Wave propagation in linear electrodynamics

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The Fresnel equation governing the propagation of electromagnetic waves for the most general linear constitutive law is derived. The wave normals are found to lie, in general, on a fourth order surface. When the constitutive coefficients satisfy the so-called reciprocity or closure relation, one can define a duality operator on the space of the two-forms. We prove that the closure relation is a sufficient condition for the reduction of the fourth order surface to the familiar second order light cone structure. We finally study whether this condition is also necessary.

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I. INTRODUCTION

The electromagnetic wave represents perhaps the most important classical device with the help of which one can carry out physical measurements and transmit information. The intrinsic properties and motion of material media, as well as the geometrical structure of spacetime, can affect the propagation of electromagnetic waves. In the most general setting [1,2], electromagnetic phenomena are described by the pair of two-forms H,F (called the electromagnetic excitation and the field strength, respectively) which satisfy the Maxwell equations $dH=J, dF=0$, *together* with the constitutive law $H=H(F)$. The latter relation contains crucial information about the underlying physical continuum (i.e., material medium and/or spacetime). Mathematically, this constitutive law arises either from a suitable phenomenological theory of a medium or from the electromagnetic field Lagrangian.

In general, the constitutive law establishes a nonlinear (or even nonlocal) relation between the electromagnetic excitation and the field strength. The function (or functional) $H(F)$ may depend on the polarization and magnetization properties of matter, and/or on the spacetime geometry, i.e., metric, curvature, torsion, and nonmetricity. Previously, the propagation of electromagnetic waves was analyzed for a variety of constitutive laws: for nonlinear models in Minkowski and Riemannian spacetimes [3], for electrodynamics in a Riemann-Cartan manifold [4], and also for certain nonminimal and higher derivative gravity models [5]. Numerous authors [6] discussed electromagnetic waves in Einstein-Maxwell theory. The main aim of this paper is to investigate wave propagation in Maxwell electrodynamics with the most general *linear* constitutive law. We derive the generalized Fresnel equation which determines the wave normals directly from the constitutive coefficients. This result is of interest, e.g., for various applications in crystaloptics and related domains.

Another motivation for the present work comes from the study of a deep relationship between the duality operators defined on two-forms and the conformal classes of spacetime metrics in four dimensions. Within classical Maxwell electrodynamics, Toupin, Schönberg, and others [8] have noticed that the constitutive coefficients define a duality operator, provided a certain reciprocity or closure condition is fulfilled, and gave first demonstrations of the existence of the corresponding conformal metric structure. Later these observations were rediscovered and developed in mathematics [9] and in gravity theory [10]. Recently the complete explicit solution of the closure relation has been given [11], and it was conjectured that the reciprocity condition is a necessary and sufficient condition for the standard null-cone structure for the light propagation (see also independent arguments in Ref. [12]). Here we give a partial answer to this question.

II. ELECTRODYNAMICS WITH LINEAR CONSTITUTIVE LAW

Let us consider the Maxwell equations in vacuum

$$dH=0, \quad dF=0, \tag{2.1}$$

i.e., we assume that the electric current three-form J vanishes in the spacetime region under consideration. Given the local coordinates x^i, $i=0,1,2,3$, we can decompose the exterior forms as

$$H=\frac{1}{2}H_{ij}dx^i\wedge dx^j, \quad F=\frac{1}{2}F_{ij}dx^i\wedge dx^j. \tag{2.2}$$

Following Refs. [11,13], we write the linear constitutive law in terms of the electromagnetic excitation and field strength tensors as

$$H_{ij}=\frac{1}{4}\epsilon_{ijkl}\chi^{klmn}F_{mn}, \quad i,j,\ldots=0,1,2,3. \tag{2.3}$$

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Here ϵ_{ijkl} is the Levi-Civita symbol and $\chi^{ijkl}(x)$ an even tensor density of weight $+1$ (called the constitutive tensor density) which can be decomposed according to

$$\chi^{ijkl}=f(x)\overset{\circ}{\chi}{}^{ijkl}+\alpha(x)\epsilon^{ijkl}, \quad \text{with } \overset{\circ}{\chi}{}^{[ijkl]}\equiv 0. \quad (2.4)$$

