Stabilizing textures in 3+1 dimensions with semilocality

L. Perivolaropoulos*
Institute of Nuclear Physics, N.C.S.R. Demokritos 153 10 Athens, Greece
(Received 18 January 2000; published 17 July 2000)

It is shown that textures in 3+1 dimensions can be stabilized by partial gauging (semilocality) of the vacuum manifold such that topological unwinding by a gauge transformation is not possible. This introduction of gauge fields can be used to evade Derrick's theorem without higher powers of derivatives and prevent the usual collapse of textures by the effective ''pressure'' terms induced by the gauge fields. A virial theorem is derived and shown to be a manifestation of the stability of the configuration towards collapse.

PACS number(s): 98.80.−k, 05.45.Yv

I. INTRODUCTION

Texture-like topological defects where the topological charge emerges by integrating over the whole physical space (not just the boundary) have played an important role in both particle physics and cosmology. Typical examples are the Skyrmion [1] which offers a useful effective model for the description of the nucleon and the global texture [2] where an instability towards collapse of the scalar field configuration has been used to construct an appealing mechanism for structure formation in the universe.

A typical feature of this class of scalar field configurations are instabilities towards field rescalings which usually lead to collapse and subsequent decay to the vacuum via a localized highly energetic event in space-time. The property of collapse is a general feature of global field configurations in 3+1 dimensions and was first described by Derrick [3] (see also Ref. [4] for recent generalizations). This feature (valid only when gauge fields are not present [3]) is particularly useful in a cosmological setup because it provides a natural decay mechanism which can prevent the dominance of the energy density of the universe by texture-like defects. At the same time, this decay mechanism leads to a high energy event in space-time that can provide the primordial fluctuations for structure formation.

In the particle physics context where a topological defect predicted by a theory can only be observed in accelerator experiments if it is at least metastable, the above instability is an unwanted feature. A usual approach to remedy this feature has been to consider effective models where non-renormalizable higher powers of scalar field derivatives are put by hand. This has been the case in QCD where chiral symmetry breaking is often described using the low energy ''pion dynamics'' model. Texture-like configurations occur here and as Skyrme first pointed out they may be identified with the nucleons (Skyrmions) [1]. Here textures are stabilized by non-renormalizable higher derivative terms in the quantum effective action. However no one has ever found such higher derivative terms with the right sign to stabilize the Skyrmion.

An alternative approach to stabilize texture-like configurations is the introduction of gauge fields [5–7] which can be shown to induce pressure terms in the scalar field Lagrangian thus balancing the effects of Derrick-type collapse. Numerical evidence for the existence of stable closed finite energy gauged knotted vortices (gauged Hopf textures) was presented in Ref. [8]. In the case of complete gauging of the vacuum manifold however, it is possible for the texture configuration to relax to the vacuum manifold by a continuous gauge transformation that can remove all the gradient energy (the only source of field energy for textures) from the non-singular texture-like configuration [2]. This mechanism of decay via gauge fields is not realized in singular defects where the topological charge emerges from the boundaries. In these defects, singularities, where the scalar field is 0, cannot be removed by continuous gauge transformations.

Recent progress in semilocal defects has indicated that physically interesting models can emerge by a partial gauging of the vacuum manifold of field theories. This partial gauging (semilocality) can lead to new classes of stable defect solutions that can persist as metastable configurations in more realistic models where the gauging of the vacuum is complete but remains non-uniform. A typical example is the semilocal string [9] whose embedding in the standard electroweak model has led to the discovery of a class of metastable (2+1)-dimensional field configurations in this model [10].

In the context of texture-like configurations, the concept of semilocality can lead to an interesting mechanism for stabilization. In fact the semilocal gauge fields are unable to lead to relaxation of the global field gradient energy because they cannot act on the whole target space. They do however induce pressure terms in the Lagrangian that tend to resist the collapse induced by the scalar sector. Therefore they have the features required for the construction of stable texture-like configurations in renormalizable models without the adhoc use of higher powers of derivatives.

