Partially quenched chiral perturbation theory and the replica method

P. H. Damgaard and K. Splittorff
The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
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We describe a novel framework for partially quenched chiral perturbation theory based on the replica method. The computational rules are exceedingly simple. We illustrate these rules by computing the partially quenched chiral condensate to one-loop order. By considering arbitrary chiral *k*-point functions we show explicitly to one-loop order the equivalence between this method and the one based on supersymmetry. It is possible to go smoothly from the conventional replica method to a supersymmetric variant by choosing the number of valence quarks to be negative.

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I. INTRODUCTION

The question of non-perturbative analytical predictions for quenched or partially quenched lattice gauge theory computations has been thoroughly studied in the context of effective chiral Lagrangians [1–4]. So far the most systematic framework has been the supersymmetric formulation of Bernard and Golterman [1,2], which builds on an idea first introduced in the context of staggered lattice fermions [5]. Here one introduces k additional quark species (of conventional statistics) on top of the N_f physical ''sea'' quarks, and k ''ghost'' quarks of opposite statistics to cancel the effects of the additional quarks. When N_f is taken to vanish this gives the fully quenched theory, while for N_f non-zero it gives the partially quenched theory. Both are accessible to a study by Monte Carlo techniques in lattice gauge theory. The chiral flavor symmetry group is in that formulation extended to a super Lie group which in perturbation theory can be taken as $SU(N_f+k|k)$. (For this reason it is commonly known as the supersymmetric method although it, as applied, has nothing to do with space-time supersymmetry, but rather is a graded symmetry.) Based on the usual assumption of spontaneous chiral symmetry breaking (here extended to the super group case) the effective low-energy theory of the lowest-lying hadronic excitations is that of a chiral Lagrangian, now with fields living on the coset of super Lie groups. This effective Lagrangian can be studied by the conventional methods of chiral perturbation theory. In what follows we denote fully and partially quenched chiral perturbation theory by QChPT and PQChPT, respectively.

The supersymmetric framework has also proven to be an efficient means of deriving analytical results for the soft part of the Dirac operator spectrum in finite volume, by taking an appropriate discontinuity of the partially quenched chiral condensate [6–8]. This has brought earlier results derived entirely from universal random matrix theory [9] (for a very recent comprehensive review, see Ref. [10]) in direct contact with the effective Lagrangian of QCD. In particular, a series of very compact relations that described general *k*-point spectral correlation functions of low-lying Dirac operator eigenvalues in terms of effective partition functions with additional quark species [11] can now be understood as due to the cancelling pairs of fermionic and bosonic valence quarks. When taking the same discontinuity near the origin in PQChPT it has also been shown that one recovers among other terms the analytical prediction for the slope of the spectral density of the Dirac operator at the origin [6,7], a formula first derived in the QCD case by Smilga and Stern [12]. The same analysis has recently been extended to the two other major chiral symmetry breaking classes by Toublan and Verbaarschot [8]. There are thus also plenty of *physical* applications of PQChPT that have nothing to do with the artifacts of the quenched approximation at all.

While the supersymmetric approach to QChPT and PQChPT has been well tested, and is by now quite well understood, it is still of interest to find alternative means of formulating the same problem. In particular, the supersymmetry itself is not fundamental and not an inherent property of QChPT and PQChPT. Indeed, it has recently been shown in the context of the finite-volume effective chiral Lagrangian related to the soft part of the Dirac operator spectrum [9] that the so-called replica method can provide a useful alternative technique [13]. Here full or partial quenching is instead achieved by adding N_v valence quarks (of usual statistics), and then taking the limit $N_v \to 0$ at the end of the calculation. In ordinary QCD perturbation theory this procedure trivially kills all valence quark loops. In the framework of the effective Lagrangian of Goldstone bosons it is far from obvious that such a procedure can be carried out explicitly. It entails an extension of the chiral symmetry group $U(N)$ to non-integer N, and integrals over such a group are not known in closed form. Nevertheless, it turns out that in series expansions the required analytical continuation can be carried out explicitly [13], and results agree with what was earlier established by the supersymmetric method [6,7]. This suggests that also conventional QChPT and PQChPT can be performed by simply taking the limit $N_v \to 0$. In this paper we shall show that this is indeed the case. We shall give the very simple Feynman rules, and explain the intimate relationship to QChPT and PQChPT in the supersymmetric formulation. As a simple illustration we show how to derive the partially quenched chiral condensate to one-loop order using this replica method. This fully or partially quenched chiral condensate is a particularly convenient observable on which to test the non-perturbative finite-volume scaling results discussed above [14,15]. The way partially quenched chiral perturbation theory smoothly matches this regime has been explained in Ref. [6].

