Monopoles, dyons, and black holes in the four-dimensional Einstein-Yang-Mills theory

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A continuum of monopole, dyon, and black hole solutions exists in the Einstein-Yang-Mills theory in asymptotically anti–de Sitter space. Their structure is studied in detail. The solutions are classified by non-Abelian electric and magnetic charges and the Arnowitt-Deser-Misner mass. The stability of the solutions which have no node in non-Abelian magnetic fields is established. There exist critical spacetime solutions which terminate at a finite radius, and have universal behavior. The moduli space of the solutions exhibits a fractal structure as the cosmological constant approaches zero.

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I. INTRODUCTION

For a long time, it was believed that no regular particle-like stable solutions (solitons) with finite mass can exist in self-gravitating systems unless the stability is guaranteed topologically. The Einstein theory in vacuum and the Einstein-Maxwell system do not admit solitons. It came as quite a surprise when Bartnik and McKinnon (BK) found globally regular solutions to the SU(2) Einstein-Yang-Mills (EYM) theory without scalar fields [1]. It was unexpected to find that self-gravitating Yang-Mills systems produced solitons. Unfortunately, the BK solutions were shown to be unstable against linear perturbations [2]. Later, other fields such as Higgs scalar fields and dilaton fields were included in the EYM action, but, with the exception of the Skyrmions, all turned out to be unstable (see [3] for a review).

Interest in the BK solutions was renewed with the discovery of black hole solutions to the EYM equations [4,5]. These non-Abelian black holes apparently violate the no-hair conjecture [6]. But these non-Abelian black hole solutions are also unstable, and again other fields were added in the hope of achieving stability without success (see Refs. [3], [7]) for a review).

We stress that it is a surprise that there are static solutions to the Einstein-Yang-Mills equations at all. There are no static solutions to the Yang-Mills equations in four-dimensional flat space. We can see this with a simple argument given by Deser [8]. The conservation of the canonical energy momentum tensor, $\partial_\nu T^{\mu\nu} = 0$, implies that for a static field configuration $\partial_j T_i^j = 0$. The total divergence of the quantity $x^i T_i^j$ must vanish to maintain finite energy and regularity, $\int d^{d-1}x\, \partial_j(x^i T_i^j) = 0$. But $\partial_j(x^i T_i^j) = T_j^j + x^i \partial_j T_i^j = T_j^j$ so that

$$\int d^{d-1} T_i^i = \int d^{d-1}x \left[\frac{1}{2}(5-d)F_{ij}^2 + (d-3)F_{0i}^2 \right] = 0. \tag{1}$$

Since the integrand above is positive definite for $d=4$, F_{ij} and F_{0i} must vanish. Thus there are no regular static solutions.

The argument above cannot be extended to curved space-time. The conservation law $T^{\mu\nu}_{\ ,\nu} = 0$ leads to

$$\int d^{d-1}x \sqrt{-g}\, T_j^j = -\int d^{d-1}x \sqrt{-g}\, x^k \Gamma_{k\mu\nu} T^{\mu\nu} \neq 0. \tag{2}$$

The failure of Deser's simple argument in curved space implies the possibility of having static solutions in curved space. Gravity supplies the attractive force needed to balance the repulsive force of Yang-Mills gauge interactions. Indeed, any solution to SU(2) EYM equations in asymptotically Minkowski space which is regular asymptotically is also regular for all $r > 0$ [9].

The particlelike and black hole solutions were later studied in a cosmological context. The behavior of static solutions to the Einstein-Yang-Mills equations depends considerably on the sign of the cosmological constant. The solutions can be separated into two families; $\Lambda \geq 0$ and $\Lambda < 0$. The solutions where $\Lambda = 0$ are the BK solutions. Their asymptotically de Sitter analogs ($\Lambda > 0$) were discovered independently by Volkov *et al.* and Torii *et al.* [10]. The BK solutions and the cosmological extensions all share similar behavior, and are unstable [11,12]. (See Ref. [3] for a review.) Recently, asymptotically anti–de Sitter black hole solutions [13] and soliton solutions [14,15] were found which are strikingly different from the BK-type solutions. In particular, the asymptotically anti–de Sitter (AdS) EYM equations have solutions where the field strengths are nonzero everywhere. These solutions were also shown to be stable against spherically symmetric linear perturbations. These solutions are the only EYM solutions solutions that are stable. This discovery would be very important to cosmology if the universe was ever in a phase where the cosmological constant is negative.

