't Hooft tensors as Kalb-Ramond fields of generalized monopoles in all odd dimensions: $d=3$ and $d=5$

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(Received 20 January 2000; published 7 July 2000)

Rank $d-1$ antisymmetric tensor fields in d Euclidean dimensions, known as Kalb-Ramond fields, can describe monopole-like solutions. In $d=3$ dimensions this Kalb-Ramond monopole is the (singular) Dirac monopole, which in turn can be described by the (regular) 't Hooft-Polyakov monopole, via the 't Hooft tensor construction. This construction is extended to arbitrary odd dimensions by performing the $d=5$ case explicitly, exploiting the (regular) ''monopoles'' of generalized Georgi-Glashow models and identifying their 't Hooft tensors as the Kalb-Ramond fields. The relevant ''magnetic charges'' are expressed as topological invariants.

PACS number(s): 14.80.Hv, 02.40.−k

I. INTRODUCTION

In *three* Euclidean space dimensions, a Dirac [1] magnetic monopole is a singular static solution of $U(1)$ Maxwell electromagnetism with nonvanishing magnetic flux over a sphere surrounding the monopole. Dirac's original description of such an object involved a string-like singularity, extending from the location of the monopole to infinity. An important step in the description of the Dirac monopole which avoids the ''Dirac string'' involves Wu-Yang [2] fields on overlapping coordinate patches (which in mathematical terms correspond to a nontrivial $U(1)$ fiber bundle over S^2), but the magnetic field strength still exhibits a singularity at the location of the monopole.

't Hooft [3] showed that the soliton solution of the $d=3$ Georgi-Glashow (GG) model, which is known as the 't Hooft-Polyakov monopole, can be interpreted as a regular realization of the Dirac magnetic monopole, identifying the unbroken $U(1)$ subgroup of the soliton with Maxwell magnetism and the topological soliton charge with magnetic charge. This is done using the 't Hooft tensor which is identified with the magnetic $U(1)$ curvature of the 't Hooft–Polyakov [3,4] monopole outside the core. In a regular gauge, the 't Hooft tensor supports a description of the topology and hence the magnetic charge in terms of the Higgs field only [5], whereas in singular gauge, the 't Hooft tensor restores the Wu-Yang description of the Dirac monopole.

Dirac's monopole construction in Maxwell theory can be extended to theories involving higher rank antisymmetric field strength tensors generalizing the usual rank two Maxwell tensor. Such (Abelian) antisymmetric tensor fields were considered long ago [6] and in the context of duality transformations in Abelian spin systems on the lattice were first introduced by Wegner [7,8]. They were later introduced by many authors [9–12] in the context of string theory [9]. More recently such fields have played a major role also in supersymmetric field theories of gravity and strings theories [13,14].

In the present work we restrict ourselves to rank $d-1$ antisymmetric tensor field strengths in d Euclidean dimensions, called *Kalb-Ramond* (KR) fields. The KR theories constructed from these (Abelian) field strengths afford a generalization of the Dirac monopole construction, and they support monopole-like configurations which were first discussed by Savit [15], Orland [16] and Pearson [17], who studied the phase structure in these theories. These higher dimensional KR monopoles were also studied by Nepomechie [18] in the continuum, where a Wu-Yang type construction allows the introduction of KR potentials for the monopoles without string singularities. The Dirac quantization condition involving this generalized magnetic charge and the generalized electric charge pertaining to the corresponding (higher form) electric field, was also established [18,19].

We note that Orland's work [16], and subsequent more recent works by Quevedo and Trugenberger [20] and by Ellwanger [21], are based on the work of Polyakov [22] on compact QED in $2+1$ dimensions, giving a particular generalization of the latter to higher dimensions. It is interesting that one of our main motivations in the present work was another generalization of Polyakov's work [22] in $2+1$ dimensions, namely the construction of a dilute Coulomb gas of instantons, to the case of $3+1$ dimensions employing a generalized non-Abelian Higgs model. As it turns out the construction of a dilute gas in this way is only possible in *odd* dimensions, as will be shown below, and discussed further in Sec. V.

It is our intention in the present work, to supply regular solitonic realizations of KR monopoles and to generalize the construction of the 't Hooft tensor to higher dimensions. This is done concretely for dimension $d=5$, in a sufficiently gen-

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eral framework which points clearly to the systematic generalization to *all odd dimensions*.

To realize our objective, there are two main ingredients needed. The first is a natural generalization of the GG model in higher dimensions, which supports solitonic solutions stabilised by a topological charge. The second is the construction of a generalization of the 't Hooft electromagnetic tensor, which describes the Abelian field strength to be identified with the higher dimensional KR field. The first of these, namely the generalized GG (GGG) models, are readily obtained by subjecting members of the hierarchy of Yang-Mills (YM) models in $4p$ dimensions [23] to dimensional descent. Examples of the resulting Higgs models in various dimensions can be found in Refs. [24]. The second, namely the definition of 't Hooft tensors in all odd dimensions, is entirely new. We have done this concretely in $d=5$, and shown that such a 't Hooft tensor can be implemented in *odd* dimensions only. The construction relies on the proper definition of the CP densites that present the lower bounds on the relevant GGG models. These CP densities are obtained by subjecting the original Chern-Pontryagin (CP) densities in $4p$ dimensions, to dimensional descent [25].

In Sec. II, we consider the gauge and Higgs fields, along with the gauge group, its representation and the Higgs multiplets, that we need if we are to satisfy the requisite topological properties necessary for the desired constructions. This includes also a description of the *regular* and *singular* (Dirac) gauges. Section III is divided into two subsections. In the first we discusses the candidates for the generalized GG models that can be employed, while in the second, the corresponding CP charge densities descended from the CP densities in higher dimensions are discussed. Section IV, in which we present the main results, is divided into three subsections. In the first, we prove that the CP charge equals the Higgs field winding number in regular gauge in all dimensions, and that the CP charge can be interpreted as magnetic KR [18] charge in odd dimensions. In the second subsection, we use the Dirac gauge to construct Wu-Yang type KR potentials for the CS forms and interpret the solitonic solutions to the odd dimensonal models as regular realizations of KR monopoles. This motivates the nomenclature ''GGG monopoles'' for these regular solutions. In the third subsection we construct the generalized 't Hooft tensors which we identify with the KR field strengths, and show in which sense these 't Hooft tensors contain the (odd dimensional) results of the two previous subsections. Section V is devoted to a discussion of our results. In Appendix A we give a brief description of KR fields and KR monopole constructions in all dimensions, that is relevant to the present work. In Appendix B we list the action or energy functionals of the four simplest GGG models, as well as the relevant CP densities and their Chern-Simons (CS) forms.

II. GAUGE GROUPS AND TOPOLOGY

Our primary considerations in this work are the topological properties of gauged Higgs systems, and their relation to KR monopoles in odd dimensions. To this end we set up the topological framework by selecting the required gauge groups, their representations, as well as the representations of the Higgs fields. Irrespective of the detailed dynamics, which is discussed in the next section, we can impose the finite action or energy conditions which are expected to lead to topologically nontrivial configurations, and which yield the asymptotic fields. We will discuss these asymptotic fields both in the *regular* and the *singular* (Dirac) gauges.

In d Euclidean dimensions, we consider a d vector multiplet Higgs field ϕ^σ which we write in isovector representation

$$\Phi=\phi^\sigma\varepsilon_{d+1}\gamma_\sigma \tag{2.1}$$

where $\{\gamma_1,\ldots,\gamma_d\}$ are the Euclidean gamma matrices in d dimensions, and ε_{d+1} is an ''anti-Hermitian factor.'' In even dimensions, $d=2M$, there exists a chiral matrix $\gamma_{2M+1}=\gamma_{d+1}=i\gamma_1\cdots\gamma_{2M}$ which is used as anti-Hermitean factor, $\varepsilon_{2M+1}=\gamma_{2M+1}$, whereas no chiral matrix exists in odd dimensions $d=2M+1$, and the imaginary unit i is used instead, $\varepsilon_{2M+2}=i$, hence

$$\varepsilon_{d+1}:=\begin{cases}\gamma_{d+1} & (d \text{ even}),\\ i & (d \text{ odd}).\end{cases} \tag{2.2}$$

The Higgs fields under consideration are gauged with $SO(d)$ gauge potentials $\mathbf{A}$ taking values in the $so(d)$ algebra with anti-Hermitian generators $\gamma_{\mu\nu}=-\frac{1}{4}[\gamma_\mu,\gamma_\nu]$,

$$\mathbf{A}=A_\mu^{[\rho\sigma]}\gamma_{\rho\sigma}\mathrm{d}x^\mu. \tag{2.3}$$

Boldface letters here denote forms, in components $\mathbf{A}=\mathrm{A}_\mu\mathrm{d}x^\mu$. The corresponding field strength is $\mathbf{F}=\mathrm{d}\mathbf{A}+\mathbf{A}\wedge\mathbf{A}=\frac{1}{2}\mathrm{F}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$.

Both Higgs and gauge fields can be assumed to be regular on $\mathbb{R}^d$, a property wich is not destroyed by regular *gauge transformations* given by $SO(d)$-valued functions $\mathrm{g}=\exp(R^{\mu\nu}\gamma_{\mu\nu})$ on $\mathbb{R}^d$ with

$$\Phi\mapsto{}^{\mathrm{g}}\Phi:=\mathrm{g}\Phi\mathrm{g}^{-1} \tag{2.4}$$

$$\mathbf{A}\mapsto{}^{\mathrm{g}}\mathbf{A}:=\mathrm{g}\mathbf{A}\mathrm{g}^{-1}+\mathrm{g}\mathrm{d}\mathrm{g}^{-1}. \tag{2.5}$$

Physical quantities, in particular the energy functional defining a concrete theory, have to be invariant under regular gauge transformations.

The existence of an energy functional yields further constraints on the fields considered, because it involves an integral which has to converge for a given field configuration. Therefore, finite energy configurations are characterized by a particular asymptotic behavior of the fields $(\Phi,\mathbf{A})$. Denoting the asymptotic fields (i.e., the leading terms of an asymptotic $1/r$ expansion of Φ and $\mathbf{A}$) by $\bar{\Phi}$ and $\bar{\mathbf{A}}$, appropriate finite energy conditions for a large class of models, including those discussed in Sec. III, read

$$\bar{\Phi}^2=-\mathbb{1} \tag{2.6}$$

$$\bar{\mathbf{D}}\bar{\Phi}:=\mathrm{d}\bar{\Phi}+[\bar{\mathbf{A}},\bar{\Phi}]=0. \tag{2.7}$$

Conditions (2.6) and (2.7) anticipate the general features of the generalized GG models to be introduced in Sec. III A below.

If the fields $(\Phi,\mathbf{A})$ are regular on $\mathbb{R}^d$, then the asymptotic fields labeled by overbars, $(\bar{\Phi},\bar{\mathbf{A}})$, are defined on S^{d-1}, and Eq. (2.7) can be solved for asymptotic gauge potential,

$$\mathrm{d}\bar{\Phi}+[\bar{\mathbf{A}},\bar{\Phi}]=0\Rightarrow\bar{\mathbf{A}}=-\frac{1}{4}[\bar{\Phi},\mathrm{d}\bar{\Phi}], \tag{2.8}$$

hence the asymptotic configurations in regular gauge are determined by the Higgs field alone which at infinity yields a mapping

$$\bar{\Phi}:S^{d-1}\rightarrow S^{d-1}. \tag{2.9}$$

Therefore, any finite energy configuration in regular gauge can be topologically classified in terms of the homotopy group

$$\Pi_{d-1}(S^{d-1})\cong\mathbb{Z} \tag{2.10}$$

using the (integer) *winding number* of the (asymptotic) Higgs field

$$\mathcal{W}^{(d)}\{\Phi\}=\frac{1}{W_d}\int_{S^{d-1}}\mathrm{tr}[\varepsilon_{d+1}\bar{\Phi}\underbrace{\mathrm{d}\bar{\Phi}\wedge\ldots\wedge\mathrm{d}\bar{\Phi}}_{d-1}] \tag{2.11}$$

as topological invariant, where W_d is a normalization constant.

