Convergence of the expansion of the Laplace-Borel integral in perturbative QCD improved by conformal mapping

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The optimal conformal mapping of the Borel plane was recently used to accelerate the convergence of the perturbation expansions in QCD. In this work we discuss the relevance of the method for the calculation of the Laplace-Borel integral expressing formally the QCD Green functions. We define an optimal expansion of the Laplace-Borel integral in the principal value prescription and establish conditions under which the expansion is convergent.

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I. INTRODUCTION

A suitable method for accelerating the convergence of power series is based on conformal mappings. As is known, a power series converges inside a circle passing through the nearest singularity of the function to be approximated. Some time ago, in Ref. [1], it was shown that, if the position of the singularities of the expanded function is known, one can reach the fastest convergence rate by expanding in powers of the function that conformally maps the *whole* holomorphy domain onto a unit disk. In addition, the convergence region extends over the whole holomorphy domain. In a recent paper [2] we applied the technique proposed in [1] to the Borel transform of the Green functions in perturbative QCD. As discussed in [2], the Borel plane is very suitable for applying the method, since some information about the singularities of the Borel transform is available from the study of certain classes of Feynman digrams and from nonperturbative arguments. By the technique of conformal mapping, this additional information can to a certain extent be incorporated even into the lowest-order terms. In this way, the convergence of the perturbative expansion is improved, allowing one in particular to approximately predict the next-order perturbative terms from the calculated low-order ones [2].

In Ref. [2] the expansion in powers of the optimal conformal mapping variable[1] was also used to calculate the Borel-Laplace integral, which is supposed to give, with a certain prescription of treating the infrared renormalons, the Borel summation of the large orders in the Green functions. The numerical results on mathematical models discussed in [2] indicate that the power expansion in the optimal variable makes also the calculation of the Borel integral convergent, in addition with a very high convergence rate. However, only qualitative arguments explaining the results were given, and the problem of whether the improved expansion of the integral is convergent or signs of divergence might appear at large orders remained open. In the present paper we address this problem and investigate the convergence of the expansion of the Borel integral in perturbative QCD, improved by the use of conformal mapping.

II. OPTIMAL EXPANSION OF THE LAPLACE-BOREL INTEGRAL

We study the following integral

$$I(a)=\int_0^\infty \mathrm{e}^{-u/a}B(u)\,\mathrm{d}u, \tag{1}$$

where $B(u)$ is assumed to be analytic near $u=0$, where it can be expanded as a Taylor series

$$B(u)=\sum_{n=0}^{\infty} b_n u^n \tag{2}$$

converging inside a circle of nonvanishing radius. The function $I(a)$ is of interest for the Borel summation of the Green functions in perturbative QCD; note, however, that the integral on the right hand side of Eq. (1) is ill defined if $B(u)$ has singularities along the positive real semiaxis, which is the case of QCD. First, the singularities of $B(u)$ (renormalons of either kind and instantons) make the expansion (2) badly divergent along the integration path. Second, the function $B(u)$, being singular at positive u, is not uniquely defined along the integration path. We shall discuss both these problems below in this paper.

We shall consider for illustration the Adler function $D(s)$ of the massless QCD vacuum polarization, which can be expressed formally as [3–6]

$$D(s)=1+\frac{1}{\pi\beta_0}I(a), \tag{3}$$

[1]The conformal mapping that maps the whole holomorphy domain of the expanded function onto the unit disk will be called optimal. In this case, the singularities are mapped onto the boundary circle, and the requirement of holomorphy implies convergence of the power series at every point of the disk, which is the map of the holomorphy domain.

