Nine-parameter electrovac metric involving rational functions

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(Received 12 April 2000; published 27 July 2000)

An analytically extended nine-parameter family of the electrovac rational function solutions of the Einstein-Maxwell equations generalizing the Chen-Guo-Ernst class of hyperextreme spacetimes is presented. The general four-soliton asymptotically flat solution possessing the equatorial symmetry and involving five independent real parameters is derived in a concise analytical form and its relevance to the equilibrium problem of two extreme particles is discussed.

PACS number(s): 04.20.Jb, 04.40.Nr, 97.60.Lf

I. INTRODUCTION

One of the reasons the rational function solutions of the Einstein-Maxwell equations, which include, e.g., all the black hole solutions [1], the Tomimatsu-Sato (TS) metrics [2], the Plebański-Demiański solution [3], have received extensive study in the literature is their comparatively simple mathematical structure permitting a far-reaching investigation of the physical and geometrical properties. The application of generating techniques to constructing new asymptotically flat rational function solutions had been started by Kinnersley and Chitre [4] who had generated a five-parameter vacuum field generalizing the $\delta=2$ TS solution. Later on, a large family of stationary vacuum spacetimes was obtained by Hoenselaers, Kinnersley, and Xanthopoulos [5]; two particular asymptotically flat two-parameter specializations [6] of this family, similar to the Kerr metric, reduce to the Schwarzschild solution in the limit of zero total angular momentum.

It is important to note that whereas the solution-generating techniques worked well in the pure vacuum case, their application to the electrovac case via the Alekseev-Cosgrove [7] or Bäcklund transformations [8] could only result in the hyperextreme solutions without the black hole limits (see, e.g., Ref. [9] for a discussion of principal difficulties of generating the subextreme electrovac spacetimes). As a result, the electrovac rational function solutions obtainable from the nonrational ones through limiting procedures were also restricted beyond the extreme case. The above said is true for instance in the case of the Chen-Guo-Ernst electrovac solution [10] which is a nine-parameter rational function field resulting from the double-Cosgrove solution [11] by means of the Kramer-Neugebauer-type limiting procedure [12], and it involves oblate spheroidal coordinates which are characteristic of the hyperextreme spacetimes. It can be mentioned that Chen, Guo, and Ernst presented their solution only in terms of the Ernst complex potentials [13] and did not calculate the corresponding metric coefficients; neither did they attempt to find the analytical continuation of their result to get the physically more interesting subextreme case, apparently because of the absence of the general recipes for carrying out such a continuation.

The objective of our paper is the construction of the analytically extended nine-parameter rational function solution of the Einstein-Maxwell equations which would contain the Chen-Guo-Ernst field as a special (hyperextreme) branch, and to give the explicit formulas for the whole set of the metric coefficients corresponding to that solution. Section II of the paper is devoted to the derivation of the general formulas defining the nine-parameter rational function metric, and in Sec. III we shall consider the physically most interesting particular five-parameter metric which is asymptotically flat and, in addition, is symmetric with respect to the equatorial plane. For the latter metric we shall work out a concise analytical representation in terms of only four factors of spheroidal coordinates, the structure of which turns out to be very similar to that of the $\delta=2$ TS solution. As an interesting application of this five-parameter metric we shall consider in Sec. IV the equilibrium problem of two charged, magnetized, rotating extreme particles. Conclusions are given in Sec. V.

II. THE NINE-PARAMETER SOLUTION AND CORRESPONDING METRIC FUNCTIONS

For our purposes we shall use Sibgatullin's method [14] which is free from the difficulties inherent in the Alekseev-Cosgrove transformation, thus permitting the construction of the analytically extended solutions of the Einstein-Maxwell equations equally applicable to the treatment of either subextreme or hyperextreme cases. Recall that according to this method, the gravitational $\mathcal{E}$ and electromagnetic Φ complex potentials, to the determination of which reduces Ernst's formulation of the axisymmetric electrovac problem [13], can be found from the integrals

$$\mathcal{E}(z,\rho)=\frac{1}{\pi}\int_{-1}^{1}\frac{e(\xi)\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}},$$

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$$\Phi(z,\rho)=\frac{1}{\pi}\int_{-1}^{1}\frac{f(\xi)\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}} \tag{1}$$

which involve the unknown function $\mu(\sigma)$ satisfying the singular integral equation

$$⨍_{-1}^{1}\frac{\mu(\sigma)[e(\xi)+\tilde{e}(\eta)+2f(\xi)\tilde{f}(\eta)]d\sigma}{(\sigma-\tau)\sqrt{1-\sigma^2}}=0 \tag{2}$$

and the normalizing condition

$$\int_{-1}^{1}\frac{\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}}=\pi. \tag{3}$$

The functions $e(\xi)$ and $f(\xi)$ are the locally holomorphic continuations of the axis data $e(z):=\mathcal{E}(z,\rho=0)$, $f(z):=\Phi(z,\rho=0)$ into the complex plane $z+i\rho$, ρ and z being the Weyl-Papapetrou cylindrical coordinates which enter Eqs. (1)–(3) as parameters; $\xi=z+i\rho\sigma$, $\eta=z+i\rho\tau$, $\sigma,\tau\in[-1,1]$; $\tilde{e}(\eta):=\overline{e(\bar{\eta})}$, $\tilde{f}(\eta):=\overline{f(\bar{\eta})}$, an overbar denoting the complex conjugation.

Our choice of $e(z)$ and $f(z)$ is that of the usual four-soliton case, i.e.,

$$e(z)=1+\frac{e_1}{z-\beta_1}+\frac{e_2}{z-\beta_2},\quad f(z)=\frac{f_1}{z-\beta_1}+\frac{f_2}{z-\beta_2}, \tag{4}$$

where e_l, β_l, and f_l, $l=1,2$, are arbitrary complex parameters. However, since we are looking for a solution expressible in rational functions of spheroidal coordinates, the algebraic equation

$$e(z)+\overline{e(z)}+2f(z)\overline{f(z)}=0 \tag{5}$$

which plays an important role in Sibgatullin's method, should have only a pair of distinct roots of multiplicity 2, and this means that not all of the complex parameters in Eq. (4) are independent. The transition to the set of independent parameters can be carried out via the relation

