Varying speed of light cosmologies as two-dimensional dynamical systems

M. Biesiada*
Department of Astrophysics and Cosmology, University of Silesia, Uniwersytecka 4, 40-007 Katowice, Poland

M. Szydłowski†
Jagellonian University, Astronomical Observatory, Orla 171, Cracow, Poland
(Received 4 November 1999; published 25 July 2000)

We formulate the dynamics of Friedman–Robertson-Walker varying speed of light models as a two-dimensional Hamiltonian dynamical system. The shape of the potential and the existence of an energy first integral can be used to classify possible evolutions of VSL models. We show that an assumed (power-law) time dependence of the speed of light leads to a uniform evolution pattern of VSL models on two-dimensional phase space. We also formulate the criteria for solving the flatness and horizon problems in terms of the phase space and discuss the emerging patterns of evolution on respective phase portraits for all possible initial conditions. We argue that in the class of FRW VSL models filled with radiation open ($K=-1$) models with a positive cosmological constant $\Lambda>0$ are preferred from the point of view of the flatness and horizon problem avoidance.

PACS number(s): 98.80.Hw

I. INTRODUCTION

Although the so-called cosmological standard model is usually believed to be the correct picture of our world [1] it still does have some problems, among which the flatness and horizon problems are the most widely known. The existence of large-scale structure in the universe extending to the limits of our deepest surveys [2] is another mystery. Its very existence implies the necessity of seeds for this structure in the early universe. In the standard big-bang scenario they could only be a built in, rather undesirable feature of the theory. Therefore, a great amount attention has been paid toward inflationary models which, despite invoking exotic (if not hypothetical) physics, are able to provide at least hope for a consistent explanation of either the flatness or horizon problems as well as the origin of seeds for the large-scale structure. The development of early universe physics led us to expect the occurrence of phase transitions when the universe was young, hot, and dense [3].

An interesting idea was formulated some years ago by Moffat [4] who conjectured that there could actually take place a spontaneous breaking of local Lorenz invariance and diffeomorphism invariance associated with a first order phase transition in the early universe. The associated variation of the speed of light over many (30) orders of magnitude is capable of explaining the flatness and horizon problems, provides a mechanism of monopole suppression and can solve the cosmological constant problem. This idea was revived in recent papers by Albrecht and Magueijo [5] and was given further consideration by Barrow [6]. In particular the term varying speed of light (VSL) cosmology was coined in the latter work.

It is a well known fact that the outcome of physical experiments is sensitive only to dimensionless combinations of dimensional constants. Consequently, the physically meaningful variability of the speed of light would manifest itself, for example, as a variation of the fine structure constant. There have been several tests [7] of possible time variation of the fine structure constant α in the past. The results of these works indicated that the VSL period (if it exists at all) should be confined to the very early universe. On the other hand, there have been suggestions of evidence of time varying fine structure constant [8] seen in comparison of quasar spectral lines in different multiplets. Barrow and Magueijo in a recent paper [9] proposed a VSL scenario in which a varying c of the magnitude required to create an apparent change in α at the observed levels [8] was also sufficient to produce an acceleration at the level suggested by the supernova type Ia (SNIa) data [10]. In Ref. [11] it has been pointed out that position of the first acoustic peak of the cosmic microwave background (CMB) angular power spectrum is a sensitive measure of the variability of c after the epoch of last scattering and that future observations should significantly improve the current bound (from the Doppler peak) allowing for about a 4%, variation in c. Hence the exotic problem of VSL cosmologies is entering the stage of seriously confronting its predictions with the real world.

Anyway, it still remains true that the prime motivation for VSL cosmologies lies in the very early universe which is the domain of high-energy cosmology. The literature of the subject is still growing and comprises a variety of approaches: phenomenological Albrecht-Barrow-Magueijo [5,6,12], scalar-tensor approach of Clayton and Moffat [4,13], and a number of other contributions, some of them inspired by higher-dimensional brane theories [14] (see also the most recent paper [15]). Hence the VSL cosmologies are becoming a serious alternative to the inflationary scenario.

There is a widespread opinion that physically realistic models of the world should possess some kind of structural stability. A suitable tool for investigating structural stability and similar kinds of questions is provided by the theory of

*Email address: mb@imp.sosnowiec.pl; mb@server.phys.us.edu.pl
†Email address: uoszydlo@cyf-kr.edu.pl

dynamical systems. This approach is known in the literature [16] and has already been applied in our previous works, e.g., Ref. [17]. In this paper we formulate VSL Friedmann-Robertson-Walker cosmological as a two-dimensional dynamical system an we discuss on phase portraits their properties emerging in this picture.

II. BASIC EQUATIONS OF THE THEORY

In recent papers by Albrecht and Magueijo [5] and by Barrow [6] a useful framework to discuss VSL models was set up by the assumption that time variable c should not introduce changes in the curvature terms of the gravitational field equations and that Einstein's equations must hold. Because time varying c breaks the Lorenz invariance the VSL cosmology requires a specific reference frame (including specific choice of time coordinate) in which changes in the field equations are minimal and one postulates it to coincide with cosmological comoving frame.

In the case of the VSL version of the Friedmann-Robertson-Walker (FRW) models the scale factor obeys the following dynamical equations:

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G(t)\rho}{3}-\frac{kc^2(t)}{a^2(t)}, \tag{1}$$

$$\frac{\ddot{a}(t)}{a}=-\frac{4\pi G(t)}{3}\left(\rho+\frac{3p}{c^2(t)}\right). \tag{2}$$

Equation (2) is called the Raychaudhuri equation, and from the above system one can build a generalized conservation equation

$$\dot{\rho}+3\frac{\dot{a}}{a}\left(\rho+\frac{p}{c^2(t)}\right)=-\rho\frac{\dot{G}}{G}+\frac{3kc^2}{8\pi Ga^2}\frac{\dot{c}}{c} \tag{3}$$

in which time dependence of fundamental constants has been taken into account explicitly. Alternatively one can think of the Raychaudhuri equation together with the generalized conservation equation as of a fundamental system to which Eq. (1) is a first integral.

