Gravitational Berry's quantum phase

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We investigate the behavior of a scalar quantum particle in a class of space-times generated by defects. In these backgrounds the wave functions associated with the particle acquires a phase (Berry's quantum phase) when transported along a closed path surrounding the defect. We also examine this problem in the framework of Kaluza-Klein theory.

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I. INTRODUCTION

Topological defects are predicted in some unified theories of particle interactions. They may have been formed at phase transitions in the very early history of the Universe [1]. Examples of such topological defects are the domain wall [2], the cosmic string [2,3], and the global monopole [4]. In particular, cosmic strings provide a bridge between the physics in microscopic and macroscopic scales.

The appearance of topological phases in the quantum dynamics of a single particle moving freely in multiply connected space-times has been studied in a variety of physical systems. The prototype of this phase is the electromagnetic Aharonov-Bohm one [5], which appears as a phase factor in the wave function of an electron which moves around a magnetic flux line. The gravitational analogue of this effect has also been investigated and discussed [6].

The quantum phase holonomy [7] has purely geometrical origin and plays an important and fundamental role in various areas of physics. In the early 1980s Berry discovered [8] that a slowly evolving (adiabatic) quantum system retains information of its evolution when returned to its original physical state. This information corresponds to what is termed Berry's phase. The appearance of this phase has been generalized to the case of nonadiabatic [9] evolution of a quantum system. In any case, the phase depends only on the geometrical nature of the pathway along which the system evolves. This phenomenon has been investigated in several areas of physics. There are various experiments which have been reported concerning the appearance of the adiabatic and nonadiabatic geometric phases, including the observations on photons [10], neutrons [11], and nuclear spins [12]. The manifestations of Berry's quantum phase in high-energy electron diffraction in a deformed crystal, which contains a screw dislocation, has been observed [13]. In this case the equation that governs high-energy electron diffraction is shown to be equivalent to a Schrödinger equation with a time-dependent Hamiltonian.

Some works [14] concerning the investigation of Berry's phase in the context of gravitation and cosmology were done in recent years. In particular, Cai and Papini [15] obtained a covariant generalization of the Berry's phase and applied this result to problems involving weak gravitational fields. Recently Corichi and Pierri [16] studied the behavior of a quantum scalar particle in a class of stationary space-times and investigated the phase acquired by the particle when transported along a closed path surrounding a rotating cosmic string.

The aim of this paper is to obtain Berry's phase for a scalar particle in the space-times of a chiral cosmic string and a multiple chiral cosmic string in the context of Einstein theory and also in Kaluza-Klein theory and emphasize the role played by these topological defects in the geometric phase.

II. GEOMETRIC PHASE IN THE SPACE-TIME OF A CHIRAL COSMIC STRING

In this section we proceed in analogy with the treatment of Corichi and Pierri [16] in order to determine the geometric phase associated with a scalar quantum test particle induced by a chiral cosmic string. The line element which describes this space-time is given by [17]

$$ds^2=(dt+4J^t d\phi)^2-d\rho^2-\alpha^2\rho^2 d\phi-(dz+4J^z d\phi)^2. \tag{1}$$

If $J^t=0$ and $J^z=0$ the metric (1) represents a cosmic string. For $J^z=0$ this metric represents the rotating cosmic string with angular momentum J^t. For $J^t=0$ the metric describes a cosmic dislocation. This metric is locally flat as we can see performing a simple redefinition of the coordinates t and z. But this space-time is not globally flat and as a consequence it presents some surprising results connected with its global structure. As an example, we can mention the holonomies associated with the parallel transport of vectors and spinors along curves located in the plane perpendicular to the string and surrounding it. In this case the holonomies are nontrivial [18] and depend on the angular momentum J^t, on the angular deficit, given by α, and on the Burgers vector which is proportional to J^z.

The behavior of a scalar quantum particle is determined by the covariant Klein-Gordon equation

$$(\Box+M^2)\Psi=0, \tag{2}$$

where M is the mass of the particle and $\hbar=1$ and $c=1$ were chosen.

