# Thick domain walls and singular spaces

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We discuss thick domain walls interpolating between spaces with naked singularities and give arguments based on the AdS/CFT correspondence why such singularities may be physically meaningful. Our examples include thick domain walls with Minkowski, de Sitter, and anti–de Sitter geometries on the four-dimensional slice. In all of these cases we can solve the equivalent quantum mechanics problem exactly, which provides the spectrum of graviton states on these spaces. In one of the examples we discuss, the continuum states have a mass gap. We compare the graviton spectra with expectations from the AdS/CFT correspondence and find qualitative agreement. We also discuss unitary boundary conditions and show that they project out all continuum states.

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## I. INTRODUCTION

Domain walls have recently attracted renewed attention after it was pointed out in Ref. [1] that four-dimensional gravity can be realized on a thin wall connecting two slices of AdS space. From the point of view of a four-dimensional observer on the domain wall the spectrum of gravity consists of a massless graviton and a tower of Kaluza-Klein (KK) modes with continuous masses. It was shown in [1] that the KK modes give a subleading correction to the gravitational interaction between two test masses on the domain wall. In the thin wall setup there is only gravity in the bulk, and the only five-dimensional space that can appear is AdS.[1] On the other hand, in supergravity or string theory one expects to have other bulk fields including scalars.

The original proposal of [1] has been generalized in several directions. One generalization involves turning on a cosmological constant on the domain wall [2,3], which results in time-dependent cosmological scenarios. Other extensions include higher dimensional embeddings [4], models with a mass gap for the continuum modes [5,6], and realizations of domain walls in gravity coupled to scalars [7–10]. There is also an extensive literature on supergravity domain walls [11–13], but so far the construction of [1] has not been realized in supergravity. In fact it was shown to be impossible in any of the known five-dimensional supergravities [11,13,14].

The thin wall construction of [1] has the disadvantage that the curvature is singular at the location of the wall. This problem can be avoided if gravity is coupled to a scalar field. By choosing a suitable potential for the scalar, we can readily generate smooth domain wall solutions [8–10] that interpolate between two AdS spaces. However, once we have a scalar in the bulk, other space-times besides AdS, dS, and Minkowski space can appear. In this paper we will study some examples of such spaces. Specifically, we consider a class of thick domain walls in gravity coupled to scalars that interpolate between spaces with naked singularities instead of regular AdS horizons. Normally such spaces would be discarded as unphysical, but in this context there are reasons to believe that considering these spaces may be meaningful.

One reason for thinking so comes from the recent proposal [15] (see also [16,17]) that five-dimensional bulk gravity in the thin domain wall case [1] has an equivalent description in terms of a cutoff four-dimensional conformal field theory (CFT) on the domain wall, very much in the spirit of the AdS/CFT correspondence [18]. The details of this correspondence are rather unclear at present. For instance, it is not clear how to identify the CFT in the non-supersymmetric purely five-dimensional setup of [1] or how to match operators and KK modes. It is also unclear how to impose a sharp cutoff on the CFT that preserves four-dimensional Poincaré invariance. However, leaving these considerations aside, we can freely borrow results from the AdS/CFT literature on renormalization group (RG) flows in five-dimensional supergravity [19]. In RG flows to non-conformal theories (see e.g. [20,21]) the AdS horizon gets replaced with a naked singularity. This singularity is physical in the sense that the singular behavior corresponds to strong coupling effects such as confinement or screening in the boundary theory. Since the non-conformal boundary theory makes sense in the infrared, the singular behavior of the metric must be resolved either by lifting to ten dimensions [22] or via string theory. Unfortunately we are not aware of any criterion that tells us exactly which type of naked singularity has a physical interpretation. We will simply assume that the singularities in the space we consider can appear in RG flows to non-conformal theories. In the last section we will discuss the validity of this assumption. If our singularities are physical, we can think of our five-dimensional space-times as four-dimensional gravity coupled to a non-conformal field theory. Such theories are well defined, which provides a justification for considering this type of singular space. We will give a more detailed discussion of these ideas in the last section.

A second argument for considering spaces that end in singularities comes from analyzing the spectrum of gravity from a four-dimensional point of view. In the original setup

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[1]Of course one can also consider slices of dS or Minkowski space, but they do not yield a four-dimensional graviton.