After providing the Feynman rules, it becomes quite obvious how the replica method in perturbation theory is equivalent to the supersymmetric method. We illustrate a few of the counting rules by considering a chiral k-point function below. Mainly out of curiosity, we also show how a variant of the replica method that is supersymmetric can be used to provide identical results. This supersymmetric variant is however slightly more cumbersome than the conventional $N_v \to 0$ replica method, and we do not propose to use that particular variant for practical calculations.

II. THE REPLICA METHOD

As explained above, with the replica method one adds N_v valence quarks to the QCD Lagrangian, which here can be taken as any SU$(N_c \geqslant 3)$ gauge theory with N_f physical (sea) quark flavors. Depending on the applications, it can be convenient to introduce k sets of such valence quarks with k different masses m_{v_j}, each set containing N_v new quark flavors. The physical quark masses are denoted by m_f. The QCD partition function with these kN_v additional quark species reads

$$\mathcal{Z}^{(N_f+kN_v)} = \int [dA] \prod_{j=1}^{k} \det(i\not{D} - m_{v_j})^{N_v} \times \prod_{f=1}^{N_f} \det(i\not{D} - m_f) e^{-S_{\rm YM}[A]}. \tag{1}$$

This partition function can be viewed as an unnormalized average of k sets of N_v identical replicas of the following partition functions of quarks in a fixed gauge field background A_μ:

$$\mathcal{Z}_{v_j} \equiv \int [d\bar{\psi}_j d\psi_j] \exp\left[\int d^4x \bar{\psi}_j (i\not{D} - m_{v_j}) \psi_j \right] \tag{2}$$

in the sense that

$$\mathcal{Z}^{(N_f+kN_v)} = \int [dA] \prod_{j=1}^{k} [\mathcal{Z}_{v_j}]^{N_v} \prod_{f=1}^{N_f} \det(i\not{D} - m_f) e^{-S_{\rm YM}}. \tag{3}$$

Clearly, if we set $N_v = 0$ this just reproduces the original QCD partition function. But the theory extended with kN_v additional quark species in this way is a generating functional for partially quenched averages of $\bar{\psi}_j \psi_j$ and mixed averages also involving physical quark fields. One simply sets N_v to zero *after* having performed the functional differentiations

$$\chi(m_{v_1}, \ldots, m_{v_k}, m_{f_1}, \ldots, m_{f_l}, \{m_f\}) \equiv \lim_{N_v \to 0} \frac{1}{N_v^k} \frac{1}{N_f^l} \frac{\partial}{\partial m_{v_1}} \cdots \frac{\partial}{\partial m_{v_k}} \frac{\partial}{\partial m_{f_1}} \cdots \frac{\partial}{\partial m_{f_l}} \ln \mathcal{Z}^{(N_f+N_v)}. \tag{4}$$

Technically, it can be convenient to add local sources for both scalar and pseudoscalar quark bilinears $\bar{\psi}_j(x) \psi_j(x)$ and $\bar{\psi}_j(x) \gamma_5 \psi_j(x)$ and similarly for the vector and axial vector currents (for simplicity taken flavor diagonal). If needed, one can of course introduce corresponding sources in the physical quark sector. Because such terms have no bearing on our arguments presented below, we shall for simplicity omit them here.

Adapting the replica method to the chiral Lagrangian

For N_v integer, and $N_f + kN_v$ small enough, chiral symmetry is assumed to be spontaneously broken according to the standard pattern of $\mathrm{SU}_L(N_f+kN_v) \times \mathrm{SU}_R(N_f+kN_v) \to \mathrm{SU}(N_f+kN_v)$. The effective low-energy theory can therefore be described in the entirely conventional framework of a chiral Lagrangian based on SU(N_f+kN_v), with no new assumptions about the pattern of chiral symmetry breaking.[1] The cases $N_f = 0$ and $N_f = 1$ are obviously very special here. For $N_f = 1$ there is not any spontaneous breaking of chiral symmetry in the theory after taking N_v to zero, and the case $N_f = 0$ (which would correspond to full quenching) is so unusual that we shall discuss it separately.

