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# Gravity of higher-dimensional global defects

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Solutions of Einstein's equations are found for global defects in a higher-dimensional spacetime with a nonzero cosmological constant $\Lambda$. The defect has a $(p-1)$-dimensional core (brane) and a ''hedgehog'' scalar field configuration in the $n$ extra dimensions. For $\Lambda=0$ and $n>2$, the solutions are characterized by a flat brane worldsheet and a solid angle deficit in the extra dimensions. For $\Lambda>0$, one class of solutions describes spherical branes in an inflating higher-dimensional universe. Instantons obtained by a Euclidean continuation of such solutions describe quantum nucleation of the entire inflating brane-world, or of a spherical brane in an inflating higher-dimensional universe. For $\Lambda<0$, one class of solutions exhibits an exponential warp factor. It is similar to spacetimes previously discussed by Randall and Sundrum for $n=1$ and by Gregory for $n=2$.

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## I. INTRODUCTION

In recent years there has been renewed interest in ''brane-world'' models in which the universe is represented by a (3+1)-dimensional subspace (3-brane) embedded in a higher-dimensional (bulk) spacetime [1,2]. In such models, all the familiar matter fields are constrained to live on the brane, while gravity is free to propagate in the extra dimensions. Initially it was thought that realistic models required compact extra dimensions, but it has been shown by Randall and Sundrum [2] (see also [14]) that it is possible to have infinite extra dimensions and still have gravity effectively localized on the brane. This is achieved by introducing a negative cosmological constant which has the effect of ''warping'' the extra-dimensional space, so that most of the physical volume is concentrated near the brane.

In most of the recent work, including that of Randall and Sundrum, the brane is pictured as a domain wall propagating in a 5-dimensional bulk spacetime. The case of two extra dimensions has also been considered, when the branes are similar to strings and the bulk has 6 dimensions. For a gauge string, the metric outside the string core is flat with a conical deficit angle, and Sundrum [3] suggested that the extra dimensions can be compactified by introducing a sufficient number of branes, so that the total deficit angle is equal to $2\pi$. This was generalized by Chodos and Poppitz [4] to include a positive cosmological constant. Cohen and Kaplan [5] considered the case of a global sting which has a curvature singularity at a finite distance from the string core. They argued that the singularity can provide an effective compactification of the extra dimensions. Gregory [6] has shown that a non-singular global string solution exists in the presence of a negative cosmological constant. In this solution the extra dimensions are infinite and strongly warped, as in the Randall-Sundrum model. Garriga and Sasaki [7] discussed a Euclidean continuation of the Randall-Sundrum spacetime and interpreted the resulting instanton as describing quantum nucleation of a 5-dimensional brane-world from nothing.

In this paper, we shall explore a more general case of a brane carrying a global charge in a higher-dimensional spacetime with a nonzero cosmological constant $\Lambda$. We shall consider a $(p-1)$-dimensional brane in a bulk spacetime of $D=p+n$ dimensions. The physically interesting case is $p=4$, but we shall allow an arbitrary $p$ for greater generality. For $n=3$ and $\Lambda=0$, one can expect to recover the global monopole metric with a solid deficit angle [8], but for $n>3$ the defects do not have 3-dimensional analogues.

The paper is organized as follows. In the next section we introduce the scalar field and metric *ansatz* and present the corresponding Einstein's equations. Our solutions for $\Lambda\geqslant0$ and $\Lambda<0$ are given, respectively, in Secs. III and IV. Euclidean instanton solutions are discussed in Sec. V, and our conclusions are summarized in Sec. VI.

## II. EINSTEIN'S EQUATIONS

We shall use the notation $\{x^\mu\}$ with $\mu=0,\dots,p-1$ for the coordinates on the brane worldsheet, $\{\xi^a\}$ with $a=1,\dots,n$ for coordinates in the extra dimensions, and $\{X^A\}$ with $A=0,\dots,D-1$ for general coordinates in the $D$-dimensional spacetime.