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[1] there was a single massless graviton and a continuous tower of KK states. The KK states couple to matter on the thin domain wall and cause small violations of four-dimensional energy and momentum conservation. This violation of conservation laws also occurs in the presence of a naked singularity. The traditional point of view [23] posits that spaces with naked singularities are physically acceptable only if one imposes boundary conditions that guarantee four-dimensional energy and momentum conservation. These boundary conditions are usually referred to as unitary boundary conditions. Note that this point of view is rather different than the AdS/CFT inspired approach described above. In the latter case we want energy and momentum to leak out into either the AdS horizon in the setup of [1] or into the naked singularities we discuss here. This leakage corresponds to four-dimensional gravitation exciting the degrees of freedom of the non-conformal field theory. Nonetheless, we can impose unitary boundary conditions and analyze the spectrum of the KK modes in that case. It turns out that these boundary conditions remove the continuum part of the KK spectrum for the models discussed here and in [1,9]. The theory of the discrete part of the spectrum is unitary, because these modes die off rapidly enough as we approach the singularity.

The discussion so far has involved only flat domain walls with four-dimensional Poincaré invariance. Some of the solutions we study in this paper can accommodate a constant curvature on the four-dimensional slice, turning it into four-dimensional de Sitter or anti–de Sitter space. Such bent domain walls have appeared previously in cosmological thin wall solutions [2] and similar thick domain walls in four dimensions were discussed in [24]. Our solution can be viewed as a non-singular analogue of the cosmological thin wall solutions. The ambient space of bent domain walls generically has horizons or singularities. We analyze a domain wall interpolating between two singular spaces in some detail and also give an example of a thick domain wall which interpolates between spaces with regular horizons. The purpose of the second example is merely to show that such solutions exist. Unfortunately it is too complicated for an analytical treatment.

Apart from their relevance for cosmology, bent domain walls are interesting because they are generic solutions of five-dimensional gravity coupled to scalars. By generic we mean that the supersymmetry inspired first order formalism of [25,26,8] cannot be used to generate a solution. In [11,13,14] it was shown that even for flat domain walls where this ''superpotential'' formalism is applicable, there is no simple supersymmetric extension of the AdS domain wall solutions. Since the bent domain wall solutions cannot be obtained from any known first order formalism, they are probably non-supersymmetric and therefore, in a sense, generic.

This paper is organized as follows. In Sec. II we briefly review gravity coupled to a scalar, and discuss a class of solutions for both flat and bent domain walls. These domain walls interpolate between spaces with naked singularities. Our solutions are simple enough that we can solve the quantum mechanics problem exactly, both for flat and bent domain walls. In Sec. III we analyze the spectrum of metric fluctuations by studying the equivalent quantum mechanics problem with and without imposing unitary boundary conditions. In Sec. IV we discuss possible implications of our results, including various speculations on the role of thick domain walls and singular spaces in the light of the AdS/CFT correspondence.

## II. GRAVITY COUPLED TO SCALARS

The action for five-dimensional gravity coupled to a single real scalar reads

$$S=\int d^4x dr\sqrt{g}\left(-\frac{1}{4}R+\frac{1}{2}(\partial\phi)^2-V(\phi)\right). \tag{2.1}$$

We will consider metrics of the form

$$ds^2=e^{2A(r)}\left(dx_0^2-e^{2\sqrt{\bar{\Lambda}}x_0}\sum_{i=1}^{3}dx_i^2\right)-dr^2 \tag{2.2}$$

or

$$ds^2=e^{2A(r)}[e^{-2\sqrt{\bar{\Lambda}}x_3}(dx_0^2-dx_1^2-dx_2^2)-dx_3^2]-dr^2, \tag{2.3}$$

where the four-dimensional slices are de Sitter and anti–de Sitter respectively. The equations of motion following from the action and the ansatz for the metric are

$$\begin{aligned}\phi''+4A'\phi'&=\frac{\partial V(\phi)}{\partial\phi}\\ A''+\bar{\Lambda}e^{-2A}&=-\frac{2}{3}\phi'^2\\ A'^2-\bar{\Lambda}e^{-2A}&=-\frac{1}{3}V(\phi)+\frac{1}{6}\phi'^2.\end{aligned} \tag{2.4}$$

The prime denotes differentiation with respect to $r$, and we have assumed that both $\phi$ and $A$ are functions of $r$ only. The equations of motion above were obtained using the metric Eq. (2.2). Reversing the sign of $\bar{\Lambda}$ yields the corresponding equations of motion for Eq. (2.3).