The goal of this paper is to demonstrate the stabilization induced by semilocal gauge fields in the context of global textures [2] that form during the symmetry breaking $O(4)\rightarrow O(3)$. In the semilocal case discussed below the $O(3)$ subgroup of the global $O(4)$ symmetry is gauged.

II. VIRIAL THEOREM

Consider first a texture field configuration in 3+1 dimensions emerging in the context of a field theory describing a

*Email address: leandros@mail.demokritos.gr

global symmetry breaking $O(4)\rightarrow O(3)$. This is described by a four component scalar field $\vec{Q}=(Q_1,Q_2,Q_3,Q_4)$ whose dynamics is described by the potential

$$V(\vec{Q})=\frac{\lambda}{4}(\vec{Q}^2-F^2)^2. \tag{1}$$

The initial condition ansatz

$$\vec{Q}=(\sin\chi\sin\theta\sin\varphi,\sin\chi\sin\theta\cos\varphi,\sin\chi\cos\theta,\cos\chi) \tag{2}$$

with $\chi(r)$ varying between 0 and π as r goes from 0 to infinity and θ,φ spherical polar coordinates, describes a configuration that winds once around the vacuum $M_0=S^3$ as the physical space is covered. Since $\pi_3(M_0)\neq 1$ this is a nontrivial configuration which is topologically distinct from the vacuum. The energy of this configuration is of the form

$$E=\int d^3x\left[\frac{1}{2}(\vec{\nabla}\vec{Q})^2+V(\vec{Q})\right]\equiv T+V \tag{3}$$

where we have allowed for possible small potential energy excitations during time evolution. A rescaling of the spatial coordinate $r\rightarrow\alpha r$ of the field $\vec{Q}(r)$ leads to $E_\alpha=\alpha^{-1}T+\alpha^{-3}V$ which is monotonic with α and leads to collapse, highly localized energy and eventual unwinding of the configuration. These highly energetic and localized *events* in spacetime have provided a physically motivated mechanism for the generation of primordial fluctuations that gave rise to structure in the universe [2].

The possible stabilization of these collapsing configurations could lead to a cosmological overabundance and a cosmological problem similar to the one of monopoles, requiring inflation to be resolved. At the same time however it could lead to observational effects in particle physics laboratories. There are at least two ways to stabilize a collapsing texture in $3+1$ dimensions. The first is well known and includes the introduction of higher powers of derivative terms in the energy functional. These terms scale like $\alpha^p(p>0)$ with a rescaling and can make the energy minimization possible thus leading to stable *Skyrmions*. Stable *Hopfions* [11] (solitons with non-zero Hopf topological charge) have also been constructed recently by the same method.

The second method of stabilization is less known (but see Refs. [5–8]) and can be achieved by introducing gauge fields that partially cover the vacuum manifold. In particular, the simplest Lagrangian that accepts semilocal texture configurations in $3+1$ dimensions can be written as follows:

$$\mathcal{L}=-\frac{1}{4}G^a_{ij}G^a_{ij}-\frac{1}{2}D_iQ_aD_iQ_a-\frac{1}{2}D_iQ_4D_iQ_4-V \tag{4}$$

where $(i,j=1,2,3)$,

$$V=\frac{1}{2}\mu^2(Q_a^2+Q_4^2)+\frac{1}{8}\lambda(Q_a^2+Q_4^2)^2 \tag{5}$$

and $\vec{Q}\equiv(Q_a,Q_4)$ is a four component scalar field $(a=1,2,3)$ with vacuum $\vec{Q}^2=F^2$ with $F^2\equiv-2\mu^2/\lambda$. Here we are using the notation of Ref. [12], i.e.,

$$G^a_{ij}\equiv\partial_iW^a_j-\partial_jW^a_i+e\epsilon_{abc}W^b_iW^c_j \tag{6}$$

and

$$D_iQ_a=\partial_iQ_a+e\epsilon_{abc}W^b_iQ_c. \tag{7}$$

The field ansatz that describes a semilocal texture may be written as

$$\begin{aligned}Q_a&=r_aQ(r)\\ Q_4&=P(r)\\ W^a_i&=\epsilon_{iab}r_bW(r).\end{aligned} \tag{8}$$

Notice that this type of embedded texture is not semilocal in the usual sense [13]. Here the orbits of the gauge group are trivially embedded in the vacuum manifold and therefore the topology of the vacuum can change due to radiative corrections. In what follows we will neglect the effects of these radiative corrections assuming that the corresponding couplings are small.

