Plane-fronted waves in metric-affine gravity

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We study plane-fronted electrovacuum waves in metric-affine gravity theories with a cosmological constant. Their field strengths are, on the gravitational side, curvature $R_\alpha{}^\beta$, nonmetricity $Q_{\alpha\beta}$, torsion T^α and, on the matter side, the electromagnetic field strength F. Our starting point is the work by Ozsváth, Robinson, and Rózga on type N gravitational fields in general relativity as coupled to null electromagnetic fields.

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I. INTRODUCTION

Even though Einstein's general relativity appears almost fully corroborated experimentally, there are several reasons to believe that the validity of such a description is limited to macroscopic structures and to the present cosmological era. The only available finite perturbative treatment of quantum gravity, namely the theory of the quantum superstring [1], suggests that *non-Riemannian* features are present on the scale of the Planck length. On the other hand, recent advances in the study of the early universe, as represented by the inflationary model, involve, in addition to the metric tensor, at the very least a *scalar dilaton* [2] induced by a Weyl geometry, i.e. again an essential departure from Riemannian metricity [3]. Even at the classical cosmological level, a dilatonic field has recently been used to describe the presence of dark matter in the universe as well as to explain certain cosmological observations which contradict the fundaments of the standard cosmological model [4].

There is good experimental evidence that, at the present state of the universe, the geometrical structure of spacetime corresponds to a metric-compatible geometry in which nonmetricity, but not necessarily the torsion, vanishes. Consequently, the full metric-affine geometry (MAG) is irrelevant for the geometrical description of the universe today. However, during the early universe, when the energies of the cosmic matter were much higher than today, we expect scale invariance to prevail and, according to MAG, the canonical dilation (or scale) current of matter; i.e., the trace of the hypermomentum current $\Delta^\gamma{}_\gamma$ becomes coupled to the Weyl covector $Q := \frac{1}{4} g^{\alpha\beta} Q_{\alpha\beta}$. Here $Q_{\alpha\beta} := -D g_{\alpha\beta}$ is the nonmetricity of spacetime. Moreover, shear type excitations of the material multispinors (Regge trajectory type of constructs) are expected to arise, thereby liberating the (metric-compatible) Riemann-Cartan spacetime from its constraint of vanishing nonmetricity $Q_{\alpha\beta}=0$. It is therefore important to derive and investigate exact solutions of these theories which contain information about the new geometric objects such as torsion and nonmetricity (for a survey of these theories see [5]).

For restricted irreducible pieces of torsion and nonmetricity, there are similarities between the Einstein-Maxwell system and the vacuum MAG field equations [6,7]. This observation encourages us to find new solutions for MAG theories [8]. However, the coupling of the post-Riemannian structures of a metric-affine spacetime to matter is still under investigation.

The search for plane-fronted wave solutions in MAG was first restricted to its Einstein-Cartan sector [9–12]. Later, plane wave solutions with non-vanishing nonmetricity were found by Tucker and Wang [13]. Colliding waves with the appropriate metric and an excited post-Riemannian triplet are studied in [14], the corresponding generalization to the electrovac case can be found in [15].

In this paper we study plane-fronted gravitational and electromagnetic waves in metric-affine gravity theories with nonzero cosmological constant in their *triplet ansatz* sector. The plane-fronted electrovacuum-MAG waves comprise curvature, nonmetricity, torsion, and an electromagnetic field.

The plan of the paper is as follows: In Sec. II, we review the plane-fronted gravitational and electromagnetic waves in Einstein-Maxwell theory. In Sec. III, we present the plane-fronted gravitational and electromagnetic waves in MAG. In Sec. IV, we specialize to particular wave solutions. In Sec. V, we discuss the results, and in Sec. VI we give an outlook for the theory.

II. PLANE-FRONTED GRAVITATIONAL AND ELECTROMAGNETIC WAVES IN EINSTEIN-MAXWELL THEORY

In this section we summarize the main results of Ref. [16]: Using the null tetrad formalism, in a coordinate system $(\rho,\sigma,\zeta,\overline{\zeta})$ (the overbar denotes complex conjugation), the metric reads

$$ds^2 = 2(\vartheta^{\hat{0}} \otimes \vartheta^{\hat{1}} + \vartheta^{\hat{2}} \otimes \vartheta^{\hat{3}}), \tag{2.1}$$