A global defect in $n$ extra dimensions is described by a multiplet of $n$ scalar fields $\phi^a$ with a Lagrangian

$$L=\frac{1}{2}\partial_A\phi^a\partial^A\phi^a+V(\phi), \tag{1}$$

where the potential $V(\phi)$ has its minimum on the $n$-sphere $\phi^a\phi^a=\eta^2$. One can use, for example,

$$V(\phi)=\frac{\lambda}{4}(\phi^a\phi^a-\eta^2)^2. \tag{2}$$

The defect solution should have $\phi=0$ at the center of the defect and approach the radial ''hedgehog'' configuration outside the core,

$$\phi^a(\xi)=\eta\frac{\xi^a}{\xi} \tag{3}$$

 

with $\xi^2 \equiv \xi^a \xi^a$. We shall be interested only in the exterior solutions, where $V(\phi)\approx 0$ and $\phi(\xi)$ is accurately approximated by Eq. (3).

We shall adopt the following *ansatz* for the metric:

$$ds^2 = A(\xi)^2 d\xi^2 + \xi^2 d\Omega_{n-1}^2 + B(\xi)^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu, \tag{4}$$

where $d\Omega_m^2$ stands for the metric on a unit $m$-sphere, and the spherical coordinates in the extra dimensions are defined by the usual relations, $\xi^a = \{\xi\cos\theta_1,\ \xi\sin\theta_1\cos\theta_2, \dots\}$. (A different *ansatz* will be considered in Secs. III C and IV C.) The energy-momentum tensor for the field configuration (3) is then given by

$$T_\xi^\xi = -\frac{1}{2}(n-1)\frac{\eta^2}{\xi^2},$$

$$T_{\theta_b}^{\theta_a} = -\frac{1}{2}(n-3)\frac{\eta^2}{\xi^2}\delta_b^a,$$

$$T_\mu^\nu = -\frac{1}{2}(n-1)\frac{\eta^2}{\xi^2}\delta_\mu^\nu. \tag{5}$$

Our goal will be to solve Einstein's equations in $D$ dimensions,

$$R_{AB} - \frac{1}{2}G_{AB}R = \kappa^2 T_{AB} - \Lambda G_{AB}, \tag{6}$$

where $G_{AB}$ is the $D$-dimensional metric, $\Lambda$ is the cosmological constant and $T_{AB}$ is from Eq. (5). The line element (4) is a special case of the more general class of metrics,

$$ds^2 = d\tilde{s}_n^2 + B(\xi^a)^2 d\hat{s}^2, \tag{7}$$

where $d\tilde{s}_n$ depends only on the transverse coordinates $\{\xi^a\}$ and $d\hat{s}^2$ only on those on the brane $\{x^\mu\}$. For such metrics, the Ricci tensor splits in the following way:

$$R_{mn} = \tilde{R}_{mn} - p\frac{B_{;mn}}{B}, \tag{8}$$

$$R_{\mu\nu} = \hat{R}_{\mu\nu} - \hat{g}_{\mu\nu}[B\tilde{\nabla}^2 B + (p-1)(\tilde{\nabla}B)^2]. \tag{9}$$

Since $T_{\mu\nu}\propto \hat{g}_{\mu\nu}$, through Einstein's equations we have that $R_{\mu\nu}\propto\hat{g}_{\mu\nu}$ and finally, from Eq. (9) above, that $\hat{R}_{\mu\nu}\propto\hat{g}_{\mu\nu}$. That is, $\hat{R}$, the curvature associated with the metric $\hat{g}_{\mu\nu}$, must be constant. Einstein's field equations then reduce to

$$\frac{1}{A^2}\left[p(p-1)\left(\frac{B'}{B}\right)^2 + 2p\frac{n-1}{\xi}\frac{B'}{B} + \frac{(n-1)(n-2)}{\xi^2}\right] - \frac{n-1}{\xi^2}(n-2-\kappa^2\eta^2) + 2\Lambda - \frac{\hat{R}}{B^2} = 0, \tag{10}$$

$$\frac{1}{A^2}\left[2p\left(\frac{B''}{B} - \frac{A'}{A}\frac{B'}{B}\right) + p(p-1)\left(\frac{B'}{B}\right)^2 + \frac{2(n-2)}{\xi}\left(p\frac{B'}{B} - \frac{A'}{A}\right) + \frac{(n-3)(n-2)}{\xi^2}\right] - \frac{n-3}{\xi^2}(n-2-\kappa^2\eta^2) + 2\Lambda - \frac{\hat{R}}{B^2} = 0, \tag{11}$$

