Spherically symmetric dust shell and the time problem in canonical relativity

Petr Hájíček

Institute for Theoretical Physics, University of Berne, Sidlerstrasse 5, CH–3012 Berne, Switzerland

Jerzy Kijowski

Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, 02-668 Warsaw, Poland
and Department of Mathematical Methods in Physics, University of Warsaw, ul. Hoża 74, 00-682 Warsaw, Poland

The dynamics of a self-gravitating spherical dust shell is analyzed within the Hamiltonian formulation. Given any time variable on the shell, a complete reduction with respect to the momentum constraints is obtained and the true degrees of freedom of the composed ''matter + gravity'' system are found. The transition between two such formulations, corresponding to two different time parametrizations, is described as a canonical transformation in the reduced phase space. Explicit formulas for such a transformation are given. Implications for quantum gravity are discussed.

I. INTRODUCTION

Field dynamics in general relativity may be formulated in terms of a constrained Hamiltonian system in the symplectic space of Cauchy data for Einstein equations [1]. The symplectic form is degenerate when restricted to constraints. For many purposes, we would like to reduce the system with respect to this degeneracy, i.e., to pass to the quotient space whose points are equivalence classes of Cauchy data. Two such sets of data belong to the same equivalence class (i.e., represent the same point of the reduced space) if they may be connected by a one-parameter family of data, such that its tangent vector (i.e., the derivative with respect to the parameter) is symplectically ''orthogonal'' to the whole constraint subspace. Physically, equivalent data are those which give isometric solutions of Einstein equations. The explicit reduction of the theory enables us, if possible, to pass from a gauge-dependent parametrization of dynamics to the gauge-independent description, in terms of ''true degrees of freedom'' of the gravitational field.

There are, roughly, two ways to reduce a constrained Hamiltonian system. The first way consists in providing a complete system of gauge-invariants parametrizing the quotient space. This might be called the ''royal road'' to Hamiltonian reduction. It is manifestly gauge-invariant and it also leads to a gauge-invariant quantum theory. Another way is based on gauge fixing in the sense that we single out a representative in each equivalence class [2]. Experience shows that the ''royal road'' is very often too difficult for explicit calculations. On the other hand, global gauge fixing is often prevented by the Gribov problem; it leads, moreover, necessarily to a gauge-dependent quantum theory (this has been shown [3] for very general cases). A relation between gauge fixing and the ''royal way'' has been thoroughly discussed by the present authors [4].

To test these theoretical concepts, models with especially simple matter field and with high symmetry are especially useful. In particular, a self-gravitating two dimensional matter shell is an important laboratory of canonical gravity [5] and (possibly) its quantized version [6]. It was shown recently [7] that the spherically symmetric dust-shell model may be effectively reduced using a particular gauge fixing for the time variable (i.e., with respect to the Hamiltonian constraint), whereas the reduction with respect to remaining (momentum) constraints may already be performed according to the ''royal way.'' In the present paper we analyze the role of this particular gauge fixing and prove that the complete reduction may be carried out in any other time gauge. Moreover, we find explicitly the canonical variables corresponding to an arbitrary time gauge and prove that the transition between two gauges is a canonical transformation of the reduced phase space. It seems, however, that the time-gauge fixing is necessary and that we are not able to perform the entire reduction according to the ''royal way.'' We prove that transformations of the reduced phase space, which are generated by different time gauges exhaust the entire symplectomorphism group. As was recently explained [3], this might imply a strong limitation to quantization procedures based on complete Hamiltonian reduction (i.e., on true degrees of freedom) of general relativity.

II. TRUE DEGREES OF FREEDOM OF THE SPHERICALLY SYMMETRIC DUST SHELL MODEL

The nonreduced phase space of the ''shell + gravity'' system is described by the following space of functions:

$$\mathcal{P}' := \{(g_{kl}, P^{kl}, y^k, p_k)\},$$

where g_{kl} is an (asymptotically flat) Riemannian metric on the Cauchy surface Σ and P^{kl} describe its Arnowitt-Deser-Misner (ADM) momentum. The remaining objects y^k and p_k describe the position and the momentum assigned to all the points of the material. This phase space carries the symplectic form

$$\Omega' := \frac{1}{16\pi} \int_\Sigma \{\delta P^{kl}(x) \wedge \delta g_{kl}(x)\} d^3x$$

$$+ \int_Z \{\delta p_k(z) \wedge \delta y^k(z)\} d^2z, \tag{2.1}$$

where by Z we denote the two dimensional ''material space'' (an abstract collection of the idealized points of the material forming our shell). This means that without constraints, the Poisson brackets between P^{kl} and g_{kl}, and the Poisson brackets between p_k and y^k, are δ like, whereas remaining Poisson brackets vanish. Taking into account constraints, the above quantities are no longer independent, which gives rise to the degeneracy of Ω'.