Fundamental difficulty concerning the system (1)–(3) is that now it became a nonautonomous system with unknown functions of time $[G(t),c(t)]$ on the right-hand sides. In order to be specific in further analysis we adopt Barrow's power-law ansatz

$$G(t)=G_0a(t)^q, \quad c(t)=c_0a(t)^n. \tag{4}$$

Moreover, we assume the hydrodynamical energy-momentum tensor with the equation of state

$$p=\gamma\rho c^2(t), \tag{5}$$

where $0\leqslant\gamma\leqslant 1$. Well recognized special cases are $\gamma=0$, dust (pressure less matter), $\gamma=1/3$, radiation, and $\gamma=1$, stiff Zeldovich matter. The power-law ansatz (4) turns the field equations back into an autonomous system. We can now think about extensions of our baseline equations. First, one can straightforwardly include the cosmological constant Λ by introducing respective pressure p_Λ and energy density ρ_Λ:

$$\rho_\Lambda=\frac{\Lambda c^2(t)}{8\pi G(t)}, \tag{6}$$

$$p_\Lambda=-\rho_\Lambda c^2(t). \tag{7}$$

One is also able to extend the system (1)–(3) with cosmological constant to some homogeneous anisotropic models of Bianchi type I (containing flat FRW model) and Bianchi type V (containing open FRW model) having isotropic curvatures. In general the line element of such models is parametrized by three scale factors $R_1(t),R_2(t),R_3(t)$. If one introduces the geometric mean scale factor $a(t)$, i.e., $a^3(t)=R_1(t)R_2(t)R_3(t)$ and the shear anisotropy scalar σ, the field equations can be cast into the form

$$\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi G(t)\rho}{3}-\frac{kc^2(t)}{a^2(t)}+\frac{\sigma^2}{3}+\frac{\Lambda c^2(t)}{3}, \tag{8}$$

$$\frac{\ddot{a}(t)}{a}=-\frac{4\pi G(t)}{3}\left(\rho+\frac{3p}{c^2(t)}-\frac{2\sigma^2}{3}+\frac{\Lambda c^2(t)}{3}\right) \tag{9}$$

$$\dot{\rho}+3\frac{\dot{a}}{a}\left(\rho+\frac{p}{c^2(t)}\right)+\frac{\sigma}{4\pi G(t)}\left(\dot{\sigma}+3\frac{\dot{a}}{a}\sigma\right)=-\rho\frac{\dot{G}}{G}+\frac{3kc^2}{8\pi Ga^2}\frac{\dot{c}}{c}. \tag{10}$$

The postulate of the standard shear evolution equation

$$\dot{\sigma}+3\frac{\dot{a}}{a}\sigma=0 \tag{11}$$

which means that $\sigma=\Sigma/a^3(t)$, $\Sigma=\text{const}$ make the generalized conservation equation retaining its original VSL form. In the other words matter decouples from shear evolution. The main conclusion of this section is therefore a considerable generality of the formalism we use in further sections.

III. REDUCTION TO THE HAMILTONIAN SYSTEM

In this section we shall present the field equations (8) and in particular Eqs. (1)–(3) in the form of a gradient dynamical system. In order to perform this task let us define a new variable

$$X(t)=a(t)^{D(\gamma)}, \tag{12}$$

where $D(\gamma)=3(1+\gamma)/2$. The field equations are now equivalent to the following gradient system:

$$\ddot{X}(t)=-\frac{dV(X)}{dX}=-\frac{3}{4}(\gamma^2-1)\Sigma^2X^{(D-6)/D}$$
$$+X^{2n/D}\{\bar{K}X^{(D-2)/D}-\bar{\Lambda}X\}, \qquad (13)$$

where $\bar{K}=\frac{3}{4}c_0^2(1+\gamma)(1+3\gamma)K$ and $\bar{\Lambda}=\frac{3}{4}c_0^2(1+\gamma)(1+3\gamma)\Lambda$.

We shall focus our attention on the case $\Sigma=0$ corresponding to VLS version of FRW models. Moreover we shall assume negative $n<0$ [6] because this is the range for which cosmological problems can be solved.

Let us now notice that by simple time rescalling $\tau=\frac{3}{4}c_0^2(1+\gamma)(1+3\gamma)t$ one can simplify the system in the sense that coefficients $\bar{K}$ and $\bar{\Lambda}$ acquire back their original form and meaning. Denoting, by primes, derivatives with respect to τ one arrives at the system

$$X''=(-KX^{(D-2)/D}+\Lambda X)X^{2n/D} \qquad (14)$$

or, in the phase plane (X,X'),

$$X'=Y,$$

$$Y'=X^{2n/D}(-KX^{(D-2)/D}+\Lambda X). \qquad (15)$$

Finally one can cast the system into a Hamiltonian form by introducing the potential function

$$V(X)=X^{2n/D}(K_DX^{(2/D)(D-1)}-\Lambda_DX^2), \qquad (16)$$

where $K_D=KD/2(D+n-1)$ and $\Lambda_D=\Lambda D/2(D+n)$. The above functional form of the potential is valid whenever $D+n-1\neq 0$ and $D+n\neq 0$. If $D+n-1=0$ then

$$V(X)=K\ln(X)-\Lambda_DX^{2+2n/D},$$

and in a similar manner, in the case of $D+n=0$ we have

$$V(X)=K_DX^{2(n+D-1)/D}-\Lambda\ln(X).$$

Let us stress that now, the constants K_D and Λ_D do not have to take the same sign as K and Λ do.

