Twisted bundles on noncommutative $\mathbf{T}^4$ and D-brane bound states

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We construct twisted quantum bundles and adjoint sections on noncommutative $\mathbf{T}^4$, and investigate relevant D-brane bound states with non-Abelian backgrounds. We also show that the noncommutative $\mathbf{T}^4$ with non-Abelian backgrounds exhibits $SO(4,4|\mathbf{Z})$ duality and via this duality we get a Morita equivalent $\mathbf{T}^4$ on which only D0-branes exist. For a reducible non-Abelian background, the moduli space of D-brane bound states in type II string theory takes the form $\Pi_a(\mathbf{T}^4)^{q_a}/S_{q_a}$.

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I. INTRODUCTION

Recent developments in nonperturbative string theories have provided new powerful tools to understand supersymmetric gauge theories [1]. The Bogomol'nyi-Prasad-Sommerfeld (BPS) brane configurations led to many exact results on the vacuum structure of supersymmetric gauge theories. One may be interested in counting the degeneracy of D-brane bound states of type II string theory compactified on $\mathbf{R}^{1,9-d}\times X$ in which a gauge field strength F and a Neveu-Schwarz B field on the brane are nonzero. Then p-branes wrapped on a compact p-cycle $W_p\subset X$ and their bound states look like particles in the effective $\mathbf{R}^{1,9-d}$ spacetime. Moreover, the degeneracy of the bound states is the same as the number of ground states in the corresponding quantum field theory on the D-brane world volume [2].

The D-brane moduli space [3,4] can be defined as a space of Chan-Paton vector bundle E over X or a space of solutions to the equation given by

$$\delta\lambda = F_{MN}\Gamma^{MN}\xi + \eta = 0$$

for some pair of covariantly constant spinors ξ and η on $\mathbf{R}^{1,9-d}\times X$. The various Ramond-Ramond (RR) charges are given by the Mukai vector $Q=v(E)=\mathrm{Ch}(E)\sqrt{\hat{A}(X)}\in H^{2*}(X,\mathbf{Z})$ where $\mathrm{Ch}(E)=\mathrm{Tr}\exp[(1/2\pi)(F-B)]$ is the Chern character and $\hat{A}(X)=1-p_1(X)/24$ is the A-roof genus for four-dimensional manifold X. Then the supersymmetric, BPS bound states, for example (D0,D2,D4) bound states on $\mathbf{T}^4$ or $\mathbf{K3}$, are allowed by the Chern-Simons couplings [5]

$$\int_{X\times\mathbf{R}} C^{RR}\wedge Q.$$

It was shown in [6,7] that noncommutative geometry can be successfully applied to the compactification of M(atrix) theory [8] in a certain background. In those papers, it was argued that M(atrix) theory in a three-form potential background with one index along the lightlike circle and two indices along $\mathbf{T}^d$ is a gauge theory on a noncommutative torus, specifically $(d+1)$-dimensional noncommutative super Yang-Mills (NCSYM) theory. Many more discussions of M- and string-theory compactifications on these geometries followed, for example [9–15].

One obvious advantage of NCSYM theory defined on $\mathbf{T}^d$ is that the T duality, $SO(d,d|\mathbf{Z})$, of type II string theory compactified on torus becomes manifest [6,7,13–15]. The Morita equivalence between two noncommutative torus [10,11] encompasses the Nahm transformation part of T duality, not clearly observed in conventional Yang-Mills theory. Using this symmetry, it may be possible to systematically count D-brane bound states on $\mathbf{T}^4$ or $\mathbf{K3}$ as ground state configurations for the supersymmetric gauge theory.

For compactifications on $\mathbf{T}^2$ and $\mathbf{T}^3$ generic $U(N)$ bundles on it admit vanishing $SU(N)$ curvature [6,12,13]. However, for compactifications on tori of dimension 4 or larger, not all bundles allow vanishing $SU(N)$ curvature, so we have to consider more generic bundles with nonvanishing $SU(N)$ curvature. It turns out [16,17] that one can construct a twisted $SU(N)$ gauge bundle on $\mathbf{T}^4$ with fractional instanton number. However, in discussing the $U(N)$ gauge theory as D-brane dynamics, it is understood that the total instanton number is integral since the instanton number is related to D0-brane charges inside D4-branes, which should satisfy Dirac quantization due to the existence of a D6-brane in type IIA string theory [18]. In [19,20], 't Hooft solutions on twisted bundles on commutative tori were realized by D-brane configurations (D-brane bound states) wrapped on tori in type II string theory, and it was shown that U duality relates their bound states.

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In general one can consider gauge bundle on $\mathbf{T}^4$ with non-Abelian constant curvature [17]. In that case, non-Abelian backgrounds can be obviously supersymmetric for self-dual or anti-self-dual fields since the supersymmetry of D-brane world volume theory may be given by

$$\delta\lambda = F_{MN}\Gamma^{MN}\xi.$$

Thus, in order to study the BPS spectrums of NCSYM theory on the non-Abelian backgrounds, it will be useful to construct the corresponding gauge bundles. In the presence of non-Abelian backgrounds as well as Abelian backgrounds, the gauge bundle may be twisted by the background magnetic fluxes. While Abelian backgrounds universally twist the $U(N)$ gauge bundle, in the case of non-Abelian backgrounds where the magnetic fluxes in $U(N)$ are decomposed into a $U(k)$ part and a $U(l)$ part [17], the magnetic flux in the $U(k)$ part twists the $U(k)\subset U(N)$ gauge bundle and that in the $U(l)$ part does the $U(l)\subset U(N)$ gauge bundle. This causes two different deformation parameters to appear.

The Chern character maps K theory to cohomology, i.e., $\mathrm{Ch}: K^0(X)\rightarrow H^{\mathrm{even}}(X,\mathbf{Z})$ and K^1 to odd cohomology and $\mathrm{Ch}(E)=\mathrm{Ch}_0(E)+\mathrm{Ch}_1(E)+\mathrm{Ch}_2(E)$ when X is four dimensional and E is a vector bundle over X. Here $\mathrm{Ch}_0(E)$ is the rank of E, $\mathrm{Ch}_1(E)$ is the first Chern class, and $\mathrm{Ch}_2(E)$ corresponds to the instanton number. $\mathrm{Ch}_1(E)$ is the integral winding number when the torus is commutative and it is not integer anymore when the torus is noncommutative but $\mathrm{Ch}_2(E)$ still remains integral even if the torus becomes noncommutative [10,11,14]. However, D-brane charges take values in $K(X)$, the K theory of X [21], which constitutes a group of integer $\mathbf{Z}$. The (4+1)-dimensional $U(N)$ SYM theory can be interpreted as the dynamics of N D4-branes. Six magnetic fluxes are D2-branes wound around six two-cycles of $\mathbf{T}^4$. Instantons are D0-branes bound to D4-branes. Thus, even when the Neveu-Schwarz–Neveu-Schwarz (NS-NS) two-form potential background is turned on, the physical D-brane numbers should be integers. In addition, the rank, six fluxes, and instanton (altogether, eight components) make a fundamental multiplet of the Weyl spinor representation of $SO(4,4|\mathbf{Z})$ [14].