The simplest example with these topological properties are the radially symmetric configurations which in regular gauge are given by

$$\Phi_{(P)}=\varepsilon_{d+1}h(r)\gamma_\mu\hat{x}_\mu, \quad \mathbf{A}_{(P)}=\frac{1+f(r)}{r}\gamma_{\mu\nu}\hat{x}_\nu\mathrm{d}x^\mu, \tag{2.12}$$

with $x_\mu=r\hat{x}_\mu$ and $\hat{x}_\mu^2=1$. Requiring finite energy, the profile functions h and f have to satisfy $h(r\rightarrow\infty)=1$, $f(r\rightarrow\infty)=0$, whereas regularity at the origin means $h(0)=0$, $f(0)=-1$. From a topological point of view, the particular shape of $h(r)$, $f(r)$ is not important since it does not affect the asymptotic radially symmetric fields

$$\bar{\Phi}_{(P)}=\varepsilon_{d+1}\gamma_\mu\hat{x}_\mu, \quad \bar{\mathbf{A}}_{(P)}=\frac{1}{r}\gamma_{\mu\nu}\hat{x}_\nu\mathrm{d}x^\mu \tag{2.13}$$

as well as the asymptotic field strength given by

$$\bar{\mathbf{F}}_{(P)}=-\frac{1}{2r^2}(\gamma_{\mu\nu}+\hat{x}_{[\mu}\gamma_{\nu]\lambda}\hat{x}_\lambda)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu. \tag{2.14}$$

Obviously, interpreted as the mapping

$$\bar{\Phi}_{(P)}:S^{d-1}\rightarrow S^{d-1} \tag{2.15}$$

the asymptotic Higgs field of any radially symmetric configuration (2.12) has winding number 1, $\mathcal{W}^{(d)}\{\bar{\Phi}_{(P)}\}=1$. Higher winding number requires *Ansätze* with more involved, e.g., axial, symmetry properties.

Besides the regular gauge transformations discussed so far, there are also singular gauge transformations which are defined only on some subset of $\mathbb{R}^d$, the corresponding gauge transformed fields also being defined only on that subset and singular on the (generally point- or string-like) complement. Those singular gauge transformations are of particular interest because they allow the transformation of the fields to the *Dirac gauge* in which the Higgs field always points in the isospace d-direction. In particular, the asymptotic Higgs field of finite energy configurations trivializes to $\pm\varepsilon_{d+1}\gamma_d$ in the Dirac gauge which allows the characterization of two distinct Dirac gauges, *positive* and *negative*. Using this characterization, (positive or negative) Dirac gauge transformations are regular on

$$\mathbb{R}^d_\pm:=\mathbb{R}^d\backslash\{(0,\ldots,0,\mp x)|x\geqslant 0\} \tag{2.16}$$

but singular on either the positive or the negative d-axis. Therefore, the fields $({}^\pm\Phi,{}^\pm\mathbf{A})$ in positive or negative Dirac gauge are also singular on the negative or positive d-axis, respectively, commonly known as ''Dirac string.''

To describe a finite energy configuration $(\Phi,\mathbf{A})$ on $\mathbb{I}^d$ in Dirac gauge, one always needs $({}^+\Phi,{}^+\mathbf{A})$ in positive Dirac gauge, defined on $\mathbb{R}^d_+$, as well as $({}^-\Phi,{}^-\mathbf{A})$ in negative Dirac gauge, defined on $\mathbb{R}^d_-$. Both $({}^+\Phi,{}^+\mathbf{A})$ and $({}^-\Phi,{}^-\mathbf{A})$ have to be gauge equivalent to $(\Phi,\mathbf{A})$. It follows that on the overlap of the positive and negative Dirac gauge definition ranges, $\mathbb{R}^d_0:=\mathbb{R}^d_+\cap\mathbb{R}^d_-$, there exists a *transition gauge transformation* T with

$${}^{\mathrm{T}}({}^-\Phi,{}^-\mathbf{A})=({}^+\Phi,{}^+\mathbf{A}). \tag{2.17}$$

Asymptotically, the trivialization of the Higgs field in Dirac gauge no longer allows one to express the asymptotic gauge field in terms of the asymptotic Higgs field since Eq. (2.8) requires regularity of the fields on $\mathbb{R}^d$. Instead, the finite energy condition (2.7) forces the breaking of the gauge symmetry of the asymptotic gauge field according to

$$\mathrm{d}^\pm\bar{\Phi}+[{}^\pm\bar{\mathbf{A}},{}^\pm\bar{\Phi}]=0\Rightarrow[{}^\pm\bar{\mathbf{A}},\gamma_d]=0\Rightarrow{}^\pm\bar{\mathbf{A}}={}^\pm\bar{A}^{[ij]}_\mu\gamma_{ij}\mathrm{d}x^\mu \tag{2.18}$$

hence ${}^\pm\bar{\mathbf{A}}$ takes values in the $so(d-1)$ subalgebra of $so(d)$ and is defined on $S^{d-1}\backslash\{0,\ldots,0,\mp 1\}$ which is the $d-1$ dimensional sphere ''at infinity,'' excluding the south or north pole, respectively. To describe the asymptotic gauge fields on S^{d-1}, it is sufficient to consider ${}^\pm\bar{\mathbf{A}}$ on the upper or lower half spheres $S^{d-1}_\pm$, respectively, which overlap on the *equator* $S^{d-2}=S^{d-1}_+\cap S^{d-1}_-$.

In regular gauges, finite energy configurations could be topologically classified in terms of the asymptotic Higgs field winding number $\mathcal{W}^{(d)}\{\Phi\}$. In the Dirac gauge, the asymptotic Higgs fields ${}^\pm\bar{\Phi}=\pm\varepsilon_{d+1}\gamma_d$ do not carry any topological information, but the fact that one needs both positive and negative Dirac gauge to describe a single finite energy configuration now yields the topological characterization. This can be be expressed in terms of the asymptotic transition gauge transformation (2.17) which reverses the sign of the asymptotic Higgs field, hence

$$\bar{\mathrm{T}}(-\varepsilon_{d+1}\gamma_d)\bar{\mathrm{T}}^{-1}=\varepsilon_{d+1}\gamma_d \Rightarrow \{\bar{\mathrm{T}},\gamma_d\}=0. \qquad (2.19)$$

$\bar{\mathrm{T}}$ takes values in the subset $H(d)\subset SO(d)$ defined by Eq. (2.19), $H(d)\cong S^{d-2}$, and transforms ${}^{-}\mathbf{A}$ to ${}^{+}\mathbf{A}$ on the overlap of their definition ranges which is the equator S^{d-2}. This means that $\bar{\mathrm{T}}$ is topologically equivalent to a mapping

$$\bar{\mathrm{T}}: S^{d-2}\rightarrow S^{d-2}, \qquad (2.20)$$

which enables the classification of a finite energy configuration in the Dirac gauge in terms of the homotopy group

$$\Pi_{d-2}(S^{d-2})\cong \mathbb{Z}, \qquad (2.21)$$

expressed by the degree of the (asymptotic) transition gauge transformation

$$\mathcal{D}^{(d)}\{\mathrm{T}\}=\frac{1}{D_d}\int_{S^{d-2}}\mathrm{tr}[\varepsilon_d\underbrace{(\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1})\wedge\ldots\wedge(\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1})}_{d-2}] \qquad (2.22)$$

where D_d is a normalization constant.

Considering the example of the radially symmetric field configuration discussed above, the singular gauge transformations

$$\mathrm{g}_{\pm}=\frac{1}{\sqrt{2(1\pm\hat{x}_d)}}\{(1\pm\hat{x}_d)\mathbb{1}\pm\gamma_d\hat{x}_i\gamma_i\}, \qquad (2.23)$$

which are well defined on $\mathbb{R}^d_{\pm}$, respectively, transform $(\Phi_{(P)},\mathbf{A}_{(P)})$ in Eq. (2.12) to the positive or negative Dirac gauges, $({}^{\pm}\Phi_{(P)},{}^{\pm}\mathbf{A}_{(P)})={}^{\mathrm{g}_{\pm}}(\Phi_{(P)},\mathbf{A}_{(P)})$. The asymptotic fields are

$${}^{\mathrm{g}_{\pm}}(\bar{\Phi}_{(P)}):={}^{\pm}\bar{\Phi}_{(P)}=\pm\varepsilon_{d+1}\gamma_d \qquad (2.24)$$

$${}^{\mathrm{g}_{\pm}}(\bar{\mathbf{A}}_{(P)}):={}^{\pm}\bar{\mathbf{A}}_{(P)}=\frac{1}{r}\frac{1}{1\pm\hat{x}_d}\gamma_{ij}\hat{x}_j\mathrm{d}x^i, \qquad (2.25)$$

with gauge field strength,

$${}^{\pm}\bar{\mathbf{F}}_{(P)}=-\frac{1}{2r^2}\left(\gamma_{ij}+\frac{1}{1\pm\hat{x}_d}\hat{x}_{[i}\gamma_{j]k}\hat{x}_k\right)\times\mathrm{d}x^i\wedge\mathrm{d}x^j\pm\frac{1}{r^2}\gamma_{ik}\hat{x}_k\mathrm{d}x^i\wedge\mathrm{d}x^d. \qquad (2.26)$$

The explicit expressions (2.25),(2.26) show that the gauge group of the asymptotic hedgehog configuration in Dirac gauge breaks down as $SO(d)\rightarrow SO(d-1)$, whereas the asymptotic Higgs field (2.24) trivializes.

The transition gauge transformation $\mathrm{T}_{(P)}$, ${}^{\mathrm{T}_{(P)}}({}^{-}\Phi_{(P)},{}^{-}\mathbf{A}_{(P)})=({}^{+}\Phi_{(P)},{}^{+}\mathbf{A}_{(P)})$, is in this case given by

$$\mathrm{T}_{(P)}=\mathrm{g}_{+}\mathrm{g}_{-}^{-1}=\frac{1}{\sqrt{1-\hat{x}_d^2}}\gamma_d\hat{x}_i\gamma_i. \qquad (2.27)$$

Restricting $\bar{\mathrm{T}}_{(P)}=\mathrm{T}_{(P)}$ to the equator $\hat{x}_d=0$ yields

$$\bar{\mathrm{T}}_{(P)}|_{\hat{x}_d=0}=\gamma_d\hat{x}_i\gamma_i, \qquad (2.28)$$

which is a mapping $S^{d-2}\rightarrow S^{d-2}$ of degree 1, $\mathcal{D}^{(d)}\{\bar{\mathrm{T}}_{(P)}\}=1$.

III. MODELS AND TOPOLOGICAL CHARGES

In the previous section we selected the gauge group to be $SO(d)$ and the representation of the Higgs field to be the d component vector. This choice was made on topological criteria, including the possibility of having a Dirac gauge. Here, we further require that models like these must also support solitonic solutions which means that the action or energy is bounded from below by a topological charge. In any given dimension d, there are in principle an infinite number of such models, out of which it is reasonable to select the simplest one. These are all derived from members of the $4p$ dimensional YM hierarchy [23] with the gauge field in one of the two chiral representations of $SO_{\pm}(4p)$, whose action density is given by

$$\mathcal{L}^{(4p)}=\mathrm{tr}(\underbrace{\mathcal{F}\wedge\ldots\wedge\mathcal{F}}_{2p})\wedge{*}(\underbrace{\mathcal{F}\wedge\ldots\wedge\mathcal{F}}_{2p}) \qquad (3.1)$$

where $\mathcal{F}$ is the $so(4p)$ valued 2-form gauge field strength (curvature), and * denotes the Hodge dual. As in 4 dimensions, the action density (3.1) is bounded from below by the $2p$-th CP density $\mathcal{C}_{2p}$

$$\int\mathcal{L}^{(4p)}\geqslant\int\mathcal{C}_{2p}. \qquad (3.2)$$

It is known that when Eq. (3.2) is saturated, the resulting self-duality equations have both spherically [26] and axially [27] symmetric solutions.

The derived d-dimensional gauged Higgs models result from the dimensional descent of the inequality (3.2) over some $(4p-d)$-dimensional compact coset space K^{4p-d}

$$\int_{\mathbb{R}^d\times K^{4p-d}}\mathcal{L}^{(4p)}\geqslant\int_{\mathbb{R}^d\times K^{4p-d}}\mathcal{C}_{2p}. \qquad (3.3)$$

After performing the (compact) integration over the K^{4p-d} coordinates, we are left with the inequality for the d-dimensional action or energy density of the residual gauged Higgs model, bounded from below by residual CP density. We will discuss these two quantities in a little more detail in the following two subsections. Before proceeding however, we make two remarks.

