# Spherically symmetric solutions in $f(R)$ theories of gravity obtained using the first order formalism

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The solutions of a class of theories obtained when we apply the first order formalism are studied. In the linear approximation we obtain the Green function and we prove that the field is independent of the size and internal stresses of the source. We show that the solutions of the field equations for a mass point are also the exterior solutions for an arbitrary spherically symmetric mass distribution. We construct the solutions of the field equation, without any approximation, for the spherically symmetric matter distribution, and prove that the exterior solutions match correctly with the interior solutions. We also prove that one of the exterior solutions is always the Schwarzschild solution. Finally, in the same case, we show that Birkhoff's theorem is satisfied. All the above results are quite similar to general relativity but are very different from the results of the fourth order theories; then we have shown that the first order formalism theories have better classical properties than fourth order theories.

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## I. INTRODUCTION

The study of fourth order theories was originally stimulated by Eddington's suggestion on the existence of a class of theories which were observationally equivalent to Einstein's [1] because they included as one of their solutions the (exterior) Schwarzschild metric. It was noted by Pauli [3] and Buchdahl [2] that every vacuum solution (including the Schwarzschild solution) of general relativity (GR) is also a solution of any fourth order theory. More generally, every nonvacuum solution of GR, associated with a conformally invariant source, $T=0$, is also a solution of any fourth order theory [4].

Later on, attempts to quantize GR, or to regularize the stress-energy-momentum tensor of quantum fields propagating in curved spacetimes, led investigators to consider gravitational actions involving curvature squared terms [5]. Higher-derivative theories appear to enjoy better renormalizability properties than GR [6], and in modern cosmology have become standard since the Starobinsky model with curvature-squared terms leads automatically to the desired inflationary period. More recently [7], the stability and Hamiltonian formulation of these theories have been studied.

Higher order theories of gravity are the generally covariant extensions of GR when we consider in the Lagrangian density nonlinear terms in the curvature. The field equations derived by second order variation of this Lagrangian contain metric derivatives of an order higher than the second. The most general action containing the Einstein plus Gauss-Bonnet terms is (for vacuum).

$$S=\int \sqrt{-g}(R+\omega R^2+\beta R_{cd}R^{cd})d^4x, \tag{1}$$

where we have not considered surface terms since they will not contribute to the analysis of the field equations we will perform. The factors $\alpha$ and $\beta$ are new universal constants (a Riemann-squared term can be eliminated using the Gauss-Bonet identity); the term linear in $R$ is necessary for a proper Newtonian limit [4]. The early investigators emphasized that the empty space solutions of Einstein's equations $R_{ab}=0$ are solutions of the field equations derived from the Hilbert's action

$$\int d^4xR\sqrt{-g} \tag{2}$$

as well as, for example, from

$$S'=\int d^4xR^2\sqrt{-g}, \tag{3}$$

which is straightforward since

$$\delta S'=\int d^4xR(R\,\delta\sqrt{-g}+2\sqrt{-g}\,\delta R) \tag{4}$$

and thus $\delta S'=0$ if $R=0$. On other words, the fourth order equations corresponding to the Lagrangian density (1) share with GR its vacuum solutions. This may suggest that the classical test of GR are automatically satisfied through the Schwarzschild solution [8,1]. However, the empty space solutions are to be matched to interior solutions and it may well occur that the matching conditions are not satisfied [9,10]. Higher order theories have a richer set of vacuum solution than GR [11,12]; in other words, the vacuum solutions of GR are also solutions of higher order theories but the converse is in general not true. Unlike lower derivative corrections, however, it is false to assume that adding a higher derivative correction term, with a small coefficient will only perturb the original theory. The presence of an unconstrained higher derivative term, no matter how small it may naively appear, makes the new theory dramatically different from the original one. Also it was pointed out, using the weak field approximation [9], that although, e.g., the Schwarzschild solution is a solution to the empty space equations, it does not couple to a positive definite matter distribution. In fact, those solutions of purely four-derivative models (without a linear term in $R$) which do couple to a positive matter source are not asymptotically flat at infinity.