Here $f(x)$ is a dimensionful scalar function such that $\overset{\circ}{\chi}{}^{ijkl}$ is dimensionless. The pseudo-scalar constitutive function $\alpha(x)$ can be identified (on the *kinematic* level) as an Abelian axion field, whereas $f(x)$ can be interpreted as a dilaton scalar field. Note that $\overset{\circ}{\chi}{}^{ijkl}$ has the same algebraic symmetries and therefore the same number of 20 independent components as a Riemannian curvature tensor:

$$\overset{\circ}{\chi}{}^{ijkl}=-\overset{\circ}{\chi}{}^{jikl}=-\overset{\circ}{\chi}{}^{ijlk}=\overset{\circ}{\chi}{}^{klij}, \quad \overset{\circ}{\chi}{}^{[ijkl]}=0. \quad (2.5)$$

This follows from the existence and the structure of the Lagrangian for the linear electrodynamics $V_{\rm lin}=-\frac{1}{2}H\wedge F$, see Refs. [2,13]. It is convenient to adopt a more compact (essentially *bivector*) notation by defining the three-(co)vector quantities

$$\mathcal{D}^a:=\begin{pmatrix} H_{23}\\ H_{31}\\ H_{12}\end{pmatrix}, \quad \mathcal{H}_a:=\begin{pmatrix} H_{01}\\ H_{02}\\ H_{03}\end{pmatrix}$$

and

$$B^a:=\begin{pmatrix} F_{23}\\ F_{31}\\ F_{12}\end{pmatrix}, \quad E_a:=\begin{pmatrix} F_{10}\\ F_{20}\\ F_{30}\end{pmatrix}, \quad (2.6)$$

for the electric and magnetic excitations, and for the magnetic and electric field strengths, respectively. The Latin indices label now $a,b,c,\ldots=1,2,3$. The constitutive tensor is then naturally parametrized by a triplet of 3×3 matrices, $\overset{\circ}{\chi}{}^{ijkl}=\{\mathcal{A}^{ab},\mathcal{B}_{ab},\mathcal{C}^a{}_b\}$, so that the constitutive law (2.4) is finally recast into

$$\begin{pmatrix}\mathcal{H}_a\\ \mathcal{D}^a\end{pmatrix}=f(x)\begin{pmatrix}\mathcal{C}^b{}_a & \mathcal{B}_{ab}\\ \mathcal{A}^{ab} & \mathcal{C}^a{}_b\end{pmatrix}\begin{pmatrix}-E_b\\ B^b\end{pmatrix}+\alpha(x)\begin{pmatrix}-E_a\\ B^a\end{pmatrix}. \quad (2.7)$$

Here the 3×3 matrices satisfy $\mathcal{A}^{ab}=\mathcal{A}^{ba}$, $\mathcal{B}_{ab}=\mathcal{B}_{ba}$, and $\mathcal{C}_a{}^a=0$, thereby providing the algebraic properties (2.5).

III. WAVE PROPAGATION: FRESNEL EQUATION

In the theory of partial differential equations, the propagation of waves is described by Hadamard discontinuities of solutions across a characteristic (wave front) hypersurface S [7]. One can locally define S by the equation $\Phi(x^i)=\text{const.}$ The Hadamard discontinuity of any function $\mathcal{F}(x)$ across the hypersurface S is determined as the difference $[\mathcal{F}](x):=\mathcal{F}(x_+)-\mathcal{F}(x_-)$, where $x_\pm:=\lim_{\varepsilon\to 0}(x\pm\varepsilon)$ are points on the opposite sides of $S\ni x$. An ordinary electromagnetic wave is a solution of the Maxwell equations (2.1) for which the derivatives of H and F have regular discontinuities across the wave front hypersurface S.