First, the choice of the compact coset space K^{4p-d} is not important for our purposes since we are not concerned with the gauge coupling constant explicitly, so we will have in mind the simplest variant $K^{4p-d}=S^{4p-d}$ when discussing the symmetry breaking that occurs in the dimensional descent, and refer to the corresonding residual models as generalized Georgi-Glashow (GGG) models.

Secondly, we will restrict to the simplest of all possible residual system in any dimension *d*. It is clear from Eq. (3.3) that the descent to d dimensions can start from any dimension $4p>d$. The simplest systems will result, obviously, when $4p$ is the smallest number that is larger than d. We shall refer to these as the *minimal* GGG models. In our considerations below, we will always restrict ourselves to these choices. Familiar examples of such residual models are the three dimensional GG model (in the Prasad-Somerfield limit) and the Abelian Higgs model, both descended from $SO_{\pm}(4)=SU(2)$ $p=1$ (i.e., usual) YM.[1]

A. The generalized GG models and finite energy conditions

Having explained the general procedure used in the derivation of residual Higgs models above, we now discuss some properties of minimal GGG systems in $d=2,3,4,5$ which are used in the subsequent section. They arise respectively, from the dimensional reduction of the $p=1$, $SO_{\pm}(4)=SU(2)(d=2,3)$, and the $p=2$, $SO_{\pm}(8)$ $(d=4,5)$, members of the YM hierarchy, and are given explicitly in the Appendix B. We have denoted the energy or action density of the residual models by $\mathcal{E}^{(p,d)}$ and will refer to them as energy densities henceforth, since in the familiar 3 dimensional case this coincides with the definition of an energy density. We note that the $d=2$ model $\mathcal{E}^{(1,2)}$ thus obtained is nothing but the usual Abelian Higgs model, whereas $\mathcal{E}^{(1,3)}$ equals the GG model in the Prasad-Somerfield limit, but we do not make a distinction on this acount because this limit makes no difference to our considerations below, the latter being sensitive only to the asymptotic *values* rather than the detailed decays of the fields.

An important general feature of the energy densities of these residual models is, that the curvature $\mathbf{F}$, the covariant derivative of the Higgs field $\mathbf{D}\Phi$ and the ''square root'' of the Higgs selfinteraction potential $\mathrm{S}=-(\Phi^2+\mathbb{1})$ all must decay asymptotically at the same rate to satisfy *finite energy conditions*. The reason for this is easily explained. Denoting the curvatures in $4p$ dimensions that appear in Eq. (3.1) with indices $M=\mu,m$, with μ labeling the coordinates of $\mathrm{x}\in\mathbb{R}_d$ and m those of $\mathrm{y}\in S^{4p-d}$, as a result of the dimensional reduction we have, for *d odd* and *even* respectively,

$$\mathcal{F}_{MN}=\begin{cases}\mathcal{F}_{\mu\nu}=\mathrm{F}_{\mu\nu}\otimes Y(\mathrm{y})\\ \mathcal{F}_{\mu m}=\mathrm{D}_\mu\Phi\otimes Y_m(\mathrm{y}),\\ \mathcal{F}_{mn}=\mathrm{S}\otimes Y_{mn}(\mathrm{y})\end{cases}$$
$$\mathcal{F}_{MN}=\begin{cases}\mathcal{F}_{\mu\nu}=\mathrm{F}^{(+)}_{\mu\nu}\otimes Y^{(+)}(\mathrm{y})+\mathrm{F}^{(-)}_{\mu\nu}\otimes Y^{(-)}(\mathrm{y})\\ \mathcal{F}_{\mu m}=\mathrm{D}_\mu\varphi\otimes Y^{(+)}_m(\mathrm{y})-\mathrm{D}_\mu\varphi^\dagger\otimes Y^{(-)}_m(\mathrm{y})\\ \mathcal{F}_{mn}=s_+\otimes Y^{(+)}_{mn}(\mathrm{y})+s_-\otimes Y^{(-)}_{mn}(\mathrm{y})\end{cases} \tag{3.4}$$

where $s_+=\varphi\varphi^\dagger-\mathbb{1}$, $s_-=\varphi^\dagger\varphi-\mathbb{1}$, and the Y's, are $\mathrm{y}\in S^{4p-d}$ dependent tensor-spinor bases whose details do not concern us here (see for example Refs. [24] for details). In the even dimensional cases, the Higgs field Φ, and the gauge potential A_μ are composed of the corresponding fields φ, $\varphi^\dagger$ and $\mathrm{A}^{(\pm)}_\mu$ (whose curvatures are $\mathrm{F}^{(\pm)}_{\mu\nu}$) as follows:

$$\mathrm{A}_\mu=\begin{bmatrix}\mathrm{A}^{(+)}_\mu & 0\\ 0 & \mathrm{A}^{(-)}_\mu\end{bmatrix},\quad \Phi=\begin{bmatrix}0 & \varphi\\ -\varphi^\dagger & 0\end{bmatrix}. \tag{3.5}$$

What is interesting here is that the substitution of Eq. (3.4) into Eq. (3.1), which yields the residual energy density in d dimensions, results in a sum of terms each of which consists of $2p$ factors of all possible types of components listed in Eq. (3.4). It follows that each of the fields $\mathrm{F}_{\mu\nu}$, $\mathrm{D}_\mu\Phi$, and $(\Phi^2+\mathbb{1})$ must have the same asymptotic decay rate if the energy is to be finite. This means that $\mathrm{D}_\mu\Phi$ and $\mathrm{S}=-(\Phi^2+\mathbb{1})$ can be neglected in asymptotic expansions which justifies the conditions (2.6) and (2.7) defining the asymptotic fields $(\bar{\Phi},\bar{\mathbf{A}})$.

Another important property of the models under consideration is that they support nontrivial, stable finite energy solutions to which we will refer to as solitons in accord with our nomenclature $\mathcal{E}^{(p,d)}$ as energy density. In particular, the radially symmetric *Ansatz* (2.12) minimizes the energy functionals, i.e., the profile functions $f(r)$ and $h(r)$ can be chosen such that they solve the Euler-Lagrange equations of the radial subsystem (which in general requires numerical integration techniques). The corresponding radially symmetric solution is called ''hedgehog.''

B. Topological charges of generalized monopoles

Under dimensional reduction, the left hand side of Eq. (3.3) yields the residual subsystems which are the candidates for GGG models. The right hand side, which is the dimensionally reduced CP density, presents the lower bounds on the GGG energy densities,

$$\mathcal{E}^{(p,d)}\geqslant\varrho^{(p,d)}. \tag{3.6}$$

The volume integral over this CP density is called CP charge $q^{(p,d)}$,

$$q^{(p,d)}=\int_{\mathbb{R}^d}\varrho^{(p,d)}d^dx. \tag{3.7}$$

The normalization of $\varrho^{(p,d)}$ and $\mathcal{E}^{(p,d)}$ in Eq. (3.6) is chosen such that the hedgehog (2.12) has unit CP charge. $q^{(p,d)}$ is commonly also called topological charge of the GGG model

[1]If by contrast the descent to $d=3$ is started from the $p=2$ YM system, another variant of the GG model is obtained [28]. Its asymptotic properties are identical to those of the usual GG model and hence it yields nothing new in the present context. We therefore exclude all such models from consideration here.

$\mathcal{E}^{(p,d)}$ since it is closely related to the topological properties of finite energy configurations discussed in Sec. II. This is shown in Sec. IV A, making use of the most important property of the CP densities $\varrho^{(p,d)}$, namely that they are total divergences,

$$\varrho^{(p,d)}=\partial_\lambda \Omega_\lambda^{(p,d)}. \tag{3.8}$$

This was shown in numerous cases in Refs. [25], both for even and odd values of the residual dimensions d, which we do not exhibit here, except for the four examples discussed in Appendix B. $\Omega_\lambda^{(p,d)}$ is the residual CS density and $\mathbf{\Omega}^{(p,d)}=\Omega_\lambda^{(p,d)}(*\mathrm{d}x^\lambda)$ the residual CS form which we refer to as the *CS form* pertaining to the model $\mathcal{E}^{(p,d)}$, in the sense that it allows us to write the CP charge (3.7) as a surface integral over the boundary of $\mathbb{R}^d$,

$$q^{(p,d)}=\lim_{r\to\infty}\int_{S^{d-1}(r)}\hat{x}_\lambda \Omega_\lambda^{(p,d)}dS=\lim_{r\to\infty}\int_{S^{d-1}(r)}\mathbf{\Omega}^{(p,d)}. \tag{3.9}$$

We point out that these *CS forms*, are *gauge invariant in odd, and gauge variant in even, dimensions* [25]. This is clearly seen from Eqs. (B15)–(B18) of Appendix B. This property of these CS forms will be very important in our subsequent considerations.

For use in the next section, we now introduce the asymptotic expressions of CS forms $\mathbf{\Omega}^{(p,d)}$ appearing in Eq. (3.9). We denote them, again with an overbar, as

$$\overline{\mathbf{\Omega}}^{(d)}:=\mathbf{\Omega}^{(p,d)}|_{(\overline{\Phi},\overline{\mathbf{A}})}, \tag{3.10}$$

where $(\overline{\Phi},\overline{\mathbf{A}})$ are the asymptotic fields defined in Eqs. (2.6) and (2.7). We have labeled $\overline{\mathbf{\Omega}}^{(d)}$ with the residual dimension d and not with the label p that specifies the model, since it does not depend on the latter. This is because of the general structure of the asymptotic CS forms which is discussed below. Accordingly the CP charge, evaluated by the surface integral (3.9), also is independent of p and we express it as

$$q^{(d)}=\int_{S^{d-1}(r)}\overline{\mathbf{\Omega}}^{(d)}. \tag{3.11}$$

The asymptotic CS forms $\overline{\mathbf{\Omega}}^{(d)}$ inherit the important property of the $\mathbf{\Omega}^{(p,d)}$, namely that they are gauge invariant for odd d and gauge variant for even d. Accordingly we treat the two cases separately.

In $2M+1$ *(odd) dimensions*, the asymptotic CS forms for the two examples $d=3$ and $d=5$ which will be needed below, can be readily extracted from the general CS forms (B16) and (B18) given in Appendix B, using the finite energy requirements. They are

$$\overline{\mathbf{\Omega}}^{(3)}=\frac{1}{c_3}\mathrm{tr}[\overline{\Phi}\overline{\mathbf{F}}] \tag{3.12}$$

$$\overline{\mathbf{\Omega}}^{(5)}=\frac{1}{c_5}\mathrm{tr}[\overline{\Phi}\overline{\mathbf{F}}\wedge\overline{\mathbf{F}}]. \tag{3.13}$$

In general, $\mathbf{\Omega}^{(p,2M+1)}$ consists of products of $\mathbf{F}$, $\mathbf{D}\Phi$, and S. As a consequence of the finite energy conditions, its asymptotic form $\overline{\mathbf{\Omega}}^{(2M+1)}$ consists of M factors $\mathbf{F}$ and one Higgs field. By virtue of the identity

$$\mathrm{tr}[\mathbf{D}\Phi\wedge\underbrace{\mathbf{F}\wedge\ldots\wedge\mathbf{F}}_{M}]=\mathrm{d}\,\mathrm{tr}[\Phi\underbrace{\mathbf{F}\wedge\ldots\wedge\mathbf{F}}_{M}] \tag{3.14}$$

then, the asymptotic CS form in odd dimensions takes the form

$$\overline{\mathbf{\Omega}}^{(2M+1)}=\frac{1}{c_{2M+1}}\mathrm{tr}[\overline{\Phi}\underbrace{\overline{\mathbf{F}}\wedge\ldots\wedge\overline{\mathbf{F}}}_{M}] \tag{3.15}$$

where c_{2M+1} is a normalization constant. From Eq. (3.15), it is obvious that $\overline{\mathbf{\Omega}}^{(2M+1)}$ is gauge invariant.