The Green functions of the linear field equations for a Lagrangian (1) with $\beta=0$ differ from the Newtonian Green function by a Yukawa term. Then, coupling the linearized theory to a pressurized fluid distribution shows that the coefficients of the Yukawa potentials depend on the pressure and the size of the distribution. This shows that Birkhoff's theorem is not valid in these models. Therefore the theory defined by the above Lagrangian have, as we have just mentioned, some quantum and cosmological interesting property but have a great trouble with the Schwarzschild solution because it is not the one that matches to a realistic interior solution. Then, we can not say that the classical tests of general relativity are automatically satisfied, as the early investigators emphasized.

On the other hand, the Palatini approach, or first order formalism, can be applied to obtain the field equations in GR assuming the metric and the connection as independent variables. This formalism has also been applied to more general Lagrangian densities, with quadratic terms [13] or a general function of the scalar curvature, to study other geometrical theories of gravitation. More recently [14], the latter theories have been extended by including a scalar field in the Lagrangian and a connection allowing torsion [15] . One apparent conceptual advantage of these theories is that quantum fluctuations of the metric and the connection are independent of each other.

In the present work we consider those theories that are obtained from a Lagrangian density $\mathcal{L}_T(R)=f(R)\sqrt{-g}+\mathcal{L}_M$, that depends on the curvature scalar and a matter Lagrangian that does not depend on the connection, and apply Palatini's method to obtain the field equations. We prove that in the first order formalism it is true that the vacuum solution that matches with the physical interior solution and is asymptotically flat, is the Schwarzschild solution. The last statement it is proved not only in the weak field approximation but also using the full, nonlinear field equations, stemming from the general Lagrangian $\int f(R)\sqrt{-g}d^4x$.

Also, we show that there exist some Lagrangians for which Birkhoff's theorem is valid, i.e., the only vacuum spherically symmetric solution is the Schwarzschild solution, although the active mass is different to the active mass of GR for the same source.

In the second section we present the general structure of this class of theories; in the third one, we prove, in the weak field approximation, that in $f(R)$ theories using the first order formalism we do not have the drawbacks of the fourth order theories. In the last section, using the full equations, we prove, for a spherically symmetric star, that there exist some cases of $f(R)$, not only GR, where Birkhoff's theorem is also valid, i.e., there is only one vacuum solution, that is the Schwarzschild solution, and it matches with a physical interior solution. Also we prove, for any $f(R)$, with $f(R=0)=0$, that the exterior solutions match to physical interior solutions, and that one of this exterior solutions is the Schwarzschild solution.

The results of the second and third sections show that the properties of the $f(R)$ first order theories are quite different from the second order theories. The first order theories show better physical properties and are similar to GR.

## II. THE STRUCTURE OF THE THEORY

Let $\mathcal{M}$ be a manifold with metric $g_{ab}$ and a torsionless derivative operator $\nabla_a$, both considered as independent variables. Consider a Lagrangian density $\mathcal{L}=f(R)\sqrt{-g}+\mathcal{L}_M$, where the matter Lagrangian $\mathcal{L}_M$ does not depend on the connection.

Suppose we have a smooth one-parameter ($\lambda$) family of field configurations starting from given fields $g^{ab}$, $\nabla_a$, and $\psi$ (the matter fields), with appropriate boundary conditions, and denote by $\delta g^{ab}$, $\delta\Gamma^c_{ab}$, $\delta\psi$ the corresponding variations, i.e., $\delta g^{ab}=(dg^{ab}_\lambda/d\lambda)|_{\lambda=0}$, etc. Then the field equations, if we vary with respect to the metric, are

$$f'(R)R_{ab}-\frac{1}{2}f(R)g_{ab}=T_{ab}, \tag{5}$$

where $f'(R)=(df/dR)$, $(\delta S_M/\delta g^{ab})\equiv -T_{ab}\sqrt{-g}$. The variation with respect to the connection, recalling that this is fixed at the boundary, gives

$$\nabla_c[\sqrt{-g}g^{ab}f'(R)]=0. \tag{6}$$

Now, we choose Lagrangians $f(R)$ with $f'(R)$ derivable and not null for any value of $R$. Then the last equation becomes

$$\nabla_c g_{ab}=b_c g_{ab}, \tag{7}$$

where

$$b_c=-[ln f'(R)]_{,c}. \tag{8}$$

Thus, we have a Weyl conformal geometry with a Weyl field given by Eq. (8).

The vanishing of the connection in a particular frame, for example in a geodesic frame, however, does not mean that the metric is flat there, because from Eq. (7) $\partial_c g_{ab}=b_c g_{ab}$. Therefore the strong equivalence principle is in general not satisfied.