Since explicit constructions of twisted bundles and adjoint sections in the literature have been performed only for Abelian backgrounds, we will construct them for constant non-Abelian backgrounds in this paper. In Sec. II, we construct twisted bundles on noncommutative $\mathbf{T}^4$. In Sec. III, adjoint sections on the twisted bundle will be constructed. In Sec. IV, we show that the modules of D-brane bound states exhibit an $SO(4,4|\mathbf{Z})$ duality and the action of this group gives a Morita equivalent $\mathbf{T}^4$ on which only D0-branes exist. Section V is devoted to the conclusion and comments on our results. In the Appendix, we present some details of the representation of $SO(4,4|\mathbf{Z})$ Clifford algebra.

II. TWISTED QUANTUM BUNDLES ON $\mathbf{T}^4$

To define the noncommutative geometry, we understand the space is noncommutative, viz.,

$$[x^\mu,x^\nu]=-2\pi i\Theta_{\mu\nu}. \tag{1}$$

Then the noncommutative $\mathbf{T}^4$, which will be denoted by $\mathbf{T}^4_\Theta$, is generated by translation operators U_μ defined by $U_\mu = e^{ix_\mu}$ and satisfies the commutation relation

$$U_\mu U_\nu = e^{2\pi i\Theta_{\mu\nu}}U_\nu U_\mu. \tag{2}$$

Also, we introduce partial derivatives satisfying

$$[\partial_\mu,x^\nu]=\delta^\nu_\mu,\ [\partial_\mu,\partial_\nu]=0.$$

We construct quantum $U(N)$ bundles on $\mathbf{T}^4_\Theta$ following the construction of [12,13] and [17]. Start with a constant curvature connection

$$\nabla_\mu=\partial_\mu+iF_{\mu\nu}x^\nu, \tag{3}$$

where Greek indices run over spatial components only. In this paper we allow $U(N)$ gauge fields with nonvanishing $SU(N)$ curvature in order to consider non-Abelian backgrounds. Following the ansatz taken by 't Hooft [17], we take the curvature $F_{\mu\nu}$ as the Cartan subalgebra element:

$$F_{\mu\nu}=F^{(1)}_{\mu\nu}+F^{(2)}_{\mu\nu}, \tag{4}$$

where $F^{(1)}_{\mu\nu}=\mathrm{Tr}\,F_{\mu\nu}$ and $F^{(2)}_{\mu\nu}\in u(1)\subset su(N)$. The constant curvature is given by

$$\mathcal{F}_{\mu\nu}=i[\nabla_\mu,\nabla_\nu]. \tag{5}$$

And one can calculate to get

$$\mathcal{F}=(2F+2\pi F\Theta F). \tag{6}$$

Note that both F and Θ are antisymmetric 4×4 matrices.

The gauge transformations of fields in the adjoint representation of the gauge group are insensitive to the center of the group, e.g., $\mathbf{Z}_N$ for $SU(N)$. Thus, for the adjoint fields in $SU(N)$ gauge theory, it is sufficient to consider the gauge group as being $SU(N)/\mathbf{Z}_N$. However, there can be an obstruction to go from an $SU(N)/\mathbf{Z}_N$ principal fiber bundle to an $SU(N)$ bundle if the second homology group of the base manifold X, $H_2(X,\mathbf{Z}_N)$, does not vanish [22]. In order to describe such a nontrivial $U(N)$ bundle, it is helpful to decompose the gauge group into its Abelian and non-Abelian components

$$U(N)=(U(1)\times SU(N))/\mathbf{Z}_N. \tag{7}$$

It means that we identify an element $(g_1,g_N)\in U(1)\times SU(N)$ with (g_1c^{-1},cg_N), where $c\in\mathbf{Z}_N$. Therefore one can arrange the twists in $U(N)$ to be trivial by cancelling them between $SU(N)$ and $U(1)$ [19]. This requires consistently combining solutions of $SU(N)/\mathbf{Z}_N$ with $U(1)$ solutions as to cancel the total twist.

To characterize the generic $U(N)$ gauge bundle on $\mathbf{T}^4_\Theta$, we allow the gauge bundle be periodic up to gauge transformation Ω_μ, i.e.,

$$\nabla_\mu(x^\alpha+2\pi\delta^\alpha_\nu)=\Omega_\nu(x^\alpha)\nabla_\mu(x^\alpha)\Omega^{-1}_\nu(x^\alpha). \tag{8}$$

Consistency of the transition functions of the U(N) bundle requires the so-called cocycle condition

$$\Omega_\mu(x^\alpha+2\pi\delta^\alpha_\nu)\Omega_\nu(x^\alpha)=\Omega_\nu(x^\alpha+2\pi\delta^\alpha_\mu)\Omega_\mu(x^\alpha). \quad (9)$$

However, the SU(N) transition function $\tilde{\Omega}_\mu(x^\alpha)$ may be twisted as [16]

$$\tilde{\Omega}_\mu(x^\alpha+2\pi\delta^\alpha_\nu)\tilde{\Omega}_\nu(x^\alpha)=Z_{\mu\nu}\tilde{\Omega}_\nu(x^\alpha+2\pi\delta^\alpha_\mu)\tilde{\Omega}_\mu(x^\alpha), \quad (10)$$

where $Z_{\mu\nu}=e^{-2\pi i n_{\mu\nu}/N}$ is the center of SU(N).

Write $\Omega_\mu(x)$ as a product of an x-dependent part and a constant part,

$$\Omega_\mu(x)=e^{i(P^{(1)}_{\mu\nu}+P^{(2)}_{\mu\nu})x^\nu}W_\mu, \quad (11)$$

where $P^{(1)}_{\mu\nu}$ is antisymmetric and proportional to the identity in the Lie algebra of U(N) while $P^{(2)}_{\mu\nu}$ is an element of u(1)$\subset$su(N). And constant $N\times N$ unitary matrices W_μ are taken as SU(N) solutions generated by 't Hooft clock and shift matrices. For comparision, our $P^{(2)}_{\mu\nu}$ in Eq. (11) corresponds to the constant SU(N) field strength $\alpha_{\mu\nu}$ in the 't Hooft ansatz in [17] if we consider commutative $\mathbf{T}^4$.

In the case of vanishing su(N) cuvature, $F^{(2)}_{\mu\nu}=P^{(2)}_{\mu\nu}=0$, an explicit construction of gauge bundles with magnetic and electric fluxes was given in [14]. For the nonvanishing $su(N)$ curvature case, following the 't Hooft solution [17] we consider diagonal connections which break U(N) to U(k)$\times$U(l) where each block has vanishing SU(k) and SU(l) curvature. We also consider the groups U(k) and U(l) as U(k)=[U(1)$\times$SU(k)]/$\mathbf{Z}_k$ and U(l)=(U(1)$\times$SU(l))/$\mathbf{Z}_l$, respectively. Thus the twists of the SU(k) or SU(l) part can be trivialized by each U(1) part. Since the U(1) in U(N) is the direct sum of U(1) in U(k) and U(1) in U(l), the SU(N) twist tensor should be a sum of SU(k) and SU(l) twist tensors.