In $2M$ *(even) dimensions*, following similar arguments as before now applied to Eqs. (B15) and (B17), yield the asymptotic CS forms for the $d=2$ and $d=4$ dimensional models,

$$\overline{\mathbf{\Omega}}^{(2)}=\frac{1}{c_2}\mathrm{tr}[\gamma_3\overline{\mathbf{A}}]=:\frac{1}{c_2}\boldsymbol{\omega}^{(1)}[\overline{\mathbf{A}},\overline{\mathbf{F}}] \tag{3.16}$$

$$\overline{\mathbf{\Omega}}^{(4)}=\frac{1}{c_4}\mathrm{tr}\left[\gamma_5\left(\overline{\mathbf{F}}\wedge\overline{\mathbf{A}}-\frac{1}{3}\overline{\mathbf{A}}\wedge\overline{\mathbf{A}}\wedge\overline{\mathbf{A}}\right)\right]=:\frac{1}{c_4}\boldsymbol{\omega}^{(2)}[\overline{\mathbf{A}},\overline{\mathbf{F}}]. \tag{3.17}$$

In Eqs. (3.16) and (3.17), we have introduced a new symbol $\boldsymbol{\omega}^{(M)}$ used in Ref. [30], for $M=1$ and $M=2$, because is will be useful in the work of subsequent sections for general M. In general, $\mathbf{\Omega}^{(p,2M)}$ consists of products of $\mathbf{F}$, $\mathbf{D}\Phi$, S, and of $\mathbf{A}$. As a consequence of the finite energy conditions, its asymptotic form $\overline{\mathbf{\Omega}}^{(2M)}$ consists only of the Higgs independent terms, and equals the trace of the chiral $SO(2M)$ matrix γ_{2M+1} times the products of $\mathbf{A}$ and $\mathbf{F}$ that appear in the trace of the CS form of the (chiral) $SO_\pm(2M)$ Yang-Mills fields without a Higgs field. Using $[\mathbf{A},\gamma_{2M+1}]=0$ one can show [30] that this is an exact form,

$$\mathrm{tr}[\gamma_{2M+1}\underbrace{\mathbf{F}\wedge\ldots\wedge\mathbf{F}}_{M}]=\mathrm{d}\boldsymbol{\omega}^{(M)}[\mathbf{A},\mathbf{F}]. \tag{3.18}$$

Unlike the corresponding expression (3.15) which can be expressed for all $d=2M+1$, the corresponding expressions in even dimensions have a more complicated dependence on $d=2M$, abbreviated by $\boldsymbol{\omega}^{(M)}$. These are easy to find explicitly in any given case and for $d=6$ and $d=8$ can be found in the first item in Refs. [25]. One can show that it is this term which contributes the leading asymptotic behavior of the CS form, hence the asymptotic CS from in even dimensions can be written as

$$\overline{\mathbf{\Omega}}^{(2M)}=\frac{1}{c_{2M}}\boldsymbol{\omega}^{(M)}[\overline{\mathbf{A}},\overline{\mathbf{F}}]. \tag{3.19}$$

Equations (3.15) and (3.19) show that the asymptotic CS form really depends on the dimension only and not on the particular model characterized by *p*, the member of the YM hierarchy the underlying model was derived from.

We emphasise that the asymptotic CS forms in odd dimensions always involve one Higgs field in addition to an antisymmetric curvature *M*-form, while in even dimensions the Higgs field is absent and is replaced by the chiral $SO(d)$ matrix.

IV. KALB-RAMOND FIELDS AND 't HOOFT TENSORS

This section is divided in three subsections. In the first we evaluate the CP charge in the regular gauge identifying it with the winding number of the Higgs field $\mathcal{W}^{(d)}\{\Phi\}$, and finding that in odd dimensions this gives rise to the description of magnetic KR [18] charge. The second subsection is devoted to the corresponding considerations in the Dirac gauge. There, we have shown that the CP charge in even dimensions cannot be evaluated as a surface integral in the Dirac gauge, while that is possible in odd dimensions. It was also shown that the CP charge results from a surface integral over Wu-Yang type KR potentials, and that it can be computed as the degree $\mathcal{D}^{(d)}\{\mathrm{T}\}$ of the transition gauge transformation between positive and negative Dirac gauges. Finally in the third subsection, we have given the prescription to construct 't Hooft tensors for all odd dimensional monopoles as a natural extension of the content of the previous two subsections.

A. Regular gauge

The CP charge can be evaluated directly by simply inserting a field configuration in regular gauge into the surface integral (3.11). For the hedgehog field configurations $(\Phi_{(P)},\mathbf{A}_{(P)})$, (2.12), the asymptotics (2.13) yield

$$\overline{\mathbf{\Omega}}^{(d)}_{(P)} := \mathbf{\Omega}^{(p,d)}\big|_{(\overline{\Phi}_{(P)},\overline{\mathbf{A}}_{(P)})}$$

$$= \frac{1}{\kappa_{d-1}}\frac{\hat{x}_\mu}{r^{d-1}}(*\mathrm{d}x^\mu) \Rightarrow \int_{S^{-1}} \overline{\mathbf{\Omega}}^{(d)}_{(P)} = 1, \qquad (4.1)$$

in which κ_{d-1} denotes surface of S^{d-1}. This direct evaluation does not, however, demonstrate the topological nature of the CP charge, i.e., its relation to the topological invariants of a finite energy configuration discussed in Sec. II.

In regular gauge, a relation between the CP charge $q^{(d)}$ and the winding number of the Higgs field $\mathcal{W}^{(d)}\{\Phi\}$ is easy to find. Using Eq. (2.8) to rewrite the asymptotic gauge fields in terms of the asymptotic Higgs fields,

$$\overline{\mathbf{A}} = -\frac{1}{2}\overline{\Phi}\mathrm{d}\overline{\Phi}, \quad \overline{\mathbf{F}} = -\frac{1}{4}\mathrm{d}\overline{\Phi}\wedge\mathrm{d}\overline{\Phi} \qquad (4.2)$$

and inserting Eq. (4.2) into Eqs. (3.12), (3.13), (3.16), (3.17) we find

$$\overline{\mathbf{\Omega}}^{(2)} = -\frac{1}{2c_2}\mathrm{tr}[\gamma_3\overline{\Phi}\mathrm{d}\overline{\Phi}] \qquad (4.3)$$

$$\overline{\mathbf{\Omega}}^{(3)} = -\frac{1}{4c_3}\mathrm{tr}[\overline{\Phi}\mathrm{d}\overline{\Phi}\wedge\mathrm{d}\overline{\Phi}] \qquad (4.4)$$

$$\overline{\mathbf{\Omega}}^{(4)} = \frac{1}{6c_4}\mathrm{tr}[\gamma_5\overline{\Phi}\mathrm{d}\overline{\Phi}\wedge\mathrm{d}\overline{\Phi}\wedge\mathrm{d}\overline{\Phi}] \qquad (4.5)$$

$$\overline{\mathbf{\Omega}}^{(5)} = \frac{1}{16c_5}\mathrm{tr}[\overline{\Phi}\mathrm{d}\overline{\Phi}\wedge\mathrm{d}\overline{\Phi}\wedge\mathrm{d}\overline{\Phi}\wedge\mathrm{d}\overline{\Phi}]. \qquad (4.6)$$

By virtue of Eqs. (4.3)–(4.6), the CP charge $q^{(d)}$ can be identified with the winding number of the Higgs field defined by Eq. (2.11),

$$q^{(d)} = \mathcal{W}^{(d)}\{\Phi\}, \qquad (4.7)$$

in both even and odd dimensions. This justifies our description of CP charges as topological charges in the previous section. We stress that Eq. (4.2) is valid in regular gauge only, therefore, Eq. (4.7) cannot be used to express the CP charge in Dirac gauge by performing a singular gauge transformation, according to $\mathcal{W}^{(d)}\{^{\pm}\Phi\} = 0 \neq q^{(d)}$.

Because a factor of one Higgs field appears in the odd dimensional asymptotic CS forms (3.15), it follows that these are *closed* forms, namely that

$$\mathrm{d}\overline{\mathbf{\Omega}}^{(2M+1)} = \frac{1}{c_{2M+1}}\mathrm{tr}[\overline{\mathbf{D}}\overline{\Phi}\ \underbrace{\overline{\mathbf{F}}\wedge\ldots\wedge\overline{\mathbf{F}}}_{M}] = 0. \qquad (4.8)$$

We should emphasize that this result follows from the presence in Eq. (4.8) of $\overline{\mathbf{D}\Phi}$ which is vanishing asymptotically according to Eq. (2.7).

In even dimensions however, the expression corresponding to Eq. (4.8) is

$$\mathrm{d}\overline{\mathbf{\Omega}}^{(2M)} = \frac{1}{c_{2M}}\mathrm{d}\boldsymbol{\omega}^{(M)}[\overline{\mathbf{A}},\overline{\mathbf{F}}]$$

$$= \frac{1}{c_{2M}}\mathrm{tr}[\gamma_{2M+1}\underbrace{\overline{\mathbf{F}}\wedge\ldots\wedge\overline{\mathbf{F}}}_{M}] \neq 0, \qquad (4.9)$$

which is *nonvanishing*, i.e., $\overline{\mathbf{\Omega}}^{(2M)}$ *is not a closed form*.

Being a *closed form*, the asymptotic CS form in $2M+1$ dimensions can be identified with a *Kalb-Ramond field strength* (cf. Appendix A) on S^{2M} which supports magnetic flux equal to the CP charge, $Q_m = q^{(2M+1)}$.

This allows us to interpret the CP charge $q^{(d)}$ solitons of the GGG model with energy density $\mathcal{E}^{(p,2M+1)}$ [e.g., the hedgehog (2.12) with $q^{(d)}=1$] as the KR monopole (A11) with magnetic charge $Q_m = q^{(d)}$. In other words, we can describe the singular KR monopole field as the $2M+1$ dimensional (regular) soliton of the appropriate GGG model.

B. Dirac gauge

The situation in Dirac gauge is different from that in regular gauge in so far as it is not possible to evaluate the CP charge in *even dimensions* using the surface integral (3.11), a fact already emphasized in Ref. [29]. For example, inserting the hedgehog (2.12) into the asymptotic CS form $\bar{\mathbf{\Omega}}^{(2M)}$ yields

$$\int_{S^{2M-1}} {}^{\pm}\bar{\mathbf{\Omega}}^{(2M)}_{(P)}=0, \tag{4.10}$$

where ${}^{\pm}\bar{\mathbf{\Omega}}^{(2M)}_{(P)}$ is the asymptotic CS form in the positive or negative Dirac gauge for the hedgehog field configuration given by Eqs. (2.24) and (2.25). This is because of the gauge variance of $\bar{\mathbf{\Omega}}^{(2M)}_{(P)}$ under the large gauge transformations (2.23) which take the regular hedgehog to positive or negative Dirac gauge. We will demonstrate this in detail in $2M=2$ and $2M=4$ and will then give the general case.

In $2M=2$ dimensions where

$${}^{\pm}\bar{\mathbf{\Omega}}^{(2)}_{(P)}=\frac{1}{c_2}\mathrm{tr}[\gamma_3 {}^{\pm}\bar{\mathbf{A}}_{(P)}] \tag{4.11}$$

we find

$${}^{\pm}\bar{\mathbf{\Omega}}^{(2)}_{(P)}=\bar{\mathbf{\Omega}}^{(2)}_{(P)}+\frac{1}{c_2}\mathrm{tr}[\gamma_3 \mathrm{g}_{\pm}\mathrm{dg}_{\pm}^{-1}]=0. \tag{4.12}$$

It turns out that the two terms in Eq. (4.12) simply cancel out.