with $a=\beta_0\alpha_s(-s)$, where $\alpha_s(-s)$ is the running coupling, and $\beta_0=(33-2n_f)/12\pi$ is the first coefficient of the β function. Expression (3) formally reproduces the renormalization-group-improved expansion of the Adler function

$$D(s)=1+\sum_{n=1}^{\infty} D_n\left(\frac{\alpha_s(-s)}{\pi}\right)^n \tag{4}$$

by taking coefficients b_n in the expansion (2) of the form

$$b_n=\frac{1}{n!}\frac{D_{n+1}}{(\pi\beta_0)^n}. \tag{5}$$

We consider also Minkowskian quantities, such as the hadronic decay rate of the τ lepton, R_τ, which can be expressed formally as [5]

$$R_\tau=3(1+\delta_{\rm EW})\left[1+\frac{1}{\pi\beta_0}\int_0^\infty \mathrm{d}\,u \times\exp\left(-\frac{u}{\beta_0\alpha_s(m_\tau^2)}\right)B(u)\,F(u)\right]. \tag{6}$$

Here δ_{EW} is an electroweak correction, $B(u)$ is the Borel transform of the Adler function, and

$$F(u)=\frac{-12\sin(\pi u)}{\pi\,u(u-1)(u-3)(u-4)}. \tag{7}$$

The extra factor $\sin(\pi u)$ in the Laplace-Borel integral is generic for Minkowskian quantities. Note that, strictly speaking, expressions (3) and (6) are not equivalent to the Borel summation method, which requires an analytic continuation of $B(u)$ from the convergence disk to an infinite strip of nonvanishing width, bisected by the real positive semiaxis. This condition is not satisfied in QCD because of instantons and infrared renormalons, which produce branch points of $B(u)$ located along the real positive semiaxis.

We shall be concerned with the evaluation of the integral (1) for complex a of the general form $a=|a|e^{i\psi}$, where $\psi=\arg a$ is the phase of a. In the case of the Adler function, with the running coupling at one loop in the V scheme $\alpha_s^{(V)}(-s)=1/[\beta_0\ln(-s/\Lambda_V^2)]$, and writing $-s=|s|\,e^{i(\phi-\pi)}$, we have

$$\frac{1}{a}=\ln\frac{|s|}{\Lambda_V^2}-i(\pi-\phi). \tag{8}$$

Outside the Landau region, i.e., for $|s|>\Lambda_V^2$, we have $\cos\psi>0$, so that

$$|\psi|<\frac{\pi}{2}, \tag{9}$$

and ψ is related to the momentum plane variable s by

$$\psi=\arctan\left[(\pi-\phi)/\ln\frac{|s|}{\Lambda_V^2}\right]. \tag{10}$$

The phase ψ is positive for s in the upper half of the s plane, where $0<\phi<\pi$, negative for s in the lower half-plane, where $\pi<\phi<2\pi$, and 0 along the Euclidean axis.

As concerns the Minkowskian quantity (6) we combine the additional factors $\exp(\pm i\pi u)$ due to the sinus with the exponential, which amounts to taking a complex with

$$\psi=\pm\arctan[\pi\overline{a}], \tag{11}$$

where $\overline{a}=\beta_0\alpha_s(m_\tau^2)$.

As already mentioned, the Borel transform $B(u)$ has singularities in the complex plane, correlated to the factorial increase of the perturbative coefficients of the Green functions at large orders [7–9]. The precise form of the singularities is not known for the exact theory, but the position and the nature of the first renormalons can be inferred from general principles. In the case of the Adler function the first ultraviolet (UV) renormalon is situated at $u=-1$ and the first infrared (IR) one at $u=2$, and they are branch points of the type $(1+u)^{\gamma_1}$ and $(2-u)^{\gamma_2}$, respectively, with γ_1 computed in [10] and γ_2 in [7]. We mention also that the summation of the one-renormalon chains in massless QCD in the large β_0 limit gives [8]

$$B(u)=\frac{32e^{-Cu}}{3(2-u)}\sum_{k=2}^{\infty}\frac{(-1)^k k}{[k^2-(1-u)^2]^2}; \tag{12}$$

i.e., all the singularities are poles [C is a scheme-dependent constant, with $C=-5/3$ in the modified minimal subtraction ($\overline{\rm MS}$) scheme, and $C=0$ in the V scheme] [3,6].