In terms of the (co)vector components, we have on the characteristic hypersurface S:

$$[\mathcal{D}^a]=0, \quad [\partial_i\mathcal{D}^a]=d^a q_i, \quad [\mathcal{H}_a]=0, \quad [\partial_i\mathcal{H}_a]=h_a q_i, \quad (3.1)$$

$$[B^a]=0, \quad [\partial_i B^a]=b^a q_i, \quad [E_a]=0, \quad [\partial_i E_a]=e_a q_i, \quad (3.2)$$

where d^a,h_a,b^a,e_a describe discontinuities of the corresponding quantities across S, and the wave covector normal to the front is given by

$$q_i:=\partial_i\Phi. \quad (3.3)$$

Equations (3.1),(3.2) represent the Hadamard geometrical compatibility conditions. Substituting Eq. (2.2) into Eq. (2.1), and using Eqs. (2.6) and (3.1),(3.2), we find

$$q_0 d^a-\epsilon^{abc}q_b h_c=0, \quad q_0 b^a+\epsilon^{abc}q_b e_c=0, \quad (3.4)$$

$$q_a d^a=0, \quad q_a b^a=0, \quad (3.5)$$

where ϵ^{abc} is the three-dimensional Levi-Civita symbol. In this system only the six equations (3.4) are independent. Assuming that $q_0\neq 0$, one finds that Eqs. (3.5) are trivially satisfied if one substitutes Eq. (3.4) into them. (Note that the characteristics with $q_0=0$ do not have intrinsic meaning for the evolution equations, since they obviously depend on the arbitrary choice of coordinates.)

Differentiating Eq. (2.7) and using the compatibility conditions (3.1),(3.2), we find additionally six algebraic equations

$$\begin{pmatrix}h_a\\ d^a\end{pmatrix}=f(x)\begin{pmatrix}\mathcal{C}^b{}_a & \mathcal{B}_{ab}\\ \mathcal{A}^{ab} & \mathcal{C}^a{}_b\end{pmatrix}\begin{pmatrix}-e_b\\ b^b\end{pmatrix}+\alpha(x)\begin{pmatrix}-e_a\\ b^a\end{pmatrix}. \quad (3.6)$$

Note that the constitutive coefficients and their first derivatives are assumed to be continuous across S.

We can now substitute d^a and h_a from Eq. (3.6) into the first equation (3.4), which gives

$$f(x)q_0(-\mathcal{A}^{ab}e_b+\mathcal{C}^a{}_b b^b)+\alpha(x)q_0 b^a = f(x)\epsilon^{abc}q_b(-\mathcal{C}^d{}_c e_d+\mathcal{B}_{cd}b^d)-\alpha(x)\epsilon^{abc}q_b e_c. \quad (3.7)$$

The terms proportional to the axion field $\alpha(x)$ drop out completely due to Eq. (3.4), and then one can also remove the common dilaton factor $f(x)$ on both sides of the equation. [We assume $f(x)\neq 0$, since otherwise there is no hyperbolic evolution system.] Finally we substitute b^a in terms of e_b from the second equation (3.4), and after some rearrangements one finds

$$(q_0^2\mathcal{A}^{ab}+q_0 q_d[\mathcal{C}^a{}_c\epsilon^{cdb}+\mathcal{C}^b{}_c\epsilon^{cda}]+q_e q_f\epsilon^{aec}\epsilon^{bfd}\mathcal{B}_{cd})e_b = 0. \quad (3.8)$$

This homogeneous algebraic equation has a nontrivial solution when

$$\mathcal{W}:=\det|q_0^2\mathcal{A}^{ab}+q_0q_d[\mathcal{C}^a{}_c\epsilon^{cdb}+\mathcal{C}^b{}_c\epsilon^{cda}]+q_eq_f\epsilon^{aec}\epsilon^{bfd}\mathcal{B}_{cd}|=0. \tag{3.9}$$

This is a Fresnel equation which is central in the wave propagation analysis. It determines the geometry of the wave normals in terms of the constitutive coefficients $\mathcal{A},\mathcal{B},\mathcal{C}$. A direct calculation yields the general result

$$\begin{aligned}\mathcal{W}=q_0^2(&q_0^4M+q_0^3q_aM^a+q_0^2q_aq_bM^{ab}\\&+q_0q_aq_bq_cM^{abc}+q_aq_bq_cq_dM^{abcd})\\=0,&\end{aligned} \tag{3.10}$$

where we have denoted

$$M:=\det\mathcal{A},\quad M^a:=2\epsilon_{bcd}\mathcal{A}^{ab}\mathcal{C}^c{}_e\mathcal{A}^{ed}, \tag{3.11}$$