Any solution of the spherically symmetric matter shell model is composed of an external piece of the Schwarzschild space and an internal piece of the Minkowski space. These pieces are stitched together along the shell's history. The results may be summarized as follows [7]. Take any time foliation which coincides with the Schwarzschild fixed time $\{t=\text{const}\}$ outside of the shell (inside of the shell it may be arbitrary). At each instant t the true degrees of freedom of the above ''matter + gravity'' system may be described by two (mutually conjugate) quantities: $\rho = R^2 := (1/4\pi)s$, where s is the surface of the shell, and the hyperbolic angle μ between the Schwarzschild fixed time surface outside of the shell and the Minkowski fixed time surface inside of the shell. More precisely: at each point of the shell we take a normalized vector $\mathbf{u}$, orthogonal to the Schwarzschild constant time surface and a normalized vector $\mathbf{v}$, orthogonal to the Minkowski constant time surface. The angle between these surfaces is defined by the scalar product $(\mathbf{u}|\mathbf{v})$ between these vectors, according to the formula

$$\cosh \mu := (\mathbf{u}|\mathbf{v}).$$

It was proved [7] that, after solving the constraints, all the gauge degrees of freedom in Eq. (2.1) drop out and the above symplectic form reduces to

$$\Omega := \tfrac{1}{2}\,\delta\mu \wedge \delta\rho. \tag{2.2}$$

Moreover, the total Hamiltonian of the system (numerically equal to the ADM mass at infinity) may be expressed explicitly in terms of the above canonical variables by the following formula

$$H(\mu,\rho) = \frac{1}{2}\sqrt{\rho}\left\{1 - \left(\cosh\mu - \sqrt{\frac{M^2}{\rho} + \sinh^2\mu}\,\right)^2\right\}, \tag{2.3}$$

where the constant M denotes the total mechanical (rest frame) mass of the shell.

Hamiltonian (2.3) generates the dynamics of our system by the standard Hamiltonian equations

$$\frac{1}{2}\dot{\rho} = \frac{\partial H}{\partial \mu}, \tag{2.4}$$

$$\frac{1}{2}\dot{\mu} = -\frac{\partial H}{\partial \rho}. \tag{2.5}$$

Suppose now that we have any explicit solution $[\rho(t),\mu(t)]$ of these equations. It was shown [7] how to reconstruct (uniquely, up to an isometry) the entire spacetime, which is a solution of the combined ''Einstein equations + matter equations'' system. We may say that Eqs. (2.2) and (2.3) contain the complete, gauge-independent description of the physical system in question.

The Hamiltonian $H(\mu,\rho)$ can be also obtained from the super-Hamiltonian that has been derived independently [8–10]: one has to choose the Schwarzschild time t as the time variable and solve the super-Hamiltonian constraint equation for the conjugate momentum. The Schwarzschild time, of course, diverges where the shell crosses the horizon. Still, one can construct a complete shell dynamics from $H(\mu,\rho)$ by carefully matching the trajectories from both sides of the horizon. Rather, we are going to discuss in the sequel how to pass to an arbitrary (and everywhere regular) time gauge.