Another new feature of the EYM theory in AdS is the existence of dyon solutions. If $\Lambda \geq 0$ the electric part of the gauge fields is forbidden [16] if the Arnowitt-Deser-Misner (ADM) mass is to remain finite. Scalar fields must be added to the theory in order for the boundary conditions at infinity to permit the electric fields and maintain a finite ADM mass [18,19].

Recently a tremendous amount of interest has evolved in field theories in AdS space. There is the AdS/CFT (conformal field theory) correspondence [17] which states that con-

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formal field theories in d dimensions (R_d) are described in terms of supergravity or string theory on the product space of AdS_{d+1} and a compact manifold. There are intimate relations between data on the boundary R_d of AdS_{d+1} and data in the bulk AdS_{d+1}. In the present paper we are examining the Einstein-Yang-Mills theory in asymptotically AdS space. The boundary in space must be playing a crucial role for the existence of stable monopole and dyon solutions, more detailed analysis of which is, however, left for future investigation. We also note that in the three-dimensional AdS space there exist nontrivial black holes [20] and monopole-instanton solutions [21].

When the value of the cosmological constant Λ is varied, the space of monopole and dyon solutions, the moduli space, also changes. With a finite negative Λ, solutions exist in continuum. They are classified in a finite number of families, or branches. With a vanishing or positive Λ solution exists only in a discrete set, but there are infinitely many. One natural question emerging is how these finite number of branches of solutions in continuum become infinitely many discrete points as $\Lambda < 0$ approaches 0. There is a surprising hidden feature in this limit. We find a fractal structure in the moduli space, which seems to explain the transition.

In the next section the general formalism is given and the equations of motion are derived with a spherically symmetric ansatz. Conserved charges in the Yang-Mills theory is defined in Sec. III. Some general no-go theorems are derived from sum rules in Sec. IV. The new soliton solutions in asymptotically anti–de Sitter space are explained in Sec. V. The critical spacetime which has universality near the edge of the space is also examined. Black hole solutions which have both magnetic and electric non-Abelian charges are presented in Sec. VI. The dependence of the moduli space on the cosmological constant Λ is investigated in Sec. VII where the fractal structure is revealed when Λ approaches zero from the negative side. The detailed analysis of the stability of the monopole solutions is presented in Sec. VIII. The subtle boundary condition in the problem requires elaboration of the previous argument presented in the $\Lambda = 0$ and $\lambda > 0$ cases.

II. GENERAL FORMALISM

In non-Abelian gauge theory, the field equations have solutions which exhibit a magnetic charge. In the 't Hooft–Polyakov monopole solution

$$\Phi_a = \frac{x_a}{er^2}H(er), \quad A_a^0 = 0, \quad A_a^i = -\epsilon_{aij}\frac{x_j}{er^2}[1-K(er)],$$
(3)

where Φ_a is a triplet Higgs scalar field. Its stability is guaranteed by the topology of the triplet Higgs scalar field [22]. The U(1) magnetic charge takes a quantized value, $4\pi/e$. Dyon solutions were obtained [23] starting with the above ansatz (3) but with a nonzero value for A_a^0 [i.e., $A_a^0 = (x_a/er^2)J(er)$].

In this paper we look for monopole and dyon solutions in the Einstein-Yang-Mills theory without scalar fields:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}(R-2\Lambda) - \frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a\right].$$
(4)

The Einstein and Yang-Mills equations are given by

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}(R-2\Lambda) = 8\pi GT^{\mu\nu},$$

$$F^{\mu\nu}{}_{;\nu} + e[A_\mu, F^{\mu\nu}] = 0.$$
(5)

We suspect that the gravity provides an attractive force to balance the equation.