$$\frac{1}{A^2}\left[2(p-1)\left(\frac{B''}{B} - \frac{A'}{A}\frac{B'}{B}\right) + \frac{2(n-1)}{\xi}\left((p-1)\frac{B'}{B} - \frac{A'}{A}\right) + (p-1)(p-2)\left(\frac{B'}{B}\right)^2 + \frac{(n-1)(n-2)}{\xi^2}\right] - \frac{n-1}{\xi^2}(n-2-\kappa^2\eta^2) + 2\Lambda + \frac{2-p}{p}\frac{\hat{R}}{B^2} = 0, \tag{12}$$

supplemented by the equation for the metric on the $(p-1)$-brane,

$$\hat{R}_{\mu\nu} = \frac{\hat{R}}{p}\hat{g}_{\mu\nu}. \tag{13}$$

It can be shown that only two of the three equations (10)–(12) are independent.

We have been able to find several classes of solutions to this set of equations. We shall discuss separately the cases of positive and negative $\Lambda$ and consider both Lorentzian and Euclidean versions of the metric.

## III. SOLUTIONS WITH $\Lambda \geqslant 0$

### A. Class I

The first class of solutions is obtained with the ansatz $A(\xi) = B(\xi)^{-1}$, which is the same as the one used for a global monopole in [8]. With this ansatz, Einstein's equations are considerably simplified. The two independent equations can be written as

$$(p+1)\frac{A'}{A^3}\xi - \frac{n-2}{A^2} + (n-2) - \kappa^2\eta^2 - \frac{2\Lambda}{n+p-2}\xi^2 = 0, \tag{14}$$

$$-\frac{A''}{A^3} + (p+2)\left(\frac{A'}{A^2}\right)^2 - \frac{(n-1)}{\xi}\frac{A'}{A^3} - \frac{\hat{R}}{p}A^2 + \frac{2\Lambda}{n+p-2} = 0, \tag{15}$$

and we find the following solution:

$$A^{-2}(\xi) = B^2(\xi) = 1 - \frac{\kappa^2\eta^2}{n-2} - \frac{2\Lambda}{(n+p-2)(n+p-1)}\xi^2 \tag{16}$$

$$\hat{R} = \frac{2\Lambda p(p-1)}{(n+p-1)(n+p-2)}\left(1 - \frac{\kappa^2\eta^2}{n-2}\right). \tag{17}$$

The solution (16) is only valid for $n>2$. For $n=3$ and $\Lambda=0$, the transverse to the brane part of the solution coincides with the metric of a global monopole. In higher dimensions, $n>3$, the form of the metric is quite similar: the defect introduces a solid angle deficit in extra dimensions. This is remarkable, considering the fact that defect solutions are quite different for $n=1,2$.

With an appropriate rescaling of the coordinates, the $\Lambda=0$, $n\geqslant 3$ solution can be written as

$$ds^2=d\xi^2+\left(1-\frac{\kappa^2\eta^2}{n-2}\right)\xi^2 d\Omega_{n-1}^2+\eta_{\mu\nu}dx^\mu dx^\nu, \quad (18)$$

where $\eta_{\mu\nu}$ is the Minkowski metric. As the symmetry breaking scale $\eta$ is increased, the solid angle deficit grows and eventually consumes the entire solid angle at the critical value

$$\eta_c=(n-2)^{1/2}\kappa^{-1}. \quad (19)$$

One expects that the transverse dimensions in this case have the geometry of an infinite cylinder whose cross sections are $(n-1)$-spheres of a fixed radius. This expectation will be verified in Sec. III C.

We next consider solutions with $\Lambda>0$. Requiring that the right-hand side of Eq. (16) is positive, we should have $\kappa^2\eta^2/(n-2)<1$, so that $\hat{R}>0$, and $\xi$ should be constrained to the interval $0<\xi<\xi_m$ where $\xi_m$, defined by the condition $B(\xi_m)=0$, is

$$\xi_m^2=\frac{(n+p-2)(n+p-1)}{2\Lambda}\left(1-\frac{\kappa^2\eta^2}{n-2}\right). \quad (20)$$

Another form of the metric can be obtained using the transformation:

$$\xi=\xi_m\sin\chi, \quad (21)$$

which gives

$$ds^2=K[d\chi^2+\alpha^2\sin^2\chi d\Omega_{n-1}^2+\cos^2\chi d\hat{s}_+^2], \quad (22)$$

$$\hat{R}=p(p-1), \quad (23)$$

where

$$K=(n+p-1)(n+p-2)/2|\Lambda|, \quad (24)$$

$$\alpha^2=\left|1-\frac{\kappa^2\eta^2}{n-2}\right|, \quad (25)$$

and $\chi$ varies in the interval $0<\chi<\pi$. The absolute value signs on the right-hand sides of Eqs. (24),(25) are introduced for later use.