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with the coframe

$$\vartheta^{\hat{0}}=\frac{1}{p}d\zeta,\quad \vartheta^{\hat{1}}=\frac{1}{p}d\overline{\zeta},\quad \vartheta^{\hat{2}}=-d\sigma,\quad \vartheta^{\hat{3}}=\left(\frac{q}{p}\right)^{2}(s\,d\sigma+d\rho), \tag{2.2}$$

where the structural functions p, q, and s are given as follows:

$$p(\zeta,\overline{\zeta})=1+\frac{\lambda_{\rm cosm}}{6}\zeta\overline{\zeta}, \tag{2.3}$$

$$q(\sigma,\zeta,\overline{\zeta})=\left(1-\frac{\lambda_{\rm cosm}}{6}\zeta\overline{\zeta}\right)\alpha(\sigma)+\zeta\overline{\beta}(\sigma)+\overline{\zeta}\beta(\sigma), \tag{2.4}$$

$$s(\rho,\sigma,\zeta,\overline{\zeta})=-\frac{\lambda_{\rm cosm}}{6}\rho^2\alpha^2(\sigma)-\rho^2\beta(\sigma)\overline{\beta}(\sigma)+\rho\,\partial_\sigma(\ln|q|)+\frac{p}{2q}H(\sigma,\zeta,\overline{\zeta}). \tag{2.5}$$

Here α, β, and H are arbitrary functions.

Let $\widetilde{R}_{\alpha\beta}$ denote the Riemannian part of the curvature 2-form. Then we can subtract out the irreducible scalar curvature piece

$$^{(6)}\widetilde{R}_{\alpha\beta}:=-\frac{1}{12}(e_\nu\rfloor e_\mu\rfloor\widetilde{R}^{\nu\mu})\,\vartheta_\alpha\wedge\vartheta_\beta \tag{2.6}$$

(see [17]) and can define the 2-form

$$S_{\alpha\beta}:=\widetilde{R}_{\alpha\beta}-{}^{(6)}\widetilde{R}_{\alpha\beta}={}^{(1)}\widetilde{R}_{\alpha\beta}+{}^{(4)}\widetilde{R}_{\alpha\beta}=C_{\alpha\beta}+{}^{(4)}\widetilde{R}_{\alpha\beta}. \tag{2.7}$$

Here e_α denotes the (vector) frame dual to the coframe ϑ^α. If the Einstein vacuum field equations (with or without a cosmological constant) are satisfied—in this specific case ${}^{(4)}\widetilde{R}_{\alpha\beta}=0$—then $S_{\alpha\beta}$ becomes the Weyl conformal curvature 2-form $C_{\alpha\beta}:={}^{(1)}\widetilde{R}_{\alpha\beta}$. Moreover, we will introduce the propagation 1-form $k:=k_\mu\,\vartheta^\mu$ which inherits the properties of the geodesic, shear-free, expansion-free and twistless null vector field k^μ representing the propagation vector of a plane-fronted wave.

The gravitational and null electromagnetic fields are subject to the radiation conditions

$$S_{\alpha\beta}\wedge k=0,\quad (e^\alpha\rfloor k)S_{\alpha\beta}=0, \tag{2.8}$$

and

$$F\wedge k=0,\quad (e^\alpha\rfloor k)\,e_\alpha\rfloor F=0. \tag{2.9}$$

In the following we will solve the Einstein-Maxwell equations (for the notion compare [8])

$$\frac{1}{2}\eta_{\alpha\beta\gamma}\wedge\widetilde{R}^{\beta\gamma}+\lambda_{\rm cosm}\,\eta_\alpha=\kappa\Sigma_\alpha^{\rm Max},$$
$$dF=0,$$
$$d{}^*F=0, \tag{2.10}$$

where $\Sigma_\alpha^{\rm Max}$ represents the energy-momentum 3-form of the Maxwell field given by

$$\Sigma_\alpha^{\rm Max}:=\frac{1}{2}[(e_\alpha\rfloor F)\wedge{}^*F-(e_\alpha\rfloor{}^*F)\wedge F]. \tag{2.11}$$

Writing the electromagnetic field as

$$F=\frac{1}{2}F_{ab}\,dx^a\wedge dx^b=f(\zeta,\sigma)\,d\zeta\wedge d\sigma+\overline{f(\zeta,\sigma)}\,d\overline{\zeta}\wedge d\sigma, \tag{2.12}$$

with $f(\zeta,\sigma)$ an arbitrary function of its arguments, one finds, for the energy-momentum 3-form of the electromagnetic field as a nonvanishing component,

$$\Sigma_{\hat{2}}^{\rm Max}=-2\,p^2 f\overline{f}\vartheta^{\hat{0}}\wedge\vartheta^{\hat{1}}\wedge\vartheta^{\hat{2}}, \tag{2.13}$$

in agreement with the result for T_{ab} mentioned in Ref. [16], Eq. (3.7).

The surfaces of constant σ are the wave fronts of the electromagnetic waves. The above conditions (2.8),(2.9) restrict the function $\alpha(\sigma)$ to the real domain whereas $\beta(\sigma)$ can be complex valued.