The series (2) converges only inside the circle $|u|<R$, passing through the nearest singularity ($R=1$ for the Adler function). Since the integration range in Eq. (1) extends far outside this region, by inserting Eq. (2) into Eq. (1) and integrating term by term one obtains a divergent expansion [11–13]. By the technique of conformal mappings one extends the domain of convergence of a series beyond the limit imposed by the first singularity. In [2] we used the optimal conformal mapping

$$w=w(u)=\frac{\sqrt{1+u}-\sqrt{1-u/2}}{\sqrt{1+u}+\sqrt{1-u/2}}, \tag{13}$$

with the inverse $u(w)=8w/(3-2w+3w^2)$. The transformation (13) preserves the origin and maps the complex u plane, cut along the real axis for $u>2$ and for $u<-1$, onto the interior of the circle $|w|<1$, all the singularities of the Borel transform being mapped onto the boundary $|w|=1$. The expansion in powers of w,

$$B(u)=\sum_{n=0}^{\infty} c_n\,w^n, \tag{14}$$

is called optimal because it converges inside the circle $|w|<1$, i.e., *in the whole domain of holomorphy of* $B(u)$ (which is the doubly cut complex u plane in our case), up to points close to the branch cuts produced by renormalons. As was already pointed out above, this power expansion yields,

when compared with other conformal mappings, *the fastest large-order convergence rate* (see a proof in [1]). In practice, as discussed in [2], the expansion (14) is obtained by suitably reorganizing the summation of the original series (2). More precisely, consider the expansion of each u^n in powers of w, truncated at a finite order N. In particular, in our case this expansion has the general form

$$u_N^n = \sum_{j=n}^{N} c_{nj} w^j, \tag{15}$$

with the coefficients c_{nj} obtained by expressing u in terms of w [in our case $u(w)$ is given explicitly after formula (13)]. Starting now with the expansion (2) truncated at finite order N, and replacing each u^n by its approximant u_N^n, one obtains a truncated expansion of the function B in powers of w, which in the limit $N \to \infty$ gives Eq. (14).

By inserting the optimal expansion (14) into the integral (1) we obtain the formal development

$$I(a) = \sum_{n=0}^{\infty} c_n I_n(a), \tag{16}$$

with

$$I_n(a) = \int_0^{\infty} \mathrm{e}^{-u/a} w^n \, \mathrm{d}u. \tag{17}$$

In the present work we shall adopt Eq. (16) as the optimal expansion of the Laplace-Borel integral. We point out that in the physical case this seems to be a natural definition. Indeed, when attempting to make the Borel summation of a perturbation expansion in QCD, one starts with a finite sum of the form

$$I_N(a) = \sum_{n=0}^{N} b_n \int_0^{\infty} \mathrm{e}^{-u/a} u^n \, \mathrm{d}u. \tag{18}$$

By replacing here the powers u^n with the approximations (15), we replace $I_N(a)$ by an expansion of the form

$$\sum_{n=0}^{N} c_n I_n(a), \tag{19}$$

which in the limit $N \to \infty$ leads to Eq. (16).

Actually, as mentioned above, the conditions for the Borel summation are, because of the instantons and infrared renormalons, not satisfied. Therefore, Eq. (16) can be considered as a definition of $I(a)$, provided that (i) the integration path in expressions such as Eq. (1) or (17) is consistently defined and (ii) the series (16) is convergent. Let us devote a brief discussion to these conditions.