Having in mind a possibly non-trivial role played by the flavor singlet meson, the lowest-order effective chiral Lagrangian is taken to be the usual $\mathcal{O}(p^2)$ expression

$$\mathcal{L} = \frac{F^2}{4} \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) - \frac{\Sigma}{2} \mathrm{Tr} \mathcal{M}(U + U^\dagger) + \frac{\mu^2}{2N_c} \Phi_0^2 + \frac{\alpha}{2N_c} \partial_\mu \Phi_0 \partial^\mu \Phi_0 . \tag{5}$$

Here the field $U \equiv \exp[i\sqrt{2}\Phi/F]$ is an element of SU$(N_f + kN_v)$, and we have kept the flavor-singlet field $\Phi_0 \equiv \mathrm{Tr}\Phi$. As in the supersymmetric method [1], it proves convenient to work in a ''quark basis'' where Φ_{ij} corresponds to $\bar{\psi}_i \psi_j$. With all external sources set to zero, this gives a simple propagator for the ''off-diagonal'' mesons corresponding to $\Phi_{ij} \sim \bar{\psi}_i \psi_j , i \neq j$:

$$D_{ij}(p^2) = \frac{1}{p^2 + M_{ij}^2}, \tag{6}$$

while for the ''diagonal'' mesons $\Phi_{ii} \sim \bar{\psi}_i \psi_i$ the propagator can be written in the form [2]

[1]The reader might worry about the assumption that $N_f + kN_v$ should be taken small enough for the theory to support spontaneous chiral symmetry breaking. Actually, there will be no new constraint from this. We simply analyze the chiral Lagrangian for arbitrary $N_f + kN_v$ even though this Lagrangian is only the low-energy theory of QCD for $N_f + kN_v$ sufficiently small. However, we take the limit $N_v \to 0$ in the end. Then we must meet only the usual constraint that the number of *physical* light quarks N_f should be small enough to lead to spontaneous chiral symmetry breaking.

TABLE I. The propagators for replica PQChPT, $SU(N_f+kN_v)$, and supersymmetric PQChPT, $SU(N_f+k|k)$. The sign ϵ_i is defined as $\epsilon_i \equiv 1$ for $i=1,\ldots,N_f+k$ and $\epsilon_i \equiv -1$ for $i=N_f+k+1,\ldots,N_f+2k$. Note that $\mathcal{F}$ coincides in the partially quenched limit of the two approaches.

Propagator	Replica PQChPT	Supersymmetric PQChPT
$D_{ij}(p^2)$	$\frac{1}{p^2+M_{ij}^2}$	$\frac{\epsilon_i}{p^2+M_{ij}^2}$
$G_{ij}(p^2)$	$\frac{\delta_{ij}}{(p^2+M_{ii}^2)}-\frac{(\mu^2+\alpha p^2)/N_c}{(p^2+M_{ii}^2)(p^2+M_{jj}^2)\mathcal{F}(p^2)}$	$\frac{\epsilon_i\delta_{ij}}{(p^2+M_{ii}^2)}-\frac{(\mu^2+\alpha p^2)/N_c}{(p^2+M_{ii}^2)(p^2+M_{jj}^2)\mathcal{F}(p^2)}$
$\mathcal{F}(p^2)$	$1+\frac{\mu^2+\alpha p^2}{N_c}\left(\sum_{j=1}^{k}\frac{N_v}{p^2+M_{v_jv_j}^2}+\sum_{f=1}^{N_f}\frac{1}{p^2+M_{ff}^2}\right)$	$1+\frac{\mu^2+\alpha p^2}{N_c}\sum_{f=1}^{N_f+2k}\frac{\epsilon_i}{p^2+M_{ii}^2}$

$$G_{ij}(p^2)=\frac{\delta_{ij}}{(p^2+M_{ii}^2)}-\frac{(\mu^2+\alpha p^2)/N_c}{(p^2+M_{ii}^2)(p^2+M_{jj}^2)\mathcal{F}(p^2)}. \quad (7)$$