Here we take the generator σ in u(1)$\subset$su(N) as

$$\sigma=\begin{pmatrix} l\,\mathbf{1}_k & 0 \\ 0 & -k\,\mathbf{1}_l \end{pmatrix}, \quad (12)$$

where the $k\times k$ matrix $\mathbf{1}_k$ is the identity in U(k) and the $l\times l$ matrix $\mathbf{1}_l$ is that in U(l). Then we take the SU(N) connection to be proportional to σ. Since the U(N) gauge field in Eq. (3) contains only the matrix σ and the identity matrix $\mathbf{1}_N$ in U(N) and so commutes with W_μ, in checking Eq. (8), W_μ are irrelevant in our situation and we have

$$P=2\pi F(\mathbf{1}_N+2\pi\Theta F)^{-1}=2\pi(\mathbf{1}_N+2\pi F\Theta)^{-1}F, \quad (13)$$

where $P_{\mu\nu}=P^{(1)}_{\mu\nu}+P^{(2)}_{\mu\nu}$. From the ansatz of Ω_μ, Eq. (11), and the cocycle condition (9), we obtain the following commutation relation for W_μ:

$$W_\mu W_\nu=e^{-2\pi i M_{\mu\nu}/N}W_\nu W_\mu, \quad (14)$$

where M is given by

$$M=M^{(1)}+M^{(2)}=N(2P-P\Theta P). \quad (15)$$

Here, the integral matrix $M^{(1)}_{\mu\nu}$ is coming from the trace part of U(N), and $M^{(2)}_{\mu\nu}$ which is also integral is proportional to σ.

We now construct the solutions in the manner of 't Hooft for bundles with a constant curvature background (4) on $\mathbf{T}^4_\Theta$. The greatest common divisor of ($M_{\mu\nu}$,N) is invariant under SL(4,$\mathbf{Z}$) and we take it as q. Also, we assume the twist matrix M and the flux P have the form of q copies of U(n) matrices $\mathbf{m}$ and $\tilde{P}$ defined by

$$\mathbf{m}=n(2\tilde{P}-\tilde{P}\Theta\tilde{P}), \quad P=\mathbf{1}_q\otimes\tilde{P}, \quad (16)$$

where $\mathbf{1}_q$ is a q-dimensional identity matrix. In other words,

$$N=q\,n, \; M=q\,\mathbf{1}_q\otimes\mathbf{m}, \quad (17)$$

where n is the reduced rank. In this case, it is convenient to consider transition functions Ω_μ and W_μ as the following block diagonal form [13]:

$$\Omega_\mu=\mathbf{1}_q\otimes\omega_\mu, \quad W_\mu=\mathbf{1}_q\otimes\tilde{W}_\mu, \quad (18)$$

where ω_μ and $\tilde{W}_\mu$ belong to U(n) and SU(n), respectively. Thus we will consider only one copy described by U(n) transition functions ω_μ.

Let us define the SU(n) matrices U and V as follows:

$$U_{kl}=e^{2\pi i(k-1)/n}\,\delta_{k,l}, \quad V_{kl}=\delta_{k+1,l}, \; k,l=1,\ldots,n, \quad (19)$$

so that they satisfy $UV=e^{-2\pi i/n}VU$. For $\mathbf{T}^4_\Theta$ with vanishing SU(n) curvature where we can put $F^{(2)}_{\mu\nu}=P^{(2)}_{\mu\nu}=0$, there are solutions of the form

$$\tilde{W}_\mu=U^{a_\mu}V^{b_\mu}, \quad (20)$$

where a_μ and b_μ are integers. In order for the U(n) twists to be trivial as in Eq. (9), the SU(n) twists $n_{\mu\nu}$ should be balanced with the U(1) fluxes $\mathbf{m}_{\mu\nu}=m_{\mu\nu}\mathbf{1}_n$. Thus, Eq. (14) gives

$$n_{\mu\nu}=m_{\mu\nu}=a_\mu b_\nu-a_\nu b_\mu \bmod n. \quad (21)$$

In the case of commutative $\mathbf{T}^4$, 't Hooft solutions with nonvanishing SU(n) curvature are described by breaking U(n) to U(k)$\times$U(l) so that background gauge fields reside along the diagonals of the U(k) and U(l) [17]. Here we have taken n as $n=k+l$. For $\mathbf{T}^4_\Theta$, we now adopt a 't Hooft type solution given by

$$\tilde{W}_\mu=U_1^{a_\mu}V_1^{b_\mu}U_2^{c_\mu}V_2^{d_\mu}, \quad (22)$$

where a_μ, b_μ, c_μ, and d_μ are integers to be determined. The matrices $U_{1,2}$ and $V_{1,2}$ acting in the two subgroup SU(k) and SU(l) satisfy the following commutation rules:

$$U_1V_1=e^{-2\pi i/k\mathbf{I}_k}V_1U_1,$$

$$U_2V_2=e^{-2\pi i/l\mathbf{I}_l}V_2U_2,$$

$$[U_1,U_2]=[U_1,V_2]=[V_1,U_2]=[V_1,V_2]=0, \tag{23}$$

where $n\times n$ matrices $\mathbf{I}_k$ and $\mathbf{I}_l$ have the forms, respectively,

$$\mathbf{I}_k=\begin{pmatrix}\mathbf{1}_k & 0\\ 0 & 0\end{pmatrix},\quad \mathbf{I}_l=\begin{pmatrix}0 & 0\\ 0 & \mathbf{1}_l\end{pmatrix}. \tag{24}$$

As discussed above, the triviality of the U(n) twists requires a balance between the SU(n) twists $n_{\mu\nu}$ and the U(1) fluxes $\mathbf{m}^{(1)}_{\mu\nu}$, which leads to the identification $n_{\mu\nu}\mathbf{1}_n=\mathbf{m}^{(1)}_{\mu\nu}$. Similarly, since each block has vanishing SU(k) or SU(l) curvature, the fluxes $m^{(k)}_{\mu\nu}$ in U(k) and $m^{(l)}_{\mu\nu}$ in U(l) have to cancel the twists $n^{(k)}_{\mu\nu}$ in SU(k) and $n^{(l)}_{\mu\nu}$ in SU(l), respectively, which leads us an identification as in Eq. (21):

$$n^{(k)}_{\mu\nu}=m^{(k)}_{\mu\nu},\ n^{(l)}_{\mu\nu}=m^{(l)}_{\mu\nu}. \tag{25}$$