If $\bar{\Lambda}=0$, the four-dimensional slices in Eqs. (2.2),(2.3) are Minkowski, space, and we can use the first order formalism of [26,8] to generate solutions. The fields and the potential can be parametrized by a single function $W(\phi)$ as

$$\begin{aligned}\phi'&=\frac{1}{2}\frac{\partial W(\phi)}{\partial\phi},\quad A'=-\frac{1}{3}W(\phi),\\ V(\phi)&=\frac{1}{8}\left(\frac{\partial W(\phi)}{\partial\phi}\right)^2-\frac{1}{3}W(\phi)^2.\end{aligned} \tag{2.5}$$

For $\bar{\Lambda}\neq 0$ there is no known first order formalism, so we need to solve the equations of motion directly. In general it is

difficult to find tractable solutions, because highly non-linear combinations of $A(r)$ and its derivatives appear in the equations of motion.

For domain walls with four-dimensional de Sitter slices, the equations of motion provide some model independent information. Assuming we have a reflection symmetric domain wall we can choose $A(0)=1$ and $A'(0)=0$. The second equation in Eqs. (2.4) then implies that $A''$ is negative and $A(r)$ diverges to $-\infty$ faster than for $\bar{\Lambda}=0$. For $\bar{\Lambda}=0$ such symmetric domain walls interpolate between AdS spaces which have regular horizons infinitely far away from $r=0$. For $\bar{\Lambda}\neq 0$ we expect a horizon or a singularity at a finite distance $r=r^*$. For domain walls with $AdS_4$ slices there does not seem to be a similar argument. We discuss an example with a naked singularity at $r=r^*$ first.

A class of solutions with naked singularities is given by $A(r)=n\log[d\cos(cr)]$. Unfortunately the expressions for the scalar field and the potential are simple only if $n=1$, so we will focus on that case. Other choices for $n$ are equally valid, but there is no closed form expression for $\phi$ and the potential. Nevertheless, these cases can be analyzed numerically. By picking suitable units for $\bar{\Lambda}$ we can set $d=1$. The complete solution to the equations of motion is then given by

$$\begin{aligned} A(r)&=\log[\cos(cr)]\\ \phi(r)&=\frac{1}{c}\sqrt{\frac{3}{2}(c^2-\bar{\Lambda})}\log\left(\frac{1+\tan(cr/2)}{1-\tan(cr/2)}\right)\\ V(\phi)&=\frac{3}{4}\cosh^2\left(\frac{c\phi}{\sqrt{\frac{3}{2}(c^2-\bar{\Lambda})}}\right)\\ &\quad\times\left[3\bar{\Lambda}+c^2-4c^2\tanh^2\left(\frac{c\phi}{\sqrt{\frac{3}{2}(c^2-\bar{\Lambda})}}\right)\right]. \end{aligned} \tag{2.6}$$

This solution has only two adjustable parameters, $c,\bar{\Lambda}$, which determine the location of the singularities, the curvature of the four-dimensional slice, and the thickness of the wall. The metric with this choice of $A(r)$ has a naked singularity at $r^*=\pm\pi/2c$. However, the singularity is very similar to the one encountered in the AdS flow to $N=1$ super Yang-Mills (SYM) theory [21]. This may indicate that it can be resolved either by lifting the five-dimensional geometry to ten dimensions or by string theory. The scalar diverges at the singularity. If we think of it as a modulus from some compactification manifold, this divergence can indicate that the compactification manifold shrinks to zero size or becomes infinitely large, so that the five dimensional truncation becomes invalid. There are some examples where singularities in five dimensions actually correspond to non-singular ten dimensional geometries [22].

In the limit $c^2=\bar{\Lambda}$ our solution simplifies dramatically. The scalar vanishes and the potential becomes constant. In fact, this limit of our solution is $dS_5$ written in unusual coordinates. Note that our solution is valid only for $c^2\geqslant\bar{\Lambda}$. The curvature of the four dimensional slice imposes the constraint $r^*\leqslant\pi/2\sqrt{\bar{\Lambda}}$ on the location of the horizon.

By changing the sign of $\bar{\Lambda}$ we obtain a solution for a domain wall with $AdS_4$ slices. In that case there is no constraint on the location of the horizons or, conversely, the value of $\bar{\Lambda}$.

Finally we should point out that the first order formalism of [26,8] does not apply here. For instance, using Eq. (2.5) we can compute $W(\phi)$ from the expression for $\phi$. The potential computed from $W$ has the same form as the potential above, but the coefficients do not agree. It would be very interesting to either find a first order formalism for $\bar{\Lambda}\neq 0$ or show that it does not exist.