(i) As concerns the integration contour, let us notice that the expansion (16) has not a precise mathematical sense with $I_n(a)$ defined by Eq. (17), because the integration path runs along the positive real semiaxis, where the w^n have cuts. We shall adopt, as in [2], the generalized principal value (PV) prescription, defining the $I_n^{PV}(a)$ as

$$I_n^{PV}(a) = \frac{1}{2} \int_{\mathcal{C}_+} \mathrm{e}^{-u/a} [w(u)]^n \, \mathrm{d}u + \frac{1}{2} \int_{\mathcal{C}_-} \mathrm{e}^{-u/a} [w(u)]^n \, \mathrm{d}u \tag{20}$$

for $n = 0,1,2, \ldots$, where $\mathcal{C}_+$ ($\mathcal{C}_-$) are lines parallel to the real positive axis, slightly above (below) it. While the PV prescription does not always give the expected results [14], in QCD it has the advantage that it reproduces, to a larger extent than other choices, the momentum plane analyticity properties of the Green functions derived from the general principles of field theory. In particular, as discussed in [15], the Adler function calculated with the PV prescription has no unphysical singularities in the region $|s| > \Lambda^2$. The functions $I_n^{PV}(a)$, $n = 1,2 \ldots$, are chosen so as to share some of the known properties with the unknown $I^{PV}(a)$. This makes them suited for the definition of $I^{PV}(a)$ by means of the expansion

$$I^{PV}(a) = \sum_{n=0}^{\infty} c_n I_n^{PV}(a). \tag{21}$$

(ii) The convergence of the series (21) for complex a is not *a priori* obvious. Indeed, the expansion (14) converges at points $|w| < 1$, therefore in the neigborhood of the integration axis, but not necessarily on the boundary. One might therefore expect that the boundary singularities could manifest in a dramatic way for very large orders N, making the series (21) divergent, like in the case of the original expansion (2). In [2] we investigated mathematical models with $B(u)$ having a few number of isolated branch point singularities and real values of a. The numerical results confirm that the expansion (2) in powers of u gives results which deviate dramatically from the exact value for large N, which is typical for a divergent expansion. On the other hand, the improved series (14) in powers of the optimal variable led to results improving continuously with increasing N, and no signs of divergence appeared even at very high N. In the next section we shall discuss the convergence of the optimal expansion of the Laplace-Borel integral, bringing analytic arguments which explain the numerical results obtained in [2].

III. CONVERGENCE OF THE OPTIMAL EXPANSION

We investigate the expansion (21) with the functions $I_n^{PV}(a)$ defined by means of the PV prescription (20). We consider in our discussion analytic functions B of real type, i.e., which satisfy $B^*(u) = B(u^*)$, where u^* is the complex conjugate of u. Therefore, the coefficients b_n in the expansion (2), as well as the coefficients c_n in the expansion (14), are real.

The contribution to Eq. (20) of the integral along the contour $\mathcal{C}_+$ can be written as

$$I_n^+(a) = \int_{\mathcal{C}_+} \mathrm{e}^{-F_n(u)} \, \mathrm{d}u, \tag{22}$$

where

$$F_n(u)=\frac{u}{a}-n\ln w(u). \tag{23}$$

We evaluate the integral (22) for large n by applying the method of steepest descent [11,14]. The saddle points are given by the equation

$$\frac{w'(u)}{w(u)}=\frac{1}{an}, \tag{24}$$

which has four solutions, having at large n the form

$$\frac{1+i}{2^{1/4}}\sqrt{an},\quad \frac{1-i}{2^{1/4}}\sqrt{an},\quad \frac{-1+i}{2^{1/4}}\sqrt{an},\quad \frac{-1-i}{2^{1/4}}\sqrt{an}. \tag{25}$$

Of interest for the evaluation of Eq. (22) is the point

$$u_0=2^{-1/4}(1+i)\sqrt{an}=|u_0|e^{i\alpha}, \tag{26}$$

with

$$|u_0|=2^{1/4}\sqrt{|a|n},\quad \alpha=\frac{\pi}{4}+\frac{\psi}{2}, \tag{27}$$

which is situated in the first quadrant of the u plane. Indeed, since the phase ψ of the parameter a satisfies the condition (9), then $\mathrm{Re}\,u_0>0$ and $\mathrm{Im}\,u_0>0$.