Following the identification (25), one can solve the total SU(n) twists $n_{\mu\nu}$ in terms of two sets of twists $n^{(k)}_{\mu\nu}$ and $n^{(l)}_{\mu\nu}$, and the SU(n) fluxes $\mathbf{m}^{(2)}_{\mu\nu}$ as in [17]. Using Eqs. (23), Eq. (14) gives

$$\frac{n_{\mu\nu}}{n}\mathbf{1}_n=\frac{n^{(k)}_{\mu\nu}}{k}\mathbf{I}_k+\frac{n^{(l)}_{\mu\nu}}{l}\mathbf{I}_l-\frac{\mathbf{m}^{(2)}_{\mu\nu}}{n}. \tag{26}$$

Taking the trace on the above equation, we get

$$n_{\mu\nu}=n^{(k)}_{\mu\nu}+n^{(l)}_{\mu\nu}, \tag{27}$$

where

$$n^{(k)}_{\mu\nu}=a_\mu b_\nu-a_\nu b_\mu \bmod k,$$

$$n^{(l)}_{\mu\nu}=c_\mu d_\nu-c_\nu d_\mu \bmod l. \tag{28}$$

Recall that the Pfaffians given by twists $n^{(k)}_{\mu\nu}$ and $n^{(l)}_{\mu\nu}$ satisfy

$$\frac{1}{8}\epsilon^{\mu\nu\alpha\beta}n^{(k)}_{\mu\nu}n^{(k)}_{\alpha\beta}=0 \bmod k,\quad \frac{1}{8}\epsilon^{\mu\nu\alpha\beta}n^{(l)}_{\mu\nu}n^{(l)}_{\alpha\beta}=0 \bmod l \tag{29}$$

due to the triviality of the SU(k) and SU(l) parts. However, the total SU(n) twists may satisfy

$$\frac{1}{8}\epsilon^{\mu\nu\alpha\beta}n_{\mu\nu}n_{\alpha\beta}\neq 0 \bmod n, \tag{30}$$

since it is not trivial in this construction. And the zero-brane charge is given by

$$C=k\cdot Pf(n^{(k)}/k)+l\cdot Pf(n^{(l)}/l)=C^{(k)}+C^{(l)}, \tag{31}$$

which is an integer, due to the triviality of each sector [19]. Therefore, our construction corresponds to D-brane bound states involved with a (4,2,2) or (4,2,2,0) system depending on the value of C in the language of [19]. The (4,2,2) system is a bound state of four-branes and two-branes with nonzero intersection number but no zero-branes. The (4,2,2,0) system is a bound state of four-, two-, and zero-branes with nonzero two-brane intersection number.

For an explicit construction of these systems, we may choose

$$n^{(k)}_{34}=n^{(l)}_{12}=0,\ n^{(k)}_{12}\neq 0,\ n^{(l)}_{34}\neq 0$$

for (4,2,2) and

$$n^{(k)}_{12}=p^{(k)},\ n^{(k)}_{34}=k,\ n^{(l)}_{12}=l,\ n^{(l)}_{34}=p^{(l)}$$

for (4,2,2,0). Here, the zero-brane charge in the (4220) case is given by $p^{(k)}+p^{(l)}$. Notice that in this construction, the (4,2,2) system can be contained in the (4,2,2,0) system as a special case.

Since some work in this direction in the vanishing SU(N) curvature case [14] was already done via van Baal construction [23], below we also show how we can construct a (4,2,2,0) system in the manner of van Baal in our case.

Equation (14) is covariant under SL(4,**Z**). Using this symmetry we can always make the matrix $m=m^{(k)}+m^{(l)}$ to a standard symplectic form by performing a SL(4,**Z**) transformation R:

$$m=Rm_0R^T, \tag{32}$$

where we choose m_0 as

$$m_0=\begin{pmatrix}0 & m_1+m_3 & 0 & 0\\ -m_1-m_3 & 0 & 0 & 0\\ 0 & 0 & 0 & m_2\\ 0 & 0 & -m_2 & 0\end{pmatrix}. \tag{33}$$

Since $m_0=m_0^{(k)}+m_0^{(l)}$, we take the matrices $m_0^{(k)}$ and $m_0^{(l)}$ as

$$m_0^{(k)}=\begin{pmatrix}0 & m_1 & 0 & 0\\ -m_1 & 0 & 0 & 0\\ 0 & 0 & 0 & m_2\\ 0 & 0 & -m_2 & 0\end{pmatrix},$$

$$m_0^{(l)}=\begin{pmatrix}0 & m_3 & 0 & 0\\ -m_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{34}$$

Here we have taken a simple U(l) solution for convenience.

Since we consider a special diagonal connection which breaks U(n) to U(k)×U(l) and each block has vanishing SU(k) or SU(l) curvature, the twisted bundle can be decomposed into a U(k) part and a U(l) part and the construction in [23] can be applied to each part separately. Introduce $q_i=\gcd(m_i,k)$, $l_0=\gcd(m_3,l)$ $(i=1,2)$ and $k_i=k/q_i$, $l_1=l/l_0$. In [23], it was shown that twist-eating solutions of the type

$$\tilde{W}_\mu \tilde{W}_\nu = e^{-2\pi i \mathbf{m}_0^{\mu\nu}/n} \tilde{W}_\nu \tilde{W}_\mu, \tag{35}$$

where $\mathbf{m}_0^{\mu\nu}/n = (m_{0\,\mu\nu}^{(k)}/k)\mathbf{I}_k + (m_{0\,\mu\nu}^{(l)}/l)\mathbf{I}_l$, can only exist if $k_1 k_2 | k$. We thus write $k = k_1 k_2 k_0$. When this restriction is satisfied, it is straightforward to check that the following solution satisfies Eq. (35):

$$\tilde{W}_1 = U_{k_1}^{m_1/q_1} \otimes \mathbf{1}_{k_2} \otimes \mathbf{1}_{k_0} \oplus U_{l_1}^{m_3/l_0} \otimes \mathbf{1}_{l_0},$$

$$\tilde{W}_2 = V_{k_1} \otimes \mathbf{1}_{k_2} \otimes \mathbf{1}_{k_0} \oplus V_{l_1} \otimes \mathbf{1}_{l_0},$$

$$\tilde{W}_3 = \mathbf{1}_{k_1} \otimes U_{k_2}^{m_2/q_2} \otimes \mathbf{1}_{k_0} \oplus \mathbf{1}_l,$$

$$\tilde{W}_4 = \mathbf{1}_{k_1} \otimes V_{k_2} \otimes \mathbf{1}_{k_0} \oplus \mathbf{1}_l, \tag{36}$$

where the SU(k_i) matrices U_{k_i} and V_{k_i} are defined as

$$(U_{k_i})_{ab} = e^{2\pi i(a-1)/k_i} \delta_{a,b}, \quad (V_{k_i})_{ab} = \delta_{a+1,b},$$

$$a, b = 1, \ldots, k_i,$$

$$(U_{l_1})_{cd} = e^{2\pi i(c-1)/l_1} \delta_{c,d}, \quad (V_{l_1})_{cd} = \delta_{c+1,d},$$

$$c, d = 1, \ldots, l_1, \tag{37}$$

so that they satisfy $U_{k_i} V_{k_i} = e^{-2\pi i/k_i} V_{k_i} U_{k_i}$ and $U_{l_1} V_{l_1} = e^{-2\pi i/l_1} V_{l_1} U_{l_1}$.