The positive-curvature metric $d\hat{s}_+^2$ can be given by any solution of Eq. (13) with $\hat{R}$ from Eq. (23). In this paper we shall assume it to be the $p$-dimensional de Sitter space,

$$d\hat{s}_+^2=(-dt^2+\cosh^2 t d\Omega_{p-1}^2). \quad (26)$$

This is the case of highest symmetry, when all points on the brane worldsheet are equivalent.

The transverse part of the solution (22) describes an $n$-sphere with a solid angle deficit. This may create an impression that the extra dimensions are compactified with a fixed compactification radius $\sim\sqrt{K}$. However, this impression is misleading. In the limit $\eta\to 0$, the deficit angle vanishes, and the metric (22),(26) becomes that of $(p+n)$-dimensional de Sitter space written in somewhat unfamiliar coordinates [9]. With a more familiar form of de Sitter metric,

$$ds^2=K(-dt^2+\cosh^2 t d\Omega_{p+n-1}^2), \quad (27)$$

it is clear that all dimensions are equally large and expanding. Spatial sections of the universe are $(p+n-1)$-spheres, and spatial sections of the brane are $(p-1)$-spheres of the same radius. So the brane is wrapped around the universe along one of the ''big circles.'' Both the brane and the universe expand exponentially with time. A nonzero $\eta$ introduces a deficit angle, but does not change the qualitative character of the spacetime.

We note that de Sitter space also appears to be static in the coordinates

$$ds^2=K(-\cos^2\psi\;\; dt^2+d\psi^2+\sin^2\psi\, d\Omega_{p+n-2}^2). \quad (28)$$

The reason for this is well known: this coordinate system does not cover the whole spacetime; it covers only the interior of a sphere of radius equal to the de Sitter horizon. Our solution (22),(26) uses a mixed representation in which the metric has a static form like Eq. (28) in the transverse dimensions and an expanding form like Eq. (27) on the brane. The coordinate system in Eq. (22) covers the region from $\chi=0$ to the horizon surface $\chi=\pi/2$ where the determinant of the metric vanishes, indicating a coordinate singularity.

The metric (22) is somewhat similar to the dilatonic string solution found by Dando and Gregory [10]. They interpreted their solution as describing a string-antistring pair in a universe with compact static transverse dimensions. Our interpretation of Eq. (22) is quite different, and we believe a similar interpretation should also apply to the Dando-Gregory solution.

The induced metric on the brane is $ds_p^2=Kd\hat{s}^2$, and the curvature of the brane worldsheet is

$$R_p=\frac{2\Lambda p(p-1)}{(n+p-1)(n+p-2)}. \quad (29)$$

This shows that the curvature of the brane is determined only by the cosmological constant $\Lambda$, while the symmetry breaking scale $\eta$ affects only the deficit angle in the extra dimensions.

The solution (22) has curvature singularities at $\chi=0,\pi$ (since $T_A^B$ is singular there), but these singularities are rather mild, and the metric coefficients are non-singular. One should remember that Eq. (22) gives a solution only in the exterior region outside the defect core. But since the metric is well behaved at $\chi\to 0,\pi$, one can expect that it gives a reasonably accurate representation of the full spacetime in the limit when the defect thickness $\delta$ can be neglected, $\delta \ll \xi_m$.

### B. Class II

We find another solution to Eqs. (10)–(12) by considering a different ansatz: $B(\xi)=\xi$. Again the equations simplify considerably and it is possible to find an analytic solution:

$$A^{-2}(\xi)=\frac{n-2-\kappa^2\eta^2}{n+p-2}-\frac{2\Lambda}{(n+p-2)(n+p-1)}\xi^2, \qquad B(\xi)=\xi \tag{30}$$

$$\hat{R}=p(n-2-\kappa^2\eta^2). \tag{31}$$

As in the previous case, for $\Lambda>0$ we have from the condition $A^2>0$ that $n-2-\kappa^2\eta^2>0$, and thus $\hat{R}>0$ and $\xi$ is constrained to the interval $0<\xi<\xi_m$ with