The function H, for a combined gravitational and electromagnetic wave, has to satisfy the equation

$$H_{,\zeta\overline{\zeta}}+\frac{\lambda_{\rm cosm}}{3p^2}H=\frac{2\kappa p}{q}f\overline{f}. \tag{2.14}$$

In order to solve this non-homogeneous equation, one observes that a complex combination of an arbitrary holomorphic function $\Phi=\Phi(\zeta,\sigma)$ of the form $\Phi_{,\zeta}-(\lambda_{\rm cosm}/3)(\overline{\zeta}/p)\Phi$ is the general complex solution to the corresponding homogeneous equation (2.14). Thus, the real $H_{\rm h}$ solution to the homogeneous equation is given by

$$H_{\rm h}=\Phi_{,\zeta}-\frac{\lambda_{\rm cosm}}{3}\frac{\overline{\zeta}}{p}\Phi+\overline{\Phi}_{,\overline{\zeta}}-\frac{\lambda_{\rm cosm}}{3}\frac{\zeta}{p}\overline{\Phi}. \tag{2.15}$$

This structure sheds light on how to find the general solution of the non-homogeneous equation. Let us look for a particular solution $H_{\rm p}$ of the form

$$H_{\rm p}=\mu_{,\zeta}-\frac{\lambda_{\rm cosm}}{3}\frac{\overline{\zeta}}{p}\mu+\overline{\mu}_{,\overline{\zeta}}-\frac{\lambda_{\rm cosm}}{3}\frac{\zeta}{p}\overline{\mu}, \tag{2.16}$$

where $\mu=\mu(\sigma,\zeta,\overline{\zeta})$, such that the function

$$H_{(1)}:=\mu_{,\zeta}-\frac{\lambda_{\rm cosm}}{3}\frac{\overline{\zeta}}{p}\mu \tag{2.17}$$

satisfies the equation

$$H_{(1),\zeta\bar{\zeta}}+\frac{\lambda_{\rm cosm}}{3}\frac{H_{(1)}}{p^2}=\frac{\kappa p}{q}f\bar{f}. \tag{2.18}$$

Then it follows that μ itself is subject to

$$(\mu_{,\bar{\zeta}})_{,\zeta\zeta}-\frac{\lambda_{\rm cosm}}{3}\left(\frac{\bar{\zeta}}{p}\mu_{,\bar{\zeta}}\right)_{,\zeta}=\frac{\kappa p}{q}f\bar{f}, \tag{2.19}$$

with the general solution

$$\mu=\kappa\int^{\bar{\zeta}}d\bar{\zeta}p^2\int^{\zeta}\frac{d\zeta'}{p^2}\int^{\zeta'}d\zeta''\frac{p}{q}f\bar{f}. \tag{2.20}$$

For any given function f one integrates for μ and, by using Eq. (2.16), one obtains $H_{\rm p}$. The general H is constructed simply by adding the homogeneous solution $H_{\rm h}$ to $H_{\rm p}$:

$$H=H_{\rm h}+H_{\rm p}. \tag{2.21}$$

The general solution H is characterized by the self-dual part of the conformal Weyl 2-form

$${}^{+}C_{\alpha\beta}:=\frac{1}{2}(C_{\alpha\beta}+i\,{}^{*}C_{\alpha\beta}), \tag{2.22}$$

the trace-free Ricci 1-form

$$\not{\tilde{R}}_{\alpha}:=e_{\beta}\rfloor\tilde{R}_{\alpha}{}^{\beta}-\frac{1}{4}\tilde{R}\vartheta_{\alpha}, \tag{2.23}$$

the Ricci scalar

$$\tilde{R}:=e_{\alpha}\rfloor e_{\beta}\rfloor\tilde{R}^{\alpha\beta}, \tag{2.24}$$

and the electromagnetic 2-form F. The ansatz (2.1)–(2.5) yields

$${}^{+}C_{\hat{2}\hat{0}}=-{}^{+}C_{\hat{0}\hat{2}}=\frac{1-i}{4}p\,q\left(H_{,\zeta\zeta}+\frac{\lambda_{\rm cosm}}{3}\frac{\bar{\zeta}}{p}H_{,\zeta}\right)\vartheta^{\hat{0}}\wedge\vartheta^{\hat{2}}, \tag{2.25}$$

$${}^{+}C_{\hat{2}\hat{1}}=-{}^{+}C_{\hat{1}\hat{2}}=\frac{1+i}{4}p\,q\left(H_{,\bar{\zeta}\bar{\zeta}}+\frac{\lambda_{\rm cosm}}{3}\frac{\zeta}{p}H_{,\bar{\zeta}}\right)\vartheta^{\hat{1}}\wedge\vartheta^{\hat{2}}, \tag{2.26}$$

$$\not{\tilde{R}}_{\hat{2}}=p\,q\left(H_{,\zeta\bar{\zeta}}+\frac{\lambda_{\rm cosm}}{3p^2}H\right)\vartheta^{\hat{2}}=2\kappa\,p^2f\bar{f}\,\vartheta^{\hat{2}}, \tag{2.27}$$

$$\tilde{R}=4\lambda_{\rm cosm}, \tag{2.28}$$

$$F=dA=-d\left[\left(\int^{\zeta}f(\zeta',\sigma)d\zeta'+\int^{\bar{\zeta}}\overline{f(\bar{\zeta}',\sigma)}d\bar{\zeta}'\right)\vartheta^{\hat{2}}\right]. \tag{2.29}$$

The Weyl 2-form could be written still a bit more compactly according to

$$p\,q\left(H_{,\zeta\zeta}+\frac{\lambda_{\rm cosm}}{3}\frac{\bar{\zeta}}{p}H_{,\zeta}\right)=\partial_\zeta\left[q^2\,\partial_\zeta\left(\frac{p}{q}H\right)\right], \tag{2.30}$$

but the form given above is more practical if a certain function H is explicitly given and calculations need to be done.