The solution above has the virtue that we can solve the equations of motion analytically, but as we pointed out, it has naked singularities at a finite distance from the center of the domain wall. This behavior is not generic. It is easy to pick $A(r)$ such that we get regular horizons instead of singularities. One such example with three free parameters is

$$\begin{aligned} e^{A(r)}=(r^2-r^{*2})\Bigg[&\frac{\sqrt{\bar{\Lambda}}}{2r^*}\left(1+\frac{1}{4r^{*2}}(r^2-r^{*2})\right)\\ &+c(r^2-r^{*2})^2\Bigg], \end{aligned} \tag{2.7}$$

but the solutions for $\phi(r)$ and the potential have to be obtained numerically. We simply mention this example to show that such solutions exist, but we will not investigate it further in this paper.

In the limit $\bar{\Lambda}=0$ four-dimensional Poincare invariance is restored and we can write down the solution for all $n$:

$$\begin{aligned} A(r)&=n\log[\cos(cr)]\\ \phi(r)&=\sqrt{\frac{3n}{2}}\log\left(\frac{1+\tan(cr/2)}{1-\tan(cr/2)}\right)\\ V(\phi)&=\frac{3nc^2}{4}\left[\cosh^2\left(\sqrt{\frac{2}{3n}}\phi\right)\right.\\ &\quad\left.-4n\sinh^2\left(\sqrt{\frac{2}{3n}}\phi\right)\right]. \end{aligned} \tag{2.8}$$

Note that in this case the solution can be parametrized by $W(\phi)$, so we could have found it using the first order formalism of [26,8]. Since the form of $A(r)$ is the same as in the previous example, this solution also has naked singularities at $r^*=\pm\pi/2c$, but unlike in the previous example there is no limit in which they disappear. If $n=1/4$, the potential is constant and near the singularity we have $g_{00}=e^{2A}\sim\sqrt{r^*-r}$. This is the behavior found in [19] for general flows assuming that the potential can be neglected near the singularity.

Since this solution has four-dimensional Poincaré invariance, we expect to find a massless graviton and a tower of KK excitations. This solution is simple enough that we can

give a complete solution of the equivalent quantum mechanics problem for $n=1$. The $n=2$ case is also tractable at the level of the example in [9]. We will comment on the differences between these two examples in Sec. IV.

### III. GRAVITON FLUCTUATIONS

The solutions in the previous section provide backgrounds in which the fluctuations of the metric exhibit interesting behavior. It is difficult to analyze the metric fluctuations in general, since they couple to fluctuations of the scalar field. However, it was shown in [8] that there is a sector of the metric fluctuations that decouples from the scalar and satisfies a simple wave equation.

A general metric fluctuation takes the form

$$ds^2=e^{2A(r)}(g_{ij}+h_{ij})dx^i dx^j-dr^2, \tag{3.1}$$

where we have made a gauge choice, and $g_{ij}$ is the four-dimensional dS, AdS or Minkowski metric. The fluctuation $h_{ij}$ is taken to be small, so the linearized Einstein equation provides the equation of motion for it. As shown in [8] the transverse traceless part of the metric fluctuation, $\overline{h}_{ij}$, satisfies the same equation of motion as a five-dimensional scalar. It turns out that transforming to conformally flat coordinates simplifies this wave equation considerably. In terms of $z=\int dr e^{-A(r)}$ the metric takes the form

$$ds^2=e^{2A(z)}(g_{ij}dx^i dx^j-dz^2), \tag{3.2}$$

and if the four-dimensional slices are $dS_4$, the transverse traceless parts of the metric fluctuation satisfy

$$\left(\partial_z^2+3A'(z)\partial_z-\partial_{x_0}^2-3\sqrt{\overline{\Lambda}}\partial_{x_0}+e^{-2\sqrt{\overline{\Lambda}}x_0}\sum_{a=1}^{3}\partial_{x_a}\right)\overline{h}_{ij}=0. \tag{3.3}$$

This equation can be simplified further by rewriting the metric fluctuation as $\overline{h}_{ij}=e^{-3(A+\sqrt{\overline{\Lambda}}x_0)/2}\rho_k(x)\psi_{ij}(z)$, where $\rho_k(x)$ satisfies $g^{ij}\partial_i\partial_j\rho_k(x)=-k^2\rho_k(x)$. Dropping the indices on $\psi$ we finally find

$$(-\partial_z^2+V_{QM}-k^2)\psi=0 \tag{3.4}$$

with

$$V_{QM}=-\frac{9\overline{\Lambda}}{4}+\frac{9}{4}A'^2(z)+\frac{3}{2}A''(z). \tag{3.5}$$

Note that for $\overline{\Lambda}\neq 0$ there is an extra constant piece in the potential. This implies that the quantum mechanics problem does not factorize as in [8]. The argument given there now constrains $k^2+9\overline{\Lambda}/4$ to be positive. We will come back to this point when we analyze the solutions of this Schrödinger equation.