III. ADJOINT SECTIONS ON TWISTED BUNDLES

According to the correspondence between a compact space X and the C^* algebra $C(X)$ of continuous functions on X, the entire topological structure of X is encoded in the algebraic structure of $C(X)$. Continuous sections of a vector bundle over X can be identified with projective modules over the algebra $C(X)$. Thus, in order to find the topological structure of the twisted bundle constructed in the previous section, it is necessary to construct the sections of the bundle on $\mathbf{T}_\Theta^4$. Furthermore as noted in [6], if D_μ and D'_μ are two connections, then the difference $D_\mu - D'_\mu$ belongs to the algebra of endomorphisms of the $\mathbf{T}_\Theta^4$ module. Thus an arbitrary connection D_μ can be written as the sum of a constant curvature connection ∇_μ and an element of the endomorphism algebra:

$$D_\mu = \nabla_\mu + A_\mu.$$

From the relation (8), we see that A is also an adjoint section. Thus the algebra of adjoint sections can be regarded as the moduli space of constant curvature connections.

In this section we will analyze the structure of the adjoint sections on the twisted bundles on $\mathbf{T}^4$, closely following the method taken by Brace *et al.* [13] and Hofman and Verlinde [14]. According to the decomposition (17), we take the adjoint sections of U(N) as the form

$$\Phi(x^\mu) = \mathbf{1}_q \otimes \tilde{\Phi}(x^\mu). \tag{38}$$

The sections $\tilde{\Phi}$ on the twisted bundle of the adjoint representation of U(n) are n-dimensional matrices of functions on $\mathbf{T}_\Theta^4$ which is generated by Eq. (2), endomorphisms of the module, and satisfy the twisted boundary conditions

$$\tilde{\Phi}(x^\mu + 2\pi\delta_\nu^\mu) = \omega_\nu \tilde{\Phi}(x^\mu) \omega_\nu^{-1}. \tag{39}$$

Suppose that the general solution for the n-dimensional matrices $\tilde{\Phi}(x^\mu)$ has the following expansion:

$$\tilde{\Phi}(x^\mu) = \sum_{n_1 \cdots n_4 \in \mathbf{Z}} \tilde{\Phi}_{n_1 \cdots n_4} Z_1^{n_1} Z_2^{n_2} Z_3^{n_3} Z_4^{n_4}. \tag{40}$$

We also try to find the solutions of the following form:

$$Z_\mu = e^{i x_\nu X^\nu{}_\mu / n} \prod_{\alpha=1}^{6} \Gamma_\alpha^{s_\alpha^\mu}, \tag{41}$$

where s_α^μ ($\alpha = 1, \ldots, 6$) are integers and X is a matrix to be determined. Here, according to the basis taken in Eq. (36), we define the SU(n) matrices Γ_α as follows:

$$\Gamma_1 = U_{k_1} \otimes \mathbf{1}_{k_2} \otimes \mathbf{1}_{k_0} \oplus \mathbf{1}_l,$$

$$\Gamma_2 = V_{k_1} \otimes \mathbf{1}_{k_2} \otimes \mathbf{1}_{k_0} \oplus \mathbf{1}_l,$$

$$\Gamma_3 = \mathbf{1}_{k_1} \otimes U_{k_2} \otimes \mathbf{1}_{k_0} \oplus \mathbf{1}_l,$$

$$\Gamma_4 = \mathbf{1}_{k_1} \otimes V_{k_2} \otimes \mathbf{1}_{k_0} \oplus \mathbf{1}_l,$$

$$\Gamma_5 = \mathbf{1}_k \oplus U_{l_1} \otimes \mathbf{1}_{l_0},$$

$$\Gamma_6 = \mathbf{1}_k \oplus V_{l_1} \otimes \mathbf{1}_{l_0}. \tag{42}$$

One can directly check that the solution (40) is compatible with the boundary condition (39) if the matrix X is taken as

$$X = Q\mathbf{N},$$

where Q and the integer matrix $\mathbf{N}$ are defined as

$$Q^{-1} = \mathbf{1}_n - \tilde{P}\Theta, \tag{43}$$

$$\frac{\mathbf{N}^\mu{}_\nu}{n} = \frac{N^{(k)\mu}{}_\nu}{k} \mathbf{I}_k + \frac{N^{(l)\mu}{}_\nu}{l} \mathbf{I}_l, \tag{44}$$

and

$$N^{(k)\mu}{}_\nu = (-m_1 s_2^\mu, q_1 s_1^\mu, -m_2 s_4^\mu, q_2 s_3^\mu) \bmod k,$$

$$N^{(l)\mu}{}_\nu = (-m_3 s_6^\mu, l_0 s_5^\mu, l\delta_3^\mu, l\delta_4^\mu) \bmod l.$$

Let $\mathcal{F} = \mathbf{1}_q \otimes \tilde{\mathcal{F}}$. Using Eqs. (6), (13), and (16), the following identity can be derived:

$$Q^2 = \mathbf{1}_n + 2\pi\tilde{\mathcal{F}}\Theta = (\mathbf{1}_n - \mathbf{m}\Theta/n)^{-1}$$
$$= Q^{(k)2}\mathbf{I}_k + Q^{(l)2}\mathbf{I}_l, \tag{45}$$

where

$$Q^{(k)^2}=(1-m^{(k)}\Theta/k)^{-1},$$

$$Q^{(l)^2}=(1-m^{(l)}\Theta/l)^{-1}.$$

Using the identity, the constant curvature (6) can be rewritten as

$$\tilde{\mathcal{F}}=\frac{1}{2\pi}(n\mathbf{1}_n-\mathbf{m}\Theta)^{-1}\mathbf{m}=\frac{1}{2\pi}\mathbf{m}(n\mathbf{1}_n-\Theta\mathbf{m})^{-1}. \quad (46)$$

Then, using the relation [13]

$$\int_{\mathbf{T}^4}d^4x\,\mathrm{Tr}\Phi(x)=(2\pi)^4(k|\det Q^{(k)}|^{-1}\,\mathrm{Tr}_q\Phi^{(k)}_{0000}+l|\det Q^{(l)}|^{-1}\mathrm{Tr}_q\Phi^{(l)}_{0000}),$$

where $\Phi^{(k)}_{0000}$ and $\Phi^{(l)}_{0000}$ are the zero modes of the expansion (40), one can check that, as it should be, the zero-brane charge C in Eq. (31) is equal to

$$C=\frac{1}{8\pi^2}\int_{\mathbf{T}^4}d^4x\,\mathrm{Tr}\,\mathcal{F}\wedge\mathcal{F}. \quad (47)$$