From Eq. (5) we obtain

$$f'(R)R-2f(R)=T, \tag{9}$$

which define $R(T)$, and we suppose the function $f(R)$ is such that $R(T)$ would be differentiable respect to the variable $T$. Therefore $b_c$ is determined by $T$ and its derivatives except in the case $f(R)=\omega R^2$, for which $Rf'-2f\equiv 0$, and then we must consistently have $T\equiv 0$. It is important to note that $b_c$ has solution only if $T$ is differentiable in $\mathcal{M}$; this condition on $T$, for the existence of solution, is not necessary in other theories, as GR or fourth order theories.

Therefore, the field equations (5) can be written as

$$G_{ab}+\frac{1}{2}\Lambda(T)g_{ab}=\kappa(T)T_{ab}(g), \tag{10}$$

with $\Lambda(T)=R(T)-f[R(T)]/f'[R(T)]$ and $\kappa(T)=1/f'[R(T)]$, and both of them continuous. In the last equation we have made explicit the dependency of $T_{ab}$ on the metric. We see that, within the first order formalism, the field equations (10) are, formally, those of GR with a cosmological constant and a gravitational constant which depend on the trace of the stress energy momentum tensor.

The connection solution to Eq. (7) is

$$\Gamma^a_{bc} = C^a_{bc} - \frac{1}{2}(\delta^a_b b_c + \delta^a_c b_b - g_{bc} b^a), \tag{11}$$

where $C^a_{bc}$ are the Christoffel symbols (metric connection). Then we have to solve only Eq. (10).

In order to compare the results with the fourth order gravity, we can write the geometrical tensors in terms of the Christoffel symbols plus contributions from the vector $b_a$. The Riemann and Ricci tensors, and the scalar curvature are

$$R^a_{bcd} = \Gamma^a_{db,c} - \Gamma^a_{cb,d} + \Gamma^a_{cf}\Gamma^f_{db} - \Gamma^a_{df}\Gamma^f_{cb} = R^{0a}_{bcd} + b_{b,[d}\delta^a_{c]} + \delta^a_b b_{[c,d]} - b^a_{,[d} g_{c]b} + \frac{1}{2}(\delta^a_{[c} b_{d]} b_b + b_f b^f \delta^a_{[d}\delta_{c]b} + b^a g_{b[d} b_{c]}) \tag{12}$$

$$R_{ab} = R^0_{ab} + \frac{3}{2} D_a b_b - \frac{1}{2} D_b b_a + \frac{1}{2} g_{ab} D\cdot b + \frac{1}{2} b_a b_b - \frac{1}{2} g_{ab} b^2 \tag{13}$$

$$R = R^0 + 3D\cdot b - \frac{3}{2} b^2, \tag{14}$$

where $R^{0a}_{bcd}$, $R^0_{ab}$, $R^0$, and $D_c$ are the Riemann, Ricci, scalar curvature and covariant derivative, defined from the metric connection, respectively. For a quadratic $f(R)$ [16] it is easy to rewrite the Lagrangian density in the form of a metric compatible fourth order term plus a noncompatible addition that includes a massive vector field $b_a$ with coupling to itself and the curvature $R^0$. This last term is absent in the second order formalism. Thus, we may expect that any action, other than the Hilbert action one, may not necessarily yield the same physics, in the first order formalism as compared to the second order formalism.

In particular, the field equations (5) and (7) are of second order, while in the second order formalism the field equations are of fourth order. This difference will be apparent in the next section, when we study the weak field approximation, and obtain the Green function for the basic field equation, which would be different to the corresponding Green function in the fourth order theory [4].

From Eq. (13) we obtain the skewsymmetric part of the Ricci tensor in the form

$$R_{[ab]} = \partial_a b_b - \partial_b b_a. \tag{15}$$

Then Eq. (8) gives $R_{(ab)} = R_{ab}$. Thus, the Ricci tensor is actually symmetric in this theory.