Now, let us consider the Klein-Gordon equation in the metric (1) which is given by

$$\left\{ \partial_t^2 - \frac{1}{\rho}\partial_\rho(\rho\partial_\rho) - \frac{1}{\alpha^2\rho^2}[(4J^t\partial_t - \partial_\phi)^2 + (4J^z\partial_z - \partial_\phi)^2 + 32J^tJ^z\partial_t\partial_z + \alpha^2\rho^2(\partial_z^2 - M^2)] \right\}\Psi(t,\rho,\phi,z) = 0. \tag{3}$$

The background described by Eq. (1) is time independent and symmetric under z translations, therefore the solution of Eq. (3) can be written as

$$\Psi_n(t,\rho,\phi,z) = \exp(-iE_nt)\exp(ik_nz)\psi_n(\rho,\phi), \tag{4}$$

where E_n are the eigenvalues of energy and k_n are the wave vectors in the z direction. Using the Dirac phase factor method we can write $\psi_n(\rho,\phi)$ as

$$\psi_n(\rho,\phi) = \exp\left(-4i\int_{\phi_0}^{\phi}(E_nJ^t - k_nJ^z)d\phi \right)\psi_0(\rho,\phi), \tag{5}$$

with $\psi_0(\rho,\phi)$ being a solution of the equation

$$\left[\frac{1}{\rho}\partial_\rho(\rho\partial_\rho) + \frac{1}{\alpha^2\rho^2}\partial_\phi^2 - (M^2 - E_n^2 - k_n^2) \right]\psi_0(t,\rho,\phi,z) = 0, \tag{6}$$

which comes out from the Klein-Gordon equation in the metric (1) for $J^t=0$ and $J^z=0$. In this way we find the solution of a more complicated equation [Eq. (3)] from the solution of a simpler one [Eq. (6)].

Now we will investigate Berry's phase in the space-time of a chiral cosmic string. For this case the geometric phase angle does depend on the spectral label just as in the rotating cosmic string case [19]. Therefore, each different eigenmode labeled by n acquires a different geometric phase, and as a consequence the appropriate treatment of this problem is obtained using the non-Abelian generalization [19] of the Berry's phase. In order to compute this phase let us confine the quantum system to a perfectly reflecting box such that the wave packet is nonzero only in the interior of the box and is given by a superposition of different eigenfunctions. The vector that localizes the box in relation to the defect is called $\vec{R}$. This vector is oriented from the origin of the coordinate system (localized on the defect) to the center of the box. Call R_i the components of $\vec{R}$, given by $R_i=(R_0,\phi_0,z_0)$ and such that $R_0>4J^t/\alpha$. This condition imposed on R_0 allows us to get rid of two problems: the multivaluedness of the wave function and the existence of closed timelike curves.

From Eqs. (4) and (5) we conclude that when $J^t=J^z=0$, the wave function has the form $\psi_n(\vec{x}-\vec{R})$, where $\vec{x}$ localizes the particle relative to the center of the box. If we consider $J^t\neq 0$ and $J^z\neq 0$, then the wave function is sensitive to these parameters and can be obtained by the Dirac phase factor given by Eq. (5), inside the box. Let the box be transported around a closed curve C threaded by the defect. Since the space-time is axisymmetric we can transport the box along the rotational Killing vector field $R^a=(\partial/\partial\phi)^a$.

Due to degeneracy of the energy eigenvalues, in order to compute Berry's geometric phase, it is necessary to use the non-Abelian version of the corresponding connection [19] given by

$$A_n^{IJ} = \langle \psi_n^I(\vec{x}-\vec{R})|\nabla_R\psi_n^J(\vec{x}-\vec{R})\rangle, \tag{7}$$

where I and J stand for possible degeneracy labels.

The inner product in Eq. (7) may be evaluated using by using the Dirac phase factor as follows:

$$\langle \psi^I(x_i-R_i)|\nabla_R\psi_n^J(x_i-R_i)\rangle = i\oint_\Sigma dS\,\psi_n^{*I}(x_i-R_i)[4(k_nJ^z - E_nJ^t)\psi_n^J(x_i-R_i) + \nabla_R\psi_n^J(x_i-R_i)]. \tag{8}$$

Computing the integrand we get the result

$$\langle \psi^I(x_i-R_i)|\nabla_R\psi_n^J(x_i-R_i)\rangle = 4i(E_nJ^t - k_nJ^z)\delta_{IJ}. \tag{9}$$

Berry's phase can be obtained from expression (9) and is given by

$$\gamma_n(C) = 8\pi(E_nJ^t - k_nJ^z), \tag{10}$$

where the labels I,J, and δ_{IJ} have been omitted. This reproduces the results of Corrichi and Pierri [16] and Mostafazadeh [19] in the case of a spinning cosmic string. As pointed out in [16], the effect can be observed by an interference of the wave function associated with the particle in the transported box and another corresponding to a particle in a box that followed the orbits along the timelike Killing vector field t^a.