Here $M_{ij}^2 \equiv (m_i+m_j)\Sigma/F^2$ and

$$\mathcal{F}(p^2)\equiv 1+\frac{\mu^2+\alpha p^2}{N_c}\left(\sum_{j=1}^{k}\frac{N_v}{p^2+M_{v_jv_j}^2}+\sum_{f=1}^{N_f}\frac{1}{p^2+M_{ff}^2}\right). \quad (8)$$

Note that N_v enters as a parameter due to the mass degeneracy of the valence quarks in each of the k sets. This is exactly what is required in order to apply the replica method. We remark that the unusual form of the propagator (7) just stems from using the quark basis and including the flavor singlet field $\Phi_0 = \mathrm{Tr}\Phi$, and not from any peculiarities of partial quenching. Although we borrow the result (7) from Ref. [2], it is also unrelated to the supersymmetry of the method discussed there.

By including the Φ_0 field in the Lagrangian we have kept open the possibility of studying various expansion schemes (see, e.g., the second reference of [1]). The Φ_0 terms affect only G_{ij}. For G_{ii} the flavor-singlet Φ_0 can give rise to double poles in the partially quenched limit, but the appearance of such double poles is not special to the replica method. Indeed such double poles are also present in the supersymmetric formulation where a thorough study has been done [1–3]. As we prove in the next section the two formulations have equivalent perturbative expansions. The appearance of the double pole in the replica method is therefore completely analogous to the case of the supersymmetric formulation. In particular, we note that also in the replica formalism the case $N_f=0$ is quite special since in that case $\mathcal{F}(p^2)$ simply becomes unity, and the double pole in G_{ii} is unavoidable. Moreover, in just that case there is no decoupling as the scale μ is sent to infinity.

In Table I we give the explicit relation between the Feynman rules based on the replica method, and those based on the supersymmetric formulation. The supersymmetry Feynman rules are supplemented by the standard relative minus sign between boson and fermion loops. Despite the additional minus signs in the Feynmann rules of the supersymmetric formulation, the Green functions are identical in the two formulations. As we show below, the signs due to *combinatorics* in the replica method match those arising from statistics and the supertrace in the supersymmetric formulation.

III. THE EQUIVALENCE BETWEEN REPLICA AND SUPERSYMMETRIC PQChPT

In this section we formulate the equivalence between the generating functional of PQChPT in the replica and supersymmetric formulations. The equivalence proof is by default restricted to perturbation theory (expressed in terms of the Feynman rules), and we can in principle not make any statements at the non-perturbative level. But this is as it should be, as our whole framework in any case is restricted to chiral perturbation theory. The Lagrangian itself contains an infinitely long string of interactions that become relevant with increasing loop order, and we shall only demonstrate the equivalence at the one-loop level. However, seeing how the equivalence proof proceeds, it is pretty obvious how to generalize this to arbitrarily high order.

Our claim is: The generating functional of replica PQChPT for N_f+kN_v flavors with k sets of N_v mass-degenerate quarks is in perturbation theory equivalent to the generating functional of supersymmetric PQChPT for N_f+k fermionic and k bosonic quarks.

By *equivalence* between the $SU(N_f+kN_v)$ and the $SU(N_f+k|k)$ generating functionals is meant that the chiral expansions are equivalent order by order. Of course, the respective limits, $N_v \to 0$ and mass degeneracy between the k bosons and k of the fermions, are to be introduced at the end of the calculations. While we believe that this statement is true we will as mentioned above only address the equivalence at the one-loop level. At this one-loop level the contributions from the $\mathcal{O}(p^4)$ chiral Lagrangian act as counter terms and we can base the discussion on the Lagrangian of Eq. (5).

