\sinh p(i\chi+\pi)}{2p}((2-p^2)(1+\coth^2\chi)+(4+p^2)\mathrm{cosech}^2\chi)\Big]\end{aligned}$$

$$\begin{aligned}P^{-7/2}_{-1/2+ip}(-\cosh\chi)=&\sqrt{\frac{2}{-\pi\sinh\chi}}(1+p^2)^{-1}(4+p^2)^{-1}(9+p^2)^{-1}\Big[\cosh p(\pi+i\chi)(p^2-11-15\mathrm{cosech}^2\chi)\\&-6\frac{i\sinh p(i\chi+\pi)}{p}((1-p^2)\coth^3\chi+(p^2+\tfrac{3}{2})\coth\chi\,\mathrm{cosech}^2\chi)\Big].\end{aligned} \tag{A12}$$

In the text, we have extracted the factors $e^{\pm p\pi}$ in these expressions in order to make contact with the usual description of the tensor correlator in terms of tensor harmonics on H^3. The coefficient functions of the bitensor $W^{L(p)}_{iji'j'}(\chi)$ in our final result (31) for the tensor correlator are

$$w_1=\frac{\mathrm{cosech}^5\chi}{4\pi^2(p^2+1)}\left[\frac{\sin p\chi}{p}(3+(p^2+4)\sinh^2\chi-p^2(p^2+1)\sinh^4\chi)-\cos p\chi(3/2+(p^2+1)\sinh^2\chi)\sinh 2\chi\right]$$

$$w_2=\frac{\mathrm{cosech}^5\chi}{4\pi^2(p^2+1)}\left[\frac{\sin p\chi}{p}(3+12\cosh\chi-3p^2(1+2\cosh\chi)\sinh^2\chi+p^2(p^2+1)\sinh^4\chi)\right.$$

$$\left.+\cos p\chi(-12-3\cosh\chi+2(p^2-2)\sinh^2\chi+2(p^2+1)\cosh\chi\sinh^2\chi)\sinh\chi\right] \tag{A13}$$

$$w_3=\frac{\mathrm{cosech}^5\chi}{4\pi^2(p^2+1)}\left[\frac{\sin p\chi}{p}(3-3p^2\sinh^2\chi+p^2(p^2+1)\sinh^4\chi)+\cos p\chi(-3/2+(p^2+1)\sinh^2\chi)\sinh 2\chi\right].$$

As mentioned before, the bitensor $W^{L(p)}_{iji'j'}$ equals the sum (A2) of the rank-two tensor eigenmodes with eigenvalue $\lambda_p=-(p^2+3)$ of the Laplacian on H^3. For $\chi\to 0$ these functions converge and they exponentially decay at large geodesic distances.

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