III. BERRY'S PHASE IN THE SPACE-TIME OF MULTIPLE CHIRAL STRINGS

In a recent paper Gal'tsov and Letelier [17] obtained the solution for multiple chiral strings, whose line element in Cartesian coordinate is given by

$$ds^2 = \left[dt - \sum_{i=1}^{N} A_i(W_i^1dy - W_i^2dx) \right]^2 - e^{-4V}(dx^2+dy^2) - \left[dz - \sum_{i=1}^{N} B_i(W_i^1dy - W_i^2dx) \right]^2, \tag{11}$$

where

$$A_i = 4J_i^t, \quad B_i = 4J_i^z, \tag{12}$$

with J_i^t and J_i^z corresponding to the angular momentum and torsion of the ith chiral cone, respectively, and W_i^1 and W_i^2 are given by

$$W_i^1=\frac{x-x_i}{|\vec{\rho}-\vec{\rho_i}|^2},\quad W_i^2=\frac{y-y_i}{|\vec{\rho}-\vec{\rho_i}|^2}. \tag{13}$$

In this section we compute the Berry's quantum phase associated with a scalar particle in the space-time of N parallel chiral cosmic strings. In this case the direct calculation of the Dirac phase from the solution of the Klein-Gordon equation is complicated. In order to avoid this difficult we shall adopt an inductive derivation based on the fact that the phase factor acquired by a vector when parallel transported in the space-time corresponding to a multiple chiral cosmic string is affected only by the chiral strings inside the curves along which the vector is parallel transported [18]. Then, let us calculate the Dirac phase factor for two chiral cones, for three, and successively for N chiral cones. We consider firstly two chiral cones, the one localized at ρ_1 and the other at ρ_2. We perform the transport of the box that contains the particle along a closed loop C_1 around the chiral cone. In this case, the Dirac phase factor is identical the one given by Eq. (5) and can be written as

$$\psi_n^1(\rho,\phi)=\exp\left[-4i(E_nJ_1^t-k_nJ_1^z)\int_{\phi_0}^{\phi}d\phi\right]\psi_0(\rho,\phi). \tag{14}$$

Now, we transport the state $\psi_n^1(\rho,\phi)$ around the cone, localized at ρ_2, along the loop C_2, which results in

$$\psi_n^2(\rho,\phi)=\exp\left[-4i(J_2^tE_n-k_nJ_2^z)\int_{\phi_0}^{\phi}d\phi\,\psi_n^1(\rho,\phi)\right]. \tag{15}$$

Substituting Eq. (14) into Eq. (15), we obtain the following result:

$$\psi_n^2(\rho,\phi)=\exp\left\{-4i[(J_2^t+J_1^t)E_n - k_n(J_2^z+J_1^z)]\int_{\phi_0}^{\phi}d\phi\right\}\psi_0(\rho,\phi). \tag{16}$$

The generalization of this result for N chiral cones localized at $\rho_1,\rho_2,\rho_3,\ldots,\rho_N$ is given by

$$\psi_n^{1,2,\ldots,N}(\rho,\phi)=\exp\left\{-4i\left[\sum_{j=1}^{N}(J_j^tE_n-k_nJ_j^z)\right]\times\int_{\phi_0}^{\phi}d\phi\right\}\psi_0(\rho,\phi). \tag{17}$$

Therefore, we can compute Berry's connection using the wave function given in Eq. (17) which results in

$$A_n^{IJ}=-4i\sum_{i=1}^{N}(J_i^tE_n-k_nJ_i^z)dR^2\delta_{IJ}, \tag{18}$$

where dR^2 is the polar angle associated with the center of the box. In this way, Berry's phase can be obtained from Eq. (18) and is given by

$$\gamma_n(C)=8\pi\sum_{i=1}^{N}(J_i^tE_n-k_nJ_i^z), \tag{19}$$

where we have omitted the labels I,J corresponding to different eigenmodes, for convenience. This result gives the phase of one particle in the box that performs one circuit C around multiple chiral cones.