These equations were written assuming that the four-dimensional slice is $dS_4$. The analogous equation for $AdS_4$ slices can be obtained by the analytic continuation $x_0\rightarrow ix_3$, $x_3\rightarrow ix_0$, and $\sqrt{\overline{\Lambda}}\rightarrow i\sqrt{\overline{\Lambda}}$. We are now ready to turn to specific examples.

The metric for the solution in Eq. (2.6) can be transformed to the conformally flat form, Eq. (3.2), with $A(z)=-\log[\cosh(cz)]$. Using Eq. (3.5), we find for the potential

$$V_{QM}=\frac{9(c^2-\overline{\Lambda})}{4}-\frac{15c^2}{4}\frac{1}{\cosh^2(cz)}. \tag{3.6}$$

The shape of this potential is rather different than the one found in [9] for a thick domain wall interpolating between AdS spaces. The most important differences are that our potential asymptotes to a non-zero constant for $z\rightarrow\pm\infty$ and that there is no potential barrier separating the asymptotic region from the interior of the domain wall. Since the potential asymptotes to a constant, we will find plane wave solutions for sufficiently large $k^2$, but these solutions are separated from the discrete modes by a mass gap. These general observations can be made precise. The Schrödinger equation with this potential has a general solution

$$\psi=a_2F_1\left(-\epsilon-\frac{3}{2},1-\epsilon+\frac{3}{2},1-\epsilon,\frac{1}{2}(1-x)\right)(1-x^2)^{-\epsilon/2}+b_2F_1\left(\epsilon-\frac{3}{2},1+\epsilon+\frac{3}{2},1+\epsilon,\frac{1}{2}(1-x)\right)(1-x^2)^{\epsilon/2}, \tag{3.7}$$

where $x=\tanh(cz)$ and $\epsilon^2=-k^2/c^2+9(c^2-\overline{\Lambda})/4c^2$. To find the discrete part of the spectrum we set $a=0$ to ensure that $\psi$ is regular at $z=\infty$ $(x=1)$. If $\epsilon-3/2=-n\in\mathbb{Z}_0^+$, the solution is also finite as $z\rightarrow-\infty$ $(x=-1)$, so the discrete part of the spectrum is given by $\epsilon_n=3/2-n$, $n=0,1$, or

$$k_n^2=\frac{9(c^2-\overline{\Lambda})}{4}-c^2\left(\frac{3}{2}-n\right)^2, \tag{3.8}$$

and the corresponding wave functions are

$$\psi_0(z)\sim\frac{1}{\cosh^{3/2}(cz)},\quad \psi_1(z)\sim\frac{\sinh(cz)}{\cosh^{3/2}(cz)}. \tag{3.9}$$

For $\overline{\Lambda}=0$ we find the expected massless graviton with $k^2=0$ and an excited state with $k^2=2c^2$. For $\overline{\Lambda}>0$ the four-dimensional slice is dS and at least the lowest $k^2$ is negative. If the four-dimensional metric is AdS $(\overline{\Lambda}<0)$, all $k^2$ are positive. These results may appear surprising at first sight. However, we should keep in mind that the notion of mass is somewhat murky in AdS and dS spaces. In both of these cases, $k^2$ is a constant that appears in the separation of variables. It should not be confused with a four-dimensional mass. If we put $\psi_0$ and $\rho_k$ for the lowest $k^2$ into the expression for $\overline{h}_{ij}$, we find that the metric fluctuation always satisfies the four-dimensional wave equation $D_lD^lh_{ij}=0$, $l=0,1,2,3$. There are several definitions of mass in dS and AdS, so it is not clear if these fields are massless, but fields

that satisfy this wave equation never signal an instability. It is worth mentioning that the negative values of $k^2$ are possible because the factorization argument of [8] constrains the combination $k^2+9\overline{\Lambda}/4$ to be positive.