Near the saddle point $F_n(u)$ can be expanded as

$$F_n(u)=F_n(u_0)+\frac{1}{2}F_n''(u_0)(u-u_0)^2+\cdots. \tag{28}$$

By using the expansion of $w(u)$ for large u in the upper half-plane [$w\approx\zeta(1-i\sqrt{2}/u)$, where $\zeta=(\sqrt{2}+i)/(\sqrt{2}-i)$], we obtain, after a straightfoward calculation,

$$e^{-F_n(u_0)}\approx\zeta^n\left(1-\frac{2^{3/4}i}{(1+i)\sqrt{an}}\right)^n e^{-2^{-1/4}(1+i)\sqrt{n/a}}$$
$$\approx\zeta^n e^{-2^{3/4}(1+i)\sqrt{n/a}} \tag{29}$$

and

$$F_n''(u_0)\approx\frac{2^{1/4}(1-i)}{\sqrt{na^3}}=|F_n''(u_0)|e^{i\beta}, \tag{30}$$

where

$$|F_n''(u_0)|=\frac{2^{3/4}}{\sqrt{n|a|^3}},\quad \beta=-\frac{\pi}{4}-\frac{3\psi}{2}.$$

Therefore Eq. (22) becomes

$$I_n^+(a)\approx\zeta^n e^{-2^{3/4}(1+i)\sqrt{n/a}}\times\int_{\mathcal{C}_+}\exp\left(-\frac{|F_n''(u_0)|}{2}e^{i\beta}(u-u_0)^2\right)du. \tag{31}$$

In order to evaluate the integral we first rotate the contour $\mathcal{C}_+$ in the trigonometric direction in the upper half-plane, until it becomes a line passing through the origin and the saddle point u_0. The rotation is possible since $B(u)$ (and therefore also w) has no singularities outside the real axis, and the arc of the circle at infinity gives a vanishing contribution, as can be easily verified. Along the rotated line $u=e^{i\alpha}t$, where α is the phase of u_0 defined in Eq. (27) and t is real, so the integral in Eq. (31) becomes

$$e^{i\alpha}\int_0^\infty\exp\left(-\left[\frac{|F_n''(u_0)|}{2}e^{i(2\alpha+\beta)}(t-|u_0|)^2\right]\right)dt. \tag{32}$$

Since $\cos(2\alpha+\beta)>0$ for ψ satisfying the condition (9), the integration axis lies in the two valleys near the saddle point u_0. Therefore it can be deformed into the path of steepest descent through u_0, without passing outside the valleys. We take the integral along the path going to infinity,

$$u-u_0\approx\sqrt{2/|F_n''(u_0)|}\,e^{-i\beta/2}\rho, \tag{33}$$

with real ρ. The phase of $(u-u_0)^2$ exactly compensates the phase of $F_n''(u_0)$, making the exponent of the integrand in Eq. (31) real. The integrand can be written as $e^{-\rho^2}$ and the integral done explicitly gives

$$I_n^+(a)\approx\zeta^n e^{-2^{3/4}(1+i)\sqrt{n/a}}\frac{e^{-i\beta/2}}{\sqrt{|F_n''(u_0)|/2}}\frac{\sqrt{\pi}}{2}, \tag{34}$$

i.e., up to a constant independent of n:

$$I_n^+(a)\approx n^{1/4}\zeta^n e^{-2^{3/4}(1+i)\sqrt{n/a}}. \tag{35}$$

It is important to note that the path of steepest descent must not cross the real axis, where $B(u)$ has singularities. From Eq. (33) this implies $-\beta/2=\pi/8+3\psi/4>0$ which is written as

$$\psi>-\frac{\pi}{6}. \tag{36}$$

The evaluation of the integral along the contour $\mathcal{C}_-$ in Eq. (20) proceeds in a similar way. The saddle point of interest is