The asymptotic behavior of the field functions $Q(r)$, $W(r)$ and $P(r)$ can be obtained by demanding finite energy, single-valueness of the fields and non-trivial topology. Thus we obtain the following asymptotics

$$\begin{aligned}rQ(r)&\rightarrow 0\\ -er^2W(r)&\rightarrow 0\\ P(r)&\rightarrow F\end{aligned} \tag{9}$$

for $r\rightarrow\infty$ and

$$\begin{aligned}rQ(r)&\rightarrow 0\\ -er^2W(r)&\rightarrow 0\\ P(r)&\rightarrow -F\end{aligned} \tag{10}$$

for $r\rightarrow 0$. We now rescale the coordinate r as $r\rightarrow r/eF$ and define the rescaled fields

$$\begin{aligned}q(r)&\equiv rQ(r)F\\ p(r)&\equiv P(r)F\\ w(r)&\equiv -W(r)/er^2.\end{aligned} \tag{11}$$

Using these rescalings, the energy of the semilocal texture ansatz (8) may be written as

$$4\pi e^{-1}F\int_0^\infty dr\mathcal{E} \tag{12}$$

where the energy density $\mathcal{E}$ may be expressed in terms of the rescaled fields as

$$\mathcal{E}=w'^2+\frac{2w^2}{r^2}\left(1-w+\frac{w^4}{4}\right)+\frac{1}{2}r^2(q'^2+p'^2)$$
$$+q^2(1-w)^2+\frac{1}{8}r^2\beta(q^2+p^2-1)^2 \tag{13}$$

where $\beta\equiv\lambda/e^2$. By varying the energy density with respect to the field functions w, q and p it is straightforward to obtain the field equations for these functions. These may be written as

$$w''-\frac{2w}{r^2}+\frac{3w^2}{r^2}-\frac{w^3}{r^2}+q^2(1-w)=0$$
$$(r^2q')'-2q(1-w)^2-\frac{\beta}{2}r^2q(p^2+q^2-1)=0$$
$$(r^2p')'-\frac{\beta}{2}r^2p(p^2+q^2-1)=0 \tag{14}$$

with boundary conditions

$$r\to 0: w\to 0, \quad q\to 0, \quad p\to 1$$
$$r\to\infty: w\to 0, \quad q\to 0, \quad p\to -1. \tag{15}$$

The stability towards collapse of the semilocal texture that emerges as a solution of the system (14) with the boundary conditions (15) can be studied by examining the behavior of the energy after a rescaling of the spatial coordinate r to $r\to\alpha r$. For global textures, this rescaling leads to a monotonic increase of the energy with α (Derrick's theorem) indicating instability towards collapse. It will be shown that this instability is not present in the field configuration of the semilocal texture due to the outward pressure induced by the gauge field which has the same effect as the higher powers of derivatives present in the Skyrmion.

The rescaling $r\to\alpha r$ modifies the total energy as follows:

$$E\to\alpha E_1+\alpha^{-1}E_2+\alpha^{-3}E_3 \tag{16}$$

where

$$E_1=A\int_0^\infty dr\left[w'^2+\frac{2w^2}{r^2}\left(1-w+\frac{w^4}{4}\right)\right]$$
$$E_2=A\int_0^\infty dr\left[\frac{1}{2}r^2(q'^2+p'^2)+q^2(1-w)^2\right]$$
$$E_3=A\int_0^\infty dr\left[\frac{1}{8}r^2\beta(q^2+p^2-1)^2\right] \tag{17}$$