It is worthwhile to mention the existence of a conformally flat solution given by

$$H=\frac{1}{p}(u+\bar{v}\zeta+v\bar{\zeta}+w\zeta\bar{\zeta}), \tag{2.31}$$

where u, v, w are arbitrary and u, w real functions of σ. The subbranch of the studied metric with constant curvature arises form the above expression by setting $w=-(\lambda_{\rm cosm}/6)u$.

If the electromagnetic field is switched off, one arrives at the non-twisting type N solutions of García and Plebański [18].

III. PLANE-FRONTED GRAVITATIONAL AND ELECTROMAGNETIC WAVES IN MAG

In this section we generalize the type N gravitational and electromagnetic waves to the metric-affine gravity theories. We will present exact solutions of the field equations belonging to the Lagrangian

$$L=V_{\rm MAG}+V_{\rm Max}, \tag{3.1}$$

where $V_{\rm Max}=-(1/2)F\wedge{}^{*}F$ is the Lagrangian of the Maxwell field and $F=dA$ is the electromagnetic field strength. The MAG Lagrangian considered here reads (a more general MAG Lagrangian can be found in [8])

$$\begin{aligned}V_{\rm MAG}=\frac{1}{2\kappa}\Big[&-a_0R^{\alpha\beta}\wedge\eta_{\alpha\beta}-2\lambda_{\rm cosm}\,\eta\\&+T^{\alpha}\wedge{}^{*}\left(\sum_{I=1}^{3}a_I\,{}^{(I)}T_{\alpha}\right)\\&+2\left(\sum_{I=2}^{4}c_I\,{}^{(I)}Q_{\alpha\beta}\right)\wedge\vartheta^{\alpha}\wedge{}^{*}T^{\beta}\\&+Q_{\alpha\beta}\wedge{}^{*}\left(\sum_{I=1}^{4}b_I\,{}^{(I)}Q^{\alpha\beta}\right)\\&+b_5({}^{(3)}Q_{\alpha\gamma}\wedge\vartheta^{\alpha})\wedge{}^{*}({}^{(4)}Q^{\beta\gamma}\wedge\vartheta_{\beta})\Big]\\&-\frac{1}{2\rho}R^{\alpha\beta}\wedge{}^{*}(z_4\,{}^{(4)}Z_{\alpha\beta}),\end{aligned} \tag{3.2}$$

where

$$a_0,\ldots,a_3,b_1,\ldots,b_5,c_2,c_3,c_4,z_4 \tag{3.3}$$

are dimensionless coupling constants, and κ is the weak and ρ the strong gravitational coupling constant. The cosmological constant is denoted by $\lambda_{\rm cosm}$. The signature of spacetime is $(-+++)$, the volume 4-form $\eta := {}^*1$, the 2-form $\eta_{\alpha\beta} := {}^*(\vartheta_\alpha \wedge \vartheta_\beta)$.

The two MAG field equations for electromagnetic matter are given by [5]

$$DH_\alpha - E_\alpha = \Sigma_\alpha^{\rm Max}, \tag{3.4}$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta = 0, \tag{3.5}$$

with $\Sigma_\alpha^{\rm Max}$ as defined in Eq. (2.11). It can be alternatively written as

$$\Sigma_\alpha^{\rm Max} = e_\alpha \rfloor V_{\rm Max} + (e_\alpha \rfloor F) \wedge H. \tag{3.6}$$

For the torsion and nonmetricity field configurations, we concentrate on the simplest non-trivial case *with* shear. According to its irreducible decomposition [5], the nonmetricity contains two covector pieces, namely the dilation piece

$$^{(4)}Q_{\alpha\beta} = Q\, g_{\alpha\beta} \tag{3.7}$$

and the proper shear piece

$$^{(3)}Q_{\alpha\beta} = \frac{4}{9}\left(\vartheta_{(\alpha} e_{\beta)} \rfloor \Lambda - \frac{1}{4} g_{\alpha\beta} \Lambda\right),$$

$$\text{with } \Lambda := \vartheta^\alpha e^\beta \rfloor \not{Q}_{\alpha\beta}. \tag{3.8}$$

Accordingly, our ansatz for the nonmetricity reads

$$Q_{\alpha\beta} = {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}. \tag{3.9}$$