Let us first consider the sea sector. (The term sea sector is used when only sea quark masses are involved in differentiations of the generating functional.) For this sector both methods are equivalent to SU(N_f) ChPT. In the replica formulation the contributions from the valence quarks at one-loop to any of the correlators

$$\chi(m_{f_1},\ldots,m_{f_l},\{m_f\})\equiv\frac{1}{N_f^k}\frac{\partial}{\partial m_{f_1}}\cdots\frac{\partial}{\partial m_{f_k}}\ln\mathcal{Z}^{(N_f+N_v)}, \tag{9}$$

are necessarily proportional to positive powers of N_v. Hence the dependence on the valence quarks vanishes as $N_v\rightarrow 0$, leaving the sea sector equivalent to SU(N_f) ChPT. The analogous statement in supersymmetric PQChPT was proven in Ref. [2]. This equivalence was formulated as three theorems in that reference. At the risk of making some oversimplifications we state them compactly as follows:

(I) The sea sector of SU($N_f+k|k$) PQChPT is equivalent to SU(N_f) ChPT.

(II) The super-η' is equivalent to the conventional η' of SU(N_f) ChPT.

(III) The double pole of G_{ii} arise at a given fermionic quark mass if and only if all fermionic quarks with this mass are paired up by bosonic quarks.

In the supersymmetric formalism theorem I is established by noting that k of the fermionic quarks and the k bosonic quarks only appear as virtual loops in the sea sector. Since these $2k$ quarks are paired up in masses the virtual loops cancel explicitly. This cancellation is also responsible for establishing theorem (II) in the supersymmetric formalism, only now it takes place in the quark loop corrections to the η'-propagator. Finally theorem (III) follows directly from the structure of the last term in G_{ii}. We emphasize here that the obvious analogs of both theorems (I) and (II) are completely trivial in the present replica formalism. Theorem (III), when re-stated in the language of the replica formalism, stipulate under what circumstances the potential double pole of G_{ii} is canceled: By inspection this occurs when $M_{ii}=M_{ff}$ for at at least one physical meson labeled by ff. The proof of theorem (III) is then almost identical in the replica and supersymmetric formulations. In the phrasing of Refs. [1–3] the double poles can only occur at mass scales that are completely quenched.

In the remaining quark sectors the equivalence is far less trivial. However, the supersymmetric bosonic Green functions equal the fermionic ones up to a well defined sign. So we can focus on the sectors involving fermionic valence quarks.

The equivalence in these sectors is not just of academic interest. As mentioned in the Introduction, differentiations with respect to valence quark masses may be related to physical quantities. For instance the partially quenched chiral condensate for the valence quarks,

$$\Sigma(m_v,\{m_f\})\equiv\lim_{N_v\rightarrow 0}\frac{1}{N_v}\frac{\partial}{\partial m_v}\ln\mathcal{Z}^{(N_f+N_v)}, \tag{10}$$

can be used to determine the Dirac spectral density. This density is given by the discontinuity of the partially quenched chiral condensate across a cut on the imaginary axis [6]:

$$\begin{aligned}\rho(\lambda;\{m_f\})&=\frac{1}{2\pi}\mathrm{Disc}|_{m_v=i\lambda}\Sigma(m_v,\{m_f\})\\&=\frac{1}{2\pi}\lim_{\epsilon\rightarrow 0}[\Sigma(i\lambda+\epsilon,\{m_f\})-\Sigma(i\lambda-\epsilon,\{m_f\})].\end{aligned} \tag{11}$$

[This identification holds when one considers $\Sigma(m_v,\{m_f\})$ as a function of a *real* mass m_v, and then replaces $m_v\rightarrow i\lambda\pm\epsilon$.]

In the valence sector and the mixed sector the equivalence is established in two steps. *First*, notice that the propagator (7) of replica PQChPT for $N_v=0$ is identical to the one for the fermionic sector of the corresponding supersymmetric PQChPT in the limit where each of the boson masses is paired up with a fermion mass; see Table I. (This equivalence holds trivially for the off-diagonal quark anti-quark propagators.) *Second*, the signs arising from combinatorics in the replica method is exactly matched by the opposite signs of boson and fermion loops occurring in the supersymmetric formulation.

In order to see exactly how the signs come to match in the two approaches, we explicitly give the derivation of the k-point function in the valence sector. The generalization to the mixed sector follows in complete analogy.

A. The one-point function in the valence sector

In this first example we give the contributions to the valence quark mass dependent chiral condensate defined in Eq. (10). We show how the cancellations that occur exactly match those of the supersymmetric formulation. It turns out that this simple 1-point function actually is ideally suited for illustrating the equivalence between the replica method and the supersymmetric method, as all essential properties of the propagators and of the combinatorics come into play.