$$\begin{aligned}M^{ab}:=\mathcal{B}_{cd}(\mathcal{A}^{ab}\mathcal{A}^{cd}-\mathcal{A}^{ac}\mathcal{A}^{bd})-\mathcal{A}^{cd}\mathcal{C}^a{}_c\mathcal{C}^b{}_d\\+4\mathcal{A}^{ac}\mathcal{C}^b{}_d\mathcal{C}^d{}_c-2\mathcal{A}^{ab}\mathcal{C}^c{}_d\mathcal{C}^d{}_c,\end{aligned} \tag{3.12}$$

$$M^{abc}:=2\epsilon^{cde}[\mathcal{B}_{df}(\mathcal{A}^{ab}\mathcal{C}^f{}_e-\mathcal{A}^{af}\mathcal{C}^b{}_e)+\mathcal{C}^a{}_e\mathcal{C}^b{}_f\mathcal{C}^f{}_d] \tag{3.13}$$

$$M^{abcd}:=\epsilon^{cef}\epsilon^{dgh}\mathcal{B}_{fh}[\tfrac{1}{2}\mathcal{A}^{ab}\mathcal{B}_{eg}-\mathcal{C}^a{}_e\mathcal{C}^b{}_g]. \tag{3.14}$$

Note that only the completely symmetric parts $M^{(a_1\cdots a_p)}$, $p=2,3,4$, contribute to the Fresnel equation. Since $q_0\neq 0$, one can delete the first factor in Eq. (3.10), and thus we finally find that the wave covector q_i lies, in general, on a fourth order surface. This is different from the light *cone* (i.e., second order) structure which arises only in a particular case. In the next section we demonstrate that the latter corresponds to the closure condition. Earlier, the relation between the fourth- and the second-order wave geometry was studied by Tamm [16] for a special case of the linear constitutive law.

IV. THE CLOSURE RELATION AS A SUFFICIENT CONDITION

The linear constitutive law defines a duality operator when the constitutive coefficients satisfy the ''reciprocity'' or ''closure'' relation [8,11]:

$$\frac{1}{4}\epsilon_{ijmn}\epsilon_{pqrs}\mathring{\chi}^{mnpq}\mathring{\chi}^{rskl}=-\delta_{ij}^{kl}, \tag{4.1}$$

or in terms of the 3×3 matrices

$$\mathcal{A}^{ac}\mathcal{B}_{cb}+\mathcal{C}^a{}_c\mathcal{C}^c{}_b=-\delta_b^a,\quad \mathcal{C}^{(a}{}_c\mathcal{A}^{b)c}=0,\quad \mathcal{C}^c{}_{(a}\mathcal{B}_{b)c}=0. \tag{4.2}$$

The general solution of the closure condition (4.1),(4.2) reads [11]

$$\mathcal{A}^{ab}=\frac{1}{\det\mathcal{B}}(k^2\mathcal{B}^{ab}-k^ak^b)-\mathcal{B}^{ab}, \tag{4.3}$$

$$\mathcal{C}^a{}_b=\mathcal{B}^{ad}\epsilon_{dbc}k^c=\frac{1}{\det\mathcal{B}}\epsilon^{adc}\mathcal{B}_{db}k_c. \tag{4.4}$$

Here k^a is an arbitrary three-vector, $k_b:=\mathcal{B}_{ab}k^a$, $k^2:=\mathcal{B}_{ab}k^ak^b$, and $\mathcal{B}^{ab}$ denotes the inverse matrix to $\mathcal{B}_{ab}$.

Starting from Eqs. (4.3),(4.4), the direct calculation yields

$$M=-\frac{1}{\det\mathcal{B}}\left(1-\frac{k^2}{\det\mathcal{B}}\right)^2, \tag{4.5}$$

$$M^a=\frac{1}{\det\mathcal{B}}4k^a\left(1-\frac{k^2}{\det\mathcal{B}}\right), \tag{4.6}$$

$$M^{ab}=-\frac{1}{\det\mathcal{B}}4k^ak^b+2\mathcal{B}^{ab}\left(1-\frac{k^2}{\det\mathcal{B}}\right), \tag{4.7}$$