The solutions of the Schrödinger equation, Eq. (3.4), also include a continuous spectrum of eigenfunctions with $\epsilon^2 \leqslant 0$ that asymptote to plane waves as $z\to\pm\infty$. Formally these solutions can be obtained from Eq. (3.7) by substituting $\epsilon\to i\kappa$. For $x\to 1$ $(z\to\infty)$ we find the asymptotic behavior $\psi(z)\sim a e^{-ic\kappa z}+b e^{ic\kappa z}$, plane waves as expected. It is easy to show that Eq. (3.7) also asymptotes to a plane wave for $x\to -1$. To summarize, the solutions to the Schrödinger equation, Eq. (3.4), consist of two normalizable states with discrete eigenvalues and a continuum of states that asymptote to plane waves at infinity.

To proceed in our discussion, we will now specialize to the case $\overline{\Lambda}=0$, so we can interpret $k^2$ as a four-dimensional mass. In this limit our solution describes a thick domain wall interpolating between two spaces with naked singularities. As mentioned in the Introduction, we can appeal to the AdS/CFT correspondence and think of modes propagating in the fifth direction as excitations of some non-conformal field theory, which should render the four-dimensional theory unitary. We will comment on this relation in the last section.

This is in harmony with the approach of [1], which is to accept small violations of unitarity in the theory on the four-dimensional slice at $z=0$. In our case this theory contains a massless graviton, one massive state, and a continuum of modes with a mass gap of size $m_{gap}=3c/2$. At very low energies, none of these massive modes can be excited, and an observer at $z=0$ sees pure four-dimensional gravity. At higher energies the massive state can be excited, giving some corrections to Newton's law, and finally at energies larger than the gap the whole continuum of modes can be excited. Since there is a mass gap in the theory, the corrections to Newton's law will always be negligible at sufficiently long distances. Violations of unitarity occur only if modes that can travel out to the singularities can be excited, i.e. only at energies above the mass of the lightest continuum mode. Both the corrections to Newton's law and the way unitarity is violated are rather different in the thin wall scenario of [1]. There the contribution of the KK modes is suppressed because in the quantum mechanics description they need to tunnel through a potential barrier. The resulting suppression of these modes at $z=0$ turns out to be sufficient to make the violations of unitarity too small to detect in present day experiments.

Another way of dealing with the violations of various conservation laws in the four-dimensional theory is to impose unitary boundary conditions [23]. These boundary conditions ensure that no supposedly conserved quantities disappear into the singularity. This approach was used in [27] to render a specific naked singularity harmless.

We will simplify our discussion by introducing a new massless scalar field, $\Phi$, which satisfies the same equation of motion as the metric fluctuation. This scalar field should not be confused with the scalar in the solutions in the previous section. We will discuss the unitary boundary conditions in terms of this scalar because that simplifies the argument somewhat. Since the metric fluctuations and the scalar satisfy the same equation of motion, the results should carry over to metric fluctuations.

For $\overline{\Lambda}=0$ the metric, Eq. (2.2), has a number of Killing vectors. It will be sufficient for our purposes to consider only the ones generating four-dimensional translations. These Killing vectors are given by $\xi_i^\mu=\delta_i^\mu$, where $i$ is a four-dimensional index. To construct currents from these Killing vectors we need the stress tensor for a massless scalar:

$$T_{\mu\nu}=\frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi-\frac{1}{2}g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\Phi\partial^\alpha\Phi\right). \tag{3.10}$$

The currents $J^\mu=T^{\mu\nu}\xi_\nu^i$ satisfy conservation laws of the form

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}J^\mu)=0, \tag{3.11}$$

which express the conservation of four-dimensional energy and momentum. To ensure that these quantities are conserved in the presence of a singularity, we demand that the flux into the singularity vanish:

$$\lim_{z\to\infty}\sqrt{g}J^z=\lim_{z\to\infty}\sqrt{g}g^{zz}\frac{1}{2}\partial_i\Phi\partial_z\Phi=0. \tag{3.12}$$

The solutions for $\Phi$ take the same form as the solutions for $\overline{h_{ij}}$, i.e., $\Phi\sim e^{-3A(z)/2}\psi(z)$ with $\psi$ given in Eq. (3.7). Using the asymptotic form $\psi(z)\sim a\sin(c\kappa z)+b\cos(c\kappa z)$, we find for the flux

$$\lim_{z\to\infty}e^{3A(z)/2}\psi(z)\partial_z[e^{3A(z)/2}\psi(z)]$$
$$\sim\psi(z)[(3b+2a\kappa)\cos(c\kappa z)+(3a-2b\kappa)\sin(c\kappa z)], \tag{3.13}$$

which does not vanish for any choice of $a,b,\kappa$ except $a=b=0$. This implies that the unitary boundary conditions eliminate all continuum modes from the spectrum. It is easy to check that the two discrete modes do not generate any flux into the singularity, so the unitary spectrum consists of these two modes. We expect similar results if we impose unitary boundary conditions in either the thin wall setup of [1] or the thick wall versions in [8–10]. In those cases only the four-dimensional massless graviton survives, and the continuum of KK states are projected out.