To evaluate the one-point function we need to introduce just one set of replica fermions. Explicitly performing the differentiation of the generating functional, see Eq. (10), or alternatively counting the number of realizations of quark flow diagrams we have, to one loop,

$$\begin{aligned}\frac{\Sigma(m_v,\{m_f\})}{\Sigma}=\lim_{N_v\rightarrow 0}\frac{1}{N_v}\Bigg[N_v-\frac{1}{F^2}\Bigg(&N_v\sum_{f=1}^{N_f}\Delta(M_{vf}^2)\\&+N_v(N_v-1)\Delta(M_{vv}^2)\\&+N_v\frac{1}{V}\sum_p G_{vv}(p^2)\Bigg)\Bigg],\end{aligned} \tag{12}$$

where

$$\Delta(M_{ij}^2)\equiv\frac{1}{V}\sum_p \frac{1}{p^2+M_{ij}^2}\equiv\frac{1}{V}\sum_p D_{ij}(p^2) \tag{13}$$

is a one-loop integral of the standard diagonal propagator for the off-diagonal mesons, $\Phi_{ij}\sim\bar{\psi}_i\psi_j$, $i\neq j$. (We write everything in finite-volume notation, having also in mind applications of the kind discussed in Refs. [6–8].) The first term in $(1/V)\Sigma_p G_{vv}(p^2)$ is simply $\Delta(M_{vv}^2)$. For arbitrary N_v this term is seen to cancel against the term just before G_{vv}. In the $N_v\to 0$ limit we also get rid of the term proportional to N_v, leaving simply

$$\frac{\Sigma(m_v,\{m_f\})}{\Sigma}=1-\frac{1}{F^2}\left(\sum_{f=1}^{N_f}\Delta(M_{vf}^2)-\frac{1}{V}\sum_p\frac{(\mu^2+\alpha p^2)/N_c}{(p^2+M_{vv}^2)(p^2+M_{vv}^2)\mathcal{F}(p^2)}\right). \tag{14}$$

This is completely analogous to the result obtained in the supersymmetric formulation. In that case a similar cancellation takes place between the first term in $(1/V)\Sigma_p G_{vv}(p)$ and the loop of the meson built up by the fermionic and bosonic valance quark. It is also instructive to trace the cancellation of valence quark loops. In the supersymmetric formulation this cancellation occurs because of a matching boson loop, while in the present formulation it is due to *the lack of a replica fermion*. Pictorially speaking, this lack of a replica fermion acts like a boson.

B. The *k*-point function in the valence sector

As for the condensate, the k-fold derivative, $k\geqslant 2$, of $\ln\mathcal{Z}^{(N_f+kN_v)}$ with respect to each of the valence quark masses is related to the spectral k-point function. The evaluation of the k-fold derivative is quite simple but we need to treat the case $k=2$ separately. The reason is simple: The product

$$\sum_{j,k=1}^{N_f+2N_v}\Phi_{ij}(x_1)\Phi_{ji}(x_1)\Phi_{lk}(x_2)\Phi_{kl}(x_2),\quad i\neq l \tag{15}$$

occurring in the two point function includes two connected terms, namely

$$\Phi_{v_1v_1}(x_1)\Phi_{v_2v_2}(x_2)\Phi_{v_2v_2}(x_2)\Phi_{v_1v_1}(x_1)$$

and

$$\Phi_{v_1v_2}(x_1)\Phi_{v_2v_1}(x_2)\Phi_{v_1v_2}(x_2)\Phi_{v_2v_1}(x_1).$$

For $k>2$ there is no connected analogue of the latter "crossed diagram," since k of the indices must be different (we differentiate with respect to different masses). The 2-point function is thus different from higher k-point functions because meson loops correspond to just quark-antiquark lines. In terms of the propagators the two-point function is[2]

$$\frac{\chi(m_{v_1},m_{v_2},\{m_f\})}{\Sigma^2}=\lim_{N_v\to 0}\frac{1}{N_v^2}\frac{1}{F^4}\times\left(N_v^2\frac{1}{V}\sum_p D_{v_1v_2}(p^2)D_{v_2v_1}(p^2)+N_v^2\frac{1}{V}\sum_p G_{v_1v_2}(p^2)G_{v_2v_1}(p^2)\right), \tag{16}$$

whereas for $k>2$ there is no crossed diagram, and we are left with

$$\frac{\chi(m_{v_1},\ldots,m_{v_k},\{m_f\})}{\Sigma^k}=\lim_{N_v\to 0}\frac{1}{N_v^k}(-1)^k\frac{1}{F^{2k}}N_v^k\frac{1}{V}\sum_p G_{v_1v_2}(p^2)\cdots G_{v_kv_1}(p^2). \tag{17}$$