$$M^{abc}=-4\mathcal{B}^{b(a}k^{c)}, \tag{4.8}$$

$$M^{(abcd)}=-(\det\mathcal{B})\mathcal{B}^{(ab}\mathcal{B}^{cd)}. \tag{4.9}$$

Substituting all this into the general Fresnel equation (3.10), we find

$$\begin{aligned}\mathcal{W}&=-\sigma q_0^2\left[\frac{q_0^2}{\sqrt{|\det\mathcal{B}|}}\left(1-\frac{k^2}{\det\mathcal{B}}\right)-\frac{2q_0(q_ak^a)}{\sqrt{|\det\mathcal{B}|}}\right.\\&\qquad\left.-\sqrt{|\det\mathcal{B}|}(q_aq_b\mathcal{B}^{ab})\right]^2\\&=-\sigma q_0^2(q_iq_jg^{ij})^2.\end{aligned} \tag{4.10}$$

Here $\sigma=\mathrm{sgn}(\det\mathcal{B})$, and g^{ij} is the (inverse) four-dimensional metric which arises from the duality operator and the closure relation [11,13]

$$g^{00}=\frac{1}{\sqrt{|\det\mathcal{B}|}}\left(1-\frac{k^2}{\det\mathcal{B}}\right), \tag{4.11}$$

$$g^{0a}=-\frac{k^a}{\sqrt{|\det\mathcal{B}|}}, \tag{4.12}$$

$$g^{ab}=-\sqrt{|\det\mathcal{B}|}\mathcal{B}^{ab}. \tag{4.13}$$

This metric g_{ij} (defined up to a conformal factor) always has the Lorentzian signature, although it is not necessarily interpretable as a spacetime metric (this is a so called *optical metric*, in general; see, e.g., Ref. [14]). As shown in Ref. [13], the constitutive tensor density (2.4) can be rewritten in terms of this metric as

$$\chi^{ijkl}=f(x)\sqrt{-g}(g^{ik}g^{jl}-g^{jk}g^{il})+\alpha(x)\epsilon^{ijkl}. \tag{4.14}$$

Thus we indeed recover the null cone $q_i q^i = q_i q_j g^{ij} = 0$ structure for the propagation of electromagnetic waves from our general analysis: provided the constitutive matrices satisfy the closure relation (4.1),(4.2), the quartic surface (3.10) degenerates to the null cone for the induced metric g_{ij}.

It is worthwhile to note that the Fresnel equation (3.10) can be rewritten in an explicitly covariant form

$$G^{ijkl} q_i q_j q_k q_l = 0, \quad i,j,\ldots = 0,1,2,3, \tag{4.15}$$

where the fourth order totally symmetric tensor density G^{ijkl} is constructed as the cubic polynomial of the components of the constitutive tensor

$$G^{ijkl} := \frac{1}{4!} \overset{\circ}{\chi}{}^{mnp(i} \overset{\circ}{\chi}{}^{j|qr|k} \overset{\circ}{\chi}{}^{l)stu} \epsilon_{mnrs} \epsilon_{pqtu}. \tag{4.16}$$

(Here the total symmetrization is extended only over the four indices i,j,k,l with all the summation indices excluded.) Tamm [16] has introduced a similar ''fourth-order metric'' for the particular case of the linear constitutive law.

V. THE CLOSURE RELATION AS A NECESSARY CONDITION

It was conjectured [11,13] that the closure relation is not only sufficient, but also a necessary condition for the reduction of the quartic geometry (3.10) to the null cone. The complete proof of this conjecture requires a rather lengthy algebra and will be considered elsewhere. Here we demonstrate the validity of the necessary condition in a particular case when the matrix $\mathcal{C} = 0$.

Putting $\mathcal{C}^a{}_b = 0$, we find from Eqs. (3.11)–(3.14) that $M^a = 0$ and $M^{abc} = 0$, whereas

$$M^{ab} = \mathcal{B}_{cd}(\mathcal{A}^{ab}\mathcal{A}^{cd} - \mathcal{A}^{ac}\mathcal{A}^{bd}), \tag{5.1}$$

$$M^{(abcd)} = (\det \mathcal{B}) \mathcal{A}^{(ab}\mathcal{B}^{cd)}. \tag{5.2}$$