Now let us calculate the commutation relations satisfied by Z_μ's, which are generators of the algebra of functions on a new torus, denoted by $\mathbf{T}^4_{\Theta'}$. From the explicit form (41), the commutation relation of the generators Z_μ's can be found as

$$Z_\mu Z_\nu=e^{2\pi i\Theta'_{\mu\nu}}Z_\nu Z_\mu, \quad (48)$$

where

$$\Theta'=n^{-2}\mathbf{N}^TQ^T\Theta Q\mathbf{N}-n^{-1}\mathbf{L}, \quad (49)$$

and the integer matrix $\mathbf{L}$ is defined by

$$\frac{\mathbf{L}_{\mu\nu}}{n}=\frac{L^{(k)}_{\mu\nu}}{k}\mathbf{I}_k+\frac{L^{(l)}_{\mu\nu}}{l}\mathbf{I}_l, \quad (50)$$

$$L^{(k)}_{\mu\nu}=q_1(s_1^\mu s_2^\nu-s_1^\nu s_2^\mu)+q_2(s_3^\mu s_4^\nu-s_3^\nu s_4^\mu)\ \mathrm{mod}\ k,$$

$$L^{(l)}_{\mu\nu}=l_0(s_5^\mu s_6^\nu-s_5^\nu s_6^\mu)\ \mathrm{mod}\ l.$$

The deformation parameters $\Theta'_{\mu\nu}$ on $\mathbf{T}^4_{\Theta'}$ given by Eq. (49) can be decomposed into a U(k) part and a U(l) part:

$$\Theta'_{\mu\nu}=\Theta'^{(k)}_{\mu\nu}\mathbf{I}_k+\Theta'^{(l)}_{\mu\nu}\mathbf{I}_l. \quad (51)$$

Here, $\Theta'^{(\iota)}$ ($\iota=k$ or l) can be rewritten as a fractional transformation [13]

$$\Theta'^{(\iota)}=\Lambda_0^{(\iota)}(\Theta)\equiv(A_\iota\Theta+B_\iota)(C_\iota\Theta+D_\iota)^{-1}, \quad (52)$$

where

$$\Lambda_0^{(\iota)}=\begin{pmatrix}A_\iota & B_\iota\\ C_\iota & D_\iota\end{pmatrix} \quad (53)$$

and the four-dimensional matrices are defined by

$$A_\iota=n_\iota^{-1}(N_\iota^T+L_\iota N_\iota^{-1}m_{0\,\iota}),\quad B_\iota=-L_\iota N_\iota^{-1},$$

$$C_\iota=-N_\iota^{-1}m_{0\,\iota},\quad D_\iota=n_\iota N_\iota^{-1}, \quad (54)$$

with the notation $n_k=k$, $n_l=l$. One can check that each $\Lambda_0^{(\iota)}$ is an element of SO(4,4|**Z**), which is a T-duality group of the type II string theory compactified on $\mathbf{T}^4$:

$$\Lambda_0^{(\iota)\,T}J\Lambda_0^{(\iota)}=J,$$

$$J=\begin{pmatrix}0 & \mathbf{1}_4\\ \mathbf{1}_4 & 0\end{pmatrix}. \quad (55)$$

For (4,2,2) or (4,2,2,0) backgrounds where the magnetic fluxes take the form of diagonal matrices breaking the gauge group to U(k)×U(l), Eq. (51) implies that the moduli space for the D-brane bound states is described by two noncommutative parameters $\Theta'^{(k)}$and $\Theta'^{(l)}$. Thus we expect that it takes the form $(\mathbf{T}^4_{\Theta'^{(k)}})^p/S_p\times(\mathbf{T}^4_{\Theta'^{(l)}})^q/S_q$ with p and q determined by ranks and fluxes [3,4].

IV. SO(4,4|Z) DUALITY AND MORITA EQUIVALENCE

In this section we analyze the bound states with nonzero D0-brane charge, $C\neq 0$, corresponding to the (4,2,2,0) system. For the given fluxes m_0 in Eqs. (34), we take the integral matrices $L^{(k)}$ and $L^{(l)}$ to be as close to the inverses of $m_0^{(k)}$ and $m_0^{(l)}$ as possible, respectively:

$$L^{(k)}=\begin{pmatrix}0 & -q_1b_1 & 0 & 0\\ q_1b_1 & 0 & 0 & 0\\ 0 & 0 & 0 & -q_2b_2\\ 0 & 0 & q_2b_2 & 0\end{pmatrix},$$

$$L^{(l)}=\begin{pmatrix}0 & -l_0b_3 & 0 & 0\\ l_0b_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad (56)$$

where b_1, b_2, and b_3 are integers such that $a_1k-b_1m_1=q_1$, $a_2k-b_2m_2=q_2$, and $a_3l-b_3m_3=l_0$, respectively. Here, we define $\tilde{m}_i=m_i/q_i$ and $\tilde{m}_3=m_3/l_0$, so that $a_ik_i-b_i\tilde{m}_i=1$ and $a_3l_1-b_3\tilde{m}_3=1$. Then the set of integers s_α^μ in Eq. (50) can be chosen to satisfy Eqs. (56):

$$s_1^\mu=(0,1,0,0),\ s_2^\mu=(b_1,0,0,0),$$

$$s_3^\mu=(0,0,0,1),\ s_4^\mu=(0,0,b_2,0),$$

$$s_5^\mu=(0,1,0,0),\ s_6^\mu=(b_3,0,0,0). \quad (57)$$

Also, for the above given set, the matrices $N^{(k)}$ and $N^{(l)}$ are given by

$$N^{(k)}=\begin{pmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_1 & 0 & 0 \\ 0 & 0 & q_2 & 0 \\ 0 & 0 & 0 & q_2 \end{pmatrix}, \quad N^{(l)}=l_0\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & l_1 & 0 \\ 0 & 0 & 0 & l_1 \end{pmatrix}. \tag{58}$$

From Eqs. (54), the SO(4,4|**Z**) transformations $\Lambda_0^{(\iota)}$ in Eq. (53) can be found as

$$\Lambda_0^{(k)}=\begin{pmatrix} a_1\mathbf{1}_2 & 0 & b_1\varepsilon & 0 \\ 0 & a_2\mathbf{1}_2 & 0 & b_2\varepsilon \\ -\tilde{m}_1\varepsilon & 0 & k_1\mathbf{1}_2 & 0 \\ 0 & -\tilde{m}_2\varepsilon & 0 & k_2\mathbf{1}_2 \end{pmatrix}, \tag{59}$$

$$\Lambda_0^{(l)}=\begin{pmatrix} a_3\mathbf{1}_2 & 0 & b_3\varepsilon & 0 \\ 0 & \mathbf{1}_2 & 0 & 0 \\ -\tilde{m}_3\varepsilon & 0 & l_1\mathbf{1}_2 & 0 \\ 0 & 0 & 0 & \mathbf{1}_2 \end{pmatrix}, \tag{60}$$

where $\mathbf{1}_2$ and ε are 2×2 identity and antisymmetric ($\varepsilon^{12}=-\varepsilon^{21}=1$) matrices, respectively. Since the general solution for an aritrary matrix m in Eq. (32) is obtained by the SL(4,**Z**) transformation R, the corresponding SO(4,4|**Z**) transformations Λ_ι can be given by the set (Rm_0R^T, RN, L) [13]. With Eq. (53), the SO(4,4|**Z**) transformation Λ_ι can be found as

$$\Lambda_\iota=\Lambda_0^{(\iota)}\begin{pmatrix} R^T & 0 \\ 0 & R^{-1} \end{pmatrix}. \tag{61}$$