We observe that in both cases the N_v dependence is such that the limit $N_v\to 0$ becomes trivial. The corresponding expressions in the supersymmetric formalism are identical. Note that sea fermion and "ghost" loops only appear in the one-point function.

IV. FROM REPLICAS TO SUPERSYMMETRY

Interestingly, in perturbation theory it is possible to use a peculiar variant of the replica method that is supersymmetric. This is because all N_v dependence in the propagators and vertices is entirely parametric. We can thus make replicas of an arbitrary real number of valence quarks. Moreover, partial quenching can be achieved not only by taking $N_v\to 0$, but also by taking N_v to any fixed number of quarks N_v', and re-interpreting the remaining N_f+N_v' as physical quarks (of which it just happens that at least N_v' are degenerate in mass). Because the N_v dependence is parametric in perturbation theory, we can trivially go one step further and consider a partially quenched theory of N_f physical fermions as the limit $N_v\to -\tilde{N}_v$ of a theory based on $N_f+\tilde{N}_v+N_v$ quarks, out of which the $\tilde{N}_v+N_v$ quarks are degenerate in mass $\tilde{m}_v=m_v$. This corresponds to considering the effective theory of a fundamental partition function that is partially supersymmetric (for simplicity considering only one such set of replica quarks):

[2]This chiral 2-point function has been analyzed in the supersymmetric formulation by Osborn, Toublan, and Verbaarschot (private communication).

$$\mathcal{Z}^{(N_f+\tilde{N}_v+N_v)}|_{N_v=-\tilde{N}_v} = \int [dA]\,\det(i\not{D}-m_v)^{N_v} \prod_{f=1}^{N_f+\tilde{N}_v} \det(i\not{D}-m_f)e^{-S_{\rm YM}[A]}|_{N_v=-\tilde{N}_v}$$
$$= \int [dA]\frac{\det(i\not{D}-\tilde{m}_v)^{\tilde{N}_v}}{\det(i\not{D}-m_v)^{\tilde{N}_v}} \prod_{f=1}^{N_f} \det(i\not{D}-m_f)e^{-S_{\rm YM}[A]}. \tag{18}$$

At this level the partition function is exactly as the starting point of the supersymmetric method. However, when we consider the effective partition function in terms of the Goldstone bosons, the working rules are entirely different. We keep our Feynman rules of Table I, and just remember to take the limit $N_v\to-\tilde{N}_v$ in the end. The fact that this procedure works is of course a direct consequence of the fact that in perturbation theory we can get bosons from fermions by letting the number of (degenerate) species go from positive to negative (also the ''statistics'' sign of closed fermion loops relative to closed boson loops comes out right in this way).

It is instructive to see how this supersymmetric variant of the replica method works in detail. Consider again our prototype of a Green function, that of the partially quenched chiral condensate. Using the notation of above, we find

$$\frac{\Sigma(m_v,\{m_f\})}{\Sigma} \equiv \lim_{\substack{N_v\to-\tilde{N}_v\\ m_v\to\tilde{m}_v}} \frac{\partial}{\partial m_v}\ln\mathcal{Z}^{(N_f+\tilde{N}_v+N_v)}$$
$$= \lim_{\substack{N_v\to-\tilde{N}_v\\ m_v\to\tilde{m}_v}} \frac{1}{N_v}\left[N_v-\frac{1}{F^2}\left(N_v\left[\sum_{f=1}^{N_f}\Delta(M_{vf}^2)+\tilde{N}_v\Delta(M_{v\tilde{v}}^2)\right]+N_v(N_v-1)\Delta(M_{vv}^2)+N_v\frac{1}{V}\sum_p G_{vv}(p^2)\right)\right], \tag{19}$$

where $M_{v\tilde{v}}^2\equiv(m_v+\tilde{m}_v)\Sigma/F^2$, and $G_{vv}(p^2)$ is as in Table I, except for the obvious change that now

$$\mathcal{F}(p^2)\equiv 1+\frac{\mu^2+\alpha p^2}{N_c}\left(\frac{N_v}{p^2+M_{vv}^2}+\frac{\tilde{N}_v}{p^2+M_{\tilde{v}\tilde{v}}^2}+\sum_{f=1}^{N_f}\frac{1}{p^2+M_{ff}^2}\right). \tag{20}$$