In $2M=4$ dimensions, where

$${}^{\pm}\bar{\mathbf{\Omega}}^{(4)}_{(P)}=\frac{1}{c_4}\mathrm{tr}[\gamma_5({}^{\pm}\bar{\mathbf{F}}_{(P)}\wedge{}^{\pm}\bar{\mathbf{A}}_{(P)}-\tfrac{1}{3}{}^{\pm}\bar{\mathbf{A}}_{(P)}\wedge{}^{\pm}\bar{\mathbf{A}}_{(P)}\wedge{}^{\pm}\bar{\mathbf{A}}_{(P)})] \tag{4.13}$$

we find

$$\begin{aligned}{}^{\pm}\bar{\mathbf{\Omega}}^{(4)}_{(P)}&=\bar{\mathbf{\Omega}}^{(4)}_{(P)}+\frac{1}{c_4}\Big\{\mathrm{d}\,\mathrm{tr}[\gamma_5\bar{\mathbf{A}}_{(P)}\wedge\mathrm{dg}_{\pm}^{-1}]\\&\quad-\frac{1}{3}\mathrm{tr}[\gamma_5(\mathrm{g}_{\pm}\mathrm{dg}_{\pm}^{-1})\wedge(\mathrm{g}_{\pm}\mathrm{dg}_{\pm}^{-1})\wedge(\mathrm{g}_{\pm}\mathrm{dg}_{\pm}^{-1})]\Big\}\\&=0\end{aligned} \tag{4.14}$$

where again the sum of terms in Eq. (4.14) cancel out.

Finally in general where, with the notation of Eq. (3.19)

$${}^{\pm}\bar{\mathbf{\Omega}}^{(2M)}_{(P)}=\frac{1}{c_{2M}}\boldsymbol{\omega}^{(M)}[{}^{\pm}\bar{\mathbf{A}}_{(P)},{}^{\pm}\bar{\mathbf{F}}_{(P)}] \tag{4.15}$$

the CS form in Dirac gauge can be rewritten as

$${}^{\pm}\bar{\mathbf{\Omega}}^{(4)}_{(P)}=\bar{\mathbf{\Omega}}^{(4)}_{(P)}+\frac{1}{c_{2M}}\{\mathrm{d}\boldsymbol{\alpha}^{(M)}[\bar{\mathbf{A}},\mathrm{g}]+\boldsymbol{\omega}^{(M)}[\mathrm{g}_{\pm}\mathrm{dg}_{\pm}^{-1},0]\}. \tag{4.16}$$

Here, we have used the transformation properties of $\boldsymbol{\omega}^{(M)}$ in the notation of Ref. [30]. Note that the general expression (4.16) is exactly of the same form as the special case (4.14) for $M=2$, while the $M=1$ case (4.12) is of a simpler form. Note also that in Eq. (4.16), the exact form $\mathrm{d}\boldsymbol{\alpha}^{(M)}[\bar{\mathbf{A}},\mathrm{g}]$ does not contribute in the surface integral (4.10), in which case the other two terms cancel out.

In *odd dimensions* on the other hand, the surface integral of ${}^{\pm}\bar{\mathbf{\Omega}}^{(2M+1)}$ over S^{2M} does yield the correct value for the CP charge [29], like in the regular gauge. In this case the asymptotic CS form of a finite energy configuration $({}^{\pm}\Phi,{}^{\pm}\mathbf{A})$ in Dirac gauge, can be given explicitly for the general case. As a consequence of the asymptotic form of the Higgs field in this gauge, ${}^{\pm}\bar{\Phi}=\pm i\gamma_{2M+1}$, and Eq. (3.15),

$${}^{\pm}\bar{\mathbf{\Omega}}^{(2M+1)}=\pm\frac{i}{c_{2M+1}}\mathrm{tr}[\gamma_{2M+1}\underbrace{{}^{\pm}\bar{\mathbf{F}}\wedge\ldots\wedge{}^{\pm}\bar{\mathbf{F}}}_{M}]. \tag{4.17}$$

This quantity is defined on the sphere $S^{2M}\backslash\{(0,\ldots,0,\mp1)\}$ excluding the south or north pole, respectively. As a consequence of the manifest gauge invariance of Eq. (4.17),

$$\bar{\mathbf{\Omega}}^{(2M+1)}={}^{\pm}\bar{\mathbf{\Omega}}^{(2M+1)}. \tag{4.18}$$

Another important property of the CS forms $\bar{\mathbf{\Omega}}^{(2M+1)}$, besides their gauge invariance, is that they are closed forms. This was used in Sec. IV A to interpret them as of KR field strengths on S^{2M}. Moreover in this case, namely in the Dirac gauge, the expression (4.17) makes it possible to express the KR fields as *curls* of KR potentials on the simply connected regions $S^{2M}_{\pm}$. Following Nepomechie [18], we denote these KR potentials by ${}^{\pm}\mathsf{B}^{(2M+1)}$,

$$\bar{\mathbf{\Omega}}^{(2M+1)}=\mathrm{d}{}^{\pm}\mathsf{B}^{(2M+1)} \tag{4.19}$$

in the notation of Appendix B. We will construct these Wu-Yang type KR potentials explicitly in three and five dimensions first and will then present the general case.

In $2M+1=3$ dimensions, the gauge group of ${}^{\pm}\bar{\mathbf{F}}$ breaks down to $SO(2)$ by virtue of $[\gamma_3,{}^{\pm}\bar{\mathbf{F}}]=0$, as discussed in Sec. II. We then have

$${}^{\pm}\bar{\mathbf{\Omega}}^{(3)}=\pm\frac{i}{c_3}\mathrm{tr}[\gamma_3{}^{\pm}\bar{\mathbf{F}}]=\pm\frac{i}{c_3}\mathrm{d}\,\mathrm{tr}[\gamma_3{}^{\pm}\bar{\mathbf{A}}], \tag{4.20}$$

which yields

$${}^{\pm}\mathsf{B}^{(3)}=\pm\frac{i}{c_3}\mathrm{tr}[\gamma_3{}^{\pm}\bar{\mathbf{A}}]. \tag{4.21}$$

In $2M+1=5$ dimensions, ${}^{\pm}\bar{\mathbf{F}}$ has $SO(4)$ gauge symmetry by virtue of $[\gamma_5,{}^{\pm}\bar{\mathbf{F}}]=0$, from which it follows that

$$\begin{aligned}{}^{\pm}\bar{\mathbf{\Omega}}^{(5)}&=\pm\frac{i}{c_5}\mathrm{tr}[\gamma_5{}^{\pm}\bar{\mathbf{F}}\wedge{}^{\pm}\bar{\mathbf{F}}]\\&=\pm\frac{i}{c_5}\mathrm{d}\,\mathrm{tr}\left[\gamma_5\left({}^{\pm}\bar{\mathbf{F}}\wedge{}^{\pm}\bar{\mathbf{A}}-\frac{1}{3}{}^{\pm}\bar{\mathbf{A}}\wedge{}^{\pm}\bar{\mathbf{A}}\wedge{}^{\pm}\bar{\mathbf{A}}\right)\right],\end{aligned} \tag{4.22}$$

thus

$$^{\pm}\mathbf{B}^{(5)}=\pm\frac{i}{c_5}\mathrm{tr}\left[\gamma_5\left({}^{\pm}\bar{\mathbf{F}}\wedge{}^{\pm}\bar{\mathbf{A}}-\frac{1}{3}{}^{\pm}\bar{\mathbf{A}}\wedge{}^{\pm}\bar{\mathbf{A}}\wedge{}^{\pm}\bar{\mathbf{A}}\right)\right]. \tag{4.23}$$

We now note that the forms on the right hand sides of Eqs. (4.20) and (4.22) coincide with the forms $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$ in the corresponding even dimensional ($d=2,4$) cases (3.16) and (3.17). In general $2M+1$ dimensions, $^{\pm}\bar{\mathbf{F}}$ is a $SO(2M)$ gauge field strength according to Eq. (2.18). This enables us to write the KR potentials of the asymptotic CS forms in general $2M+1$ dimensions with the help of $\boldsymbol{\omega}^{(M)}$ [which was used in Eq. (3.19) in the context of even dimensional CS forms] as follows:

$$^{\pm}\mathbf{B}^{(2M+1)}=\pm\frac{i}{c_{2M+1}}\boldsymbol{\omega}^{(M)}[{}^{\pm}\bar{\mathbf{A}},{}^{\pm}\bar{\mathbf{F}}]. \tag{4.24}$$

The KR potentials $^{\pm}\mathbf{B}^{(2M+1)}$ can be used to rewrite the CP charge surface integral corresponding to Eq. (3.11), over S^{2M}, as an integral over the equator S^{2M-1}. First splitting the integral (3.11) into two integrals over the upper and lower half spheres $S^{2M}_{\pm}$, respectively, one can use the gauge invariance of the odd dimensional CS forms, Eq. (4.18), to evaluate the CP charge $q^{(2M+1)}$ without integrating over the singularities in the Dirac gauge. Finally, using the KR potentials (4.19) and applying Stoke's theorem on $\partial S^{2M}_{+}=-\partial S^{2M}_{-}=S^{2M-1}$ yields

$$q^{(2M+1)}=\int_{S^{2M}_{+}}{}^{+}\bar{\boldsymbol{\Omega}}^{(2M+1)}+\int_{S^{2M}_{-}}{}^{-}\bar{\boldsymbol{\Omega}}^{(2M+1)}$$
$$=\int_{S^{2M-1}}({}^{+}\mathbf{B}^{(2M+1)}-{}^{-}\mathbf{B}^{(2M+1)}). \tag{4.25}$$

According to the discussion in Appendix A 2, the integral over the *transition form*

$$\mathbf{U}^{(2M+1)}={}^{+}\mathbf{B}^{(2M+1)}-{}^{-}\mathbf{B}^{(2M+1)} \tag{4.26}$$

in Eq. (4.25) determines the KR magnetic charge of the Wu-Yang type KR potentials $^{\pm}\mathbf{B}^{(2M+1)}$ which enables us again to identify the CP charge $q^{(2M+1)}$ with magnetic KR charge Q_m.

In the context of the GGG models, the transition form $\mathbf{U}^{(2M+1)}$ is a composite field which can be written explicitly in terms of the asymptotic gauge potentials $^{\pm}\bar{\mathbf{A}}$ and the asymptotic transition gauge transformation $\bar{\mathrm{T}}$ which takes the negative to the positive Dirac gauge, $^{\bar{\mathrm{T}}}({}^{-}\bar{\Phi},{}^{-}\bar{\mathbf{A}})=({}^{+}\bar{\Phi},{}^{+}\bar{\mathbf{A}})$. Again, we first present this result in the three and five dimensional case, giving the general odd dimensional result afterwards.

In *three dimensions,* using $\{\bar{\mathrm{T}},\gamma_3\}=0$, (2.19), we find

$$^{+}\mathbf{B}^{(3)}=\frac{i}{c_3}\mathrm{tr}[\gamma_3^{\bar{\mathrm{T}}}({}^{-}\bar{\mathbf{A}})]={}^{-}\mathbf{B}^{(3)}+\frac{i}{c_3}\mathrm{tr}[\gamma_3\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1}], \tag{4.27}$$

yielding $\mathbf{U}^{(3)}=(i/c_3)\mathrm{tr}[\gamma_3\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1}]$. Substituting this into the equatorial integral (4.25) leads to

$$q^{(3)}=\frac{i}{c_3}\int_{S^1}\mathrm{tr}[\gamma_3\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1}]. \tag{4.28}$$

In *five dimensions* we find in the same way, using $\{\gamma_5,\bar{\mathrm{T}}\}=0$,

$$\mathbf{U}^{(5)}=\frac{i}{c_5}\left\{\mathrm{d}\,\mathrm{tr}[\gamma_5{}^{-}\bar{\mathbf{A}}\wedge\mathrm{d}\bar{\mathrm{T}}^{-1}]-\frac{1}{3}\mathrm{tr}[\gamma_5(\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1})\wedge(\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1})\wedge(\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1})]\right\}. \tag{4.29}$$

Substituting this into the equatorial (closed surface) integral (4.25), by Stokes' theorem the first term in Eq. (4.29) does not contribute and the rest yields

$$q^{(5)}=-\frac{i}{3c_3}\int_{S^3}\mathrm{tr}[\gamma_5(\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1})\wedge(\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1})\wedge(\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1})]. \tag{4.30}$$

The five dimensional example already shows the structure of the general case. Using the same notations $\boldsymbol{\alpha}^{(M)}$ and $\boldsymbol{\omega}^{(M)}$ introduced in Eq. (4.16) but taking care of the anticommutator $\{\gamma_{2M+1},\bar{\mathrm{T}}\}=0$,

$$\mathbf{U}^{(2M+1)}=\frac{i}{c_{2M+1}}\{\mathrm{d}\boldsymbol{\alpha}^{(M)}[{}^{-}\bar{\mathbf{A}},\bar{\mathrm{T}}]+\boldsymbol{\omega}^{(M)}[\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1},0]\}. \tag{4.31}$$

Substituting this into the equatorial integral (4.25), in which the term $\mathrm{d}\boldsymbol{\alpha}^{(M)}[{}^{-}\bar{\mathbf{A}},\bar{\mathrm{T}}]$ does not contribute, we can write the CP charge in the general case as

$$q^{(2M+1)}=\frac{i}{c_{2M+1}}\int_{S^{2M-1}}\boldsymbol{\omega}^{(M)}[\bar{\mathrm{T}}\mathrm{d}\bar{\mathrm{T}}^{-1},0]. \tag{4.32}$$

Finally, we see that the CP charges (4.28) and (4.30) equal, respectively, the degrees of the transition gauge transformations (2.22), in three and five dimensions. In the general case with $q^{(2M+1)}$ given by Eq. (4.32), we have

$$q^{(2M+1)}=\mathcal{D}^{(2M+1)}\{\mathrm{T}\}. \tag{4.33}$$

C. Construction of the 't Hooft tensor

In this section, we give the prescription for constructing *'t Hooft tensors* [3,5] pertaining to the GGG monopoles, in all odd dimensions. This encapsulates the results of both previous Secs. IV A and IV B simultaneously. In three dimensions, the 't Hooft tensor is identified with the residual Maxwell field of the Dirac monopole. In general odd dimensions, the Dirac monopole generalizes to KR monopoles described in terms of KR fields (which coincide with Maxwell fields in $d=3$ dimensions). The construction of 't Hooft tensors in higher dimensions then follows naturally.