The Hamiltonian of the system (15) has the form

$$H(X,Y)=\frac{Y^2}{2}+V(X) \qquad (17)$$

and of course $H=\text{const}=C$ is the first integral. It is easy to demonstrate that $C\geqslant 0$ since $K=\Lambda=0$ correspond to the Minkowski space. The system (17) is one dimensional and hence can be integrated in quadratures

$$\tau-\tau_0=\frac{1}{\sqrt{2}}\int_{X_0}^{X}\frac{d\zeta}{\sqrt{C-V(\zeta)}}, \qquad (18)$$

where $X_0=X(\tau_0)$, one can assume $\tau_0=0$ and $X_0=0$. An example of such integral for $C=0$ is

$$\tau-\tau_0=-\frac{DX\sqrt{(1-\Lambda X^{2/D}/K)(-\Lambda X^2+KX^{2(D-1)/D})}\,{}_1F_1[(1-n)/2,1/2,(3-n)/2,(\Lambda X^{2D})/K]}{\sqrt{2}(n-1)\sqrt{-\Lambda X^2+KX^{2(D-1)/D}}\sqrt{-X^{2n/D}(-\Lambda X^2+KX^{2(D-1)/D})}}, \qquad (19)$$

where ${}_1F_1$ is respective hypergeometric function.

The Hamiltonian structure of Eq. (17) can be used to classify the VSL models qualitatively in the spirit of classical Robertson's classification [18]. For this purpose let us consider the function

$$\varphi(X)=\frac{Y^2}{2}=C-V(X)$$
$$=C-X^{2n/D}(K_DX^{(2/D)(D-1)}-\Lambda_DX^2). \qquad (20)$$

The motion of the system in the configuration space of the model is confined to the region $\mathcal{D}=\{X\in R;\varphi(X)\geqslant 0\}$. The zero velocity curve $\varphi(X)=0$ can be represented as

$$\Lambda(X)=\frac{2(n+D)}{D}\left(-CX^{-(2/D)(n+D)}\right.$$
$$\left.+\frac{KD}{2(D+n-1)}X^{-(2/D)}\right) \qquad (21)$$

and the evolution of models is represented as horizontal lines lying above the $\Lambda(X)$ curve and bouncing off the "wall" of $\varphi(X)=0$. Because the zero-velocity curve is determined by the potential function $V(X)$, one should consider additionally two special cases: (i) $D+n-1=0$ then

$$\Lambda(X)=\frac{2(n+D)}{D}X^{-2(D+n)/D}[\ln(X)-C],$$

and (ii) $D+n=0$ then

$$\Lambda(X)=\frac{K_DX^{(n+D-1)/D}-C}{\ln(X)}.$$

Figures 1 and 2 show the $\Lambda(X)$ curves corresponding to $K=\pm 1$ models for several values of n. They can be used to classify (qualitatively) all possible evolutions of respective cosmological models. In the family of closed VSL FRW models ($K=+1$) displayed in Fig. 1, the curve for $n=0$ reproduces standard classical behavior which is inherited by VSL models with $n\geqslant -2$. Possible patterns of evolution in

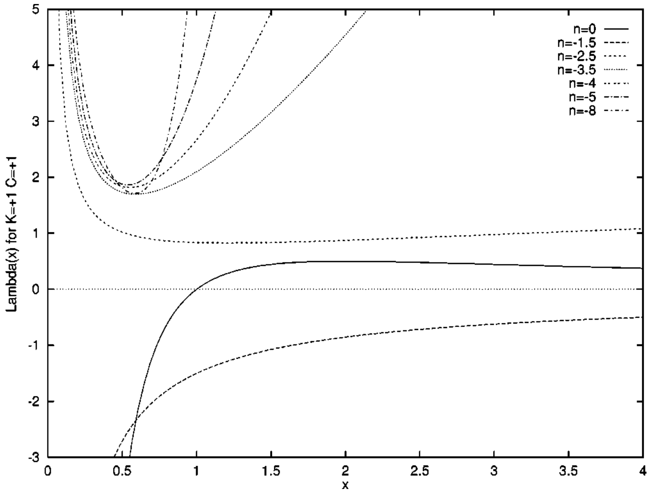

FIG. 1. $\Lambda_D(X)$ function corresponding to closed $(K=+1)$ VLS Friedmann-Robertson-Walker models with radiation $(D=2)$ for different values of n.

this class are the following. For Λ larger than the maximum $\Lambda_{\max}$ of the $\Lambda(X)$ curve the universe starts from the initial singularity and expands to infinity. For $\Lambda<\Lambda_{\max}$ two distinct evolution patterns emerge: the universe either starts from the initial singularity and ends in the final singularity or starting from the finite size expands to infinity (nonsingular universe). In the case of $\Lambda=\Lambda_{\max}$ we have either a universe expanding from the initial singularity to a static Einstein–de Sitter state or starting from finite size expands to infinity (Lemaitre-Eddington-type evolution). The VSL models with strongly negative n (lesser than -2) exist for $\Lambda\geqslant\Lambda_{\min}$ and starting from the finite size they expand to a finite sized universe. The case of $\Lambda=\Lambda_{\min}$ corresponds to the static universe. Models with $\Lambda<\Lambda_{\min}$ are forbidden [see, e.g., the phase portrait on Fig. 6(d)].