The torsion, in addition to its tensor piece, encompasses a covector and an axial covector piece. Let us choose only the covector piece as non-vanishing:

$$T^\alpha = {}^{(2)}T^\alpha = \frac{1}{3}\vartheta^\alpha \wedge T, \quad \text{with} \quad T := e_\alpha \rfloor T^\alpha. \tag{3.10}$$

Thus we are left with the three non-trivial 1-forms Q, Λ, and T. We shall assume that this triplet of 1-forms shares spacetime symmetries; that is, its members are proportional to each other [19–24]. Our ansatz for the nonmetricity is expected to require a nonvanishing post-Riemannian term quadratic in the segmental curvature. This is the term in Eq. (3.2) carrying the coupling constant z_4 (note that the enumeration of the constants stems from the general Lagragian mentioned in [8]).

We assume the following so-called *triplet ansatz* for our three 1-forms in Eqs. (3.9) and (3.10):

$$Q = k_0 \omega, \quad \Lambda = k_1 \omega, \quad T = k_2 \omega, \tag{3.11}$$

where k_0, k_1, and k_2 are constants. The triplet ansatz (3.11) reduces the *electrovacuum* MAG field equations (3.4),(3.5) to an effective Einstein-Proca-Maxwell system:

$$\frac{a_0}{2}\eta_{\alpha\beta\gamma} \wedge \tilde{R}^{\beta\gamma} + \lambda_{\rm cosm}\, \eta_\alpha = \kappa\,[\Sigma_\alpha^{(\omega)} + \Sigma_\alpha^{\rm Max}], \tag{3.12}$$

$$d\,{}^*d\omega + m^2\, {}^*\omega = 0, \tag{3.13}$$

$$dF = 0, \quad d\,{}^*F = 0. \tag{3.14}$$

These are partial differential equations in terms of the coframe ϑ^α, the triplet 1-form ω, and the electromagnetic potential 1-form A; here the tilde denotes again the Riemannian part of the curvature. The energy-momentum current of the triplet field ω reads

$$\Sigma_\alpha^{(\omega)} := \frac{z_4 k_0^2}{2\rho}\{(e_\alpha \rfloor d\omega) \wedge {}^*d\omega - (e_\alpha \rfloor {}^*d\omega) \wedge d\omega + m^2[(e_\alpha \rfloor \omega) \wedge {}^*\omega + (e_\alpha \rfloor {}^*\omega) \wedge \omega]\}; \tag{3.15}$$

the effective ‘‘mass’’ m depends, additionally, on κ and the strong gravitational coupling constant z_4/ρ see [7].

Therefore, as mentioned above, in the framework of the triplet ansatz, the electrovacuum sector of MAG reduces to an effective Einstein-Proca-Maxwell system. Moreover, by setting $m=0$, the system acquires the following constraint among the coupling constants k_0, k_1, k_2 of the triplet ansatz (3.11) and the constants of the Lagrangian (3.2):

$$-4b_4 + \frac{3}{2}a_0 + \frac{k_1}{2k_0}(b_5 - a_0) + \frac{k_2}{k_0}(c_4 + a_0) = 0. \tag{3.16}$$

The coframe we will consider is of the form (2.2); i.e., it is the same as in the general relativistic case. Note that we changed the name of the function H in s [cf. Eq.(2.5)] into $\mathcal{H}$ in order to distinguish the general relativistic from the MAG case.

Now $\mathcal{H}$, representing a combined gravitational MAG plane wave and an electromagnetic wave, has to satisfy the equation

$$\mathcal{H}_{,\zeta\bar\zeta} + \frac{\lambda_{\rm cosm}}{3} p^{-2} \mathcal{H} = \frac{2\kappa p}{q}[f\bar f + g\bar g], \tag{3.17}$$

where $f = f(\zeta,\sigma)$ and $g = g(\zeta,\sigma)$ are arbitrary functions of their arguments.

The general solution of this equation is given by $\mathcal{H}_{\rm h} + \mathcal{H}_{\rm p}$ with

$$\mathcal{H}_{\rm h} = \Phi_{,\zeta} - \frac{\lambda_{\rm cosm}}{3}\frac{\bar\zeta}{p}\Phi + \bar\Phi_{,\bar\zeta} - \frac{\lambda_{\rm cosm}}{3}\frac{\zeta}{p}\bar\Phi \tag{3.18}$$

and

$$\mathcal{H}_{\rm p} = M_{,\zeta} - \frac{\lambda_{\rm cosm}}{3}\frac{\bar\zeta}{p}M + \bar M_{,\bar\zeta} - \frac{\lambda_{\rm cosm}}{3}\frac{\zeta}{p}\bar M. \tag{3.19}$$