and $A\equiv 4\pi e^{-1}F$. The expression (16) has an extremum with respect to α which is found by demanding $\delta E/\delta\alpha|_{\alpha=1}=0$. This leads to a virial theorem connecting the energy terms E_1, E_2 and E_3 as

$$E_1=E_2+3E_3. \tag{18}$$

By considering the second variation of the energy with respect to the rescaling parameter α it is easy to see that the extremum of the energy corresponding to the semilocal texture solution is indeed a minimum. Indeed

$$\frac{\delta^2 E}{\delta\alpha^2}\Big|_{\alpha=1}=2E_2+12E_3>0. \tag{19}$$

The above analysis showing stability towards rescaling does not by itself eliminate the possibility that there are other deformations of the fields that reduce the energy. However the configuration under study can be viewed as a localized 't Hooft–Polyakov monopole [12] (which is known to be stable) with an extra field which makes the configuration *topologically non-trivial* within S^3. Thus it can be anticipated that the configuration will be stable against any type of perturbation.

III. CONCLUSION

We conclude that the semilocal texture field configuration is a local minimum of the energy functional with respect to coordinate rescalings in contrast with its global counterpart. In addition it is impossible to unwind this configuration to the vacuum by a gauge transformation because only the $O(3)$ sub-group of the full symmetry $O(4)$ is gauged and therefore it is impossible to "rotate" all four components of the scalar field $\vec{Q}$ by a gauge transformation. Therefore an initial field configuration with non-trivial $\pi_3(S^3)$ topology can *neither unwind continuously to the vacuum* (due to non-trivial topology and insufficient gauge freedom) *nor collapse* (due to the derived virial theorem). Thus the energy of the configuration will remain trapped and localized either in the form of a static configuration [if a solution to the static system (14) exists] or in the form of a localized time-dependent breather-type configuration. In both cases there can be interesting consequences for both particle physics and cosmology.

Interesting extensions of this Brief Report include the study of the cosmological effects of semilocal textures and the possibility of embedding these objects in realistic extensions of the standard model. It is also important to perform a detailed numerical construction of these objects. A detailed study including these and other issues is currently in progress.

ACKNOWLEDGMENTS

I thank T. Tomaras for useful discussions.

[1] T.H.R. Skyrme, Proc. R. Soc. London **A260**, 127 (1961).
[2] N. Turok, Phys. Scr. **T36**, 135 (1991) and references therein.
[3] G.H. Derrick, J. Math. Phys. **5**, 1252 (1964).
[4] L. Perivolaropoulos, Phys. Rev. D **46**, 1858 (1992); Nucl. Phys. **B375**, 665 (1992).
[5] C. Bachas and T.N. Tomaras, Phys. Rev. D **51**, R5356 (1995); C. Bachas, B. Rai, and T.N. Tomaras, Phys. Rev. Lett. **82**, 2443 (1999).
[6] L. Perivolaropoulos, hep-ph/9903539 1999; L. Perivolaropoulos and T. Tomaras, Phys. Rev. D **62**, 025012 (2000).
[7] M. Hindmarsh, Nucl. Phys. **B392**, 461 (1993).
[8] A. Niemi, K. Palo, and S. Virtanen, hep-th/9807228, 1998.
[9] T. Vachaspati and A. Achucarro, Phys. Rev. D **44**, 3067 (1991); M. Hindmarsh, Phys. Rev. Lett. **68**, 1263 (1992); A. Achucarro, K. Kuijken, L. Perivolaropoulos, and T. Vachaspati, Nucl. Phys. **B388**, 435 (1992).
[10] T. Vachaspati, Nucl. Phys. **B397**, 648 (1993); M. James, L. Perivolaropoulos, and T. Vachaspati, Phys. Rev. D **46**, 5232 (1992); Nucl. Phys. **B395**, 534 (1993).
[11] L. Faddeev and A.J. Niemi, Nature (London) **357**, 58 (1997); J. Gladikowski and M. Hellmund, Phys. Rev. D **56**, 5194 (1997).
[12] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974).
[13] J. Preskill, Phys. Rev. D **46**, 4218 (1992).