$$2+\sum_{l=1}^{2}\left(\frac{e_l}{z-\beta_l}+\frac{\bar{e}_l}{z-\bar{\beta}_l}\right)+2\left(\sum_{l=1}^{2}\frac{f_l}{z-\beta_l}\right)\left(\sum_{l=1}^{2}\frac{\bar{f}_l}{z-\bar{\beta}_l}\right)=\frac{2(z^2-\alpha^2)^2}{\Pi_{l=1}^{2}(z-\beta_l)(z-\bar{\beta}_l)}, \tag{6}$$

where $\pm\alpha$ are two roots of Eq. (5), and it follows from Eq. (6) that the set of independent parameters consists of α, β_l and f_l, $l=1,2$, since e_1 and e_2 are representable as combinations of these parameters, namely,

$$e_l=\frac{2(\beta_l^2-\alpha^2)^2}{(\beta_l-\bar{\beta}_l)(\beta_l-\beta_k)(\beta_l-\bar{\beta}_k)}-2f_l\left(\frac{\bar{f}_1}{\beta_l-\bar{\beta}_1}+\frac{\bar{f}_2}{\beta_l-\bar{\beta}_2}\right),$$

$$k\neq l;\ l,k=1,2. \tag{7}$$

Therefore, the number of real parameters involved in the solution we are going to construct is 9, taking into account that α can only assume real or pure imaginary values.

The unknown function $\mu(\sigma)$, according to Sibgatullin's method, should be searched for in the form

$$\mu(\sigma)=A_0+\frac{A_1}{\xi+\alpha}+\frac{A_2}{\xi-\alpha}+\frac{A_3}{(\xi+\alpha)^2}+\frac{A_4}{(\xi-\alpha)^2}, \tag{8}$$

where $A_0,\dots,A_4$ are constant coefficients (with respect to σ) which have to be found from Eqs. (2) and (3). Substituting now Eq. (8) into the latter equations, integrating and equating to zero the coefficients at the independent powers of η, one arrives at the linear algebraic system of five equations for the determination of A_0–A_4:

$$A_0+\sum_{n=1}^{2}\left[\frac{A_n}{r_n}+\frac{(z-\alpha_n)A_{n+2}}{r_n^3}\right]=1,$$

$$A_0+\sum_{n=1}^{2}\left[\frac{A_n}{\beta_l-\alpha_n}+\frac{A_{n+2}}{(\beta_l-\alpha_n)^2}\right]=0,$$

$$\sum_{n=1}^{2}\frac{1}{(\alpha_n-\bar{\beta}_l)r_n}[h_l(\alpha_n)A_n+g_l(\alpha_n)A_{n+2}]=0,\quad l=1,2,$$

$$h_l(\alpha_n):=\bar{e}_l+2\bar{f}_lf(\alpha_n),\quad f(\alpha_n):=\sum_{l=1}^{2}\frac{f_l}{\alpha_n-\beta_l},$$

$$g_l(\alpha_n):=h_l(\alpha_n)\left(\frac{z-\alpha_n}{r_n^2}-\frac{1}{\alpha_n-\bar{\beta}_l}\right)-2\bar{f}_l\sum_{k=1}^{2}\frac{f_k}{(\alpha_n-\beta_k)^2}$$
$$=\frac{\partial}{\partial\alpha_n}\left[\frac{h_l(\alpha_n)}{(\alpha_n-\bar{\beta}_l)r_n}\right](\alpha_n-\bar{\beta}_l)r_n, \tag{9}$$

with $\alpha_1:=-\alpha$, $\alpha_2:=\alpha$, $r_n:=\sqrt{\rho^2+(z-\alpha_n)^2}$.

On the other hand, from Eqs. (1), taking into account Eq. (9), we can find the dependence of $\mathcal{E}$ and Φ on the coefficients $A_0,\dots,A_4$, yielding

$$\mathcal{E}=2A_0-1,\quad \Phi=\sum_{n=1}^{2}[f(\alpha_n)A_n+\phi(\alpha_n)A_{n+2}]/r_n,$$

$$\phi(\alpha_n):=\frac{(z-\alpha_n)f(\alpha_n)}{r_n^2}-\sum_{l=1}^{2}\frac{f_l}{(\alpha_n-\beta_l)^2}=\frac{\partial}{\partial\alpha_n}\left[\frac{f(\alpha_n)}{r_n}\right]r_n. \tag{10}$$

Solving the system (9) and substituting the resulting expressions for A_0–A_4 into Eq. (10), we obtain the final formulas for $\mathcal{E}$ and Φ in terms of determinants

$$\mathcal{E}=E_+/E_-,\quad \Phi=F/E_-,$$

$$E_{\pm}=\begin{vmatrix} 1 & 1 & 1 & p_1 & p_2 \\ \pm 1 & & & & \\ \pm 1 & & \mathbb{C} & & \\ 0 & & & & \\ 0 & & & & \end{vmatrix}, \quad H=\begin{vmatrix} z & 1 & 1 & p_1 & p_2 \\ -\beta_1 & & & & \\ -\beta_2 & & \mathbb{C} & & \\ \overline{e}_1 & & & & \\ \overline{e}_2 & & & & \end{vmatrix},$$

$$F=\begin{vmatrix} 0 & f(\alpha_1) & f(\alpha_2) & \phi(\alpha_1)r_1 & \phi(\alpha_2)r_2 \\ -1 & & & & \\ -1 & & \mathbb{C} & & \\ 0 & & & & \\ 0 & & & & \end{vmatrix}, \quad I=\begin{vmatrix} f_1+f_2 & 0 & f(\alpha_1) & f(\alpha_2) & \phi(\alpha_1)r_1 & \phi(\alpha_2)r_2 \\ z & 1 & 1 & 1 & p_1 & p_2 \\ -\beta_1 & -1 & & & & \\ -\beta_2 & -1 & & \mathbb{C} & & \\ \overline{e}_1 & 0 & & & & \\ \overline{e}_2 & 0 & & & & \end{vmatrix}, \tag{11}$$