Here $M = M(\sigma,\zeta,\bar\zeta)$ is a solution of the non-homogeneous equation for $\mathcal{H}$, which is given by

$$M = \kappa \int^{\bar\zeta} d\bar\zeta p^2 \int^{\zeta} \frac{d\zeta'}{p^2} \int^{\zeta'} d\zeta'' \frac{p}{q}[f\bar f + g\bar g]. \tag{3.20}$$

For given functions f and g, one integrates Eq. (3.20) for M and obtains $\mathcal{H}_{\rm p}$ from Eq. (3.19). The general solution is obtained by adding the homogeneous solution (3.18), where Φ is an arbitrary holomorphic function of ζ and σ. The 1-form ω entering the triplet ansatz (3.11) is given by

$$\omega=-\left[\int^{\zeta}g(\zeta',\sigma)d\zeta'+\int^{\overline{\zeta}}\overline{g(\overline{\zeta'},\sigma)}d\overline{\zeta'}\right]\vartheta^{\hat{2}}, \quad (3.21)$$

where $g=g(\zeta,\sigma)$ represents an arbitrary function of the coordinates. Moreover, the electromagnetic 2-form is given by

$$F=dA=-d\left[\left(\int^{\zeta}f(\zeta',\sigma)d\zeta'+\int^{\overline{\zeta}}\overline{f(\overline{\zeta'},\sigma)}d\overline{\zeta'}\right)\vartheta^{\hat{2}}\right] \quad (3.22)$$

in terms of the arbitrary function $f=f(\zeta,\sigma)$. Inserting this ansatz into the field equations (3.12)–(3.14) yields the following additional constraints among the constants of Eq. (3.2):

$$a_0=1,\quad z_4=\frac{\rho}{2k_0}. \quad (3.23)$$

IV. PARTICULAR SOLUTIONS

For better understanding, let us look for certain families of particular solutions of our dynamical system by integrating Eq. (3.17) restricted to $\alpha=1$ and $\beta=0$. Now the coframe in terms of $p(\zeta,\overline{\zeta}),q(\zeta,\overline{\zeta})$ and $\mathcal{H}(\sigma,\zeta,\overline{\zeta})$ reads

$$\vartheta^{\hat{0}}=\frac{1}{p}d\zeta,\quad \vartheta^{\hat{1}}=\frac{1}{p}d\overline{\zeta},\quad \vartheta^{\hat{2}}=-d\sigma,$$

$$\vartheta^{\hat{3}}=\left(\frac{q}{p}\right)^2\left[\left(\frac{p}{2q}\mathcal{H}(\sigma,\zeta,\overline{\zeta})-\frac{\lambda_{\rm cosm}}{6}\rho^2\right)\times d\sigma+d\rho\right]. \quad (4.1)$$

Here p and q take the explicit form

$$p(\zeta,\overline{\zeta})=1+\frac{\lambda_{\rm cosm}}{6}\zeta\overline{\zeta},\quad q(\zeta,\overline{\zeta})=1-\frac{\lambda_{\rm cosm}}{6}\zeta\overline{\zeta}. \quad (4.2)$$

Equation (3.17) is a linear equation, therefore, one can look independently for solutions of the non-homogeneous equation for the f excitations (associated with the electromagnetic field) and for the g excitations (associated with the post-Riemannian pieces). Consequently, the addition of these solutions, corresponding to f and g, will be again a solution. For simplicity, we shall restrict ourselves to the case where $g(\zeta,\sigma)$ and $f(\zeta,\sigma)$ are polynomial functions of ζ and ζ^{-1}. Let us try the cases

$$f(\zeta,\sigma)=f_0\zeta^n,\quad n=0,\pm1,\pm2,\pm3,\ldots. \quad (4.3)$$

Then one obtains the following branches of solutions for $\mathcal{H}_{\rm p}$:

(i) $n<-1$:

$$\mathcal{H}_{\rm p}=\frac{2\kappa p f_0^2}{q}\left(\frac{(\zeta\overline{\zeta})^{1+n}}{(1+n)^2}+4\left(\frac{\lambda_{\rm cosm}}{6}\right)^{-n-1}\times\ln|q|-4\left(\frac{\lambda_{\rm cosm}}{6}\right)^{-n-1}\ln|p-1| +4\sum_{r=1}^{-n-1}\frac{\left(\frac{\lambda_{\rm cosm}}{6}\right)^{-n-r-1}}{r\,(\zeta\overline{\zeta})^r}\right)+\frac{8\kappa f_0^2(\zeta\overline{\zeta})^{n+1}}{(1+n)\,p}. \quad (4.4)$$

(ii) $n=-1$:

$$\mathcal{H}_{\rm p}=\frac{2\kappa f_0^2}{p}\left(4\,q\ln|q|+\frac{2\lambda_{\rm cosm}\zeta\overline{\zeta}}{3}\ln(\zeta\overline{\zeta})+\frac{q}{2}\ln^2(\zeta\overline{\zeta})\right). \quad (4.5)$$