$$\xi_m^2=\frac{(n+p-1)(n-2-\kappa^2\eta^2)}{2\Lambda}. \tag{32}$$

As before, we redefine the radial coordinate as

$$\xi=\xi_m \sin\chi, \tag{33}$$

and the metric takes the form

$$ds^2=K[d\chi^2+\tilde{\alpha}^2\sin^2\chi(d\Omega_{n-1}^2+d\hat{s}^2)] \tag{34}$$

where $K$ is given by Eq. (24),

$$\tilde{\alpha}^2=\left|\frac{n-2-\kappa^2\eta^2}{n+p-2}\right|, \tag{35}$$

$d\hat{s}^2$ stands for a $p$-dimensional spacetime of constant curvature $\hat{R}=p(p-1)$, and $\chi$ takes values in the interval $0<\chi<\pi$.

An unphysical feature of the solution (34) is that the deficit angle does not vanish even for $\eta=0$, that is, in the absence of a defect. We have verified that the curvature invariant $R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau}$ diverges at $\chi=0,\pi$ for $\eta=0$. These singularities appear to be unrelated to the defect, and we dismiss class-II solutions as unphysical.

### C. Class III

As we mentioned in Sec. III A, the ‘‘conical’’ geometry of the extra dimensions is expected to degenerate into a cylinder at some critical value of the symmetry breaking scale $\eta$. In order to verify this expectation, we introduce the following ansatz:

$$ds^2=d\xi^2+C^2d\Omega_{n-1}^2+B(\xi)^2d\hat{s}^2, \tag{36}$$

where $C$ is a constant radius of the $(n-1)$-spheres. This is again of the form (7), so Eqs. (8), (9) can be used, and we obtain

$$-p\frac{B''}{B}-\frac{2\Lambda}{n+p-2}=0, \tag{37}$$

$$\frac{n-2-\kappa^2\eta^2}{C^2}-\frac{2\Lambda}{n+p-2}=0, \tag{38}$$

$$\frac{1}{p}\frac{\hat{R}}{B^2}-\frac{B''}{B}-(p-1)\left(\frac{B'}{B}\right)^2-\frac{2\Lambda}{n+p-2}=0. \tag{39}$$

For $\Lambda=0$, Eq. (38) gives $\eta=(n-2)^{1/2}\kappa^{-1}$, which agrees with the critical value (19). From Eq. (37), $B'=\text{const}$, and it follows from Eq. (39) that the worldsheet curvature $\hat{R}$ can be either positive or zero. For $\hat{R}=0$, $B=\text{const}$, and the solution is

$$ds^2=d\xi^2+C^2d\Omega_{n-1}^2+\eta_{\mu\nu}dx^\mu dx^\nu. \tag{40}$$

The radius of the cylinder $C$ is arbitrary; we expect it to be determined by matching to an appropriate interior solution in the defect core, with the complete geometry being that of a ‘‘cigar.’’

For $\hat{R}>0$ and with a suitable normalization of the radial coordinate, the solution can be written as

$$ds^2=C^2d\Omega_{n-1}^2+d\chi^2+\chi^2d\hat{s}_+^2 \tag{41}$$

with $d\hat{s}_+^2$ from Eq. (26). It can be shown that the last two terms in the metric (41) describe a $(p+1)$-dimensional Minkowski space in unfamiliar coordinates [11]. This metric is therefore equivalent to Eq. (40).

For $\Lambda>0$, Eq. (38) gives

$$C^2=(n+p-2)(n-2-\kappa^2\eta^2)/2\Lambda, \tag{42}$$

and we find a solution of the form

$$ds^2=C^2d\Omega_{n-1}^2+\omega^{-2}(d\chi^2+\sin^2\chi d\hat{s}_+^2), \tag{43}$$

where

$$\omega=\sqrt{\frac{2\Lambda}{p(n+p-2)}}. \tag{44}$$

The last two terms in the metric (43) describe a $(p+1)$-dimensional de Sitter space. Note that, in contrast to the $\Lambda=0$ case, solutions now exist for all values of $\eta<\eta_c$, while for $\eta=\eta_c$ the solution becomes singular, with $C=0$. This shows that the flat cylindrical solution (40) with $\eta=\eta_c$ is unstable with respect to the introduction of an arbitrarily small cosmological constant $\Lambda$.