We first repeat the construction in three dimensions, using the notation developed above and pointing out the main steps as a guideline for the five dimensional case presented subsequently. This construction can be systematically extended to all odd dimensions.

In *three dimensions*, the asymptotic CS form (3.12) can be written as

$$\bar{\mathbf{\Omega}}^{(3)}=\frac{1}{c_3}\,\mathrm{tr}[\bar{\Phi}\bar{\mathbf{F}}]=\frac{1}{c_3}\,\mathrm{tr}[\bar{\Phi}(\mathrm{d}\bar{\mathbf{A}}+\bar{\mathbf{A}}\wedge\bar{\mathbf{A}})]. \tag{4.34}$$

The plan now is to add an asymptotically vanishing, gauge invariant, expression which cancels the second (highest order in $\bar{\mathbf{A}}$'s) term in Eq. (4.34), such that the result consists of an exact form plus a term depending on the Higgs fields only. To be concrete, one can check the identity

$$0=\mathrm{tr}[\bar{\Phi}\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}]=\mathrm{tr}[\bar{\Phi}(\mathrm{d}\bar{\Phi}\wedge\mathrm{d}\bar{\Phi}+4\bar{\mathbf{A}}\wedge\bar{\mathbf{A}})-4\,\mathrm{d}\bar{\Phi}\wedge\bar{\mathbf{A}}] \tag{4.35}$$

which is asymptotically zero by virtue of $\bar{\mathbf{D}}\bar{\Phi}=0$. Multiplying the identity (4.35) by the normalization factor $-(4c_3)^{-1}$, and adding this to the asymptotic CS form (4.34), we have the desired definition of the usual 't Hooft tensor

$$\mathbf{H}^{(3)}:=\bar{\mathbf{\Omega}}^{(3)}-\frac{1}{4c_3}\,\mathrm{tr}[\bar{\Phi}\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}] \tag{4.36}$$

$$=\mathrm{d}\mathbf{B}^{(3)}-\frac{1}{4c_3}\,\mathrm{tr}[\bar{\Phi}\mathrm{d}\bar{\Phi}\wedge\mathrm{d}\bar{\Phi}] \tag{4.37}$$

with

$$\mathbf{B}^{(3)}=\frac{1}{c_3}\,\mathrm{tr}[\bar{\Phi}\bar{\mathbf{A}}]. \tag{4.38}$$

In the definition equation (4.37), we have used the symbol $\mathbf{H}^{(3)}$ for the 't Hooft tensor, which is also the symbol for the KR field strength introduced in Appendix A. This is because $\bar{\mathbf{\Omega}}^{(3)}$ on the left hand side of Eq. (4.34) describes both these quantities, and the 't Hooft tensor can be identified with the KR field strength. The CP charge now equals the surface integral of this KR field strength $\mathbf{H}^{(3)}$, which by virue of Eq. (A9), defines magnetic KR charge,

$$q^{(3)}=\int_{S^2}\bar{\mathbf{\Omega}}^{(3)}=\int_{S^2}\mathbf{H}^{(3)}:=Q_m\,. \tag{4.39}$$

In the *regular* gauge, where the surface integral is over the entire (closed) S^2, the term $\mathrm{d}\mathbf{B}^{(3)}$ in Eq. (4.37) does not contribute. It follows that the only contribution comes from the second term

$$q^{(3)}=\int_{S^2}\mathbf{H}^{(3)}=-\frac{1}{4c_3}\int_{S^2}\mathrm{tr}[\bar{\Phi}\mathrm{d}\bar{\Phi}\wedge\mathrm{d}\bar{\Phi}]=\mathcal{W}^{(3)}[\Phi]. \tag{4.40}$$

This reproduces the result (4.7) of Sec. IV A in three dimensions.

In the *Dirac* gauge, the surface integral is carried out over the sphere with a hole around the negative or positive (south or north) Dirac string singularity, and the Higgs field is a constant. As a consequence of the constancy of the Higgs field, the term $\mathrm{tr}[\bar{\Phi}\mathrm{d}\bar{\Phi}\wedge\mathrm{d}\bar{\Phi}]$ in Eq. (4.37) vanishes so that the only contribution to the surface integral comes from the residue of the singularity in the first term. This is the (closed S^1) line integral of

$$\mathbf{B}^{(3)}|_{(\pm\bar{\Phi},\pm\bar{\mathbf{A}})}=\pm\frac{i}{c_3}\,\mathrm{tr}[\gamma_3^{\pm}\bar{\mathbf{A}}]={}^{\pm}\mathbf{B}^{(3)}, \tag{4.41}$$

in the notation of Eq. (4.21). Here ${}^{\pm}\mathbf{B}^{(3)}$ are the KR potentials of the KR field strength $\mathbf{H}^{(3)}=\mathrm{d}^{\pm}\mathbf{B}^{(3)}$ on $S^2_{\pm}$. This reproduces the results of Sec. IV B.

In *five dimensions*, the asymptotic CS form (3.13) can be written as

$$\begin{aligned}\bar{\mathbf{\Omega}}^{(5)}&=\frac{1}{c_{(5)}}\,\mathrm{tr}(\bar{\Phi}\bar{\mathbf{F}}\wedge\bar{\mathbf{F}})\\&=\frac{1}{c_{(5)}}\,\mathrm{tr}[\bar{\Phi}(\mathrm{d}\bar{\mathbf{A}}\wedge\mathrm{d}\bar{\mathbf{A}}+\mathrm{d}\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\\&\quad+\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\mathrm{d}\bar{\mathbf{A}}+\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\bar{\mathbf{A}})].\end{aligned} \tag{4.42}$$

As in the three dimensional case, the plan now is to add an asymptotically vanishing, gauge invariant, expression which cancels the fourth (highest order in $\bar{\mathbf{A}}$'s) term in Eq. (4.42), such that the result consists of an exact form plus a term depending on the Higgs fields only. Unlike in the three dimensional case where there is only one such candidate, cf. $\mathrm{tr}[\bar{\Phi}\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}]$ in Eq. (4.35), now there are two, namely $\mathrm{tr}[\bar{\Phi}\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}]$ and $\mathrm{tr}[\bar{\Phi}\bar{\mathbf{F}}\wedge\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}]$. It only remains to find the relative numerical coefficients of these two terms, such that they result in the elimination of the said term. To be concrete, one can check the identity

$$\begin{aligned}0&=\mathrm{tr}[\bar{\Phi}(\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}-8\bar{\mathbf{F}})\wedge\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}]\\&=\mathrm{tr}[\bar{\Phi}\mathrm{d}\bar{\Phi}\wedge\mathrm{d}\bar{\Phi}\wedge\mathrm{d}\bar{\Phi}\wedge\mathrm{d}\bar{\Phi}]-16\,\mathrm{tr}[\bar{\Phi}\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}]\\&\quad-8\,\mathrm{d}\,\mathrm{tr}\left[\bar{\Phi}\left(\mathrm{d}\bar{\Phi}\wedge\mathrm{d}\bar{\Phi}+\mathrm{d}\bar{\Phi}\wedge\bar{\mathbf{A}}\bar{\Phi}+\frac{1}{3}\bar{\mathbf{A}}\bar{\Phi}\wedge\bar{\mathbf{A}}\bar{\Phi}\right)\wedge\bar{\mathbf{A}}\right]\\&\quad-8\,\mathrm{tr}[\bar{\Phi}(\mathrm{d}\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}+\bar{\mathbf{A}}\wedge\mathrm{d}\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}+\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\mathrm{d}\bar{\mathbf{A}})]\\&\quad+8\,\mathrm{tr}[\mathrm{d}\bar{\Phi}(\mathrm{d}\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}+\bar{\mathbf{A}}\wedge\mathrm{d}\bar{\mathbf{A}}+\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\bar{\mathbf{A}})].\end{aligned} \tag{4.43}$$

Multiplying the identity (4.43) by the normalization factor $(16c_3)^{-1}$, and adding this to the asymptotic CS form (4.42), we have the definition of the 't Hooft tensor in five dimensions,

$$\mathbf{H}^{(5)}:=\bar{\mathbf{\Omega}}^{(5)}+\frac{1}{16c_5}\,\mathrm{tr}[\bar{\Phi}(\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}-8\bar{\mathbf{F}})\bar{\mathbf{D}}\bar{\Phi}\wedge\bar{\mathbf{D}}\bar{\Phi}] \tag{4.44}$$

$$=d\mathbf{B}^{(5)}+\frac{1}{16c_5}\mathrm{tr}[\bar{\Phi}d\bar{\Phi}\wedge d\bar{\Phi}\wedge d\bar{\Phi}\wedge d\bar{\Phi}] \tag{4.45}$$

with

$$\mathbf{B}^{(5)}=\frac{1}{2c_{(5)}}\mathrm{tr}\Big[\bar{\Phi}\Big(d\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}+\bar{\mathbf{A}}\wedge d\bar{\mathbf{A}}+\bar{\mathbf{A}}\wedge\bar{\mathbf{A}}\wedge\bar{\mathbf{A}} -\frac{1}{3}\bar{\mathbf{A}}\wedge\bar{\Phi}\bar{\mathbf{A}}\bar{\Phi}\wedge\bar{\mathbf{A}}\Big)+d\bar{\Phi}\wedge(\bar{\Phi}d\bar{\Phi}+\bar{\Phi}\bar{\mathbf{A}}\bar{\Phi})\wedge\mathbf{A}\Big] \tag{4.46}$$

Again, we have used the symbol $\mathbf{H}^{(3)}$ in the definition equation (4.37) which is also the symbol for the KR field strength introduced in Appendix A, defining the KR field strength describing the monopoles of the GGG models as the 't Hooft tensor (4.45). The CP charge then is the surface integral of the KR field strength $\mathbf{H}^{(5)}$ over S^4 in analogy to Eq. (4.39),

$$q^{(5)}=\int_{S^4}\bar{\mathbf{\Omega}}^{(5)}=\int_{S^4}\mathbf{H}^{(5)}=Q_m. \tag{4.47}$$

In the *regular* gauge, where the surface integral is over the entire (closed) S^4, the term $d\mathbf{B}^{(5)}$ in Eq. (4.45) does not contribute. It follows that the only contribution comes from the second term

$$q^{(5)}=\int_{S^4}\mathbf{H}^{(5)} =\frac{1}{16c_5}\int_{S^4}\mathrm{tr}[\bar{\Phi}d\bar{\Phi}\wedge d\bar{\Phi}\wedge d\bar{\Phi}\wedge d\bar{\Phi}] =\mathcal{W}^{(5)}[\Phi]. \tag{4.48}$$

This reproduces the result (4.7) of Sec. IV A in five dimensions.