This situation should be contrasted with the example encountered in [27]. In that case the potential in the quantum mechanics problem diverges near the singularity. This results in an infinite tower of eigenfunctions with discrete eigenvalues. The potential in our example asymptotes to a constant near the singularity, so we get a continuum of plane wave states. It turns out to be impossible to satisfy the no-flux condition with these solutions, which implies that all of these states are projected out.

We close this section with a brief comment on the solutions with $\bar{\Lambda}=0$ and $n>1$. For $n=2$ the transformation to conformally flat coordinates is given by $z=\tan(cr)/c$ and the conformal factor reads $A(z)=-\log(1+c^2z^2)$. The potential in the quantum mechanics problem is given by

$$V_{QM}=-3c^2\frac{1-4c^2z^2}{(1+c^2z^2)^2}. \tag{3.14}$$

This potential appeared previously in [10] and a similar potential was discussed in detail in [9]. Unlike the $n=1$ potential, this potential asymptotes to zero for large $z$. There is one discrete bound state at threshold and a continuum of states that asymptote to plane waves as $z\to\pm\infty$. We can repeat the analysis above for this potential with essentially the same result. Imposing unitary boundary conditions eliminates the continuous spectrum, leaving only the four-dimensional graviton, while invoking the AdS/CFT correspondence allows us to retain the continuum. In that case the entire fifth dimension gets reinterpreted as a non-conformal field theory on the four-dimensional slice, and any bulk excitations should be viewed as excitations of this field theory.

## IV. DISCUSSION AND SPECULATIONS

In this paper we worked out an example of a thick domain wall that interpolates between two spaces with naked singularities. Our example is simple enough that we can compute the spectrum of the graviton KK modes exactly. It is possible to extend this solution to domain walls with cosmological constants in the four-dimensional slices. These domain walls can be viewed as non-singular analogues of the bent thin domain walls that appeared in the literature as cosmological extensions of the setup in [1].

There are other reasons for considering this type of thick wall. Thick walls with four-dimensional Minkowski slices can be obtained from a first order ''superpotential'' formalism, but to find bent solutions one needs to solve the nonlinear equations of motion directly. In this sense the bent walls are more generic than the flat examples.

The traditional approach to rendering naked singularities harmless consists of imposing unitary boundary conditions on modes propagating in the bulk. If we take this approach for our solution or for solutions of the type studied in [1,9], we project out all continuum modes, leaving only the four-dimensional graviton and other discrete modes if any exist.

The AdS/CFT correspondence offers another point of view. Most of this section will be devoted to comments and speculations about this correspondence in domain wall settings of the type studied here and in [1,9]. We would like to emphasize that unlike in the original AdS/CFT correspondence [18,28], there is at present no precise recipe for relating five-dimensional gravity to a boundary field theory in domain wall space-times of the type studied in [1]. Without such a recipe, our comments are necessarily of a very speculative nature, but we hope that some of them may lead to a firmer understanding of this correspondence in time.

Before discussing the thick domain walls studied here and in [9], let us briefly review how the AdS/CFT correspondence is expected to work in the scenario of [1]. The setup in [1] consists of a thin domain wall separating the horizon parts of two AdS spaces. Usually the $\mathbb{Z}_2$ symmetry of this space-time is gauged, so that the two slices of AdS are identified. The location of the domain wall cuts off the AdS space at some finite radial distance. Gravity in the slice of AdS space is expected to have a dual description as a strongly coupled cutoff CFT on the domain wall.