## IV. SOLUTIONS WITH $\Lambda<0$

The solutions (16) and (30) given in the previous section also allow for negative values of $\Lambda$. There are actually three different possibilities, since now $n-2-\kappa^2\eta^2$ can be either positive, negative, or zero.

### A. Class I

For $\Lambda<0$ and depending on the sign of $n-2-\kappa^2\eta^2$, we can define a new radial coordinate $\chi$ as

$$\xi=\sqrt{K}\alpha\sinh\chi,\quad \sqrt{K}e^{\chi},\quad \sqrt{K}\alpha\cosh\chi \tag{45}$$

for $n-2-\kappa^2\eta^2$ less, equal and greater than zero, respectively. The range for the new coordinate is $0\leqslant\chi<\infty$ in the first case and $-\infty<\chi<\infty$ in the other two. Then we can write the metric as

$$\hat{R}<0:\quad ds^2=K[d\chi^2+\alpha^2\sinh^2\chi d\Omega_{n-1}^2+\cosh^2\chi d\hat{s}_-^2], \tag{46}$$

$$\hat{R}=0:\quad ds^2=K[d\chi^2+e^{2\chi}(d\Omega_{n-1}^2+d\hat{s}_0^2)], \tag{47}$$

$$\hat{R}>0:\quad ds^2=K[d\chi^2+\alpha^2\cosh^2\chi d\Omega_{n-1}^2+\sinh^2\chi d\hat{s}_+^2]. \tag{48}$$

Here, $d\hat{s}_\pm^2$ is the metric on a space of constant curvature satisfying Eq. (13) with $\hat{R}=\pm p(p-1)$, and $d\hat{s}_0^2$ is a Ricci-flat metric. In the case of negative curvature, we can choose, for example, the anti–de Sitter space

$$d\hat{s}_-^2=[-dt^2+\sin^2 t(d\psi^2+\sinh^2\psi d\Omega_{p-2}^2)]. \tag{49}$$

Flat space metric can be used for $d\hat{s}_0^2$, and the de Sitter metrics (26) can be used for the constant positive curvature space $d\hat{s}_+^2$.

For $\hat{R}<0$, the defect is located at $\chi=0$. For $\hat{R}=0$ it is removed to $\chi=-\infty$, and for $\hat{R}>0$ there is no defect at all. In the latter case, there is a minimum radius for the $(n-1)$-spheres in the extra dimensions, $r_{min}=K\alpha$. We thus have a wormhole connecting a monopole configuration at $\chi>0$ with an antimonopole configuration at $\chi<0$.

### B. Class II

For the solutions defined by expressions (30) we find a similar situation. With a new coordinate $\chi$ defined as in Eq. (45), but with $\alpha$ replaced by $\tilde{\alpha}$, we have

$$\hat{R}>0:\quad ds^2=K[d\chi^2+\tilde{\alpha}^2\sinh^2\chi(d\Omega_{n-1}^2+d\hat{s}_+^2)] \tag{50}$$

$$\hat{R}=0:\quad ds^2=K[d\chi^2+e^{2\chi}(d\Omega_{n-1}^2+d\hat{s}_0^2)] \tag{51}$$

$$\hat{R}<0:\quad ds^2=K[d\chi^2+\tilde{\alpha}^2\cosh^2\chi(d\Omega_{n-1}^2+d\hat{s}_-^2)]. \tag{52}$$

Once again, the metric (50) is singular at $\chi=0$ even in the absence of a defect ($\eta=0$), and we dismiss this solution as unphysical.

### C. Class III

We finally consider the cylindrical metric ansatz (36). The solutions of Eqs. (37)–(39) for $\Lambda<0$ have the form

$$\hat{R}>0:\quad ds^2=C^2d\Omega_{n-1}^2+\omega^{-2}(d\chi^2+\sinh^2\chi d\hat{s}_+^2), \tag{53}$$

$$\hat{R}<0:\quad ds^2=C^2d\Omega_{n-1}^2+\omega^{-2}(d\chi^2+\cosh^2\chi d\hat{s}_-^2), \tag{54}$$

$$\hat{R}=0:\quad ds^2=C^2d\Omega_{n-1}^2+d\chi^2+e^{\pm 2\omega\chi}d\hat{s}_0^2, \tag{55}$$