III. TRANSITION BETWEEN DIFFERENT TIME VARIABLES AS A CANONICAL TRANSFORMATION OF THE REDUCED PHASE SPACE

In our approach, a gauge condition is necessary to fix uniquely the surfaces $\{t=\text{const}\}$ and to rewrite the dynamics of our ''shell + gravity'' system in terms of the Hamiltonian dynamics. Any gauge condition $\mathcal{G}$ is a choice of a one-parameter family of such surfaces, which we may call $\Sigma_t^{\mathcal{G}}$. We want to respect the asymptotic flatness at infinity. Hence, we assume that each $\Sigma_t^{\mathcal{G}}$ tends asymptotically to an asymptotic time surface. The label t refers to the prefixed Schwarzschild time at spatial infinity, which we choose *a priori*. The results described in the previous section were obtained in a particular gauge. Suppose now that we change the gauge condition in such a way, that the intersection of the shell with the new surfaces $\Sigma_t^{\mathcal{G}}$ is the same as before. Due to standard gauge-invariance arguments of general relativity, it is obvious that the reduced phase space does not change and that $(\rho(t),\mu(t))$ remain canonical variables. This shows that, in fact, our final results do not depend upon the entire Σ's but only on the time parametrization which they introduce on the moving shell.

Let now $\mathcal{G}$ be another gauge condition. Let us now denote by $(\rho^{\mathcal{G}}(t),\mu^{\mathcal{G}}(t))$ the value of the same geometric quantities, but taken on the new $\Sigma_t^{\mathcal{G}}$ surface. We shall use these variables to parametrize the reduced phase space. As will be obvious, $(\rho^{\mathcal{G}}(t),\mu^{\mathcal{G}}(t))$ are no longer canonical variables. We will show how to describe explicitly the reduced phase-space structure in terms of them.

For this purpose let us denote by ''v'' the retardation (Verspätung) of the new time coordinate on the shell with respect to the previous one. This means that the surface $\Sigma_t^{\mathcal{G}}$ intersects the shell at the Schwarzschildian time ''$t+v(\rho,\mu)$.'' We allow the value of this retardation to depend upon the particular dynamical situation, i.e., upon the phase-space point. But we assume that both gauge conditions are ''intrinsic,'' i.e., may be formulated in terms of initial data only (as e.g., is true for the so-called β gauges [11]). This excludes an explicit dependence of the retardation function upon the time variable.

The function $v=v(\rho,\mu)$ contains the entire information about the transformation between the old variables

$(\rho(t),\mu(t))$ and the new ones $(\rho^{\mathcal{G}}(t),\mu^{\mathcal{G}}(t))$. Indeed, once we know $(\rho(t),\mu(t))$, we solve the dynamical equations and put

$$\rho^{\mathcal{G}}(t) := \rho[t+v(\rho(t),\mu(t))], \qquad (3.1)$$

$$\mu^{\mathcal{G}}(t) := \mu[t+v(\rho(t),\mu(t))]. \qquad (3.2)$$

For a generic, nonconstant function v, such a transformation *is not*, in general, canonical. We are going to show in the sequel how to calculate the canonical structure of our reduced phase space in terms of these new variables. In particular, we will show how to find the momentum canonically conjugate to the new variable $\rho^{\mathcal{G}}(t)$.

It is convenient to use for this purpose the language of *contact manifolds*. Let us observe that the entire information about dynamics may be retrieved from the three-dimensional contact space, defined as a surface $\{E=H(\rho,\mu)\}$ in the four dimensional space $\{(t,E,\rho,\mu)\}$, equipped with the standard form

$$\Psi := \tfrac{1}{2}\delta\mu\wedge\delta\rho - \delta E\wedge\delta t. \qquad (3.3)$$

The form (once symplectic on the four-dimensional phase space) becomes degenerate when restricted to the $\{E=H(\rho,\mu)\}$ subspace. The system's trajectories are uniquely defined as those, whose tangent vector belongs to this degeneracy. To prove this statement it is sufficient to parametrize the subspace with three variables (t,ρ,μ) and rewrite the form as

$$\Psi := \frac{1}{2}\delta\mu\wedge\delta\rho - \left(\frac{\partial H}{\partial\mu}\delta\mu + \frac{\partial H}{\partial\rho}\delta\rho\right)\wedge\delta t. \qquad (3.4)$$

Now, it is easy to see that any vector which annihilates the above form must be proportional to the following vector:

$$X := \frac{\partial}{\partial t} + \dot\rho\frac{\partial}{\partial\rho} + \dot\mu\frac{\partial}{\partial\mu},$$

where the coefficients $\dot\rho$ and $\dot\mu$ are given by the Hamilton equations (2.4) and (2.5).