Because the matter action must be invariant under diffeomorphisms and the matter fields satisfy the matter field equations, then $T_{ab}$ is conserved

$$D^a T_{ab} = 0. \tag{16}$$

Therefore, we may conclude that a test particle will follow the geodesics of the metric connection. Using Eqs. (8) and (9) we have

$$b_c = -\frac{f''\nabla_c T}{f'(Rf'' - f')}. \tag{17}$$

Except for the case of GR, $f''\equiv 0$, the Weyl field is nonzero wherever the trace of the energy-momentum tensor varies with respect to the coordinates. If $T$ is constant, then $R$ is also constant, $b_c = 0$ and Eq. (5) takes the form

$$G_{ab} + \frac{1}{2}\Lambda g_{ab} = \kappa T_{ab}, \tag{18}$$

where $\Lambda$ and $\kappa$ are two constants depending on $R$. All those cases with constant trace of the energy-momentum tensor are equivalent to GR for a given cosmological constant. This is the so-called [17] Universality of the Einstein equations for matter with constant $T$.

## III. THE WEAK FIELD APPROXIMATION

Writing $g_{ab} = \eta_{ab} + h_{ab}$ the linearized Ricci tensor, from Eqs. (13) and (8), is found to be

$$R_{ab} = \partial^c \partial_{(b} h_{a)c} - \frac{1}{2}\partial^c\partial_c h_{ab} - \frac{1}{2}\partial_a\partial_b h + \partial_{(a} b_{b)} + \frac{1}{2}\eta_{ab}\partial^c b_c, \tag{19}$$

where $h = \eta^{ab} h_{ab}$ and we are working in a global inertial coordinate system. As it is well known there is a gauge freedom in any geometrical theory of gravitation corresponding to the group of diffeomorphism of spacetime. In practice, these diffeomorphisms may be viewed as coordinate freedom which may be used to impose coordinate conditions. For instance, we may employ harmonic coordinates $x^a$ which satisfy

$$g^{ab}\Gamma^c_{ab} = 0. \tag{20}$$

In the linear approximation the last expression can be achieved by an infinitesimal coordinate transformation that leaves the flat metric $\eta^{ab}$ unchanged. Then, in this approximation the perturbation $h_{ab}$ and the vector field $b_a$ satisfy the gauge condition

$$\partial^c h_{ca} - \frac{1}{2}\partial_a h + b_a = 0. \tag{21}$$

In this gauge the linearized Ricci tensor simplifies to become

$$R_{ab} = \frac{1}{2}\eta_{ab}\partial^c b_c - \frac{1}{2}\partial^c\partial_c h_{ab}. \tag{22}$$

Therefore, the linearized field equations (10) are

$$\Box\left(h_{ab}-\eta_{ab}\frac{f''(0)}{f'(0)}T\right)=-\frac{2}{f'(0)}\left(T_{ab}-\frac{1}{2}\eta_{ab}T\right),\tag{23}$$

where we have assumed, for simplicity, $f(0)=0$ and we have used Eq. (8) and the first order of Eq. (9) to expressed $b_a$ in terms of the trace $T$.

In the nonrelativistic limit the operator $\Box$ reduces to $\nabla$ and the source has a Newtonian behavior, $T_{ab}\approx\rho t_a t_b$ where $t_a$ is the time direction of our global inertial coordinate system; thus, Eq. (23) becomes

$$\nabla^2\left(h_{ab}+\frac{f''(0)}{f'(0)}\eta_{ab}\rho\right)=-\frac{2}{f'(0)}\left(t_a t_b+\frac{1}{2}\eta_{ab}\right)\rho.\tag{24}$$

This operator has the Green function

$$\frac{[t_a t_b+(1/2)\eta_{ab}]}{2\pi f'(0)|\mathbf{x}-\mathbf{x}'|}-\frac{f''(0)}{f'(0)}\eta_{ab}\delta(\mathbf{x}-\mathbf{x}').\tag{25}$$

Therefore, the only solutions of Eq. (24) which are well behaved at infinity (we assume that $\rho$ is of compact support) is

$$h_{ab}=\frac{1}{2\pi f'(0)}\left(t_a t_b+\frac{1}{2}\eta_{ab}\right)V_N-\frac{f''(0)}{f'(0)}\eta_{ab}\rho(\mathbf{x}),\tag{26}$$

where $V_N$ is the Newtonian potential.

Using the properties of the spherically symmetric Newtonian potential we are able to show that, in this theory and in the linear approximation, the Schwarzschild solution is the exterior solution for the above energy-momentum tensor when it is spherically symmetric; the last statement is not true in the case of fourth order theories.

It is important to note that the solution (25) of the field equations for a mass point is also the exterior solution for an arbitrary spherically symmetric mass distribution. This important property of GR is shared by these theories, but it is not true for fourth order theories.