where

$$\omega=\sqrt{\frac{-2\Lambda}{p(n+p-2)}}, \tag{56}$$

$$C^2=-(n+p-2)[\kappa^2\eta^2-(n-2)]/2\Lambda. \tag{57}$$

Of greatest interest are the flat brane solutions (55) which generalize the solutions considered by Gregory [6] in the $n=2$ case. The geometry of the extra dimensions in the metric (55) is that of a cylinder with a cross section being an $(n-1)$-sphere of a fixed radius $C$. It would be interesting if this solution could be matched to an appropriate interior solution, so that the complete geometry is that of a ''cigar.'' Gregory [6] has argued that this is possible for $n=2$, but her analysis does not directly apply to $n\geqslant 3$.

Cigar-like defect solutions with an exponential warp factor would be of interest, since they would have features similar to those of the Randall-Sundrum geometry. If the brane is located at $\chi=0$ and the asymptotic metric is given by Eq. (55) with a negative sign in the exponential, then the volume of the extra dimensions would be finite, despite their infinite extent in the $\chi$ direction. As in the Randall-Sundrum case, most of the volume would be concentrated near the brane, and one can expect that gravitons would be effectively confined to the brane.

The right-hand side of Eq. (57) should be positive, so we must have $\kappa^2\eta^2-(n-2)>0$. While this does not give any additional information for $n=2$, this condition requires a super-Planckian symmetry breaking scale, $\eta>\kappa^{-1}$, for the defects when $n>2$.

## V. INSTANTON SOLUTIONS

Euclidean continuations of brane-world solutions are of interest, since they can be interpreted as gravitational instantons describing quantum nucleation of a brane-world. The nucleation probability is given by

$$\mathcal{P}\propto e^{\pm|S|}, \tag{58}$$

where $S$ is the instanton action. The choice of sign in the exponential is determined by the choice of boundary conditions for the wave function of the universe. The lower sign is chosen for the tunneling and Linde boundary conditions, and the upper sign for the Hartle-Hawking boundary condition [12]. For definiteness we shall adopt the tunneling boundary condition below.

For the instantons to give a nonvanishing contribution to the nucleation probability, they must have a finite action, with instantons of the smallest absolute value of the action giving the dominant contribution. The action is typically extremized for solutions of the highest symmetry, so we shall consider instantons with $d\hat{s}_+^2$, $d\hat{s}_-^2$ and $d\hat{s}_0^2$ being maximally symmetric spaces of positive, negative and zero curvature, that is, Euclidean de Sitter, anti–de Sitter, and flat spaces, respectively.

The Euclidean action for our model is given by

$$S=-\frac{1}{2\kappa^2}\int d^{(n+p)}x\sqrt{-g}[R-2\Lambda-2\kappa^2 L(\phi)], \quad (59)$$

where $R$ is the $D$-dimensional scalar curvature and $L(\phi)$ is the scalar field Lagrangian. We can eliminate $R$ by making use of Einstein's equations to obtain

$$R=2\kappa^2 L(\phi)+\frac{2(n+p)}{n+p-2}\Lambda \quad (60)$$

and

$$S=-\frac{\Lambda}{\kappa^2(n+p-2)}\int d^{n+p}x\sqrt{-g}. \quad (61)$$

For class-I and class-II solutions with $\Lambda<0$, the volume of the transverse space is infinite, and $|S|=\infty$. If cigar-like class-III solutions exist, they may have a finite transverse volume, but the action is still infinite due to the divergence of the $p$-dimensional volume of the flat brane worldsheet. Hence, we only need to consider solutions with $\Lambda>0$. In this case the curvature of the brane must be positive, $\hat{R}>0$, and thus the metric $d\hat{s}^2$ should be that of a Euclidean de Sitter space, that is, a $p$-sphere:

$$ds_E^2=K[d\chi^2+\alpha^2\sin^2\chi d\Omega_{n-1}^2 + \cos^2\chi(d\psi^2+\sin\psi^2 d\Omega_{p-1}^2)]. \quad (62)$$

One can model the nucleation of a closed universe with a brane by allowing $\psi$ to vary in the interval $[0,\pi/2]$ in the Euclidean region and then continuing it in the imaginary direction in the Lorentzian region, $\psi=\pi/2+it$. This turns Eq. (62) into the metric

$$ds^2=K[d\chi^2+\alpha^2\sin^2\chi d\Omega_{n-1}^2 + \cos^2\chi(-dt^2+\cosh^2 t d\Omega_{p-1}^2)] \quad (63)$$

describing an expanding braneworld.