We look for spherically symmetric solutions. The metric takes the form

$$ds^2 = -\frac{H}{p^2}dt^2 + \frac{dr^2}{H} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
(6)

whereas Yang-Mills fields are given [24,25], in the regular gauge, by

$$A^{(0)} = \frac{\tau^j}{2e}\left\{A_0\frac{x_j}{r}dt + A_1\frac{x_jx_k}{r^2}dx_k + \frac{\phi_1}{r}\left(\delta_{jk} - \frac{x_jx_k}{r^2}\right)dx_k\right.$$

$$\left. - \epsilon_{jkl}\frac{1-\phi_2}{r^2}x_kdx_l\right\}.$$
(7)

Here the Cartesian coordinate x^k's are related to the polar coordinates (r, θ, ϕ) as in the flat space. H, p, A_0, A_1, ϕ_1, and ϕ_2 are functions of r for monopole or dyon solutions. In the discussion of the stability of the solutions they depend on both t and r. The regularity of solutions at the origin demands that H, p are finite, whereas A_0, A_1, and $\phi_1 \to 0$ and $\phi_2 \to 1$ at $r = 0$.

A. Simplification of the static gauge field ansatz

Let $A = A_\mu dx^\mu = \frac{1}{2}\tau^a A_\mu^a dx^\mu$, where τ^α are the usual Pauli matrices. In terms of the basis in spherical coordinates $(\tau_r, \tau_\theta, \tau_\phi) = (\vec{n}_r, \vec{n}_\theta, \vec{n}_\phi)\vec{\tau}$, which satisfies $[\tau_i, \tau_j] = 2i\epsilon_{ijk}\tau_k$ ($i = r, \theta, \phi$), the ansatz (7) is written as

$$A^{(0)} = \frac{1}{2e}[A_0\tau_rdt + A_1\tau_rdr + (\phi_1\tau_\theta + (\phi_2-1)\tau_\phi)d\theta$$

$$+ (-(\phi_2-1)\tau_\theta + \phi_1\tau_\phi)\sin\theta d\phi].$$
(8)

Note that there are no singularities in this gauge. Next make a gauge transformation $A = SA^{(0)}S^{-1} - (i/e)dS \cdot S^{-1}$ where

$$S = \begin{pmatrix} +e^{i(\phi+\Omega)/2}\cos\dfrac{\theta}{2} & +e^{-i(\phi-\Omega)/2}\sin\dfrac{\theta}{2} \\ -e^{i(\phi-\Omega)/2}\sin\dfrac{\theta}{2} & +e^{-i(\phi+\Omega)/2}\cos\dfrac{\theta}{2} \end{pmatrix},$$

$$\Omega = \Omega(t,r).$$
(9)

Useful identities are

$$S\tau_r S^{-1} = \tau_3,$$

$$S\tau_\theta S^{-1} = \cos\Omega\,\tau_1 - \sin\Omega\,\tau_2,$$

$$S\tau_\phi S^{-1} = \sin\Omega\,\tau_1 + \cos\Omega\,\tau_2,$$

$$
2idS\cdot S^{-1} = -(\Omega'dr + \dot\Omega dt)\tau_3 - d\theta(\sin\Omega\,\tau_1 + \cos\Omega\,\tau_2)
$$
$$
+ d\phi(\sin\theta\cos\Omega\,\tau_1 - \sin\theta\sin\Omega\,\tau_2
$$
$$
- \cos\theta\,\tau_3). \tag{10}
$$

The new gauge potential is

$$
A = \frac{1}{2e}\{u\tau_3 dt + v\tau_3 dr + (w\tau_1 + \widetilde{w}\tau_2)d\theta
$$
$$
+ (\cot\theta\,\tau_3 + w\tau_2 - \widetilde{w}\tau_1)\sin\theta\,d\phi\}, \tag{11}
$$

where

$$u = A_0 + \Omega,$$

$$v = A_1 + \Omega',$$

$$w = +\phi_1\cos\Omega + \phi_2\sin\Omega,$$

$$\widetilde{w} = -\phi_1\sin\Omega + \phi_2\cos\Omega. \tag{12}$$

Note that the gauge transformation (9) is singular at $\theta=0$ and π. Equation (12) is the gauge potential in the singular gauge. It has a Dirac string. One can always choose $\Omega(t,r=0)=\pi/2$ with which the boundary conditions at $r=0$ are $u=v=\widetilde{w}=0$ and $w=1$. With appropriate $\Omega(t,r)$ one can set $v(t,r)=0$ or $u(t,r)=0$.