According to Eq. (26) the field exterior to a spherical body is independent of its size and internal stresses and is identical to the static spherically symmetric solution of vacuum field equations. From this point of view, apparently, these theories satisfy Birkhoff's theorem. In the next section we will see that this is not always true for the theories out of the linear approximation.

Havas [4,11] has pointed out, when he studied, in the fourth order theories, the importance of the spherically symmetric solutions for extended sources and its relation with the exterior solutions, that this problem is close to the one which caused Newton to delay publication of the Principia for two decades. The problem that concerned him was the relation between a postulated force law between two mass points and the corresponding force law between two homogeneous spheres. This problem is independent of any field equation satisfied by the force or the corresponding potential; however, if such equations, rather than elementary force laws, are taken as basic, a new question arises; namely, whether these equations admit more than one type of spherically symmetric solutions. Then Havas pointed out (and referred also to Ref. [18]) that no potential between point particles, other than one proportional to $e^{-\lambda r}/r$, yields a potential between spheres that has the same dependence on $r$, and only if $\lambda$ vanishes (the Newtonian case), it is the coefficient of the resultant potential independent of the radii.

In our case, we have the superposition of two potentials of the type $e^{-\lambda r}/r$. One corresponds to the Newtonian case ($\lambda=0$), and the other is $\delta(\mathbf{x}-\mathbf{x}')$ which is obtained as a limit of

$$\lambda^2 e^{-\lambda r},\tag{27}$$

when $\lambda\to\infty$. To obtain the delta function we have to remember that

$$\lim_{\lambda\to\infty}\int_0^\infty \lambda^2 e^{-\lambda r} r\,dr=1.\tag{28}$$

Therefore, the potential (25) between point particles has the same dependence on $r$ that the potential between spheres. On the other hand, the superposition of one potential with $\lambda=0$ and other with $\lambda\to\infty$ also gives a potential between spheres with coefficients which do not depend on the radii. The second one is an special case of $e^{-\lambda r}/r$ and corresponds to a case which was not discussed before.

## IV. SPHERICALLY SYMMETRIC SOLUTIONS AND BIRKHOFF'S THEOREM

In the case $T=0$, the scalar $R$ is any of the roots, $R_i$, of $f'(R)R-2f(R)=0$. For each root the solutions of the field equations are the solutions of GR with cosmological constant $\Lambda=R_i/2$.

From now on, we will consider only those theories which have as possible solutions, in vacuum, the solutions of GR. Then we asked that $f(R=0)=0$; with this condition one of the roots is always $R=0$ and it corresponds to the case $\Lambda=0$. Therefore, we have in vacuum a set of solutions: some of them corresponding to GR and the others to GR with cosmological constant. But there exist Lagrangian with $f(R)$ which have the property that they have only one root, then this is $R=0$. For these Lagrangians the vacuum solutions of the theory are only the solutions of GR. One example is $f(R)=aR+bR^2$.

Therefore, in vacuum the spherically symmetric solutions are the Schwarzschild metric with cosmological constants $R_i/2$, although the name is properly used only if we put $\Lambda=0$; in all the cases the solutions are static. Then Birkhoff's theorem is valid only in those theories with a unique root $R_i=0$.

When the theories have more than one root, only one of the vacuum solutions is asymptotically flat and spherically symmetric, that is the Schwarzschild metric. In the linear case we only have the linear Schwarzschild solution because in the linear approximation equation (9) has the unique root $R=-T/f'(0)$, and then for vacuum $R=0$.

In the general case we still have to prove that the exterior metric matches correctly to an interior solution which would correspond to a physical situation. In this section we will study those solutions which correspond to having spherically symmetric matter in a region of spacetime.