We can easily calculate the action for the instanton solution (62):

$$S=\frac{1}{2\kappa^2}\frac{V_p}{2}V_{(n-1)}K^{(n+p)/2}\alpha^{n-1}\int_0^\pi d\chi|\cos\chi|^p(\sin\chi)^{(n-1)} \times\frac{4\Lambda}{n+p-2} \quad (64)$$

$$=\frac{4}{\kappa^2}\frac{\sqrt{\pi}^{(n+p+1)}}{\Gamma[(n+p-1)/2]}K^{(n+p-2)/2}\alpha^{n-1}. \quad (65)$$

where $V_k$ stands for the volume of a k-sphere of unit radius, that is, $V_k=2\pi^{(k+1)/2}/\Gamma[(k+1)/2]$.

Apart from nucleation of the entire brane-world, the instanton (62) can also describe nucleation of spherical branes in an inflating $(n+p)$-dimensional de Sitter space. The situation here is very similar to the nucleation of circular loops of string and of spherical domain walls in a (3+1)-dimensional de Sitter space, as discussed by Basu *et al.* [13]. The nucleation rate is given by

$$\Gamma\propto e^{-B} \quad (66)$$

with

$$B=S-S_0, \quad (67)$$

where $S$ is the instanton action and $S_0$ is the action for the Euclidean de Sitter space without a brane. From Eq. (65) we have

$$B=\frac{4}{\kappa^2}\frac{\pi^{(n+p+1)/2}}{\Gamma[(n+p-1)/2]}K^{(n+p-2)/2}(1-\alpha^{n-1}). \quad (68)$$

The initial radius of the brane is $r=\sqrt{K}$. After nucleation, it is stretched by the exponential expansion of the universe.

## VI. CONCLUSIONS

In this paper we have found a number of solutions describing global defects in a higher-dimensional space. We assumed that the core of the defect is centered on a $(p-1)$-dimensional brane and concentrated on the case when the number of extra dimensions is $n\geqslant 3$.

In the absence of a cosmological constant, we found that for all $n\geqslant 3$ the defect solution is very similar to that for a global monopole [8]. The brane worldsheet is flat, and there is a solid angle deficit in the extra dimensions. This is rather surprising, considering the fact that solutions are very different for $n=1$ and $n=2$. The maximal solid angle deficit is reached at the critical value $\eta_c=(n-2)^{1/2}\kappa^{-1}$, when the transverse metric becomes that of a cylinder.

For a positive cosmological constant, $\Lambda>0$, our solutions describe spherical branes in an inflating higher-dimensional universe. In the limit $\eta\to 0$, when the gravitational effect of the defect can be neglected, the universe can be pictured as an expanding $(p+n-1)$-dimensional sphere with a brane wrapped around it in the form of a sphere of lower dimensionality $(p-1)$. A nonzero $\eta$ introduces a deficit angle in the dimensions orthogonal to the brane worldsheet. It is interesting that the expansion rate of the universe (and of the brane) is independent of the symmetry breaking scale $\eta$ and is determined only by $\Lambda$, while the deficit angle is determined by $\eta$ and independent of $\Lambda$. Gravitational instantons obtained by a Euclidean continuation of this class of solutions have the geometry of a $(p+n)$-sphere with the brane represented by a maximal $p$-sphere and with a deficit solid angle in the dimensions transverse to the brane. These instantons can be interpreted as describing quantum nucleation either of the entire brane-world, or of a spherical brane in an inflating $(p+n-1)$-dimensional universe.

Another class of solutions has curvature singularities even in the absence of a defect ($\eta=0$), and we have dismissed such solutions as unphysical.

The third class of solutions has the geometry of a $(p+1)$-dimensional de Sitter space, with the remaining $(n-1)$ dimensions having the geometry of a cylinder.

We have also found 3 classes of solutions for $\Lambda<0$. The first two are essentially analytic continuations of the positive-$\Lambda$ solutions. The third class is similar to Randall-Sundrum $(n=1)$ and Gregory $(n=2)$ solutions, exhibiting an exponential warp factor. If solutions of the third class can be matched to appropriate interior solutions in the defect core, one may be able to use them as a basis for realistic brane-world models.

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