(iii) $n>-1$:

$$\mathcal{H}_{\rm p}=\frac{8\kappa f_0^2 q}{p}\left(\frac{\lambda_{\rm cosm}}{6}\right)^{-n-1}\left(\ln|q|+\sum_{r=1}^{n}\frac{\binom{n}{r}}{r}\times([p-2)^r-(-1)^r]\right)+\frac{2\kappa f_0^2(\zeta\overline{\zeta})^{n+1}}{p\,(n+1)^2}[4(n+1)+q]. \quad (4.6)$$

Similarily one can proceed with solutions for g:

$$g(\zeta,\sigma)=g_0\,\zeta^l,\quad l=0,\pm1,\pm2,\pm3,\ldots. \quad (4.7)$$

The form of the different branches of $\mathcal{H}_{\rm p}$ does not change, but the substitution $n\to l$ and $f_0\to g_0$ should be performed. Therefore, one can obtain different branches of solutions by combining the f branches with the g branches of $\mathcal{H}_{\rm p}$.

For these particular classes one can choose $\mathcal{H}_{\rm h}$ as displayed in Eq. (3.18). Given $g(\zeta,\sigma)$ and $f(\zeta,\sigma)$ it is straightforward to evaluate the 1-form ω of Eq. (3.21) and the electromagnetic 2-form of Eq. (3.22).

This solution was checked by means of the computer algebra system REDUCE [25] by applying its EXCALC package [26] for treating exterior differential forms [27].

V. DISCUSSION

We investigated plane-fronted electrovacuum-MAG waves with cosmological constant in the triplet ansatz sector of the theory. These waves carry curvature, nonmetricity, torsion, and an electromagnetic field. Apart from the cosmological constant, the solutions contain four wave parameters, given by the functions $\alpha(\sigma)$, $\beta(\sigma)$, $\overline{\beta}(\sigma)$ and $\partial_\sigma\ln|q(\sigma,\zeta,\overline{\zeta})|$. Our plane-fronted wave solutions are given in terms of three arbitrary complex functions, i.e. $\Phi(\sigma,\zeta)$ associated with the Riemannian part, $g(\sigma,\zeta)$ related to the non-Riemannian triplet, and $f(\sigma,\zeta)$ corresponding to the Maxwell field. In this way, we generalize the plane-fronted

TABLE I. Ozsváth-Robinson-Rózga waves in MAG.

Ansatz for coframe $\vartheta^{\hat{0}},\vartheta^{\hat{1}},\vartheta^{\hat{2}},\vartheta^{\hat{3}}$	Eq. (2.2)
Arbitrary functions in coframe	$\alpha(\sigma)$, $\beta(\sigma)$
MAG Lagrangian $V_{\rm MAG}$ and non-vanishing coupling constants	Eqs. (3.2),(3.3) with Eqs. (3.16),(3.23)
Triplet ansatz for nonmetricity and torsion	$Q\sim T\sim\Lambda\sim\omega$; (cf. Eq. (3.11)
Energy-momentum current of the Maxwell field	Eq. (2.11) resp. Eq. (2.13)
Energy-momentum current of the triplet field	Eq. (3.15)
Field equations	Eqs. (3.12)–(3.14)
Arbitrary function governing the vacuum solution $\mathcal{H}_{\rm h}$	$\Phi(\sigma,\zeta)$; cf. Eq. (3.18)
Arbitrary function in the electromagnetic 2 form F	$f(\sigma,\zeta)$; cf. Eq. (3.22)
Arbitrary function in the triplet 1 form ω	$g(\sigma,\zeta)$; cf. Eq. (3.21)
Solution for the electromagnetic 2 form F	$F\sim -d(\int f\, d\zeta+\overline{f}\, d\overline{\zeta})\ \vartheta^{\hat{2}}$
Solution for the triplet 1-form ω	$\omega\sim -(\int g\, d\zeta+\overline{g}\, d\overline{\zeta})\ \vartheta^{\hat{2}}$
Solution for function $\mathcal{H}(\sigma,\zeta,\overline{\zeta})$ entering coframe	Eqs. (3.18)–(3.20)

electrovacuum Ozsváth-Robinson-Rózga waves. In brief, the solution reads as in Table I.

The final form of T^α and $Q_{\alpha\beta}$ in terms of $g(\zeta',\sigma)$ reads

$$T^\alpha=-\frac{k_2}{3}\left[\int^\zeta g(\zeta',\sigma)d\zeta'+\int^{\overline{\zeta}}\overline{g(\overline{\zeta'},\sigma)d\overline{\zeta'}}\right]\vartheta^\alpha\wedge\vartheta^{\hat{2}}, \tag{5.1}$$

$$Q_{\alpha\beta}=-\frac{4k_1}{9}\vartheta_{(\alpha}e_{\beta)}\rfloor\left[\int^\zeta g(\zeta',\sigma)d\zeta'+\int^{\overline{\zeta}}\overline{g(\overline{\zeta'},\sigma)d\overline{\zeta'}}\right]\vartheta^{\hat{2}} +g_{\alpha\beta}\left(\frac{k_1}{9}-k_0\right)\left[\int^\zeta g(\zeta',\sigma)d\zeta' +\int^{\overline{\zeta}}\overline{g(\overline{\zeta'},\sigma)d\overline{\zeta'}}\right]\vartheta^{\hat{2}}. \tag{5.2}$$