A straightforward calculation leads to the field strength $F=dA-ieA\wedge A$:

$$
F = \frac{1}{2e}\{(\dot v - u')\tau_3 dt\wedge dr + [(\dot w - u\widetilde{w})\tau_1 + (\dot{\widetilde{w}} + uw)\tau_2]dt\wedge d\theta
$$
$$
- [(uw + \dot{\widetilde{w}})\tau_1 + (u\widetilde{w} - \dot w)\tau_2]dt\wedge\sin\theta d\phi + [(w' - v\widetilde{w})\tau_1 + (\widetilde{w}' + wv)\tau_2]dr\wedge d\theta
$$
$$
+ [(w' - v\widetilde{w})\tau_2 + (-\widetilde{w}' - vw)\tau_1]dr\wedge\sin\theta d\phi - (1 - w^2 - \widetilde{w}^2)\tau_3 d\theta\wedge\sin\theta d\phi\}. \tag{13}
$$

The configurations where $v=u=0$, $w=\widetilde{w}=$const, and $w^2 + \widetilde{w}^2 = 1$ are pure gauge.

B. Equations of motion

In the general spherically symmetric metric (6) tetrads are

$$
e_0 = \frac{\sqrt{H}}{p}dt, \quad e_1 = \frac{1}{\sqrt{H}}dr, \quad e_2 = rd\theta, \quad e_3 = r\sin\theta d\phi. \tag{14}
$$

In the tetrad basis $F_{ab} = (e_a)_\mu(e_b)_\nu F^{\mu\nu}$ and the energy-momentum tensors are $T_{ab} = F_{ac}^{(i)}F_b^{c(i)} - \frac{1}{4}\eta_{ab}F_{de}^{(i)}F^{de(i)}$. The nonvanishing components of the Yang-Mills equations (5) are

$$
[pr^2(u' - \dot v)]' - 2\frac{p}{H}\{w(uw + \dot{\widetilde{w}}) + \widetilde{w}(u\widetilde{w} - \dot w)\} = 0,
$$

$$
[pr^2(u' - \dot v)]_{,t} - 2\frac{H}{p}\{-\widetilde{w}w' + \widetilde{w}'w + v(w^2 + \widetilde{w}^2)\} = 0,
$$

$$
\left(\frac{H}{p}(w' - \widetilde{w}v)\right)' - \left(\frac{p}{H}(\dot w - u\widetilde{w})\right)_{,t} + \frac{p}{H}u(uw + \dot{\widetilde{w}})
$$
$$
+ \frac{w(1 - w^2 - \widetilde{w}^2)}{pr^2} - \frac{H}{p}v(\widetilde{w}' + wv) = 0,
$$

$$
\left(\frac{H}{p}(\widetilde{w}' + wv)\right)' - \left(\frac{p}{H}(\dot{\widetilde{w}} + uw)\right)_{,t} + \frac{p}{H}u(u\widetilde{w} - \dot w)
$$
$$
+ \frac{\widetilde{w}(1 - w^2 - \widetilde{w}^2)}{pr^2} + \frac{H}{p}v(w' - \widetilde{w}v) = 0. \tag{15}
$$

The nonvanishing components of the energy-momentum tensor are given by

$$T_{00} = \frac{1}{e^2}(A + B),$$

$$T_{11} = \frac{1}{e^2}(-A + B),$$

$$T_{22} = T_{33} = \frac{1}{e^2}A,$$

$$T_{01} = -\frac{1}{e^2}C, \tag{16}$$

where

$$A = \frac{1}{2}p^2(\dot v - u')^2 + \frac{1}{2r^4}(1 - w^2 - \widetilde{w}^2)^2,$$

$$B = \frac{p^2}{r^2 H}\{(uw+\dot{\tilde{w}})^2 + (\dot{w}-u\tilde{w})^2\} + \frac{H}{r^2}\{(w'-\nu\tilde{w})^2$$

$$+ (\tilde{w}'+\nu w)^2\},$$

$$C = \frac{2p}{r^2}\{(\tilde{w}'+\nu w)(uw+\dot{\tilde{w}}) + (w'-\nu\tilde{w})(\dot{w}-u\tilde{w})\}. \quad (17)$$