We claim that choosing our new gauge condition $\mathcal{G}$ is equivalent to choosing the variable

$$T := t - v \qquad (3.5)$$

as a new time parameter. Indeed, for a given spacetime event (t,ρ,μ) in spacetime, where t is our previous (e.g., Schwarzschildian) time, we want to choose, among all the new surfaces $\Sigma_T^{\mathcal{G}}$, the only one which passes through this particular event. Due to our previous considerations, it is the one which corresponds to $T := t-v$. In our new gauge we will label our event by this particular value of time (this is why we call v a *retardation* and not an *acceleration*).

Let us, therefore, rewrite our form Ψ in terms of the new time T. At this point it is convenient to choose the energy E as one of the independent parameters, rather than μ, and to treat the latter as the function $\mu=\mu(E,\rho)$ obtained by solving Eq. (2.3) with respect to μ. In this way we obtain

$$\Psi = \frac{1}{2}\delta\mu\wedge\delta\rho - \delta E\wedge\delta T + \delta E\wedge\left(\frac{\partial v}{\partial E}\delta E + \frac{\partial v}{\partial\rho}\delta\rho\right)$$

$$= \frac{1}{2}\left(\delta\mu + 2\frac{\partial v}{\partial\rho}\delta E\right)\wedge\delta\rho - \delta E\wedge\delta T. \qquad (3.6)$$

If we now define the function

$$V(E,\rho) := 2\int\frac{\partial v}{\partial\rho}(E,\rho)\ dE + a(\rho)$$

$$= \int\frac{1}{R}\frac{\partial v}{\partial R}(E,R)\ dE + a(\rho) \qquad (3.7)$$

[an additive constant $a(\rho)$ is arbitrary], then the variable

$$\widetilde\mu := \mu + V[H(\rho,\mu),\rho], \qquad (3.8)$$

is the momentum canonically conjugate to $\rho^{\mathcal{G}}(t)$ because of the following formula:

$$\Psi = \tfrac{1}{2}\delta\widetilde\mu\wedge\delta\rho - \delta E\wedge\delta T. \qquad (3.9)$$

In this way we obtain immediately the canonical structure of the reduced phase space in terms of any gauge, as soon as we know its retardation function v with respect to any gauge which we already know. This allows us to consider the dynamics of our system with respect to different times, not only with respect to the Schwarzschildian one. Once we construct the corresponding momentum $\widetilde\mu^{\mathcal{G}}(T) := \widetilde\mu(t)$ with $t=T+v$, and express the Hamiltonian in terms of the variables $(\widetilde\mu^{\mathcal{G}},\rho^{\mathcal{G}})$, we may even forget about the foliation $\Sigma_t^{\mathcal{G}}$ which has led us to this result: we may simply treat it as a Hamiltonian system which describes the evolution of the quantities $(\widetilde\mu,\rho)$, with respect to the new time variable T. The relations (3.5), (3.8) enable us, however, to identify solutions of these two, apparently different, Hamiltonian systems as representing the same solutions of Einstein's equations with the dust-shell matter.

Let us observe that if the retardation function depends *only* upon the energy E (i.e., if its derivative with respect to ρ vanishes) the quantity V may be put equal zero. This means that $(\mu^{\mathcal{G}},\rho^{\mathcal{G}})$ are still canonical variables and the transformation between them and the previous variables (μ,ρ) is a canonical transformation (there is still a possibility of adding a function $a(\rho)$ to the momentum, but this is *always* a canonical transformation). Because E is constant on each trajectory, we conclude that this happens when the retardation remains constant during the evolution. In such a case, even if the surface $\Sigma_t^{\mathcal{G}}$ intersects the shell at a different time T, the difference between the two times t and T, remains constant during the evolution. Such two times differ, therefore, only by an additive gauge, but they both define the same notion of the ''time lapse.'' We conclude that the entire Hamiltonian description does not change if we move between two-time parametrizations t and T which differ by such an additive gauge only.