where $p_n := (z-\alpha_n)/r_n$ and $\mathbb{C}$ is the following 4×4 matrix

$$\mathbb{C}=\begin{pmatrix} \frac{r_1}{\alpha_1-\beta_1} & \frac{r_2}{\alpha_2-\beta_1} & -\frac{r_1^2}{(\alpha_1-\beta_1)^2} & -\frac{r_2^2}{(\alpha_2-\beta_1)^2} \\ \frac{r_1}{\alpha_1-\beta_2} & \frac{r_2}{\alpha_2-\beta_2} & -\frac{r_1^2}{(\alpha_1-\beta_2)^2} & -\frac{r_2^2}{(\alpha_2-\beta_2)^2} \\ \frac{h_1(\alpha_1)}{\alpha_1-\overline{\beta}_1} & \frac{h_1(\alpha_2)}{\alpha_2-\overline{\beta}_1} & \frac{g_1(\alpha_1)r_1}{\alpha_1-\overline{\beta}_1} & \frac{g_1(\alpha_2)r_2}{\alpha_2-\overline{\beta}_1} \\ \frac{h_2(\alpha_1)}{\alpha_1-\overline{\beta}_2} & \frac{h_2(\alpha_2)}{\alpha_2-\overline{\beta}_2} & \frac{g_2(\alpha_1)r_1}{\alpha_1-\overline{\beta}_2} & \frac{g_2(\alpha_2)r_2}{\alpha_2-\overline{\beta}_2} \end{pmatrix}. \tag{12}$$

The corresponding metric functions f, γ, and ω which enter the axisymmetric line element

$$ds^2=f^{-1}[e^{2\gamma}(d\rho^2+dz^2)+\rho^2 d\varphi^2]-f(dt-\omega d\varphi)^2 \tag{13}$$

can be shown to be defined by the following expressions (see Refs. [15,16] for details of Sibgatullin's method)

$$f=\frac{E_+\overline{E}_-+\overline{E}_+E_-+2F\overline{F}}{2E_-\overline{E}_-}, \quad e^{2\gamma}=\frac{E_+\overline{E}_-+\overline{E}_+E_-+2F\overline{F}}{2K_0\overline{K}_0 r_1^4 r_2^4}, \tag{14}$$

$$\omega=\frac{2\,\mathrm{Im}\{E_+\overline{H}-\overline{E}_+G-F\overline{I}\}}{E_+\overline{E}_-+\overline{E}_+E_-+2F\overline{F}},$$

$$G=\begin{vmatrix} 0 & r_1+\alpha_1-z & r_2+\alpha_2-z & \rho^2/r_1 & \rho^2/r_2 \\ -1 & & & & \\ -1 & & \mathbb{C} & & \\ 0 & & & & \\ 0 & & & & \end{vmatrix},$$

$$K_0=\begin{vmatrix} \frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_2-\beta_1} & -\frac{1}{(\alpha_1-\beta_1)^2} & -\frac{1}{(\alpha_2-\beta_1)^2} \\ \frac{1}{\alpha_1-\beta_2} & \frac{1}{\alpha_2-\beta_2} & -\frac{1}{(\alpha_1-\beta_2)^2} & -\frac{1}{(\alpha_2-\beta_2)^2} \\ \frac{h_1(\alpha_1)}{\alpha_1-\overline{\beta}_1} & \frac{h_1(\alpha_2)}{\alpha_2-\overline{\beta}_1} & \frac{\partial}{\partial\alpha_1}\left[\frac{h_1(\alpha_1)}{\alpha_1-\overline{\beta}_1}\right] & \frac{\partial}{\partial\alpha_2}\left[\frac{h_1(\alpha_2)}{\alpha_2-\overline{\beta}_1}\right] \\ \frac{h_2(\alpha_1)}{\alpha_1-\overline{\beta}_2} & \frac{h_2(\alpha_2)}{\alpha_2-\overline{\beta}_2} & \frac{\partial}{\partial\alpha_1}\left[\frac{h_2(\alpha_1)}{\alpha_1-\overline{\beta}_2}\right] & \frac{\partial}{\partial\alpha_2}\left[\frac{h_2(\alpha_2)}{\alpha_2-\overline{\beta}_2}\right] \end{vmatrix}.$$

Formulas (9)–(14) fully define the analytically extended nine-parameter electrovac metric which generalizes the Chen-Guo-Ernst hyperextreme spacetime, the latter corresponding to the pure imaginary values of the parameter α.

It should be underlined that not all of the parameters involved in our solution are physical, and for any concrete application, the rotational monopole and magnetic monopole moments ν and μ_0, respectively, determined by the expressions

$$\nu=\mathrm{Im}\{e_1+e_2\}, \quad \mu_0=\mathrm{Im}\{f_1+f_2\}, \tag{15}$$

should be excluded [we remind that e_1 and e_2 are given by Eq. (7)]. The remaining seven arbitrary parameters can be associated for instance with the mass monopole and quadrupole moments, angular momentum dipole and octupole moments, electric charge, and electric and magnetic dipole moments (the parametrization of the four-soliton solutions in terms of the relativistic multipole moments can be found in Ref. [17]).

The cylindrical coordinates ρ and z enter the metric obtained via the functions r_1 and r_2 which means that the generalized spheroidal coordinates x and y can be introduced by the formulas

$$x=\frac{1}{2\alpha}(r_1+r_2), \quad y=\frac{1}{2\alpha}(r_1-r_2), \tag{16}$$

and the metric becomes rational in terms of these new coordinates.

To conclude this section, let us point out that the main justification for the construction of the nine-parameter rational function metric in the form chosen, and not in any other form, say, corresponding to one simple root of Eq. (5) and the other of multiplicity 3, is that our case contains solutions symmetric with respect to the equatorial plane.

III. MAGNETIZED KERR-NEWMAN METRIC POSSESSING AN ARBITRARY MASS-QUADRUPOLE MOMENT

The physically most interesting particular case arising from the electrovac metric described in the previous section is the five-parameter *asymptotically flat* solution representing exterior field of a charged, magnetized, spinning deformed mass and possessing the additional reflection symmetry with respect to the equatorial plane. This general equatorially symmetric four-soliton rational function solution can be also envisaged as the asymptotically flat magnetized generalization of the Kerr-Newman spacetime endowed with an arbitrary mass-quadrupole moment. Formulas (9)–(14) permit us to work out an elegant representation of that solution, making the latter very suitable for concrete applications.