The family of VSL open $(K=-1)$ models is shown in Fig. 2. From the formula (21) one can see that the asymptotic

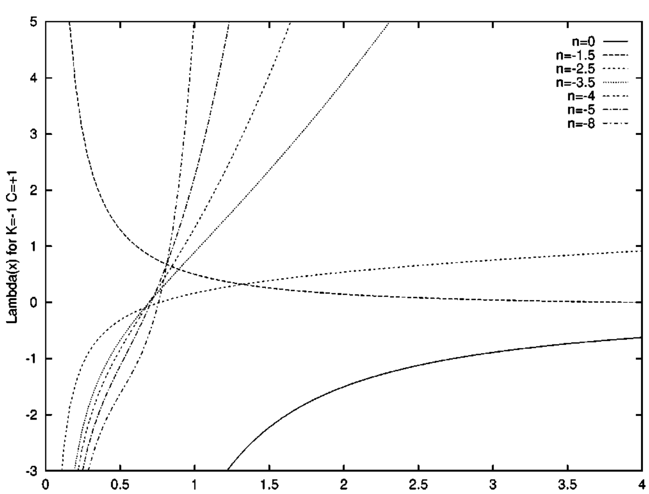

FIG. 2. $\Lambda_D(X)$ function corresponding to open $(K=-1)$ VLS Friedmann-Robertson-Walker models with radiation $(D=2)$ for different values of n.

behavior of $\Lambda(X)$ depends crucially on n. Namely, for $-1\leqslant n\leqslant 0$ $\Lambda(X)\to-\infty$ for $X\to 0$ and $\Lambda(X)\to 0$ for $X\to+\infty$. Then for $-2\leqslant n<-1$ $\Lambda(X)\to+\infty$ for $X\to 0$ and $\Lambda(X)\to 0$ for $X\to+\infty$ and finally for $n<-2$ $\Lambda(X)\to-\infty$ for $X\to 0$ and $\Lambda(X)\to+\infty$ for $X\to+\infty$. Consequently the possible evolutions of respective models are the following: for $-1\leqslant n\leqslant 0$ models with $\Lambda>\Lambda_{\max}$ start with initial singularity and expand to infinity, models with $\Lambda\leqslant\Lambda_{\max}$ either are oscillatory (with initial and final singularities) or expand from the finite size to infinity. For $-1<n\leqslant -2$ solutions with $\Lambda<0$ are forbidden and models with positive cosmological constant expand to infinity starting from the finite size. Finally VSL models with $n<-2$ start from the initial singularity and finish their evolution in the final singularity.

IV. PROPERTIES OF DYNAMICAL SYSTEMS DESCRIBING THE EVOLUTION OF VLS MODELS

We shall now investigate the system (15). Right-hand sides of this system define a smooth vector field on $\mathbf{R}^2/\{0,0\}$. If we want to have a smooth dynamical system on the whole $\mathbf{R}^2$ the regularization procedure [see below Eq. (23)] is required. In this way all possible evolutions of VSL models can be represented as curves in the phase space. We shall discuss emerging phase portraits in the case of radiative matter (for the purpose of illustration), i.e., for $D=2$.

One can notice that for $D=2$ the classical system (i.e., $n=0$ corresponding to constant speed of light) is linear and transition to the VLS model adds nonlinearity to the equations. In the classical case the system has two critical points $(X_0,Y_0)=(0,0)$ and $(X_1,Y_1)=[(K/\Lambda)^{D/2},0]$ and in general they can survive in VSL system. The first critical point is degenerate and the second one could be either center or saddle point.

One can notice that the phase-space flow generating vector field is singular (blows up to infinity) in the origin of the coordinate system (0,0). A natural way to deal with this difficulty is to introduce projective coordinates. We shall illustrate this approach for $D=2$. Let us define projective coordinates: $z=1/X$, $u=Y/X$. In these new variables the original system takes the form of autonomous dynamical system with rational right-hand sides

$$\frac{dz}{d\tau}=-uz,$$

$$\frac{du}{d\tau}=\frac{1}{z^n}(-Kz+\Lambda)-u^2. \tag{22}$$

Introducing new time parameter $dT=d\tau/z^n$ we obtain the following system, now with polynomial right-hand sides:

$$\frac{dz}{dT}=-uz^{n+1},$$

$$\frac{du}{dT}=-Kz+\Lambda-u^2z^n \tag{23}$$

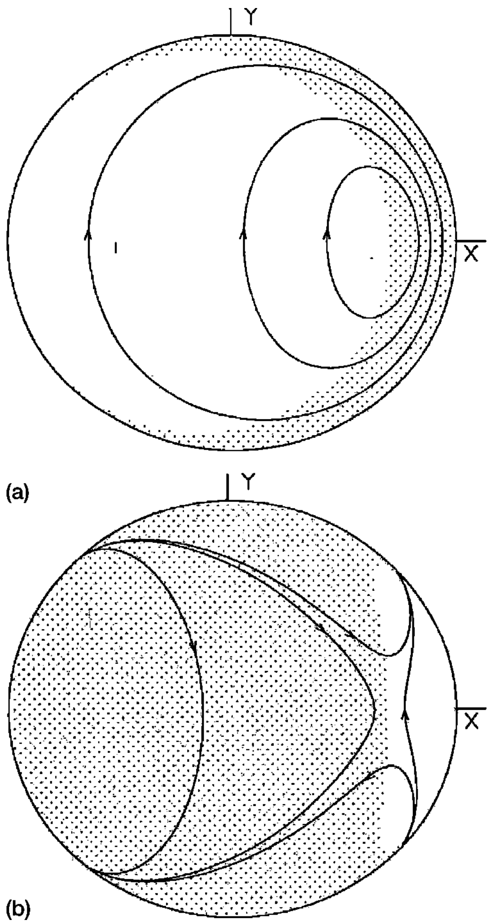

FIG. 3. (a) Phase portrait of the classical system $(n=0)$ corresponding to open FRW model with negative cosmological constant $(K=-1, \Lambda=-1)$. (b) Phase portrait of the classical system $(n=0)$ corresponding to closed FRW model with positive cosmological constant $(K=+1, \Lambda=+1)$.