Consider a conformal transformation of the metric, given by

$$\tilde{g}_{ab}=\Omega^2(x)g_{ab}. \tag{29}$$

Then, the field equation (7) changes as

$$\nabla_c\tilde{g}_{ab}=[b_c+2(\ln\Omega)_c]\tilde{g}_{ab}. \tag{30}$$

Taking into account the definition of $b_c$, Eq. (8), we can choose $\Omega(T)=C\sqrt{|f'|}$, where the constant $C$ is such that $\Omega(0)=1$. Then the field equation (30) for the transformed metric is

$$\nabla_c\tilde{g}_{ab}=0. \tag{31}$$

Therefore, the connection of our theory is the metric connection for the $\tilde{g}_{ab}$ metric. Thus, $G_{ab}=\tilde{G}^0_{ab}$ and the other field equation, Eq. (10), can be written in the form

$$\tilde{G}_{ab}+\frac{1}{2}\frac{\Lambda(T)}{\Omega^2(T)}\tilde{g}_{ab}=\kappa(T)T_{ab}(\Omega^{-2}\tilde{g}), \tag{32}$$

where we have dropped the ''0'' from $\tilde{G}^0_{ab}$, for simplicity, and taken into account the dependence of $T_{ab}$ on the metric. It is apparent from Eq. (32) that our theory is conformally equivalent to GR without minimal coupling, and with cosmological and gravitational constants that are not in fact constants, except in the linear case. We may expect to find some observational consequence of this fact, for instance in a cosmological model, if we use the present values of the cosmological parameters $q_0, H_0, (\dot{G}/G)_0$, and $(\ddot{G}/G)_0$, to estimate the order of magnitude of what may be essentially the departure of $f(R)$ from linearity.

Let us consider now a spherically symmetric spacetime manifold with a metric $\tilde{g}$ in the standard form (curvature coordinates)

$$d\tilde{s}^2=e^{\alpha(r,t)}dr^2+r^2d\omega^2-e^{\gamma(r,t)}dt^2. \tag{33}$$

The nonzero components of the Einstein tensor are

$$\tilde{G}^r_r=\frac{e^{-\alpha}}{r^2}(1+r\gamma_{,r})-\frac{1}{r^2},$$

$$\tilde{G}^\theta_\theta=\tilde{G}^\phi_\phi,$$

$$\tilde{G}^\theta_\theta=e^{-\alpha}\left(\frac{\gamma_{,rr}}{2}+\frac{\gamma^2_{,r}}{4}+\frac{\gamma_{,r}}{2r}-\frac{\alpha_{,r}}{2r}-\frac{\alpha_{,r}\gamma_{,r}}{4}\right)$$

$$-e^{-\gamma}\left(\frac{\alpha_{,tt}}{2}+\frac{\alpha^2_{,t}}{4}-\frac{\alpha_{,t}\gamma_{,t}}{4}\right),$$

$$\tilde{G}^t_t=\frac{e^{-\alpha}(1-r\alpha_{,r})}{r^2}-\frac{1}{r^2},$$

$$\tilde{G}^r_t=-e^{\gamma-\alpha}G^t_r=\frac{e^{-\alpha}\alpha_{,t}}{r}. \tag{34}$$

The nonzero components of the energy-momentum tensor with spherical symmetry are $T^r_r$, $T^\theta_\theta=T^\phi_\phi$, $T^t_t$, $T^r_t$, $T^tr$, with $e^\alpha T^r_t=-e^\gamma T^t_r$. Let us regard $T^r_r$ and $T^t_t$ as assigned, then the solutions of the field equations (32) are

$$e^{-\alpha}=1-\frac{2\tilde{\mathcal{M}}(r,t)}{r}, \tag{35}$$

$$e^\gamma=e^{-\alpha}+\exp\left(\int_o^r\hat{r}e^\alpha\kappa(T)[T^r_r(\Omega^{-2}\tilde{g})-T^t_t(\Omega^{-2}\tilde{g})]d\hat{r}\right), \tag{36}$$

where $\tilde{\mathcal{M}}(r,t)$ is

$$\tilde{\mathcal{M}}(r,t)=-\frac{1}{2}\int_0^r\hat{r}^2\left(\kappa(T)T^t_t-\frac{1}{2}\frac{\Lambda(T)}{\Omega^2(T)}\right)d\hat{r}. \tag{37}$$

The spherical symmetry of the metric is, of course, invariant under the conformal change (29) and we can write the metric $g$ in the form

$$ds^2=A(r',t)dr'^2+r'^2d\omega^2-B(r',t)dt^2, \tag{38}$$

where $r'=r/\Omega$.