On the other hand, we may choose virtually *any* variable $\widetilde\mu$ (independent from ρ) as a momentum canonically conju-

gate to ρ. Indeed, given such a variable we may always find a gauge (i.e., a time parametrization) that the transformation from (μ,ρ) to $(\tilde{\mu}^{\mathcal{G}},\rho^{\mathcal{G}})$ is canonical. For this purpose it is sufficient to express both $\tilde{\mu}$ and μ in terms of E and ρ and to treat formula (3.7) as the definition of the necessary retardation v between the two gauges: the old one and the new one which we are going to define. Differentiating (3.8) with respect to energy we obtain

$$\frac{\partial}{\partial E}(\tilde{\mu}-\mu)=:2\frac{\partial v}{\partial \rho}(E,\rho)=\frac{1}{R}\frac{\partial v}{\partial R}(E,R). \quad (3.10)$$

The left hand side is known. Once we find the function v by integrating the above equation with respect to ρ or R, it is straightforward to define a gauge $\mathcal{G}$ which has this value of the retardation with respect to our previous gauge.

The fact, that *any* variable $\tilde{\mu}$ may arise as a momentum canonically conjugate to ρ, if we only allow all possible time gauges, proves that insisting on quantization respecting gauge invariance of the model, we must allow the entire group of all symplectomorphisms of our two-dimensional reduced phase space as the symmetry group of the quantum version of the theory. At the moment, no general symplectomorphism-invariant quantization procedure is available and this may imply strong limitation to quantization procedures based on complete Hamiltonian reduction.

As an example of the above results take the function $\tilde{\mu}$ which was used to describe the motion of the shell with respect to the Minkowski time [12], calculated on the internal face of the shell:

$$\tilde{\mu}=\sinh \mu \quad \sqrt{1-\frac{2E}{R}}$$
$$=\sinh \mu\left(\cosh \mu-\sqrt{\left(\frac{M}{R}\right)^2+\sinh^2\mu}\right). \quad (3.11)$$

Solving Eq. (2.3) with respect to μ gives us

$$\cosh \mu=\frac{R^2-(\frac{1}{2}M^2+ER)}{R^2\sqrt{1-2E/R}},$$

$$\sinh \mu=\frac{\sqrt{(\frac{1}{2}M^2+ER)-M^2R^2}}{R^2\sqrt{1-2E/R}}.$$

Differentiating the first formula with respect to E and using the second formula we obtain

$$\frac{\partial \mu}{\partial E}=\frac{ER-\frac{1}{2}M^2}{(R-2E)\sqrt{\left(\frac{1}{2}M^2+ER\right)-M^2R^2}}. \quad (3.12)$$

On the other hand, the expression for $\sinh\mu$ enables us to express easily $\tilde{\mu}$ in terms of E and R:

$$\tilde{\mu}=\sinh\mu \ \sqrt{1-\frac{2E}{R}}=\frac{1}{R^2}\sqrt{(\frac{1}{2}M^2+ER)-M^2R^2}, \quad (3.13)$$

and consequently,

$$\frac{\partial \tilde{\mu}}{\partial E}=\frac{ER+\frac{1}{2}M^2}{R\sqrt{\left(\frac{1}{2}M^2+ER\right)-M^2R^2}}. \quad (3.14)$$

Finally, formula (3.10) gives us

$$\frac{\partial v}{\partial R}=\frac{M^2R-2E(\frac{1}{2}M^2+ER)}{(R-2E)\sqrt{(\frac{1}{2}M^2+ER)^2-R^2M^2}}. \quad (3.15)$$

To prove that the above quantity describes the retardation v between the Minkowski time T inside the shell and the old, external Schwarzschild time t we differentiate formula (3.5) with respect to t:

$$\dot{T}=1-\frac{\partial v}{\partial R}\dot{R} \quad (3.16)$$

using $\dot{E}\equiv0$. The derivative $\dot{R}$ may be calculated from Eq. (2.4), where the Hamiltonian is given by Eq. (2.3) (we remember that $\dot{\rho}=2R\dot{R}$). This way we obtain

$$\frac{\partial v}{\partial R}\dot{R}=\frac{M^2-(2E/R)(\frac{1}{2}M^2+ER)}{\sqrt{(\frac{1}{2}M^2+ER)^2-R^2M^2}}\cdot\frac{\sinh \mu}{\sqrt{(M/R)^2+\sinh^2 \mu}}. \quad (3.17)$$