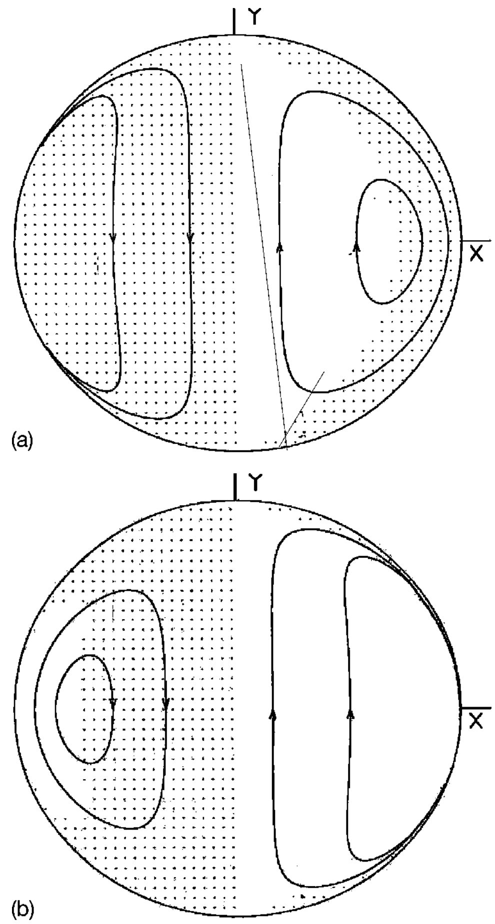

FIG. 4. (a) Phase portrait of the VSL model $(n=-1)$ corresponding to open FRW model with negative cosmological constant $(K=-1, \Lambda=-1)$. (b) Phase portrait of the VSL model $(n=-1)$ corresponding to open FRW model with positive cosmological constant $(K=-1, \Lambda=+1)$.

which is regular in the origin of the coordinate system.

The system (23) does not have critical points in the origin of coordinate system. We have one critical point $(z_0,u_0)=(\Lambda/K,0)$. Trace and determinant of linearization matrix A at this point are equal $\mathrm{Tr}\,A=-(\Lambda/K)^{n+1}$ and $\det A=K(\Lambda/K)^{n+1}$. This means that if K and Λ have the same signs then $\mathrm{Tr}\,A<0$ and $\mathrm{sgn}(\det A)=\mathrm{sgn}\,K$. Type of evolution (for $K=0$ or $K=-1$) depends on $\Delta=(\Lambda/K)^{n+1}\sqrt{1-4K}$ whereas stability of solutions is determined by the sign of $\mathrm{Tr}\,A$.

There is a great advantage of phase-space dynamical description that one is able to discuss the distribution of models with given properties. In the other words one can imagine an ansamble [19] of models starting from different initial conditions and ask questions how is given property distributed in the ansamble. We shall now formulate sufficient conditions for solving the flatness and the horizon problem in terms of phase-space relations. We understand the solution of flatness and horizon problems in a similar way as formulated in [6,5], i.e., the standard considerations are valid with the $c(t)$ function taken explicitly into account.

Let us recall that the flatness problem is solved whenever the scale factor's acceleration is positive

$$\ddot{a}(t)>0. \tag{24}$$

This condition is fulfilled in the subspace $\mathcal{D}_{\mathrm{flat}}$ of the phase space

$$\mathcal{D}_{\mathrm{flat}}=\left\{(X,Y)\colon \frac{Y^2}{2}+\frac{D}{D-1}\frac{dV(X)}{dX}<0\right\} \tag{25}$$

it means that trajectories representing the histories of VLS universes experience an accelerated expansion while staying in $\mathcal{D}_{\mathrm{flat}}$ region. Let us stress here that dotted area depicted in Figs. 3–6 denote the domain where the condition (25) is broken, i.e., $\ddot{a}\leqslant 0$ holds. One can restate the relation (25) using the Hamiltonian constraint $Y^2=2[C-V(X)]$. It is easy to see that the respective condition expressed purely in terms of configuration space, reads

$$\mathcal{D}_{\mathrm{flat}}=\left\{X\colon C-V(X)+\frac{D}{D-1}\frac{dV(X)}{dX}<0\right\}. \tag{26}$$