Having thus expressed the metric $\tilde{g}_{ab}$ in terms of $T^r_r(\Omega^{-2}\tilde{g})$, $T^t_t(\Omega^{-2}\tilde{g})$ and $T$ we get the following expressions for the metric $g_{ab}$:

$$A(r',t)=\Omega^{-2}(T)\frac{1}{(1-2\tilde{\mathcal{M}}(r',t)/r')}, \tag{39}$$

$$B(r',t)=\Omega^{-2}(T)\left[1-\frac{2\tilde{\mathcal{M}}(r',t)}{r'}+\exp\left(\int_o^{r'\Omega}\hat{r}e^\alpha\kappa(T)\right.\right.$$

$$\left.\left.\times[T^r_r(\Omega^{-2}\tilde{g})-T^t_t(\Omega^{-2}\tilde{g})]d\hat{r}\right)\right], \tag{40}$$

where

$$\tilde{\mathcal{M}}(r',t)=-\frac{1}{2\Omega}\int_0^{r'\Omega}\hat{r}^2\left(\kappa(T)T^t_t-\frac{1}{2}\frac{\Lambda(T)}{\Omega^2(T)}\right)d\hat{r}. \tag{41}$$

In the exterior region, $r>R$, the solution is

$$A(r)=\frac{1}{1-2M/r+R_i/12(r^2-R^2)}, \tag{42}$$

$$B(r)=1-\frac{2M}{r}+\frac{R_i}{12}(r^2-R^2), \tag{43}$$

where $M$ is

$$M=-\frac{1}{2}\int_0^R \hat{r}^2\left(\kappa(T)T_t^t-\frac{1}{2}\frac{\Lambda(T)}{\Omega^2(T)}\right)d\hat{r}. \tag{44}$$

In writing Eq. (43) we have reabsorbed the integral term in Eq. (40) by introducing a new time coordinate, as is usual in GR; recall also that $\Omega(0)=1$. The function $\Omega(T)$ is differentiable, then we have proved that in all cases, the exterior metrics match correctly with the interior solutions. In particular, the Schwarzschild solution $R_i=0$, which is always one of the exterior solutions, satisfies the junction conditions with a physical interior metric. On the other hand, it is clear that the active mass of the Schwarzschild solution, in these theories, it is quite different from the active mass of GR for the same source. However, as it was recently proved [19], the active mass, in these theories, is equal to the inertial mass.

It is interesting to estimate the relative difference between the active mass in GR and the active mass in our theory. To this end, we choose as a model theory $f(R)=R+\alpha R^2$. In this model we have $R=-T$. We consider a star similar to our Sun (radius $\mathcal{R}$), and work in the Newtonian approximation. Thus $T\approx-\rho$ and $M/\mathcal{R}^3\approx 4\times 10^{-28}$ cm$^{-2}$, in geometrical units ($c=G=1$). Also $M/\mathcal{R}\approx 2\times 10^{-6}$. We have $\Omega^2=f'$, $\Lambda(T)\approx\alpha T^2$, $\kappa(T)=(1-2\alpha T)^{-1}$. If we call $M'$ the active mass of GR, and use that $\rho\approx 3M'/4\pi\mathcal{R}^3$, we obtain from Eq. (44) that

$$\frac{M'-M}{M'}\approx\frac{9\alpha}{2\pi}10^{-28}\ \text{cm}^{-2}. \tag{45}$$

The constant $\alpha$ has dimensions of cm$^2$ and its value could be estimated, in principle, from the values of observational cosmological parameters in the same model theory.

## V. CONCLUSIONS

We have analyzed for the first order $f(R)$ theories of gravity the properties of the solutions in the linear case, and the properties of the spherically symmetric solutions in the nonlinear case. We have compared the results with the results for the fourth order theories.

We have identified the Green function of the field equations, in the linear approximation, and proved that the only exterior solution which is well behaved at infinity and has a physical source, is the Schwarzschild solution. Also, we have shown that the solution of the field equations for a ''mass point'' is also the exterior solution for an arbitrary spherically symmetric mass distribution. The exterior linear solution does not depend on the size and internal stresses of the source. In this approximation we have proved Birkhoff's theorem but in most cases it is not true when we work in the full theory.

In the case of the spherically symmetric solutions we have obtained the exterior and interior metrics and we have shown that they satisfy the junctions conditions, although the Birkhoff's theorem is not satisfied in all the cases. However, one of the exterior solutions is always the Schwarzschild solution and it matches correctly with an interior physical metric.

All the above statements, which are shared by GR, are not true for the theories with the same Lagrangians in the second order formalism (fourth order theories). Therefore, we may conclude that the class of theories that we have studied in this paper has a better classical behavior than the fourth order theories.

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