Under the SO(4,4|**Z**) transformation (59) or (60), the rank, six fluxes, and instanton (eight components altogether) make a fundamental multiplet of the Weyl spinor representation of SO(4,4|**Z**) and this multiplet is mapped onto Morita equivalent tori by the action of SO(4,4|**Z**) [10,11,13,14]. For convenience, the explicit construction will be performed only for the SO(4,4|**Z**) matrix (59) since, for the matrix (60), it is essentially similar, and so we will drop the index (ι) from here.

Since the vector and spinor representations of SO(4,4|**Z**) are related by

$$S^{-1}\gamma_i S=\Lambda_i^j\gamma_j, \quad i,j=1,\ldots,8, \tag{62}$$

where the gamma matrices satisfy

$$\{\gamma_i,\gamma_j\}=2J_{ij}, \tag{63}$$

the spinor representation $S(\Lambda)$ corresponding to the transformation $\Lambda=\Lambda_0\Lambda(R)$ in Eq. (61) is a product of $S(\Lambda_0)$ corresponding to Λ_0 and $S(R)$ corresponding to $\Lambda(R)$:

$$S(\Lambda)=S(\Lambda_0)S(R). \tag{64}$$

On $\mathbf{T}^4$, the rank k, six fluxes $m_{\mu\nu}$, and U(k) instanton number, $C=Pf(m_{\mu\nu})/k$, make a fundamental multiplet of the Weyl spinor representation of SO(4,4|**Z**). We write such an eight-dimensional spinor ψ as

$$\psi=k|0\rangle+\frac{1}{2}m^{\mu\nu}a_\mu^\dagger a_\nu^\dagger|0\rangle+\frac{C}{4!}\epsilon^{\mu\nu\rho\sigma}a_\mu^\dagger a_\nu^\dagger a_\rho^\dagger a_\sigma^\dagger|0\rangle, \tag{65}$$

with the fermionic Fock basis defined in the Appendix. Explicitly we take the spinor basis ψ_α $(\alpha=1,\ldots,8)$ as follows:

$$\psi_\alpha=(k,m_{34},m_{42},m_{23},m_{12},m_{13},m_{14},C). \tag{66}$$

Using the result in the Appendix, $S(R)$ acts on this spinor as

$$\psi_0=S(R)\psi=(k,m_2,0,0,m_1,0,0,\tilde{C}), \tag{67}$$

where $\tilde{C}=m_1m_2/k$. Note that the instanton number $\tilde{C}=\tilde{m}_1\tilde{m}_2k/k_1k_2$ is integral since $k_1k_2|k$ [23]. Now one can check that, using the result in the Appendix, $S(\Lambda)$ acts on this spinor as

$$\begin{aligned}\psi'&=S(\Lambda_0)S(R)\psi\\&=S(\Lambda_0)\psi_0\\&=(k_0,0,0,0,0,0,0,0).\end{aligned} \tag{68}$$

Since the transformation $S(\Lambda)$ is an isomorphism between Fock spaces described by quantum number ψ, Eq. (68) implies that the quantum torus with quantum number ψ is (Morita) equivalent to that of ψ'. Similarly, the quantum tori described by the matrix (60) will be mapped to Morita-equivalent tori with quantum number $(l_0,0,0,0,0,0,0,0)$. Thus it implies that the moduli space of the (4,2,2,0) system as well as the (4,2,2) system in U(N) super Yang-Mills theory can be mapped to D0-brane moduli space and so it takes the form $(\mathbf{T}^4_{\Theta'(k)})^{qk_0}/S_{qk_0}\times(\mathbf{T}^4_{\Theta'(l)})^{ql_0}/S_{ql_0}$. This prediction is also consistent with the fact that the moduli space for the reducible connections takes the form of a product of smaller moduli spaces [4]. For a direct generalization, one can consider a generic constant background which breaks U(N) to Π_aU(k_a). Then, we expect that the moduli space of D-brane bound states in type II string theory takes the form $\Pi_a(\mathbf{T}^4_{\Theta'(a)})^{q_a}/S_{q_a}$.

V. CONCLUSION AND COMMENTS

We studied the modules of D-brane bound states on noncommutative $\mathbf{T}^4$ with non-Abelian constant backgrounds and examined the Morita equivalence between them. We found that the quantum tori with various D-brane charges are (Morita) equivalent to those of D0-branes. For a generic constant background which breaks U(N) to $\Pi_aU(k_a)$, it was shown that the moduli space of D-brane bound states in type II string theory takes the form $\Pi_a(\mathbf{T}^4_{\Theta'(a)})^{q_a}/S_{q_a}$.

The construction in this paper has only involved constant D-brane backgrounds. The noncommutative instantons on $\mathbf{T}^4$ may share some properties with noncommutative instantons on $\mathbf{R}^4$ [24] such as the resolution of the small instanton singularity. Unfortunately explicit construction of full instanton modules seems very hard, not due to the noncommutativeness of the geometry, but rather due to the non-Abelian properties of instanton connections. It would be very nice to give a construction also for these non-Abelian instantons since it was claimed in [25] that the moduli space of the twisted little string theories of k NS5-branes at the A_{q-1} singularity [26], compactified on $\mathbf{T}^3$, is equal to the moduli space of k U(q) instantons on a noncommuative $\mathbf{T}^4$.

Some interesting problems remain. The present construction may be generalized to the noncommutative **K3** and instanton solutions on it. The instanton configurations on noncommutative $\mathbf{T}^4$ or **K3** should be relevant to the microscopic structures of D1–D5 black holes with a B_{NSNS} field background, since the counting of microscopic BPS bound states can be related to the number of massless fields parametrizing the moduli space of the bound states [27]. It is also interesting since the type IIB string theory on $\mathrm{AdS}_3\times\mathbf{S}^3\times X$ with nonzero NS-NS B field along X, where X is **K3** or $\mathbf{T}^4$, corresponds to the conformal sigma model whose target space is the moduli space of instantons on the noncommutative X [28].

Another interesting problem is the deformation quantization of M(atrix) theory on noncommutative $\mathbf{T}^4$ [15]. Although the algebra of the functions on $\mathbf{T}^4$ is deformd by the so-called * product, the fuctions can be Fourier expanded in the usual way. In that case, the * product between Fourier-expanded functions will be relatively simple. We hope to address these problems soon.