The electromagnetic potential 1-form is given by

$$A=-\left(\int^\zeta f(\zeta',\sigma)d\zeta'+\int^{\overline{\zeta}}\overline{f(\overline{\zeta'},\sigma)d\overline{\zeta'}}\right)\vartheta^{\hat{2}}. \tag{5.3}$$

It is straightforward to perform a detailed classification [28] of the plane-fronted waves in MAG by carrying through a similar analysis as the one done by Sippel and Goenner [10]. We leave this, however, for future work.

VI. OUTLOOK

The theories of modern physics generally involve a mathematical model, defined by a certain set of differential equations and supplemented by a set of rules for translating the mathematical results into meaningful statements about the physical world. In the case of gravity theories, because they deal with the most universal of the physical interactions, one has an additional class of problems concerning the influence of the gravitational field on other fields and matter. These are often studied by working within a fixed gravitational field, usually an exact solution [28]. In this context our plane-fronted wave solutions contribute to enhance our understanding of some of these questions in the framework of MAG theories, in particular the ones concerned with the gravitational radiation.

Gravitational waves [29] have traveled almost unimpeded through the universe since they were generated at times as early as 10^{-24} sec after the big bang. This radiation carries information that no electromagnetic radiation can give to us because the electromagnetic radiation is scattered countless times by the dense material surrounding the explosion, losing in the process most of the detailed information it might carry about the explosion. Beyond this, we can be virtually certain that the gravitational wave spectrum has surprises for us, clues to phenomena we never suspected. Therefore, it is not surprising, that considerable effort is nowadays being devoted to the development of sufficiently sensitive gravitational wave antennas. Moreover, observing them would provide important constraints on theories of inflation and high-energy physics.

Even though Einstein's treatment of spacetime as a Riemannian manifold appears almost fully corroborated experimentally, there are several reasons to believe that the validity of such a description is limited to macroscopic structures and to the present cosmological era. The only available finite perturbative treatment of quantum gravity, namely the theory of the quantum superstring [1], suggests that non-Riemannian features are present on the scale of the Planck length. On the other hand, recent advances in the study of the early universe, as represented by the inflationary model, involve, in addition to the metric tensor, at the very least a scalar dilaton [2] induced by a Weyl geometry, i.e. again an essential departure from Riemannian metricity [3]. Even at the classical cosmological level, a dilatonic field has recently been used to describe the presence of dark matter in the universe, as well as to explain certain cosmological observations which contradicted the fundaments of the standard cosmological model [4].

Inflation is an attractive scenario for the early universe because it makes the large scale homogeneity of the universe easy to understand. It also provides a mechanism for producing initial density perturbations large enough to evolve into galaxies as the universe expands. These perturbations are accompanied by perturbation of the gravitational field that travel through the universe, redshifting in the same way that photons do. The perturbations arise by parametric amplification of quantum fluctuations in the gravitational wave field that existed before the inflation began. The huge expansion associated with inflation puts energy into these fluctuations, converting them into real gravitational waves with classical amplitudes. Even if inflation did not occur, the perturbations that lead to galaxies must have arisen in some othe way, and it is possible that this alternative mechanism also produced gravitational waves.

It is worthwhile stressing [5] the fact that we do not believe that at the present state of the universe the geometry of spacetime is described by a metric-affine one. We rather think, and there is good experimental evidence, that the present-day geometry is metric compatible; i.e., its nonmetricity vanishes. In earlier epochs of the universe, however, when the energies of the cosmic ''fluid'' were much higher than today, we expect scale invariance to prevail—and the canonical dilation or scale current of matter, the trace of the hypermomentum current $\Delta^{\gamma}{}_{\gamma}$, is coupled, according to MAG, to the Weyl covector $Q^{\gamma}{}_{\gamma}$. By the same token, shear type excitations of the material multispinors (Regge trajectory type of constructs) are expected to arise, thereby liberating the (metric-compatible) Riemann-Cartan spacetime from its constraint of vanishing nonmetricity $Q_{\alpha\beta}=0$. Tresguerres [30] has proposed a simple cosmological model of Friedmann type which carries a metric-affine geometry at the beginning of the universe, the nonmetricity of which dies out exponentially in time. That is the kind of thing we expect.

In full, exact solutions of the type obtained may serve well as starting point for the upcoming analysis of gravitational wave astronomy data. In this sense it might contribute to our understanding of light and gravitational wave propagation in early stages of the universe. Moreover, plane wave solutions contribute to resolving some of the controversies about the existence of such gravitational radiation.

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