We can adopt these arguments and apply them to thick domain walls. Let us first consider the domain wall discussed in [9]. It can be viewed as a non-singular version of the setup in [1], since it interpolates between the horizon parts of two AdS spaces. Unlike in the thin wall case, one usually does not mod out by the $\mathbb{Z}_2$ symmetry of the geometry, so we have two independent physical AdS spaces. Since the thick domain wall smoothly connects the two slices of AdS space, there is no sharp cutoff as in the thin wall case. A possible interpretation of this is that a smooth domain wall corresponds to a soft (or softer) cutoff in the field theory. From the field theory perspective this is more desirable than the sharp cutoff imposed by a thin wall. While there is no known regularization scheme with a sharp cutoff that preserves four-dimensional Poincare invariance, we have a candidate for a soft cutoff. Dimensional regularization preserves the desired invariances and corresponds to a soft cutoff in momentum space. Thick domain walls may be more appealing from the field theory point of view, but in gravity they pose additional challenges. For instance, it is not clear where the four-dimensional field theory is supposed to reside, since the space does not have a boundary. This problem could potentially be cured by orbifolding the thick domain wall, which introduces a boundary at $z=0$. The geometry already has a $\mathbb{Z}_2$ symmetry, so orbifolding it simply identifies the two AdS spaces, but the derivative of the scalar does not vanish at $z=0$, so we need to put a source for it on the orbifold fixed point. Orbifolding the space should not affect our speculation that the thick domain wall corresponds to a soft cutoff in the CFT.

We now turn to the examples studied here. The main difference is that we are considering thick domain walls that interpolate between singular spaces. As mentioned before, such singular spaces appear in AdS flows to non-conformal theories, so we speculate that we can replace our singular five-dimensional geometry by a non-conformal four-dimensional field theory with a soft cutoff. This speculation is even harder to make precise than the previous one, because five-dimensional gravity fails near the singularity and higher dimensional gravity or string theory has to come to the rescue. Nonetheless, we will forge ahead and offer some speculations on the field theory interpretations of the singularities we studied here.

We will discuss the $\bar{\Lambda}=0$ solutions given in Eq. (2.8). For $n=1$ the spectrum consist of the massless four-dimensional graviton, an excited state with $m=\sqrt{2}c$, and a continuum of states with masses $m\geqslant m_{gap}=3c/2$. If we assume that our singular space corresponds to a non-conformal theory such as SYM theory or QCD, we can attempt to interpret this KK spectrum. A confining theory will have a strong coupling

scale, $\Lambda_{QCD}$, which sets the mass scale for the light states in the theory. We can interpret the mass gap found in the KK modes as the energy needed to produce the lightest particle in the non-conformal field theory. The presence of the mass gap in our solution at least does not automatically rule it out. Unfortunately we do not have a good interpretation for the single massive resonance in the spectrum. This state should not have a field theory counterpart, since it is localized on the domain wall and does not propagate out to the singularity. Luckily we can eliminate this state by imposing the orbifold projection discussed above for the AdS domain wall. This provides us with a boundary and eliminates this unwanted state. If a version of the AdS/CFT correspondence can be formulated at all, it is likely to be in the orbifolded case.

We also briefly discuss the solution for $n=2$. The equivalent quantum mechanics problem in that case cannot be solved completely, but we can obtain enough information to discuss this case in the light of the AdS/CFT correspondence. The spectrum in this case is very much like that found in [9]. We find a single massless graviton and a continuum of plane wave states with masses starting at zero. Unlike the case studied in [9] this space has naked singularities at a finite distance from the domain wall. These singularities imply that, if this space has a field theory interpretation, it should be in terms of a non-conformal theory. We have already discussed the confining case above. Since there is no mass gap in the $n=2$ case, we suggest that this space may correspond to a theory that is free in the infrared. Such a theory would have excitations with masses that are continuous from zero. From the original form of the AdS/CFT correspondence we expect the gravity description to break down completely if the field theory becomes weakly coupled. This makes it unlikely that the singularity for $n=2$ can be resolved by lifting to ten dimensions. The dual description should require string theory on some highly curved manifold.

Unfortunately all of our speculations follow from the assumption that we can use the AdS/CFT correspondence to gain some intuition about the domain wall space-times we studied here. To make any of our statements more precise we would need a formulation of the AdS/CFT correspondence along the lines of [28]. This is clearly a necessary ingredient if we want to study domain walls in singular space-times in a more reliable and systematic way.

As this paper was nearing completion, Refs. [29] and [30] appeared. These papers have no direct overlap with the results here, but they provide another motivation for studying singular spaces. These papers discuss an intrinsically higher dimensional approach to solving the cosmological constant problem. In their analysis they naturally encounter spaces with singularities at finite distances. While it is not clear if the spaces we discussed here can be used in that context, their results provide another piece of evidence that singular spaces may play an important role in domain wall universe scenarios.

After this paper was submitted a first order formalism for bent thick domain walls appeared in the newest version of [8].

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