IV. BERRY'S PHASE IN KALUZA-KLEIN THEORY

In this section we consider Berry's quantum phase of a scalar quantum particle induced by a space-time background in the framework of a Kaluza-Klein theory. We analyze the behavior of a scalar particle in the space-time of a magnetic chiral flux string [20] in Kaluza-Klein theory, whose line element is given by

$$ds^2=(dt+4J^td\phi)^2-d\rho^2-\alpha^2\rho^2d\phi-(dz+4J^zd\phi)^2-\left(dx^5+\frac{\Phi}{2\pi}\right)^2, \tag{20}$$

where Φ is the magnetic flux. The dynamics of a scalar quantum particle in this background is described by the following equation:

$$\left\{\partial_t^2-\frac{1}{\rho}\partial_\rho(\rho\partial_\rho)-\frac{1}{\alpha^2\rho^2}\left[\partial_\phi-4J^t\partial_t-4J^z\partial_z-\frac{\Phi}{2\pi}\partial_{x^5}\right]^2-M^2\right\}\Psi(t,\rho,\phi,z,x^5)=0. \tag{21}$$

From this point we proceed in analogy with the previous sections. The solution of this equation can be written as

$$\Psi(t,\rho,\phi,z,x^5)=e^{-iE_nt}e^{ik_nz}e^{iQx^5}\psi(\rho,\phi). \tag{22}$$

Substituting Eq. (22) into Eq. (21) we get

$$\left\{\frac{1}{\rho}\partial_\rho(\rho\partial_\rho)+\frac{1}{\alpha\rho^2}\left[\partial_\phi-4iE_nJ^t-4ik_nJ^z-\frac{\Phi Q}{2\pi}\right]^2+E_n^2-k_n^2-Q^2+M^2\right\}\varphi(\rho,\phi)=0. \tag{23}$$

For this case, the Dirac phase factor is given by

$$\psi_n(\rho,\phi)=\exp\left\{\left[-4i(E_nJ^t-k_nJ^z)-iQ\frac{\Phi}{2\pi}\right]\int_{\phi_0}^{\phi}d\phi\right\}\psi_0(\rho,\phi). \tag{24}$$

Consider that the quantum particle is in a box located at a distance R_i from the string. Then the associated connection to Berry's quantum phase is given by

$$A_n^{IJ}=\left[-4i(E_nJ^t-k_nJ^z)-i\frac{Q\Phi}{2\pi}\right]dR^2\delta_{IJ}. \tag{25}$$

Therefore, the geometric Berry's phase for this problem is

$$\gamma_n(C)=8\pi[E_nJ^t-k_nJ^z]+Q\Phi. \tag{26}$$

Note that for $J^t=J^z=0$ we obtain Berry's quantum phase corresponding to the electromagnetic Aharonov-Bohm effect [8]; for $J^z=0$ and $\Phi=0$ we get the gravitational geometric phase of Corrichi and Pierri [16]. For $\Phi=0$, we reobtain our previous result.

V. BERRY'S PHASE IN MULTIPLE MAGNETIC STRING SPACE-TIME IN KALUZA-KLEIN THEORY

Now, let us generalize the result of the previous section for the space-time generated by multiple magnetic chiral cosmic strings in Kaluza-Klein theory. Using the solutions of Azreg-Ainou and Clement [20] for the chiral cone in Kaluza-Klein theory we do the generalization for multiple chiral cones in Kaluza-Klein. Firstly, let us write Eq. (20) in a Cartesian coordinate system $x=\rho\cos\varphi, y=\rho\sin\varphi$, which reads

$$ds^2=\left(dt-4J^t\frac{xdy-ydx}{\rho^2}\right)^2-e^{-4V}(dx^2+dy^2)-\left(dz+4J^z\frac{xdy-ydx}{\rho^2}\right)^2-\left(dx^5-\frac{\Phi}{2\pi}\frac{xdy-ydx}{\rho^2}\right)^2, \tag{27}$$

where $V=2\mu\ln\rho$.

In this context, the generalization of the chiral cone to a multiple chiral cone can be obtained by introducing the parameters μ_i, J_i^t, J_i^z, and Φ_i with $i=1,2,\ldots,N$ associated with each chiral string located at the points $\vec{\rho}=\vec{\rho_i}$ of the plane $z=0$. The resulting metric has the form given by Eq. (27) with the following interchanges:

$$J^t\frac{xdy-ydx}{\rho^2}\rightarrow\sum_{i=1}^{N}J_i^t\frac{(x-x_i)dy-(y-y_i)dx}{|\vec{\rho}-\vec{\rho_i}|^2}, \tag{28}$$

$$J^z\frac{xdy-ydx}{\rho^2}\rightarrow\sum_{i=1}^{N}J_i^z\frac{(x-x_i)dy-(y-y_i)dx}{|\vec{\rho}-\vec{\rho_i}|^2}, \tag{29}$$

$$\frac{\Phi}{2\pi}\frac{xdy-ydx}{\rho^2}\rightarrow\sum_{i=1}^{N}\frac{\Phi_i}{2\pi}\frac{(x-x_i)dy-(y-y_i)dx}{|\vec{\rho}-\vec{\rho_i}|^2}, \tag{30}$$

$$V=2\mu\ln\rho\rightarrow V=\sum_{i=1}^{N}\mu_i\ln[\rho^2-2\rho\rho_i\cos(\varphi-\varphi_i)+\rho_i^2]. \tag{31}$$