Using our previous results we are able to rewrite this quantity in the following form [12]:

$$\frac{\partial v}{\partial R}\dot{R}=\frac{(2E/R)(\frac{1}{2}M^2+ER)-M^2}{\frac{1}{2}M^2-ER}, \quad (3.18)$$

which, finally leads to the following formula

$$\dot{T}=\left(1-\frac{2E}{R}\right)\frac{ER+\frac{1}{2}M^2}{ER-\frac{1}{2}M^2}. \quad (3.19)$$

The explicit value of the retardation function between the Minkowski time T and the Schwarzschild time t may be immediately obtained by integrating the right hand side of (3.15) with respect to R, using e.g., the Maple V packet. This way we obtain:

$$v(E,R) = \frac{1}{\sqrt{c}\sqrt{a+cC^2-bC}} \times \left\{ \ln(2\sqrt{c}\sqrt{a+bR+cR^2}+2cR+b)A\sqrt{a+cC^2-bC} \right.$$

$$\left. - \operatorname{arcsinh}\left(\frac{2cCR-bR+bC-2a}{(R+C)\sqrt{4ac-b^2}}\right)b\sqrt{c} - \operatorname{arcsinh}\left(\frac{2cCR-bR+bC-2a}{(R+C)\sqrt{4ac-b^2}}\right)CA\sqrt{c}\right\},$$

where $A = M^2 - 2E$, $C = 2E$, $a = \frac{1}{2}M^2$, $b = EM^2$, and $c = E^2 - M^2$.

Rewriting the canonical form in the following way: $\frac{1}{2}\delta\widetilde{\mu}\wedge\delta R^2 = \delta(R\widetilde{\mu})\wedge\delta R$, we obtain $R\widetilde{\mu}$ as the momentum canonically conjugate to the (radial) position R of the shell. The following formula may be easily checked [7]:

$$R\widetilde{\mu} = P\sqrt{1 - \frac{2E}{R}},$$

where P denotes the special relativistic (i.e., calculated with respect to the internal Minkowskian metric) total radial mechanical momentum of the shell. It is interesting to notice that under the influence of gravitational interaction this momentum gets deformed to the total "mechanical + gravitational" momentum and that this deformation consists in multiplying P by the Schwarzschildian "potential factor" $\sqrt{1-2E/R}$ only.

The transformation between the Schwarzschild-time t and Minkowski-time T that we have explicitly written down is an example of a general Bergmann-Komar type of transformation: it is explicitly dependent on the dynamical degrees of freedom (here E and R) of the system. The two quantum mechanics based on these two times cannot, therefore, be unitarily equivalent. As was recently explained [3], the time parameters must always be c numbers, whereas the transformation between them contains q numbers, and these two facts are not compatible with the unitary equivalence.

ACKNOWLEDGMENTS

[1] R. Arnowitt, S. Deser, and C. Misner, in *Gravitation: an Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), p. 227.

[2] J. Kijowski and W. M. Tulczyjew, *A Symplectic Framework for Field Theories*, Lecture Notes in Physics Vol. 107 (Springer, Berlin, 1979).

[3] P. Hájíček, in Proceedings of 19th Texas Symposium of Relativistic Astrophysics and Cosmology, Paris, 1999, gr-qc/9903089.

[4] P. Hájíček and J. Kijowski, Phys. Rev. D **61**, 024037 (2000).

[5] P. Hájíček and J. Kijowski, Phys. Rev. D **57**, 914 (1998).

[6] W. Israel, Nuovo Cimento B **44**, 1 (1966); **48**, 463 (1967); P.

Hájíček, B. S. Kay, and K. Kuchař, Phys. Rev. D **46**, 5439 (1992).

[7] J. Kijowski, Acta Phys. Pol. **29**, 1001 (1998).

[8] S. Ansoldi, A. Aurilia, R. Balbinot, and E. Spalluci, Class. Quantum Grav. **14**, 2727 (1997).

[9] P. Hájíček, Phys. Rev. D **57**, 936 (1998).

[10] V. Berezin, A. M. Bojarsky, and A. Yu. Neronov, Phys. Rev. D **57**, 1118 (1998).

[11] J. Jezierski and J. Kijowski, Phys. Rev. D **36**, 1041 (1987).

[12] P. Hájíček, B. S. Kay, and K. V. Kuchař, Phys. Rev. D **46**, 5439 (1992); P. Hájíček and J. Bičák, *ibid.* **56**, 4706 (1997).