$$u_0'=2^{-1/4}(1-i)\sqrt{an}=2^{1/4}\sqrt{|a|n}\,e^{-i(\pi/4-\psi/2)}, \tag{37}$$

which satisfies $\mathrm{Re}\,u_0>0$ and $\mathrm{Im}\,u_0<0$ for ψ in the range given in Eq. (9). Instead of Eq. (31) we have

$$I_n^-(a)\approx(\zeta^*)^n e^{-2^{3/4}(1-i)\sqrt{n/a}}\times\int_{\mathcal{C}_-}\exp\left(-\frac{|F_n''(u_0')|}{2}e^{i\beta'}(u-u_0')^2\right)du, \tag{38}$$

where $\beta'=\pi/4-3\psi/2$. We rotate the contour $\mathcal{C}_-$ in the lower half-plane up to a line passing through the point u_0', and then deform it into the steepest descent path. One can easily verify that this path does not cross the real axis for $-\pi/8+3\psi/4<0$ which is written as

$$\psi < \frac{\pi}{6}. \tag{39}$$

Collecting the terms we obtain the coefficients $I_n(a)$ in the PV prescription (20) as

$$I_n(a) \approx n^{1/4} \zeta^n e^{-2^{3/4}(1+i)\sqrt{n/a}} + n^{1/4} (\zeta^*)^n e^{-2^{3/4}(1-i)\sqrt{n/a}}. \tag{40}$$

In order to examine the convergence of the expansion (21), we consider the ratio

$$\left| \frac{c_n I_n(a)}{c_{n-1} I_{n-1}(a)} \right| \tag{41}$$

for large n. If the coefficients c_n do not grow too rapidly, i.e.,

$$|c_n| < C e^{\epsilon n^{1/2}}, \tag{42}$$

for all $\epsilon > 0$, then the expansion (21) converges for a complex in the domain

$$\mathrm{Re}[(1 \pm i) a^{-1/2}] > 0. \tag{43}$$

As we already discussed these conditions are equivalent to Eq. (9). If the coefficients c_n behave at large n like

$$|c_n| \approx e^{c n^{1/2}} \tag{44}$$

for some positive c, then the expansion (21) converges in the domain

$$\mathrm{Re}[(1 \pm i) a^{-1/2} + c] > 0, \tag{45}$$

while for coefficients c_n which grow faster than $\exp(c n^{1/2})$ the new series (21) is also divergent. We mention that such a behavior is not excluded in general for series of the form (14) with a radius of convergence equal to 1 [14].

We recall, however, that the expression (40) is valid only for ψ which satisfy the conditions (36) and (39), i.e.,

$$|\psi| < \frac{\pi}{6}, \tag{46}$$

which define a sector in the a complex plane (we recall that ψ is the phase of a). This inequality is a condition of the applicability of the steepest descent method used by us. We found therefore that the expansion (21), improved by the optimal conformal mapping of the Borel plane, is convergent if the Taylor coefficients c_n of the expansion (14) satisfy the condition (42), at least inside the sector (46) of the complex plane of a, or, if they behave like Eq. (44), in the smallest of the domains (45) and (46). For the Adler function in massless QCD, using Eq. (10) we write the condition (46) in the form

$$|\pi - \phi| < \frac{1}{\sqrt{3}} \ln \frac{|s|}{\Lambda_V^2}, \tag{47}$$

where ϕ is the phase of s and we have $|s| > \Lambda_V^2$. For the Minkowskian quantities, from Eq. (11) we obtain

$$\pi \overline{a} < \frac{1}{\sqrt{3}}, \tag{48}$$

which means in particular that for the τ-hadronic decay rate the expansion defined as in Eq. (21) is convergent for $\alpha_s(m_\tau^2) < 4/(9\sqrt{3}) \approx 0.257$.