In this particular case it is convenient to reparametrize the axis data in the form

$$e(z)=\frac{(z-m-\mathrm{i}a)(z+\mathrm{i}b)+d-\delta-ab}{(z+m-\mathrm{i}a)(z+\mathrm{i}b)+d-\delta-ab},$$

$$f(z)=\frac{qz+\mathrm{i}\mu}{(z+m-\mathrm{i}a)(z+\mathrm{i}b)+d-\delta-ab},$$

$$\delta:=\frac{\mu^2-m^2b^2}{m^2-(a-b)^2-q^2},\quad d:=\frac{1}{4}[m^2-(a-b)^2-q^2]. \tag{17}$$

The advantage of this parametrization is that the real parameters m, a, and q coincide with the total mass, total angular momentum per unit mass, and total charge, respectively, while the remaining real parameters b and μ represent the arbitrary mass-quadrupole moment Q and magnetic dipole moment $\mathcal{M}$ of the source via the formulas

$$Q=-m(d-\delta-ab+a^2),\quad \mathcal{M}=\mu+q(a-b). \tag{18}$$

Then it is easy to see that Eq. (5) will have a pair of roots

$$z_\pm=\pm\kappa,\quad \kappa:=\sqrt{d+\delta} \tag{19}$$

of the multiplicity two which, depending on the interrelation of the parameters, can assume *real* or *pure imaginary* values. The simple fraction decomposition of the data (17) gives

$$e_l=\frac{2m(\beta_l+\mathrm{i}b)}{\beta_k-\beta_l},\quad f_l=\frac{q\beta_l+\mathrm{i}\mu}{\beta_l-\beta_k},\quad l,k=1,2;\ k\neq l,$$

$$\beta_1+\beta_2=-m+\mathrm{i}(a-b),\quad \beta_1\beta_2=d-\delta+\mathrm{i}mb, \tag{20}$$

and these values for e_l, β_l, and f_l should be substituted into Eqs. (11) and (12) to obtain the form of the complex potentials $\mathcal{E}$ and Φ. A laborious computer analytical evaluation of the determinants finally yields the following concise expressions:

$$\mathcal{E}=\frac{A-2mB}{A+2mB},\quad \Phi=\frac{2C}{A+2mB},$$

$$\begin{aligned}A=&4[(\kappa^2x^2-\delta y^2)^2-d^2-\mathrm{i}\kappa^3xy(a-b)(x^2-1)]-(1-y^2)\\&\times[(a-b)(d-\delta)-m^2b+q\mu]\\&\times[(a-b)(y^2+1)+4\mathrm{i}\kappa xy],\end{aligned}$$

$$\begin{aligned}B=&\kappa x\{2\kappa^2(x^2-1)+[b(a-b)+2\delta](1-y^2)\}\\&+\mathrm{i}y\{2\kappa^2b(x^2-1)-[\kappa^2(a-b)-m^2b+q\mu-2a\delta]\\&\times(1-y^2)\},\end{aligned}$$

$$\begin{aligned}C=&2\kappa^2(x^2-1)(\kappa qx+\mathrm{i}\mu y)+(1-y^2)\{\kappa x[2q\delta+\mu(a-b)]\\&-\mathrm{i}y[q(a-b)(d-\delta)-q(m^2b-q\mu)-2\mu\delta]\},\end{aligned} \tag{21}$$

which are written in the generalized spheroidal coordinates

$$x=\frac{1}{2\kappa}(r_++r_-),\quad y=\frac{1}{2\kappa}(r_+-r_-),$$

$$r_\pm:=\sqrt{\rho^2+(z\pm\kappa)^2}. \tag{22}$$

In a similar way we can calculate the corresponding metric functions f, γ and ω using formulas (14). Apparently, the calculation of the function ω is most difficult—first, because of four different determinants involved in the calculations and, secondly, because it is practically impossible to reduce a complicated intermediate expression for ω without a solid strategy for establishing the factor structure of the whole metric. After very tedious work we have finally been able to find elegant expressions for all three metric coefficients; namely,

$$f=E/D,\quad \mathrm{e}^{2\gamma}=E/16\kappa^8(x^2-y^2)^4,\quad \omega=-(1-y^2)F/E, \tag{23}$$

$$\begin{aligned}E=&\{4[\kappa^2(x^2-1)+\delta(1-y^2)]^2+(a-b)[(a-b)(d-\delta)-m^2b+q\mu](1-y^2)^2\}^2\\&-16\kappa^2(x^2-1)(1-y^2)\{(a-b)[\kappa^2(x^2-y^2)+2\delta y^2]+(m^2b-q\mu)y^2\}^2,\end{aligned}$$

$$D=\{4(\kappa^2x^2-\delta y^2)^2+2\kappa mx[2\kappa^2(x^2-1)+(2\delta+ab-b^2)(1-y^2)]+(a-b)\times[(a-b)(d-\delta)-m^2b+q\mu](y^4-1)-4d^2\}^2+4y^2\{2\kappa^2(x^2-1)[\kappa x(a-b)-mb]-2mb\delta(1-y^2)+[(a-b)(d-\delta)-m^2b+q\mu](2\kappa x+m)(1-y^2)\}^2,$$

$$F=8\kappa^2(x^2-1)\{(a-b)[\kappa^2(x^2-y^2)+2\delta y^2]+y^2(m^2b-q\mu)\}\{\kappa mx[(2\kappa x+m)^2-2y^2(2\delta+ab-b^2)-a^2+b^2-q^2]-2\kappa^2q^2x^2-2y^2(4\delta d-m^2b^2)\}+\{4[\kappa^2(x^2-1)+\delta(1-y^2)]^2+(a-b)[(a-b)(d-\delta)-m^2b+q\mu](1-y^2)^2\}\times(4(2\kappa mbx+2m^2b-q\mu)[\kappa^2(x^2-1)+\delta(1-y^2)]+(1-y^2)\times\{(a-b)(m^2b^2-4\delta d)-(4\kappa mx+2m^2-q^2)[(a-b)(d-\delta)-m^2b+q\mu]\}).$$