From Eq. (25) one obtains that the demand of

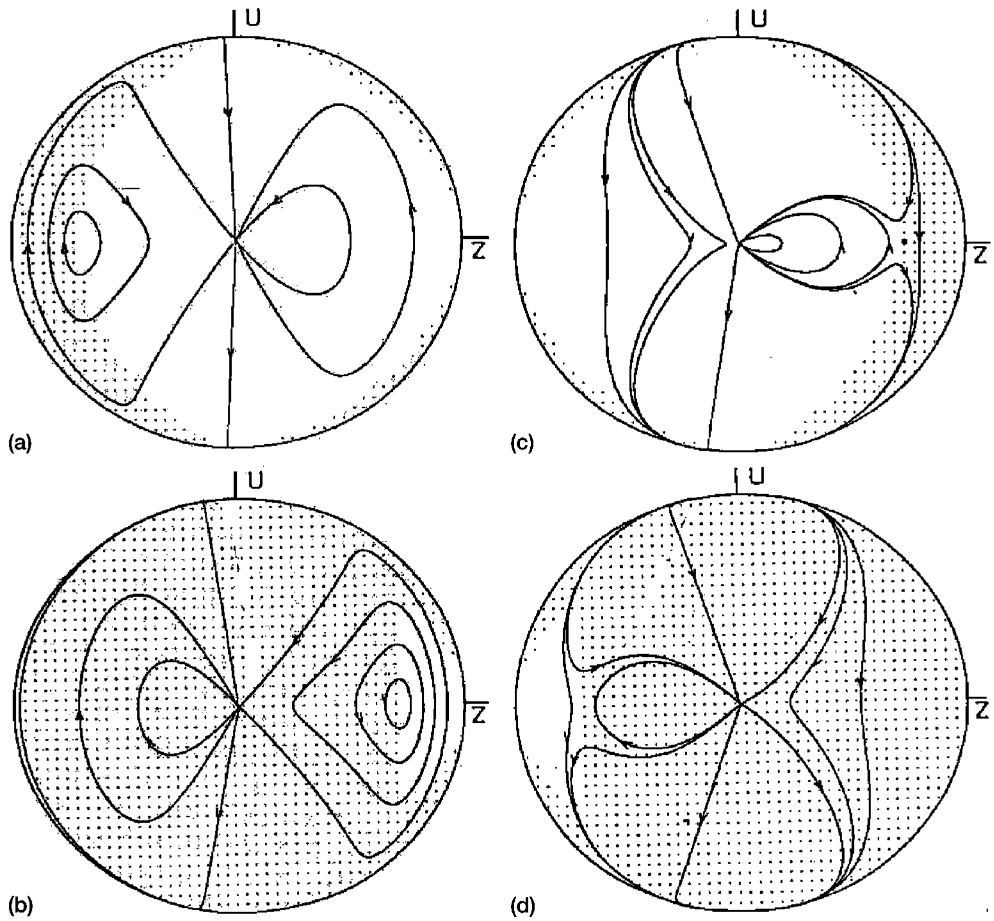

FIG. 5. (a) Phase portrait of regularized VSL model $(n=-3)$ corresponding to open FRW model with positive cosmological constant $(K=-1,\ \Lambda=+1)$. (b) Phase portrait of regularized VSL model $(n=-3)$ corresponding to open FRW model with negative cosmological constant $(K=-1,\ \Lambda=-1)$. (c) Phase portrait of regularized VSL model $(n=-3)$ corresponding to closed FRW model with positive cosmological constant $(K=+1,\ \Lambda=+1)$. (d) Phase portrait of regularized VSL model $(n=-3)$ corresponding to open FRW model with negative cosmological constant $(K=+1,\ \Lambda=-1)$.

$$\frac{dV(X)}{dX}<0, \tag{27}$$

where $V(X)$ is given by the formula (16) is the necessary but not sufficient condition for solving the flatness problem.

Another interesting question concerns the horizon problems. It is not difficult to prove the following criterion of avoiding the horizon problems in the case $D>1$.

Corollary. The FRW cosmological model does not have event horizons near the singularity if $\dot{a}(t)$ tends to a constant while $a(t)$ tends to zero. The proof is elementary and is a consequence of the observation that condition $\dot{a}<C$ implies that, as $a\rightarrow 0$, $\int_0^{a_0}(da/a)<C\int_0^{t_0}(dt/a)=C(\eta_0-\eta_{\rm sing})$ where η is conformal time. Because the left-hand side integral is divergent it means that $\eta\rightarrow-\infty$ and that there is no causally disconnected regions in this spacetime.

The above criterion can be reformulated in the language of the phase-space in the form

$$Y=C_0X^{(D-1)/D}. \tag{28}$$

For example in the VLS models with radiation $D=2$ the criterion is $Y^2/2\propto C_0X$ or equivalently $V(X)\propto(C-\gamma X)$ In general, the trajectories of models without horizons have universal behavior (28). One can distinguish the VLS models with or without horizon problems solely on the ground of their universal asymptotic behavior near singularity. Namely, if $2[C-V(X)]X^{2(1-D)/D}\rightarrow\text{const}$ while $X\rightarrow 0$ the model does not have event horizons. If $2[C-V(X)]X^{2(1-D)/D}\rightarrow\infty$ then the event horizon is present in the evolution of the model.

V. DISCUSSION OF PHASE PORTRAITS

The phase portraits of FRW VSL model with radiative matter are shown in Figs. 3–6. Shaded regions correspond to regions of the phase space for which flatness and horizon problems cannot be solved (i.e., $\ddot{a}\leqslant 0$).

The phase portraits of classical system are displayed in Fig. 3. Figure 3(a) corresponds to the open FRW model with negative cosmological constant $(K=-1,\ \Lambda=-1)$. In the physical domain there exists one critical point—a center, leading to periodic trajectories. Trajectories starting from the