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APPENDIX

To construct the spinor representation $S(\Lambda)$, we introduce fermionic operators $a_\mu^\dagger=\gamma_\mu/\sqrt{2}$ and $a_\mu=\gamma_{4+\mu}/\sqrt{2}$ satisfying anticommutation relations

$$\{a_\mu, a_\nu^\dagger\}=\delta_{\mu\nu},\ \{a_\mu^\dagger, a_\nu^\dagger\}=\{a_\mu, a_\nu\}=0,\ \mu,\nu=1,\dots,4. \tag{A1}$$

Since the SL(4,**Z**) transformation does not affect the rank and the instanton number and the SL(4,**Z**) is isomorphic to SO(3,3|**Z**), we expect, in the spinor basis (65), that the spinor representation $S(R)$ corresponding to $\Lambda(R)$ in Eq. (61) has the following form:

$$S(R)=\begin{pmatrix} 1 & 0 & 0\\ 0 & \mathrm{SO}(3,3|\mathbf{Z}) & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{A2}$$

Indeed, according to [11], the operator $\mathbf{\Lambda}(R)$ corresponding to $\Lambda(R)$ is given by

$$\mathbf{\Lambda}(R)=\exp(-a_\mu\lambda^{\mu\nu}a_\nu^\dagger),\quad (R)_{\mu\nu}=\exp(\lambda_{\mu\nu}), \tag{A3}$$

and then the spinor representation $S_{\alpha\beta}(R)$ is defined as

$$\mathbf{\Lambda}(R)|\beta\rangle=\sum_{\alpha=1}^{8}|\alpha\rangle S_{\alpha\beta}(R). \tag{A4}$$

Obviously, acting on the rank ($\beta=1$) and the instanton ($\beta=8$) basis, $S_{\alpha 1}(R)=S_{1\alpha}(R)=\delta_{\alpha 1}$ and $S_{\alpha 8}(R)=S_{8\alpha}(R)=\delta_{\alpha 8}$. After a little algebra, we can find the 6×6 matrix in Eq. (A2) denoted as $H(R)=H_3H_2H_1\in\mathrm{SO}(3,3|\mathbf{Z})$:

$$H_1=\begin{pmatrix} C_{12}^T & 0\\ 0 & C_{12}^{-1}\end{pmatrix},\ H_2=\begin{pmatrix}\mathbf{1}_3 & 0\\ \mathcal{A} & \mathbf{1}_3\end{pmatrix},\ H_3=\begin{pmatrix}\mathbf{1}_3 & \mathcal{B}\\ 0 & \mathbf{1}_3\end{pmatrix},$$

$$\mathcal{A}=\begin{pmatrix} 0 & -R_{14} & R_{13}\\ R_{14} & 0 & -R_{11}\\ -R_{13} & R_{11} & 0\end{pmatrix},\ \mathcal{B}=\begin{pmatrix} 0 & d_{14} & d_{13}\\ -d_{14} & 0 & 0\\ -d_{13} & 0 & 0\end{pmatrix}, \tag{A5}$$

where $C_{\mu\nu}$ is a 3×3 matrix formed by removing the μth row and νth column from the 4×4 matrix R, $d_{\mu\nu}=\det(C_{\mu\nu})$, and we normalized the matrix C_{12} to be SL(3,**Z**) by absorbing determinant factor in the above definition.

Next we will construct the spinor representation $S(\Lambda_0^{(k)})$ corresponding to $\Lambda_0^{(k)}$ in Eq. (59). Let us make a blockwise Gauss decomposition of $\Lambda_0^{(k)}$:

$$\begin{aligned}\Lambda_0^{(k)}&=\begin{pmatrix}\mathbf{1}_4 & 0\\ \mathcal{C} & \mathbf{1}_4\end{pmatrix}\cdot\begin{pmatrix} G & 0\\ 0 & G^{-1}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{1}_4 & \mathcal{D}\\ 0 & \mathbf{1}_4\end{pmatrix}\\ &=\Lambda_{\mathcal{C}}\cdot\Lambda_G\cdot\Lambda_{\mathcal{D}},\end{aligned} \tag{A6}$$

where the antisymmetric matrices $\mathcal{C}$, $\mathcal{D}$ and a symmetric matrix G are given by

$$\mathcal{C}=-\begin{pmatrix}\frac{m_1}{a_1}\varepsilon & 0\\ 0 & \frac{m_2}{a_2}\varepsilon\end{pmatrix},\ \mathcal{D}=\begin{pmatrix}\frac{b_1}{a_1}\varepsilon & 0\\ 0 & \frac{b_2}{a_2}\varepsilon\end{pmatrix},$$

$$G=\begin{pmatrix} a_1\mathbf{1}_2 & 0\\ 0 & a_2\mathbf{1}_2\end{pmatrix}, \tag{A7}$$

and ε is an antisymmetric 2×2 matrix. Then the corresponding spinor operator $\mathbf{\Lambda}_0^{(k)}$ will be given by

$$\mathbf{\Lambda}_0^{(k)}=\exp\left(\frac{1}{2}\mathcal{C}^{\mu\nu}a_\mu^\dagger a_\nu^\dagger\right)\exp(-h^{\mu\nu}a_\mu^\dagger a_\nu)\exp\left(\frac{1}{2}\mathcal{D}^{\mu\nu}a_\mu a_\nu\right), \tag{A8}$$

where $(G)_{\mu\nu}=\exp(h_{\mu\nu})$. Thus the representation $S(\Lambda_0^{(k)})$ can be obtained by a product of each spinor representation:

$$S(\Lambda_0^{(k)})=S(\Lambda_{\mathcal{C}})\cdot S(\Lambda_G)\cdot S(\Lambda_{\mathcal{D}}), \tag{A9}$$

where

$$S(\Lambda_0^{(k)})=\begin{pmatrix} a_1a_2 & -a_1b_2 & 0 & 0 & -a_2b_1 & 0 & 0 & b_1b_2\\ -a_1\tilde{m}_2 & a_1k_2 & 0 & 0 & b_1\tilde{m}_2 & 0 & 0 & -b_1k_2\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ -a_2\tilde{m}_1 & b_2\tilde{m}_1 & 0 & 0 & a_2k_1 & 0 & 0 & -b_2k_1\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ \tilde{m}_1\tilde{m}_2 & -\tilde{m}_1k_2 & 0 & 0 & -\tilde{m}_2k_1 & 0 & 0 & k_1k_2 \end{pmatrix}. \tag{A10}$$

Similarly,

$$S(\Lambda_0^{(l)})=\begin{pmatrix} a_3 & 0 & 0 & 0 & -b_3 & 0 & 0 & 0\\ 0 & a_3 & 0 & 0 & 0 & 0 & 0 & -b_3\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ -\tilde{m}_3 & 0 & 0 & 0 & l_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & -\tilde{m}_3 & 0 & 0 & 0 & 0 & 0 & l_1 \end{pmatrix}. \tag{A11}$$

Here we used the definition (62) in order to drop the global factors such as $1/a_1a_2$ and $1/a_3$.

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