We mention that our search for the above expressions for f, γ, and ω was facilitated by Refs. [18,19] and by the remarkable similarity of the factor structure of the formulas (23) with that of the $\delta=2$ TS solution which made it possible to apply Perjés' results [19] to our particular electrovac metric. It turns out that the functions E, D, and F from Eq. (23) can be rewritten in terms of only four factors P, R, S, T (these are analogues of Perjés' polynomials π,ρ,σ,τ, respectively); the explicit expressions are

$$E=R^2+\lambda_1\lambda_2S^2,\quad D=E+RP+\lambda_2ST,\quad F=RT-\lambda_1SP,$$

$$P:=2\{\kappa mx[(2\kappa x+m)^2-2y^2(2\delta+ab-b^2)-a^2+b^2-q^2]-2\kappa^2q^2x^2-2y^2(4\delta d-m^2b^2)\},$$

$$R:=4[\kappa^2(x^2-1)+\delta(1-y^2)]^2+(a-b)[(a-b)(d-\delta)-m^2b+q\mu](1-y^2)^2,$$

$$S:=-4\{(a-b)[\kappa^2(x^2-y^2)+2\delta y^2]+y^2(m^2b-q\mu)\},$$

$$T:=4(2\kappa mbx+2m^2b-q\mu)[\kappa^2(x^2-1)+\delta(1-y^2)]+(1-y^2)\{(a-b)(m^2b^2-4\delta d)-(4\kappa mx+2m^2-q^2)[(a-b)(d-\delta)-m^2b+q\mu]\},\tag{24}$$

where the coefficients λ_1 and λ_2, given in Perjés' notations, have the form

$$\lambda_1=\kappa^2(x^2-1),\quad \lambda_2=y^2-1.\tag{25}$$

A striking similarity of the above Eqs. (24), (25) with Perjés' formulas (2), (3) of Ref. [19] suggests that the factor structure of the TS solutions is shared by a wide class of the vacuum and electrovac rational function metrics, and we expect that this similarity will be better studied in the future. Let us consider now the limits of the solution (21) among which there are several very well-known ones, together with other solutions only recently discussed in the literature.

A. Classical limits

(1) By first setting in Eq. (21) $\mu=bq$ and then putting $b^2=a^2+q^2-m^2$, we arrive at the Kerr-Newman solution *without* any restriction on the parameters m, a and q since in this case the complex continuation $b\rightarrow ib$ is possible. Further reductions $q=0$, or $a=0$, or $q=a=0$ lead us, respectively, to the Kerr, Reissner-Nordström, or Schwarzschild solutions [1] in their analytically extended form. Therefore, *all* the black hole solutions are contained in our five-parameter metric.

(2) In the pure vacuum limit, i.e., when $q=\mu=0$, one arrives at the general reflection symmetric solution from the Kinnersley-Chitre class [4]. By further setting $b=0$, one comes to the $\delta=2$ TS solution [2].

(3) The magnetostatic limit $(a=b=q=0)$ leads to Bonnor's version of the massive magnetic dipole [20].

(4) The Kramer-Neugebauer solution for a charged massive magnetic dipole [21] follows from Eq. (21) when the parameters are chosen in the form

$$m=2M(1+\beta^2)/(1-\beta^2),\quad a=b=2\beta c/(1+\beta^2),$$

$$q=4M\beta/(1-\beta^2),\quad \mu=2Mc(1+\beta^2)/(1-\beta^2).\tag{26}$$

Note that in this case the angular momentum is proportional to the charge parameter β and to the magnetic dipole parameter c, and it vanishes when either of the latter two parameters is equal to zero.

(5) In the case $b=\mu=0$ one arrives at the charged generalization of the $\delta=2$ TS solution obtained by Ernst [22].

(6) The hyperextreme part of our solution corresponding to the pure imaginary values of κ belongs to the Chen-Guo-Ernst family of hyperextreme spacetimes [10], being probably appropriate for the description of relativistic disks [23].

B. Other limits

Let us now point out some other limits of our solution which have been considered recently in the literature.

(1) The three-parameter Manko-Ruiz solution [24] representing the stationary generalization of Bonnor's massive magnetic dipole is obtainable from Eq. (21) by setting $b=q=0$. Since in the paper [24] a concise representation of

the whole metric was not given, below we shall write out in the *prolate* spheroidal coordinates the physically most interesting subextreme part of this solution which elegantly unifies Bonnor's magnetic dipole [20] and $\delta=2$ TS [2] fields. Then, after introducing the real constants p, q and λ via the relations

$$m=2\kappa/p, \quad a=2\kappa q/p, \quad \delta=\kappa^2\lambda^2/p^2,$$
$$p^2+q^2-\lambda^2=1, \tag{27}$$

where κ is an arbitrary *real* parameter, we obtain (in the prolate spheroidal coordinates) from Eqs. (21) and (23):

$$\mathcal{E}=\frac{A-B}{A+B}, \quad \Phi=\frac{C}{A+B}, \quad f=\frac{E}{D}, \quad e^{2\gamma}=\frac{E}{p^8(x^2-y^2)^4}, \quad \omega=-\frac{2\kappa q(1-y^2)F}{pE},$$
$$A=(p^2x^2-\lambda^2y^2)^2-(p^2-\lambda^2)^2-q(p^2-2\lambda^2)(1-y^2)[q(y^2+1)+2ipxy]-2ip^3qxy(x^2-1),$$
$$B=2px[p^2(x^2-1)+\lambda^2(1-y^2)]-2iqy(p^2-2\lambda^2)(1-y^2),$$
$$C=2i\lambda\sqrt{1-q^2}[p^2y(x^2-1)+(1-y^2)(\lambda^2y-ipqx)],$$
$$E=\{[p^2(x^2-1)+\lambda^2(1-y^2)]^2+q^2(p^2-2\lambda^2)(1-y^2)^2\}^2$$
$$-4p^2q^2(x^2-1)(1-y^2)[p^2(x^2-y^2)+2\lambda^2y^2]^2,$$
$$D=\{(p^2x^2-\lambda^2y^2)^2+2px[p^2(x^2-1)+\lambda^2(1-y^2)]+q^2(p^2-2\lambda^2)(y^4-1)-(p^2-\lambda^2)^2\}^2$$
$$+4q^2y^2[p^3x(x^2-1)+(p^2-2\lambda^2)(px+1)(1-y^2)]^2,$$
$$F=4p^2(x^2-1)[p^2(x^2-y^2)+2\lambda^2y^2]\{px[(px+1)^2-q^2-\lambda^2y^2]-\lambda^2y^2(p^2-\lambda^2)\}$$
$$-(1-y^2)[2(p^2-2\lambda^2)(px+1)+\lambda^2(p^2-\lambda^2)]$$
$$\times\{[p^2(x^2-1)+\lambda^2(1-y^2)]^2+q^2(p^2-2\lambda^2)(1-y^2)^2\}. \tag{28}$$

By putting in the above formulas $q=0$ (magnetostatic limit), one arrives at the Bonnor magnetic dipole solution [20], and by setting $\lambda=0$ (stationary pure vacuum limit), one comes to the $\delta=2$ TS metric [2].