Therefore, the line element for the space-time generated by N multiple chiral magnetic flux strings can be written as

$$ds^2=\left[dt-\sum_{i=1}^{N}A_i(W_i^1dy-W_i^2dx)\right]^2-e^{-4V}(dx^2-dy^2)-\left[dz-\sum_{i=1}^{N}B_i(W_i^1dy-W_i^2dx)\right]^2-\left[dx^5-\sum_{i=1}^{N}C_i(W_i^1dy-W_i^2dx)\right]^2, \tag{32}$$

where

$$A_i=4J_i^t,\quad B_i=4J_iz,\quad C_i=\frac{\Phi_i}{2\pi} \tag{33}$$

and

$$W_i^1=\frac{x-x_i}{|\vec{\rho}-\vec{\rho_i}|^2},\quad W_i^2=\frac{y-y_i}{|\vec{\rho}-\vec{\rho_i}|^2}. \tag{34}$$

Let us return to the same procedure used in the case of multiple chiral space-time and get the Dirac phase for one, two, etc., N strings. For one magnetic flux string the Dirac phase method gives us the following expression:

$$\psi_n^1(\rho,\phi)=\exp\left\{\left[-4i(E_nJ_1^t-k_nJ_1^z)-iQ\frac{\Phi_1}{2\pi}\right]\int_{\phi_0}^{\phi}d\phi\right\}\psi_0(\rho,\phi), \tag{35}$$

Now, we transport the state $\psi_n^1(\rho,\phi)$ around the second string, localized at ρ_2, along the curve C_2. The procedure results in

$$\psi_n^2(\rho,\phi)=\exp\left\{\left[-4i(J_2^tE_n-k_nJ_2^z)-i\frac{Q\Phi_2}{2\pi}\right]\int_{\phi_0}^{\phi}d\phi\right\}\psi^1(\rho,\phi). \tag{36}$$

Substituting Eq. (36) into Eq. (34), we obtain the result

$$\psi_n^2(\rho,\phi)=\exp\left\{-4i\left[(J_2^t+J_1^t)E_n-k_n(J_2^z+J_1^z)+\frac{iQ(\Phi_1+\Phi_2)}{2\pi}\right]\int_{\phi_0}^{\phi}d\phi\right\}\psi_0(\rho,\phi). \quad (37)$$

The generalization of this result for N magnetic chiral strings localized at $\rho_1,\rho_2,\rho_3,\ldots,\rho_N$ follows the analogy with the previous case and results in

$$\psi_n^{1,2,\ldots,N}(\rho,\phi)=\exp\left\{\left[-4i\left(\sum_{i=1}^{N}(J_i^tE_n-k_nJ_i^z)\right)+\frac{iQ}{2\pi}\sum_{i=1}^{N}\Phi_i\right]\int_{\phi_0}^{\phi}d\phi\right\}\psi_0(\rho,\phi). \quad (38)$$

Using the results given by Eq. (38) and the same procedure of the previous section we get

$$\gamma_n(C)=\sum_{i}^{N}[8\pi(J_i^tE_n-k_nJ_i^z)+Q\Phi_i], \quad (39)$$

which is the geometric Berry's quantum phase for a scalar particle in the space-time generated by N magnetic flux strings in the framework of Kaluza-Klein theory. Note that for convenience we have dropped the labels I,J which are connected with the degeneracy of the energy.

VI. CONCLUSION

We have found the geometric phase for a class of space-times corresponding to topological defects using the Dirac phase factor [16] method. In all the cases the geometric phase depends on the spectral labels due to degeneracy of the energy. Therefore, in order to do the appropriate treatment of this problem we used the non-Abelian generalization of Berry's phase [19]. Following in this way we found the geometric phase of a scalar field induced by a chiral cosmic string and multiple chiral cosmic strings. In this last case we used the fact that from the global point of view the strings localized outside the curves do not affect the holonomy [18] and as a consequence do not affect the Berry's phase. Using the same approach and considerations of the previous cases we extended our calculation to the framework of the Kaluza-Klein theory obtaining similar results.

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