Consequently, Eq. (3.10) reduces to

$$\mathcal{W} = q_0^2(\det \mathcal{A} q_0^4 + q_0^2 \gamma + \det \mathcal{B} \alpha \beta), \tag{5.3}$$

where $\alpha := A^{ab} q_a q_b$, $\beta := B^{ab} q_a q_b$, and $\gamma := M^{ab} q_a q_b$. Assuming that the last equation describes a null cone, one concludes that the roots for q_0^2 should coincide and thus necessarily

$$\gamma^2 = 4 \det \mathcal{A} \det \mathcal{B} \alpha \beta. \tag{5.4}$$

Let us write $(\det \mathcal{A} \det \mathcal{B}) = s |\det \mathcal{A} \det \mathcal{B}|$, with $s = \mathrm{sgn}(\det \mathcal{A} \det \mathcal{B})$. Then Eq. (5.4) yields

$$2\sqrt{|\det \mathcal{A} \det \mathcal{B}|} \frac{\alpha}{\gamma} = s\lambda, \quad 2\sqrt{|\det \mathcal{A} \det \mathcal{B}|} \frac{\beta}{\gamma} = \frac{1}{\lambda}, \tag{5.5}$$

where λ is an arbitrary scalar factor. Recalling the definitions of α, β, γ, we then find

$$\mathcal{A}^{ab} = s\lambda^2 \mathcal{B}^{ab}. \tag{5.6}$$

Consequently, $M = \det \mathcal{A} = s\lambda^6 / \det \mathcal{B}$ and $M^{ab} = 2\lambda^4 \mathcal{B}^{ab}$, and therefore one verifies that

$$\mathcal{W} = \frac{s\lambda^2 q_0^2}{\det \mathcal{B}} (\lambda^2 q_0^2 + s q_a q_b \mathcal{B}^{ab} \det \mathcal{B})^2. \tag{5.7}$$

We immediately see that for $s = -1$ the quadratic form in Eq. (5.7) can have either the $(+---)$ signature or $(+++-)$. Similarly, for $s = 1$ the signature is either $(++++)$ or $(++--)$. Therefore, the Fresnel equation describes a correct light *cone* (hyperbolic) structure only in the case $s = -1$. Finally, one can verify that the above solutions satisfy

$$\frac{1}{4} \epsilon_{ijmn} \epsilon_{pqrs} \overset{\circ}{\chi}{}^{mnpq} \overset{\circ}{\chi}{}^{rskl} = s\lambda^2 \delta_{ij}^{kl}, \tag{5.8}$$

which for $s = -1$ reproduces the closure relation (4.1) after a trivial rescaling of the constitutive tensor density (and subsequently absorbing the factor λ into the ''dilaton'' field f).

VI. CONCLUSIONS

In this paper we have derived, extending the earlier results (see, e.g., Refs. [6,14,16]), the Fresnel equation governing the propagation of electromagnetic waves for the most general linear constitutive law. The wave covector lies, in general, on a *fourth order surface*. Such generic fourth order structure is not affected by the axionlike and dilatonlike parts of the constitutive tensor. Note, however, that the linear constitutive law $H = \alpha(x) F$ does not lead to hyperbolic evolution equations, and hence necessarily $f(x) \neq 0$.

We have proved that the closure relation (4.1) is a sufficient condition for the reduction of the fourth order surface to the familiar second order light cone structure. The corresponding family of conformally related metrics g coincides with that derived in Ref. [11], see also Ref. [13]. This result may be considered as an alternative (as compared to Urbantke's scheme [9,10]) derivation of the Lorentzian metric g from a duality operator. In terms of the Lagrangian, the closure relation is equivalent to the statement that $V_{\rm lin} = -\frac{1}{2}[f(x) F \wedge {}^*F + \alpha(x) F \wedge F]$, where the Hodge operator is defined by the metric g.

For the special case $\mathcal{C}^a{}_b = 0$ we have proved that the requirement of reduction of the fourth order Fresnel structure to a second order one implies a relation between the constitutive coefficients which is slightly weaker than the closure relation (4.1), in that it allows for an arbitrary scalar factor. The latter though can be removed by the redefinition of the dilaton field $f(x)$. Also the signature of the resulting quadratic form is not fixed, so that one has to impose hyperbolicity as a separate condition.

It is worthwhile to note that the results obtained can be directly applied to the refinement and generalization of the previous analyses of the observational tests of the equivalence principle. See, for instance, Ref. [15] where some particular cases of the Fresnel equation have been studied in this context.

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