(2) When $b=0$, our five-parameter solution reduces to the charged, magnetized $\delta=2$ TS electrovac spacetime considered in Ref. [25].

(3) By setting to zero the electric charge parameter ($q=0$), one comes to the solution [26] recently proposed for the description of the exterior field of a rotating neutron star.

(4) The last particular case which we are going to mention here is the two-parameter solution recently considered by Clément [27]; it is obtainable from Eq. (21) if the parameters are constrained by the relations [28]

$$q=0, \quad \mu=-mb, \quad \kappa^2(a-b)=m^2b \tag{29}$$

with the subsequent definitions

$$p:=2\kappa/m, \quad q:=(a-b)/m, \quad p^2+q^2=1. \tag{30}$$

It is worthwhile pointing out that the electromagnetic field in our solution is described by the electric A_4 and magnetic A_3 components of the electromagnetic four-potential, the former component being the real part of the complex potential Φ, and the latter component being the real part of Kinnersley's potential Φ_2 [29] which in our case has the form

$$\Phi_2=\frac{K}{A+2mB}-iq,$$
$$K=2\kappa^2(x^2-1)(2\kappa x[\mu(1-y^2)+q(a-b)]+m(3\mu-bq)(1-y^2)-2mbq+iy\{2q[\kappa^2(x^2-1)+\kappa mx+2d]$$
$$+(1-y^2)[\mu(a-b)+2q\delta]\})+(1-y^2)((2\kappa x+m)\{qd(a-b)(1+y^2)+\mu(2m^2+q^2)-3m^2bq+(1-y^2)$$
$$\times[\delta(2\mu+aq-bq)+q(m^2b-q\mu)]\}+m\{(a-b)[bq(a-b)-2a\mu]+(\mu-bq)(4\delta-q^2)-2q\delta(a+b)$$
$$+b[\mu(a-b)+2q\delta](1-y^2)\}-2iy\{2\kappa mb\mu x+q(d-\delta)(a-b)^2+\mu(a-b)(q^2-2\delta)+m^2b(\mu-aq)\}). \tag{31}$$

TABLE I. Particular equilibrium states of charged particles.

m	a	b	q	κ	LRS($\rho,z=0$)
4	4	14.24327	$\pm$4.19018	1.96538	3.58199
4	6	16.97784	$\pm$4.68997	2.19735	4.01295
8	2	19.87867	$\pm$6.60935	3.10759	5.63851
2	4	9.82089	$\pm$2.57108	1.20354	2.20154

With the above expression for Φ_2, the electromagnetic processes in our electrovac field can be fully described.

IV. THE EQUILIBRIUM OF TWO CHARGED, MAGNETIZED, SPINNING, EXTREME PARTICLES

The five-parameter solution considered in the previous section is the *simplest* asymptotically flat solution of the Einstein-Maxwell equations able to describe the exterior field of a charged, magnetized, deformed, spinning mass, and its relevance for example to the astrophysics of neutron stars (in the absence of the electric charge parameter) has already been discussed in Ref. [26]. Another interesting application of our metric to the consideration of which we are turning now is the analysis of the equilibrium states in a binary system of extreme identical spinning particles endowed with electric charges and magnetic dipole moments.

The interpretation of the metric (23) as describing two balancing extreme particles situated at the points $\pm\kappa$ of the symmetry axis is possible if the five parameters involved in the metric preserve the reality of κ and, in addition, cause the metric functions γ and ω to vanish on the part of the symmetry axis separating the particles, i.e.,

$$\gamma(x=1)=0, \quad \omega(x=1)=0 \tag{32}$$

(note that, by construction, γ and ω vanish on the parts of the symmetry axis exterior to the two particles). The above two conditions, the first of which, following Hoenselaers [30], can be called the balance condition, and the second one, the axis condition, in the case of the metric (23) have the form

$$4\delta^2+(a-b)[(a-b)(d-\delta)-m^2b+q\mu]\pm 4\kappa^4=0 \tag{33}$$

(the balance condition), and

$$4\delta(2\kappa mb+2m^2b-q\mu)+(a-b)(m^2b^2-4\delta d) -(4\kappa m+2m^2-q^2)[(a-b)(d-\delta)-m^2b+q\mu]=0 \tag{34}$$

(the axis condition).

We shall separately discuss the equilibrium problem for the following types of binary systems consisting of identical extreme particles: (a) a pure vacuum binary system ($q=\mu=0$), (b) an electrovac binary system of charged particles ($\mu=0$), (c) an electrovac system of magnetized parrticles ($q=0$), and (d) the general five-parameter case of charged, magnetized particles. For each equilibrium state we shall plot (in the cylindrical coordinates ρ and z) the stationary limit surface, the location of singularities arising as solutions of the equation

$$A+2mB=0, \tag{35}$$

and the magnetic lines of force.

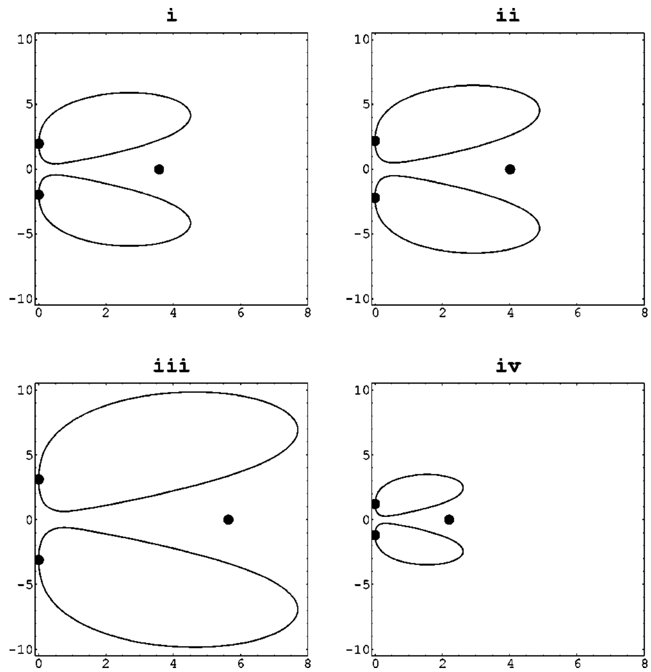

FIG. 1. Stationary limit surfaces corresponding to the equilibrium states from Table I.