In the *Dirac* gauge, the surface integral is carried out over the sphere with a hole around the negative or positive (south or north) Dirac string singularity, and the Higgs field is a constant. As a consequence of the constancy of the Higgs field, the term $\mathrm{tr}[\bar{\Phi}d\bar{\Phi}\wedge d\bar{\Phi}\wedge d\bar{\Phi}\wedge d\bar{\Phi}]$ in Eq. (4.45) vani'shes so that the only contribution to the surface integral comes from the residue of the singularity in the first term. This is the (closed S^3) integral of

$$\mathbf{B}^{(5)}|_{(\pm\bar{\Phi},\pm\bar{\mathbf{A}})}=\pm\frac{i}{c_5}\mathrm{tr}\Big[\gamma_5\Big({}^{\pm}\bar{\mathbf{F}}\wedge{}^{\pm}\bar{\mathbf{A}}-\frac{1}{3}{}^{\pm}\bar{\mathbf{A}}\wedge{}^{\pm}\bar{\mathbf{A}}\wedge{}^{\pm}\bar{\mathbf{A}}\Big)\Big] ={}^{\pm}\mathbf{B}^{(5)} \tag{4.49}$$

in the notation of Eq. (4.23). ${}^{\pm}\mathbf{B}^{(5)}$ are the KR potentials of the KR field strength $\mathbf{H}^{(5)}=d^{\pm}\mathbf{B}^{(5)}$ on $S^4_{\pm}$. This reproduces the result of Sec. IV B.

It is now clear that the prescription for the construction of the 't Hooft tensor in three and five dimensions can be systematically extended to any odd dimension. The result can be stated formally, omitting the expression corresponding to Eqs. (4.36) and (4.44) in the definitions of the 't Hooft tensors, and restricting to the result corresponding to Eqs. (4.37) and (4.45). It is

$$\mathbf{H}^{(2M+1)}=d\mathbf{B}^{(2M+1)}+\frac{1}{W_{2M+1}}\mathrm{tr}[\bar{\Phi}\ \underbrace{d\bar{\Phi}\wedge\dots\wedge d\bar{\Phi}}_{M}], \tag{4.50}$$

where $W_{2M+1}=(-4)^M c_{2M+1}$ is the normalization constant appearing in Eq. (2.11). In the Dirac gauge, the expression corresponding to Eqs. (4.41) and (4.49) is

$$\mathbf{B}^{(2M+1)}|_{(\pm\bar{\Phi},\pm\bar{\mathbf{A}})}=\pm\frac{i}{c_{2M+1}}\boldsymbol{\omega}^{(M)}[{}^{\pm}\bar{\mathbf{A}},{}^{\pm}\bar{\mathbf{F}}]={}^{\pm}\mathbf{B}^{(2M+1)} \tag{4.51}$$

which yields the KR potential (4.24) on $S^{2M}_{\pm}$ for the KR field strength $\mathbf{B}^{(2M+1)}$ which we generally define as the 't Hooft tensor of an odd dimensional GGG model.

V. SUMMARY AND DISCUSSION

We have studied the topological properties of the solitons of the generalized Georgi-Glashow (GGG) models, which we have described along with the corresponding reduced Chern-Pontryagin (CP) densities and Chern-Simons (CS) forms in the *regular* and the the *Dirac* gauges, in all dimensions. In common with the familiar three dimensional case, the topology of these monopoles can be described exclusively by the $SO(d)$ isovector Higgs fields in the *regular* gauge, identifying the CP charge with the Higgs field winding number. All this was carried out in all, even and odd, dimensions.

Just as the magnetic Maxwell field, which is the reduced CS form pertaining to the monopole of the three dimensional GG model, can be identified as the Dirac monopole field strength, we have identified the reduced CS forms, descended from higher order CP densities, as Kalb-Ramond (KR) monopole fields in all odd dimensions. The role analogous to that of the 't Hooft-Polyakov monopole is played by the solitons of the GGG models, which we have called GGG monopoles. This construction is not possible in even dimensions.

Just as in the three dimensional case, the study of the GGG monopoles and the corresponding CS forms, both in the *regular* and *Dirac* gauges, led us to define the 't Hooft tensors in all odd dimensions. These are identified with the relevant KR field strengths of the GGG monopoles. All our concrete constructions were carried out using the hedgehog (spherically symmetric) GGG monopoles in three and five dimensions, but the results hold generally.

We now discuss briefly, connections of our results with physically relevant problems.

The first is concerned with the construction of dilute gases of GGG monopoles. KR theory is linear in the sense that the sum of two solutions is again a solution to the KR equations (A2) and (A8). This allows the construction of a *dilute gas* of KR monopoles in analogy to the Coulomb gas in usual Maxwell theory. This Coulomb gas was used in Polyakov's work [22] to construct a dilute gas of 't Hooft-Polyakov monopoles of the usual GG model in three dimensions, yielding a mechanism for confinement in the resulting QCD toy model. One can thus expect that Polyakov's construction can be adopted to the GGG models in odd dimensions, but unfortunately not in even dimensions — and particularly not in the physically interesting case $d=4$, where the significance of the GGG model $\mathcal{E}^{(2,4)}$ has recently been pointed out [31]. Indeed, the construction of a dilute gas from KR monopoles in arbitrary dimensions was carried out long ago in the context of lattice field theory [15–17].

Perhaps of most topical interest is the relevance of our results to the modern concept of branes. In usual Maxwell theory, the elementary electrically charged objects are pointlike (zero dimensional), and their time evolution is described by a *worldline*. The Maxwell potential describing a magnetic field, on the other hand, is a 1-form which can be integrated along the worldline of the electric charge. This line integral, multiplied by the electric charge coupling constant e, yields the interaction energy (or, in Minkowskian spacetime, action) between the electric charge and the Maxwell magnetic field. Generalizing this construction to higher dimensions, the electrically charged objects which couple to $d-2$ form KR potentials must have a $d-2$ dimensional world volume such that the interaction between the electrically charged object and the KR field is described as the integral of the KR potential over this world volume. This means that the electrically charged object itself is a $d-3$ brane. Nepomechie [18] has shown that the existence of a KR monopoles with magnetic charge Q_m leads to a quantization condition for the electric charge of the branes which couple to the KR monopole field in the way described above. Therefore the GGG monopoles in $2M+1$ dimensions which we have discussed yield the quantization of the elementary electric charge of $2M-2$ branes in analogy to the 't Hooft-Polyakov monopole which forces the quantization of elementary electric point charges (which are 0-branes). KR fields, on the other hand, arise naturally in the context of string theories [9] which are the background of all modern brane physics, so that we expect strings and D-branes to be the correct context in which our constructions may be of some significance.

Finally, we discuss a generalization of the GGG monopoles used in this work. The dynamics of the systems studied was described by the so-called GGG models, and in particular the solitons they support in odd spatial dimensions are the GGG monopoles alluded to above. These GGG monopoles are the classical solutions to the Euler-Lagrange equations in the temporal gauge $A_0=0$. When the odd spatial dimensions are restricted to $d=4M-1$, the GGG monopoles are in fact solutions to the relevant first order Bogomol'nyi or self-duality equations. These were discussed in Ref. [32] and the analytic proof of existence given in Ref. [33]. These self-dual GGG monopoles, which are generalizations of the Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles in three ($M=1$) dimensions, are entirely suited for generalization to the corresponding ''dyons'' following the prescription used by Julia and Zee [34] in the $M=1$ case. This task can be performed systematically and will be reported elsewhere. The question here is, can such ''dyons'' be described as KR fields?

ACKNOWLEDGMENTS

We are grateful to P. Orland for informative communications. F.Z. acknowledges the Higher Education Authority (HEA), Ireland, for financial support.

APPENDIX A: KALB-RAMOND THEORY AND MAGNETIC MONOPOLES

In this appendix, we summarize the theory of static Kalb-Ramond (KR) fields and KR monopoles [15–18], following the explicit continuum constructions of Nepomechie [18].

1. Free KR theory

Kalb-Ramond (KR) theories [9] generalize Maxwell electromagnetism as they deal with higher-rank antisymmetric tensor field strength instead of the rank two Maxwell field strength tensor. In mathematical terminology, KR field strengths are real valued r-forms in $d+1$ Minkowskian spacetime dimensions. Static KR theory involving r-forms on d dimensional space $\mathbb{R}^d$ then generalizes Maxwell magnetostatics.

Here, we consider the special case $r=d-1$ in d spatial dimensions and simply refer to this special case of magnetic KR theories as the ''KR theory in d dimensions'' since it is this type of magnetic KR theory which we consider in the main part of this work. It is defined in terms of the magnetic KR field strength $(d-1)$-form

$$\mathbf{H}=\frac{1}{(d-1)!}\mathsf{H}_{\mu_1\ldots\mu_{d-1}}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{d-1}}=\tilde{\mathsf{H}}_\mu(*\mathrm{d}x^\mu),\tag{A1}$$

$$\mathsf{H}_{\mu_1\ldots\mu_{d-1}}:=\epsilon_{\nu\mu_1\ldots\mu_{d-1}}\tilde{\mathsf{H}}_\nu$$

which is closed on $\mathbb{R}^d$,

$$\mathrm{d}\mathbf{H}=0\Leftrightarrow\epsilon_{\mu_1\ldots\mu_d}\partial_{\mu_1}\mathsf{H}_{\mu_2\ldots\mu_d}=0\Leftrightarrow\partial_\mu\tilde{\mathsf{H}}_\mu=0.\tag{A2}$$

Poincaré's lemma then allows us to introduce the KR potential $(d-2)$-form

$$\mathbf{B}=\frac{1}{(d-2)!}\mathsf{B}_{\mu_1\ldots\mu_{d-2}}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{d-2}}=\frac{1}{2}\tilde{\mathsf{B}}_{\mu_1\mu_2}*(\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}),\tag{A3}$$

$$\mathsf{B}_{\mu_1\ldots\mu_{d-2}}=\frac{1}{2}\epsilon_{\mu_1\ldots\mu_{d-2}\nu_1\nu_2}\tilde{\mathsf{B}}_{\nu_1\nu_2}$$

such that

$$\mathbf{H}=\mathrm{d}\mathbf{B}\Leftrightarrow\mathsf{H}_{\mu_1\ldots\mu_{d-1}}=\partial_{[\mu_1}\mathsf{B}_{\mu_2\ldots\mu_{d-1}]}\Leftrightarrow\tilde{\mathsf{H}}_\mu=\partial_\nu\tilde{\mathsf{B}}_{\mu\nu}.\tag{A4}$$

$\mathbf{B}$ is well defined up to an exact $(d-2)$-form $\mathrm{d}\boldsymbol{\Lambda}$,

$$\boldsymbol{\Lambda}=\frac{1}{(d-3)!}\Lambda_{\mu_1\ldots\mu_{d-3}}\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{d-3}}$$
$$=\frac{1}{3!}\tilde{\Lambda}_{\mu_1\mu_2\mu_3}*(\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\mathrm{d}x^{\mu_3}),$$

$$\Lambda_{\mu_1\ldots\mu_{d-3}}=\frac{1}{3!}\epsilon_{\mu_1\ldots\mu_{d-3}\nu_1\nu_2\nu_3}\tilde{\Lambda}_{\nu_1\nu_2\nu_3} \tag{A5}$$

which means that $\mathbf{H}$ is invariant under KR transformations

$$\mathbf{B}\mapsto\mathbf{B}+\mathrm{d}\boldsymbol{\Lambda}\Leftrightarrow \mathrm{B}_{\mu_1\ldots\mu_{d-2}}\mapsto \mathrm{B}_{\mu_1\ldots\mu_{d-2}}+\partial_{[\mu_1}\Lambda_{\mu_2\ldots\mu_{d-2}]}$$
$$\Leftrightarrow\tilde{\mathrm{B}}_{\mu_1\mu_2}\mapsto\tilde{\mathrm{B}}_{\mu_1\mu_2}+\partial_\nu\tilde{\Lambda}_{\nu\mu_1\mu_2} \tag{A6}$$

and a KR energy functional is supposed to preserve this invariance.