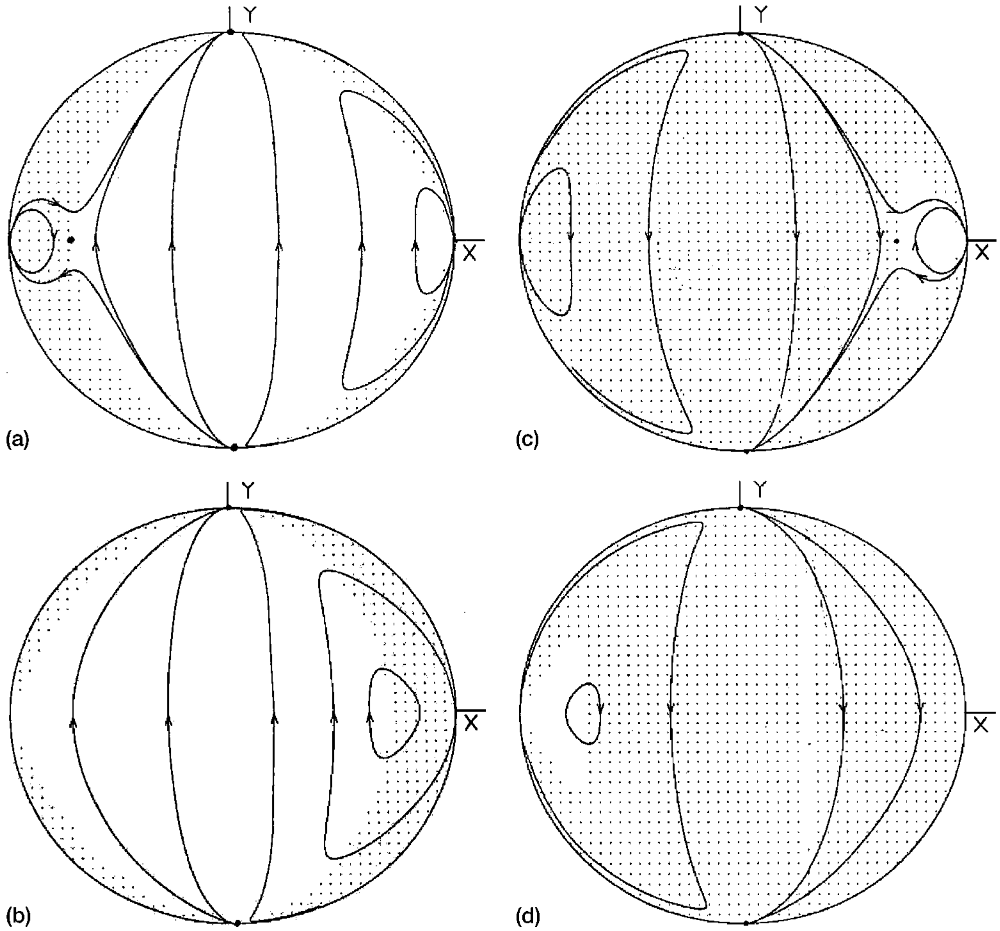

FIG. 6. (a) Phase portrait of the VSL model in (X,Y) variables for $n=-4$ corresponding to open FRW model with positive cosmological constant $(K=-1,\ \Lambda=+1)$. (b) Phase portrait of the VSL model in (X,Y) variables for $n=-4$ corresponding to open FRW model with negative cosmological constant $(K=-1,\ \Lambda=-1)$. (c) Phase portrait of the VSL model in (X,Y) variables for $n=-4$ corresponding to closed FRW model with positive cosmological constant $(K=+1,\ \Lambda=+1)$. (d) Phase portrait of the VSL model in (X,Y) variables for $n=-4$ corresponding to open FRW model with negative cosmological constant $(K=+1,\ \Lambda=-1)$.

singularity finish their evolution in the final singularity located in the same point of the phase space. At certain point $(X=X_{\max},\ Y=0)$ a maximum of the scale factor is reached. Let us note that the critical point $(X_0=K/\Lambda,\ Y=0)$ lies at the boundary of the region solving the flatness problem $\partial D_{\rm flat}$.

Figure 3(b) corresponds to the closed FRW model with positive cosmological constant $(K=+1, \Lambda=+1)$. In this case a critical point representing static Einstein solution is a saddle point located on the boundary $\partial D_{\rm flat}$. In the physical region (i.e., $X\geqslant 0, Y$, arbitrary) we have a stable node in the upper half plane $Y\geqslant 0$ or unstable node in the lower half plane $Y\leqslant 0$. Let us stress the existence of physical solutions traversing $D_{\rm flat}$ region. If $K\Lambda>0$ then the critical point lies in the physical domain $X\geqslant 0$.

In other cases (other combinations of K and Λ signs) respective phase portraits are specular reflections (with respect to OY axis) of the above ones—phase portrait of the system $K=-1$, $\Lambda=+1$ is a reflection of the portrait for $K=+1$, $\Lambda=+1$ and the portrait of the system $K=+1$, $\Lambda=-1$ is a reflection of the portrait for $K=-1$, $\Lambda=-1$

Figures 4(a) and 4(b) show the phase portraits of VSL models (corresponding to degenerate case $n=-1$) for different combinations of K and Λ. Phase portrait for the model $K=-1, \Lambda=-1$ is shown in Fig. 4(a). Comparison with classical (non-VSL) FRW model [Fig. 3(a)] makes it clear that despite of nonequivalence (in strict mathematical sense) of these phase portraits there is no qualitative difference in the behavior of trajectories in the physical domain between VSL and ordinary cosmological models. Quantitatively, in the VSL models the phase of evolution during which X changes slowly and Y rapidly, lasts longer. This phase of evolution belongs to the domain $D_{\rm flat}$ and the critical point lies at the boundary $\partial D_{\rm flat}$.

Phase portrait for the model with $K=-1$, $\Lambda=+1$ is displayed in Fig. 4(b) and is in fact a specular reflection of the portrait from Fig. 4(a) with respect to OY axis. A characteristic feature here is a long phase of fast variability of Y. Such trajectories fill the $D_{\rm flat}$ region. Critical point (center) lies in this case in nonphysical domain.

Phase portraits of regularized system are shown on Figs. 5(a)–5(d) for the VSL models with $n=-3$. Trajectories are now parametrized by a new time τ, such that $d\tau=z^n dT$. Typically there exist two critical points: $z=0$, $u=0$, and z

$=\Lambda/K,0$. The most interesting feature of the portraits is how do trajectories behave near the singularity, i.e., for $z\rightarrow\infty$. From the point of view of resolution of the flatness problem the models with negative curvature and positive cosmological constant are preferred. Closed models with negative cosmological constant do not realize the idea of flatness problem avoidance as a result of varying speed of light [Fig. 5(d)]. When the cosmological constant is positive then the idea is realized but not for $R\sim 0$. In negatively curved models, trajectories lie in the D_{flat} region even for $\Lambda<0$. However, the preferred case is that of $K=-1, \Lambda=1$ in which the D_{flat} region coincides with the physical domain. Let us also note that $u/z\approx$const as the trajectories approach critical points. It means that $Y\sim X^2$ and we obtain additionally a resolution of the horizon problem.