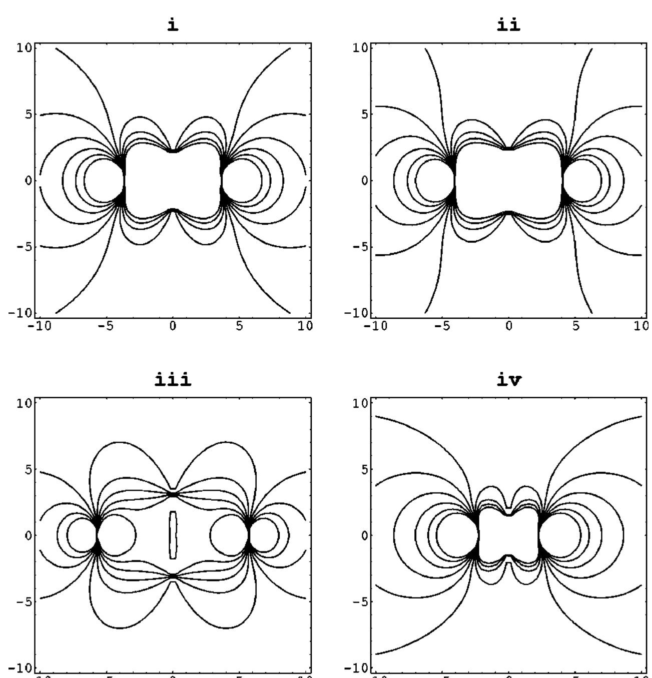

FIG. 2. Magnetic lines of force corresponding to the equilibrium states from Table I.

TABLE II. Particular equilibrium states of magnetized particles.

m	a	b	μ	κ	LRS($\rho, z=0$)
4	4	14.24259	28.95362	2.20180	3.68253
20	6	-33.26463	260.45332	6.54463	11.09293
6	10	26.67472	86.34478	3.80993	6.34455
2	-4	-9.75382	10.76928	1.34653	2.23819

(a) *The pure vacuum case.* We have been unable to find any numerical solution of the system (33)–(34) in the absence of the electromagnetic field for positive total mass, whence we tentatively conclude that the equilibrium of two extreme particles with positive masses due to the spin-spin repulsion effect [31] is impossible. In this respect it would be worthwhile noting that the equilibrium states of two extreme particles reported by Tomimatsu [32] are unphysical not because of the presence of a ring singularity pointed out by Hoenselaers [30], but because this ring singularity accompanies a constituent with negative mass, the fact that can be easily seen from the formulas (3.2) of Ref. [30] since the masses M_1 and M_2 there have opposite signs for the range of the parameter l involved in the analysis.

(b) *The stationary electrovac case with* $\mu=0$. The system (33)–(34) admits numerical roots with nonzero charge parameter, and particular equilibrium positions are given in Table I. All four equilibrium states from Table I are accompanied by a ring singularity which lies at the equatorial plane $z=0$, or $y=0$ [see the last column of Table I for the corresponding value of ρ defining the location of ring singularities (LRS)]. The respective stationary limit surfaces (Fig. 1) represent two disconnected regions which can be associated with the stationary limit surfaces of individual particles located at the points $\pm\kappa$ of the symmetry axis. From the shape of the magnetic lines of force (Fig. 2) it is clear that ring singularities are originated by the magnetic field.

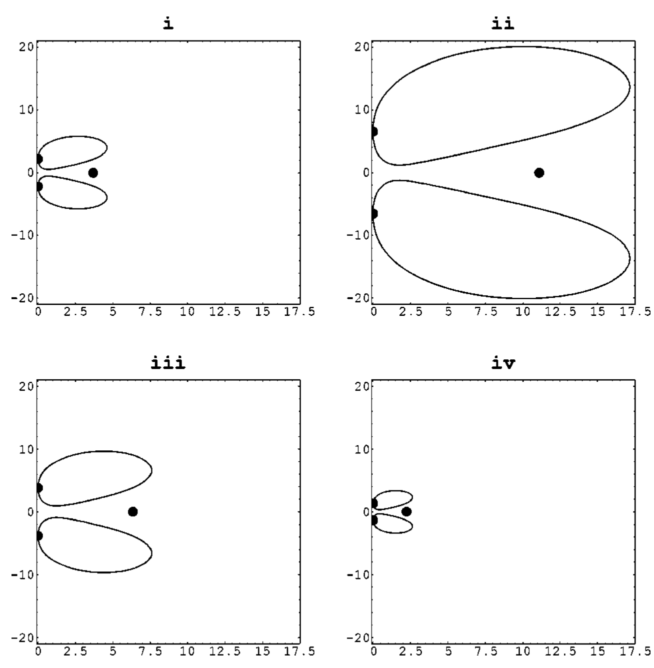

FIG. 3. Stationary limit surfaces corresponding to the equilibrium states from Table II.

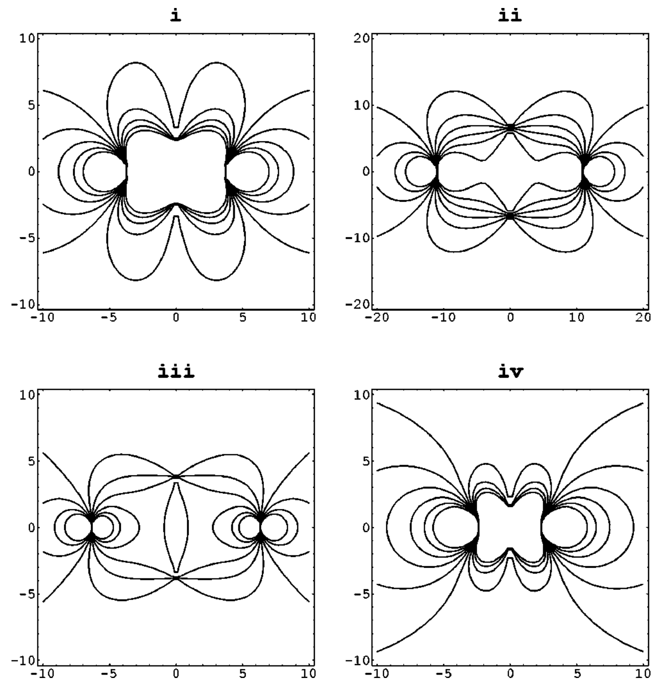

FIG. 4. Magnetic lines of force corresponding to the equilibrium states from Table II.