Taking the degenerate mass limit $\tilde{m}_v=m_v$ and letting $N_v\to-\tilde{N}_v$ we note that terms cancel out exactly as in the previous $N_v\to0$ replica method. For instance, in $\mathcal{F}(p^2)$ the terms linear in N_v and $\tilde{N}_v$ just cancel each other. In Eq. (19) the term proportional to N_v^2, which previously dropped out trivially in the $N_v\to0$ limit, is now precisely canceled by a similar term proportional to $N_v\tilde{N}_v\to-\tilde{N}_v^2$. All ''unwanted'' terms thus exactly cancel as they should, and we are left with the correct one-loop result (14). As we mentioned earlier, this example of the one-point function is actually the most instructive for illustrating the cancellations. The other k-point functions clearly proceed analogously.

Although it is thus possible to make a supersymmetric variant of the replica method, it is obviously rather pointless to do so. The simplest Feynman rules come from using just the conventional $N_v\to0$ limit. We also note that although the starting partition function (18) is identical to that forming the basis for the supersymmetric chiral Lagrangian [1,2], the effective theory one works with in the analogous supersymmetric replica scheme is of a very different nature, and has in fact here only been defined by means of the perturbative expansion.

V. CONCLUSIONS

We have shown how the replica method can be adapted to chiral perturbation theory. This provides a new and systematic realization of quenched and partially quenched chiral perturbation theory. We have demonstrated how the replica method is equivalent to the supersymmetric formulation in perturbation theory. This equivalence is quite trivial in the sector of physical quarks, and has allowed us to extend the three theorems of [2] to the present replica formulation of PQChPT. The equivalence between the replica and the supersymmetric formalisms also extends outside the sea sector. The complete agreement (at least to one-loop order) of the two approaches offers a non-trivial consistency check. In particular, the assumed extension of the standard symmetry breaking pattern to the supergroup case is avoided in the present context. The fact that results agree can be taken as independent confirmation of the validity of both approaches.

As an equivalent but nevertheless independent formulation of PQChPT the replica method illustrates the fact that supersymmetry is a technical tool for quenching rather than of fundamental nature. For practical purposes the usefulness of the replica method as compared to the supersymmetric formulation is perhaps a matter of taste. The advantage of having fewer sign-rules using the replica method is to some extent traded for the marginally simpler combinatorics in the supersymmetric formulation.

Finally, the replica method presented here gives the background and the justification for the rules observed by Colangelo and Pallante in [4]. Within the supersymmetric formulation they studied fully quenched chiral perturbation theory to one loop. Based on an explicit calculation of the divergent parts of the generating functional for both SU($k|k$) [and the additional U(1) of the Φ_0] and standard SU(N_f) chiral perturbation theory (without the Φ_0), they proposed a set of rules for writing down large parts of the SU($k|k$) generating functional from that of SU(N_v). The equivalence between the SU($k|k$) and SU($N_v \rightarrow 0$) theories (when the Φ_0 is included in both), is a special case of the general equivalence established here. This formally establishes the rules suggested in [4] and furthermore shows that the terms missing in SU($N_f \rightarrow 0$) chiral perturbation are just those produced by including the Φ_0. The procedure to compute in partially quenched chiral perturbation theory to any order is now extremely simple. One must take a usual chiral SU($N_f + N_v$) chiral Lagrangian and add the contributions from Φ_0. For example, to order p^6 the whole list of divergent contributions in the case of a degenerate SU(N_v) theory is provided in Ref. [16]. This can form the basis for a fully quenched calculation once the contributions from the flavor singlet have been included (for a discussion of the large-N_c limit, see e.g., Ref. [17]).

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