The behavior of the coefficients c_n depends on the singularities of $B(w)$. By the conformal mapping (13) all the instantons and renormalons are situated on the circle $|w| = 1$, appearing in conjugate pairs since $B(u)$ is of real type. Assuming that all the singularities are poles or branch points, c_n has the generic form

$$c_n = \frac{1}{n!} \mathrm{Re} \sum_j (-1)^{p_j} r_j p_j (p_j + 1)(p_j + 2) \cdots (p_j + n - 1) \times e^{i \alpha_j (p_j + n)}, \tag{49}$$

where $\exp(\pm i \alpha_j)$ denote the position of the renormalon in the w plane, r_j the residue, and p_j the exponent of the singularity. In Ref. [2] we investigated simple models with a finite number of singularities, and real values of the parameter a, for which the conditions of convergence are satisfied. In the physical case, one knows only that for the first UV renormalon $\alpha_1 = \pi$ and $p_1 = 2\gamma_1$, and for the first IR renormalon $\alpha_2 = 0$ and $p_2 = 2\gamma_2$. In the large β_0 case, as seen from Eq. (12), all the singularities are poles, p_j in Eq. (49) is independent of j, and r_j are known. In this case the condition of convergence, Eq. (42), is satisfied. Therefore, the optimal expansion on the Laplace-Borel integral, in the PV prescription, for the summation of one renormalon chains in the large β_0 limit, is convergent, at least in the sector of the complex a plane defined by the condition (46).

In conclusion, we investigated the expansion of the Laplace-Borel integral in perturbative QCD, improved by the analytic continuation of the Borel transform outside the perturbative convergence disk (and, simultaneously, by reaching the fastest convergence rate) by means of optimal conformal mapping [2]. The convergence properties of the new expansion depend on the strength of the singularities of the Borel transform, reflected in the behavior of the Taylor coefficients of the expansion (14). If the Taylor coefficients satisfy the condition (42), the new expansion of the Laplace-Borel integral converges in the sector of the complex plane of the coupling a defined by Eq. (46). The conditions are satisfied in the case of the resummation of one-loop renormalons in the large-β_0 limit. We mention that in the region where the series converges the function $I(a)$ must be analytic. For the Adler function in the complex momentum plane this corresponds to the region described by Eq. (47), where $D(s)$ is analytic.

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[1] S. Ciulli and J. Fischer, Nucl. Phys. **24**, 465 (1961).
[2] I. Caprini and J. Fischer, Phys. Rev. D **60**, 054014 (1999).
[3] C.N. Lovett-Turner and C.J. Maxwell, Nucl. Phys. **B452**, 188 (1995).
[4] P. Ball, M. Beneke, and V.M. Braun, Nucl. Phys. **B452**, 563 (1995).
[5] G. Altarelli, P. Nason, and G. Ridolfi, Z. Phys. C **68**, 257 (1995).
[6] M. Neubert, Phys. Rev. D **51**, 5924 (1995).
[7] A.H. Mueller, in *QCD—Twenty Years Later*, Aachen 1992, edited by P. Zerwas and H. A. Kastrup (World Scientific, Singapore, 1992).
[8] D. Broadhurst, Z. Phys. C **58**, 339 (1993).
[9] M. Beneke, Phys. Lett. B **307**, 154 (1993); Nucl. Phys. **B405**, 424 (1993).
[10] M. Beneke, V.M. Braun, and N. Kivel, Phys. Lett. B **404**, 315 (1997).
[11] H. Jeffreys, *Asymptotic Approximations* (Clarendon, Oxford, 1962).
[12] G.N. Hardy, *Divergent Series* (Oxford University Press, New York, 1949).
[13] R.B. Dingle, *Asymptotic Expansions: Their Derivation and Interpretation* (Academic, New York, 1972).
[14] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 3rd ed. (Oxford University Press, New York, 1996).
[15] I. Caprini and M. Neubert, J. High Energy Phys. **03**, 007 (1999).