(c) *The equilibrium of magnetized particles* ($q=0$). In the absence of electric charge ($q=0$), the binary system of extreme particles possesses equilibrium states due to strong magnetic fields. Table II provides one with four particular sets of parameters at which the equilibrium occurs. As in the previous case of charged particles, the ring singularities of all the equilibrium states from Table II are due to the magnetic field, which can be seen by comparing Figs. 3 and 4.

(d) *The equilibrium of charged, magnetized particles: the general case.* Clearly, since in the particular cases $\mu=0$ and $q=0$ the equilibrium between two extreme constituents exists, it should also exist in the general five-parametric case. In Table III four different numerical solutions of the system (33),(34) are given which determine the balance of charged, magnetized constituents. Here again, the equilibrium states are characterized by a ring singularity at the equatorial plane, and we have been unable to find any equilibrium position without it. It is intriguing, however, that in all the cases the ring singularity can be associated with a source of the magnetic field, say, a loop of steady current. The shapes of the

TABLE III. Particular equilibrium states of charged, magnetized particles.

m	a	b	q	μ	κ	LRS($\rho, z=0$)
6	-2	5.99726	4	26.76232	1.47443	2.18024
6	2	-8.45806	-2	28.68518	1.81894	2.92675
6	4	14.85488	4	69.43584	2.73185	4.07605
2	4	10.06336	-2	2.88325	1.26539	2.28076

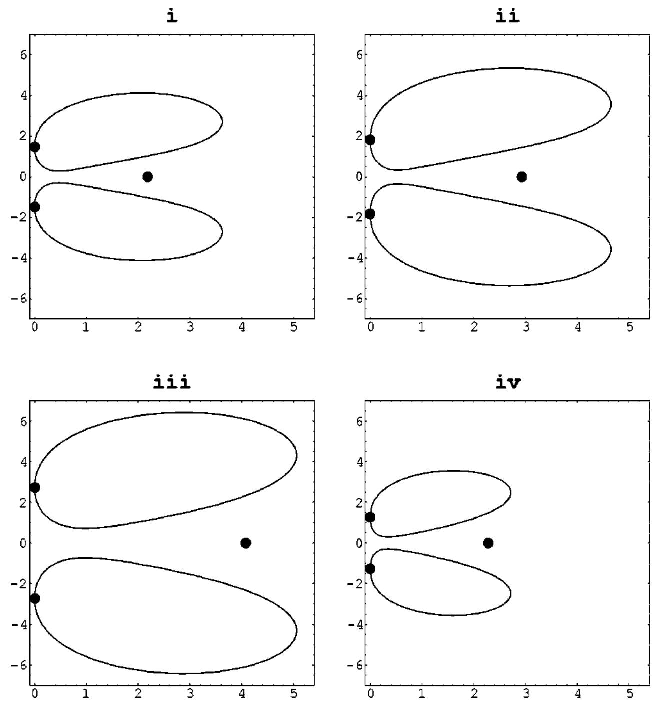

FIG. 5. Stationary limit surfaces corresponding to the equilibrium states from Table III.

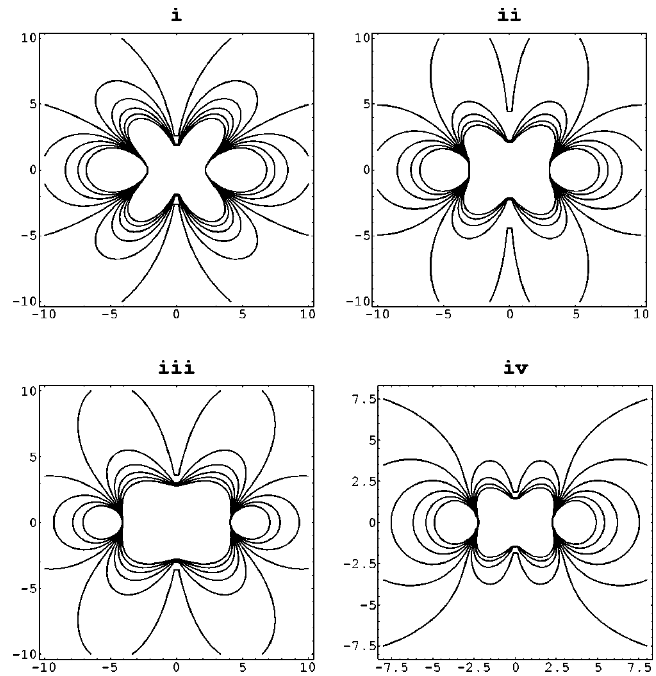

FIG. 6. Magnetic lines of force corresponding to the equilibrium states from Table III.

corresponding stationary limit surfaces and of the magnetic lines of force are plotted in Fig. 5 and Fig. 6, respectively.

V. CONCLUSIONS

The present paper illustrates well that Sibgatullin's method of the construction of exact solutions of the Einstein-Maxwell equations from the axis expressions of the Ernst complex potentials is a powerful instrument for obtaining new electrovac rational function fields which to some extent can be considered as limiting cases of the general soliton metric [16].

Furthermore, it is most important that the known black hole solutions can now be incorporated with the aid of Sibgatullin's method into the stationary electrovac spacetimes of a more general nature, thus making the newly constructed solutions astrophysically more significant. The five-parameter metric from Sec. III which is the simplest asymptotically flat rational function field describing the exterior geometry of a charged, magnetized, spinning, deformed source is of apparent astrophysical interest, and we expect that its relation for instance to the models of neutron stars already discussed in Ref. [26] (the case $q=0$) will be subjected to a more detailed analysis in the future. In the present paper the metric (23) has been used for the study of the balance problem of two extreme charged, magnetized, spinning particles, and we have found particular equilibrium states which, to our knowledge, throw some new light on the equilibrium two-body problem in general relativity.

Lastly, the results presented in our paper clearly demonstrate that the words of Chen, Guo, and Ernst [10] about the potential physical importance of the electrovac rational function solutions obtainable with the aid of the superposition techniques are true.

ACKNOWLEDGMENTS

We would like to thank Jerzy Plebański for continuing interest in our work and stimulating discussions. This work has been partially supported by Project 34222-E from CONACyt of Mexico. J.D.S-G. acknowledges financial support from Colciencias of Colombia.

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