Figures 6(a)–6(d) represent phase portraits in X,Y variables for another case $n=-4$. Comparison of respective figures support the view that evolution of trajectories is qualitatively generic for any negative value of n. The distinguished role played by open VSL models is also apparent.

VI. CONCLUSIONS

Let us assume that one takes the idea of the varying speed of light seriously as a physical effect that might have happen in a very early universe and today is confined to a very narrow range admissible by inaccuracy of existing bounds on variability of c. One of the problems[1] arising then is to see how this (admittedly *ad hoc*) modification of physics would change the evolution of standard Friedmann–Robertson-Walker cosmological models. So far only specific qualitative results are known [5,6] concerning the solution of flatness and horizon problems in VSL models. In the present work we attempted to extend this qualitative discussion in the sense that by constructing phase-space portraits of VSL cosmological models we were able to obtain a global view of their dynamics. For this purpose we have used a power law ansatz [6] for $c(t)$ function and investigated classical Einstein equations with c allowed to be a function of time.

We reduced the dynamics of VSL models to a two-dimensional Hamiltonian dynamical system with quadratic kinetic energy form and a potential function depending on the generalized scale factor. The shape of the potential and the existence of the energy integral was used to classify possible evolutions of VLS models. The possibilities comprise models evolving from the singularity to infinity, oscillatory behavior between initial and final singularity, Einstein–de Sitter–type models evolving from the singularity to the static world, Lemaitre-Eddington-type models evolving from the static Einstein solution to infinity, models expanding to infinity from the finite size and finally models starting and ending with a finite scale factor.

We have shown that assumed time dependence of the speed of light leads to a uniform evolution pattern of VSL models on the phase space. The criteria for solving the flatness and horizon problems were formulated in terms of the phase space. It is an advantage of the phase-space approach that one can trace the patterns of evolution for all possible initial conditions. We have depicted, on respective phase portraits, the regions where the flatness problem is solved. The models where the region of initial conditions leading to flatness and horizon problem avoidance is large play a distinguished role. From this perspective open $(K=-1)$ models with positive cosmological constant $\Lambda>0$ are preferred in the class of FRW VSL models filled with radiation. The formalism presented in this paper can be easily extended to the case where the matter content of the model is a mixture of different types of matter and to the case of models with shear (e.g., Bianchi type I or V).

[1]More perhaps even more important problems are discussed, e.g., in Ref. [15].

ACKNOWLEDGMENTS

The support of the Polish State Committee for Scientific Research Grant No. 2 P03D-01417 is gratefully acknowledged.

[1] M. Turner, in *The Proceedings of the Conference Particle Physics and the Universe*, edited by D.O. Caldwell (AIP, Woodbury, 1999), astro-ph/9904051.

[2] L. Pietronero and S. Labini in *Critical Dialogues in Cosmology*, edited by Neil Turok (Prineton University Press, Princeton, 1997).

[3] E. Kolb and M. Turner, *The Early Universe* (Addison-Wesley, New York, 1990).

[4] J. Moffat, Int. J. Theor. Phys. **2**, 351 (1993); Found. Phys. **23**, 411 (1993).

[5] A. Albrecht and J. Magueijo, Phys. Rev. D **59**, 043516 (1999).

[6] J.D. Barrow, Phys. Rev. D **59**, 043515 (1999).

[7] J.D. Beckenstein, Phys. Rev. D **25**, 1527 (1982); T. Darmour and F. Dyson, Nucl. Phys. **B480**, 47 (1996); M.J. Drinkwater, J.K. Webb, J.D. Barrow, and V.V. Flambaum, Mon. Not. R. Astron. Soc. **295**, 457 (1998).

[8] J.K. Webb, V.V. Flambaum, C.W. Churchill, M.J. Drinkwater, and J.D. Barrow, Phys. Rev. Lett. **82**, 884 (1999).

[9] J.D. Barrow and J. Magueijo, ''Can changing α explain Supernovae results?,'' astro-ph/9907354.

[10] Perlmutter *et al.* Nature (London) **391**, 51 (1998); B.P. Schmidt, Astrophys. J. **507**, 46 (1998).

[11] P.P. Avelino, C.J.A.P. Martins, and G. Rocha, Phys. Lett. B **483**, 210 (2000).

[12] J.D. Barrow and J. Magueijo Class. Quantum Grav. **16**, 1435 (1999); Phys. Lett. B **443**, 104 (1998).

[13] J.W. Moffat ''Varying light velocity as a solution to the problems in cosmology,'' astro-ph/9811390; M.A. Clayton and J.W. Moffat, Phys. Lett. B **460**, 263 (1999); **477**, 269 (2000).

[14] A. Albrecht, ''Cosmology with a time varying speed of light'' COSMO98 Proceedings, edited by D. Caldwell, astro-ph/9904185; P.P. Avelino and C.J.A.P. Martins, Phys.

Lett. B **459**, 486 (1999); I.T. Drummond ''Variable light-cone theory of gravity,'' gr-qc/9908058; S.H.S. Alexander, ''On the varying speed of light in a brane-induced FRW Universe,'' hep-th/9912037.

[15] B.A. Bassett, S. Liberati, C. Molina-Paris, and M. Visser, ''Geometrodynamics of variable speed of light cosmologies;'' astro-ph/0001441.

[16] M. Szydlowski, M. Heller, and Z. Golda, Gen. Relativ. Gravit. **16**, 877 (1984).

[17] M. Szydlowski and M. Biesiada, Phys. Rev. D **41**, 2487 (1990); M. Biesiada and M. Szydlowski, Phys. Lett. A **145**, 401 (1990).

[18] H.P. Robertson, Rev. Mod. Phys. **5**, 62 (1933).

[19] C.B. Collins, Commun. Math. Phys. **39**, 131 (1974); Z. Golda, M. Heller, and M. Szydlowski, Astrophys. Space Sci. **90**, 313 (1983).