Free KR theory is given by the energy functional

$$\mathrm{E}_{KR}^{(d)}[\mathbf{H}]:=\frac{1}{2}\int_{\mathbb{R}^d}\mathbf{H}\wedge*\mathbf{H}:=\frac{1}{2(d-1)!}\int_{\mathbb{R}^d}\mathrm{H}^2_{\mu_1\ldots\mu_{d-1}}$$
$$=\frac{1}{2}\int_{\mathbb{R}^d}\tilde{\mathrm{H}}^2_\mu. \tag{A7}$$

Varying $\mathrm{E}_{KR}^{(d)}$ with respect to $\mathbf{B}$ yields the magnetic KR equation

$$*\mathrm{d}*\mathbf{H}=0\Leftrightarrow\partial_\nu\mathrm{H}_{\nu\mu_1\ldots\mu_{d-2}}=0\Leftrightarrow\partial_{[\mu_1}\tilde{\mathrm{H}}_{\mu_2]}=0. \tag{A8}$$

Equations (A2) and (A8) are the d-dimensional generalization of the magnetic Maxwell equations in $d=3$ space dimensions.

2. KR monopoles in d dimensions

Assuming the existence of ''magnetic'' charges changes the mathematical structure of KR theory. From the analog of Maxwell magnetostatics, the flux of the magnetic KR field strength form through the sphere S^{d-1} equals a magnetic KR charge Q_m,

$$Q_m=\int_{S^{d-1}}\mathbf{H}. \tag{A9}$$

Since S^{d-1} is not simply connected, $\mathrm{d}\mathbf{H}=0$ does not enforce the existence of a global KR potential form. However, on the simply connected upper and lower half spheres $S_+^{d-1},S_-^{d-1}\subset S^{d-1}$, KR potential forms exist, $\mathbf{H}|_{S_\pm^{d-1}}=\mathrm{d}^\pm\mathbf{B}$. In order to yield a well-defined field strength form $\mathbf{H}$, the potential forms ${}^+\mathbf{B}$ and ${}^-\mathbf{B}$ on the overlap of their definition ranges, i.e., the equator $S_+^{d-1}\cap S_-^{d-1}=S^{d-2}$, differ by a *transition form* $\mathbf{U}={}^+\mathbf{B}|_{S^{d-2}}-{}^-\mathbf{B}|_{S^{d-2}}$ which has to be closed, $\mathrm{d}\mathbf{U}=0$. The magnetic KR charge is determined by $\mathbf{U}$,

$$Q_m=\int_{S^{d-1}}\mathbf{H}$$
$$=\int_{S_+^{d-1}}\mathrm{d}^+\mathbf{B}+\int_{S_-^{d-1}}\mathrm{d}^-\mathbf{B}$$
$$=\int_{S^{d-2}}({}^+\mathbf{B}-{}^-\mathbf{B})$$
$$=\int_{S^{d-2}}\mathbf{U}. \tag{A10}$$

In analogy to the corresponding construction for the Dirac monopole, Eq. (A10) can be called the Wu-Yang construction for magnetic KR charge. The crucial point in the construction is that $\mathbf{U}$ is not exact on the equator S^{d-2} although it is closed.

An actual solution $\mathbf{H}_{(P)}$ of the KR equations (A2) and (A8) on S^{d-1} with nonvanishing flux (A9) is called a (magnetic) *KR monopole*. One can show [18,35] that the only such solution is given by

$$\mathbf{H}_{(P)}=\frac{Q_m}{\kappa_{d-1}}\frac{\hat{x}_\mu}{r^{d-1}}(*\mathrm{d}x^\mu) \tag{A11}$$

with $\mathbf{H}_{(P)}=\mathrm{d}^\pm\mathbf{B}_{(P)}$ on $S_\pm^{d-1}$, respectively. The explicit expressions for ${}^\pm\mathbf{B}_{(P)}$, which do not concern us here, are given in Ref. [18]. Interpreting this as $(d-1)$-form on $\mathbb{R}^d$ in distributional sense (i.e., including the singularity at the origin) and using

$$\partial_\mu\left(\frac{\hat{x}_\mu}{r^{d-1}}\right)=\kappa_{d-1}\delta(\mathrm{x}) \tag{A12}$$

yields

$$\mathrm{d}\mathbf{H}_{(P)}=Q_m\delta(\mathrm{x})(*1), \tag{A13}$$

i.e., Eq. (A2) is modified by a point charge acting as source for the KR field strength.

APPENDIX B: MINIMAL GENERALIZED GEORGI-GLASHOW MODELS IN $d=2,3,4,5$ DIMENSIONS

In this appendix we give the explicit expression for the energy densities, CP densities and CS densities or forms of the minimal GGG models in $d=2,3,4,5$ dimensions.

Using $\mathrm{S}=-(\Phi^2+\mathbb{1})$, the energy densities are

$$\mathcal{E}^{(1,2)}=\frac{1}{C_{(1,2)}}\mathrm{tr}\left[-\frac{1}{4}\mathrm{F}^2_{\mu\nu}-\frac{1}{2}(\mathrm{D}_\mu\Phi)^2+\frac{1}{2}\mathrm{S}^2\right] \tag{B1}$$

$$\mathcal{E}^{(1,3)}=\frac{1}{C_{(1,3)}}\mathrm{tr}\left[-\frac{1}{4}\mathrm{F}^2_{\mu\nu}-\frac{1}{2}(\mathrm{D}_\mu\Phi)^2\right] \tag{B2}$$

$$\mathcal{E}^{(2,4)}=\frac{1}{C_{(2,4)}}\mathrm{tr}[\{\mathrm{F}_{\mu[\nu}\mathrm{F}_{\rho\sigma]}\}^2+4\{\mathrm{F}_{[\mu\nu},\mathrm{D}_{\rho]}\Phi\}^2-18(\{\mathrm{S},\mathrm{F}_{\mu\nu}\}$$
$$+[\mathrm{D}_\mu\Phi,\mathrm{D}_\nu\Phi])^2-54\{\mathrm{S},\mathrm{D}_\mu\Phi\}^2+54\mathrm{S}^4] \tag{B3}$$

$$\mathcal{E}^{(2,5)}=\frac{1}{C_{(2,5)}}\mathrm{tr}[\{F_{\mu[\nu}F_{\rho\sigma]}\}^2+\{F_{[\mu\nu},D_{\rho]}\Phi\}^2-24(\{S,F_{\mu\nu}\}+[D_\mu\Phi,D_\nu\Phi])^2-48\{S,D_\mu\Phi\}^2]. \tag{B4}$$

$C_{(p,d)}>0$ are normalization constants to be discussed below.

The corresponding residual CP densities $\varrho^{(p,d)}\leqslant\mathcal{E}^{(p,d)}$ are found to be

$$\varrho^{(1,2)}=\frac{1}{2C_{(1,2)}}i\epsilon_{\mu\nu}\,\mathrm{tr}[\gamma_3(SF_{\mu\nu}+D_\mu\Phi D_\nu\Phi)] \tag{B5}$$

$$\varrho^{(1,3)}=-\frac{1}{4C_{(1,3)}}\epsilon_{\mu\nu\rho}\,\mathrm{tr}[F_{\mu\nu}D_\rho\Phi] \tag{B6}$$

$$\varrho^{(2,4)}=\frac{18}{C_{(2,4)}}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{tr}[\gamma_5(S^2F_{\mu\nu}F_{\rho\sigma}-2S\{F_{\mu\nu},D_\rho\Phi\}D_\sigma\Phi+2(SF_{\mu\nu}+D_\mu\Phi D_\nu\Phi)(SF_{\rho\sigma}+D_\rho\Phi D_\sigma\Phi))] \tag{B7}$$

$$\varrho^{(2,5)}=-\frac{24}{C_{(2,5)}}i\epsilon_{\mu\nu\rho\sigma\tau}\,\mathrm{tr}[3SF_{\mu\nu}F_{\rho\sigma}D_\tau\Phi+2F_{\mu\nu}D_\rho\Phi D_\sigma\Phi D_\tau\Phi]. \tag{B8}$$

The volume integral (3.7) of the CP densities yields the CP charge $q^{(p,d)}$. It is usual to choose the normalization constants $C_{(p,d)}$ such that the hedgehog (2.12) has unit topological charge.

The CP charges can be written as total divergences of the CS densities $\Omega_\lambda^{(p,d)}$,

$$\varrho^{(p,d)}=\partial_\lambda\Omega_\lambda^{(p,d)}. \tag{B9}$$

The corresponding explicit expressions for the CS densities are

$$\Omega_\lambda^{(1,2)}=-\frac{1}{C_{(1,2)}}i\epsilon_{\lambda\mu}\,\mathrm{tr}\left[\gamma_3\left(A_\mu-\frac{1}{2}\Phi D_\mu\Phi\right)\right] \tag{B10}$$

$$\Omega_\lambda^{(1,3)}=-\frac{1}{4C_{(1,3)}}\epsilon_{\lambda\mu\nu}\,\mathrm{tr}[\Phi F_{\mu\nu}] \tag{B11}$$

$$\Omega_\lambda^{(2,4)}=-\frac{108}{C_{(2,4)}}\epsilon_{\lambda\mu\nu\rho}\,\mathrm{tr}\left[\gamma_5\left(F_{\mu\nu}A_\rho-\frac{2}{3}A_\mu A_\nu A_\rho-\frac{1}{2}(2\mathbb{1}+\Phi^2)\times\Phi\{F_{\mu\nu},D_\rho\Phi\}+\frac{1}{3}\Phi D_\mu\Phi D_\nu\Phi D_\rho\Phi\right)\right] \tag{B12}$$

$$\Omega_\lambda^{(2,5)}=\frac{24}{C_{(2,5)}}i\epsilon_{\lambda\mu\nu\rho\sigma}\,\mathrm{tr}[(3\mathbb{1}+\Phi^2)\Phi F_{\mu\nu}F_{\rho\sigma}-2\Phi D_\mu\Phi D_\nu\Phi F_{\rho\sigma}]. \tag{B13}$$

Note that in the $d=2$ formulas (B1), (B5) and (B10), the gauge field $A_\mu=-iA_\mu\gamma_3$ is Abelian, and the Higgs field then has only two components as seen from Eq. (2.1), or alternatively can be parametrized by a single complex field φ in Eq. (3.5). It is then obvious that Eq. (B1) pertains to the usual Abelian Higgs model.

Finally, we give the explicit expressions for the CS forms $\mathbf{\Omega}^{(p,d)}$ of the four minimal GGG models (B1)–(B4) in the language of differential forms used in this work. They are defined as Hodge duals of the CS density 1-forms with components $\Omega_\lambda^{(p,d)}$ given by Eqs. (B10)–(B18),

$$\mathbf{\Omega}^{(p,d)}=\Omega_\lambda^{(p,d)}({*}dx^\lambda), \tag{B14}$$

thus

$$\mathbf{\Omega}^{(1,2)}=-\frac{1}{C_{(1,2)}}i\,\mathrm{tr}\left[\gamma_3\left(\mathbf{A}-\frac{1}{2}\Phi\mathbf{D}\Phi\right)\right] \tag{B15}$$

$$\mathbf{\Omega}^{(1,3)}=-\frac{1}{2C_{(1,3)}}\mathrm{tr}[\Phi\mathbf{F}] \tag{B16}$$

$$\mathbf{\Omega}^{(2,4)}=-\frac{216}{C_{(2,4)}}\mathrm{tr}\left[\gamma_5\left(\mathbf{F}\wedge\mathbf{A}-\frac{1}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}-\frac{1}{2}(2\mathbb{1}+\Phi^2)\times\Phi\{\mathbf{F},\mathbf{D}\Phi\}+\frac{1}{6}\Phi\mathbf{D}\Phi\wedge\mathbf{D}\Phi\wedge\mathbf{D}\Phi\right)\right] \tag{B17}$$

$$\mathbf{\Omega}^{(2,5)}=\frac{96}{C_{(2,5)}}i\,\mathrm{tr}[(3\mathbb{1}+\Phi^2)\Phi\mathbf{F}\wedge\mathbf{F}-\Phi\mathbf{D}\Phi\wedge\mathbf{D}\Phi\wedge\mathbf{F}]. \tag{B18}$$

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