The Einstein equations reduce to

$$\frac{p'}{p} = -\frac{8\pi G}{e^2}\frac{rB}{H} - \frac{H'}{H} + \frac{1-H}{r^2} = \frac{8\pi G}{e^2}(A+B)+\Lambda,$$

$$\frac{p}{2}\left\{\left(\frac{p\dot{H}}{H^2}\right)_{,t} + \left(\frac{pH'-2p'H}{p^2}\right)'\right\} + \frac{1-H}{r^2} = \frac{16\pi G}{e^2}A,$$

$$\frac{p\dot{H}}{rH} = -\frac{8\pi G}{e^2}C. \quad (18)$$

It is convenient to introduce $m(r)$ defined by

$$H(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}. \quad (19)$$

$m(r)$ is the mass contained inside the radius r. $p(r) = \text{const}$ and $m(r) = 0$ correspond to the Minkowski, de Sitter, or anti–de Sitter space for $\Lambda = 0$, > 0, or < 0, respectively. Then the second equation in Eq. (18) becomes

$$m' = \frac{4\pi G}{e^2}r^2(A+B). \quad (20)$$

The system of the Einstein-Yang-Mills equations contains one redundant equation. The third equation in Eq. (18) follows from Eq. (15) and the rest of Eq. (18).

C. Static configurations

It is most convenient to take the $\nu = 0$ gauge for static configurations. The second equation in Eq. (15) then yields $w\tilde{w}' - w'\tilde{w} = 0$, which leads to $\tilde{w}(r) = Cw(r)$. By a further global rotation $\Omega = \text{const}$ in Eq. (12), one can set $\tilde{w} = 0$. As a result

$$A = \frac{1}{2e}\{u\tau_3 dt + w\tau_1 d\theta + (\cot\theta\tau_3 + w\tau_2)\sin\theta d\phi\}$$

$$F = \frac{1}{2e}\{-u'\tau_3 dt\wedge dr + uw dt\wedge(\tau_2 d\theta - \tau_1\sin\theta d\phi)$$

$$+ w' dr\wedge(\tau_1 d\theta + \tau_2\sin\theta d\phi) - (1-w^2)$$

$$\times\tau_3 d\theta\wedge\sin\theta d\phi\}. \quad (21)$$

Then the Einstein-Yang-Mills equations are

$$\left(\frac{H}{p}w'\right)' = -\frac{p}{H}u^2 w - \frac{w}{p}\frac{(1-w^2)}{r^2}, \quad (22)$$

$$(r^2 pu')' = \frac{2p}{H}w^2 u, \quad (23)$$

$$p' = -\frac{2v}{r}p\left[(w')^2 + \frac{u^2 w^2 p^2}{H^2}\right], \quad (24)$$

$$m' = v\left[\frac{(w^2-1)^2}{2r^2} + \frac{1}{2}r^2 p^2(u')^2\right.$$

$$\left. + H(w')^2 + \frac{u^2 w^2 p^2}{H}\right], \quad (25)$$

where $v = 4\pi G/e^2$.

These equations are solved with the given boundary conditions. Near the origin solutions must be regular so that

$$u(r) = ar + \frac{a}{5}\left\{-2b + \frac{1}{3}\Lambda + 2v(a^2+4b^2)\right\}r^3,$$

$$w(r) = 1 - br^2,$$

$$m(r) = \frac{1}{2}v(a^2+4b^2)r^3,$$

$$p(r) = 1 - v(a^2+4b^2)r^2, \quad (26)$$

where a and b are arbitrary constants. The boundary conditions at the origin of the EYM equations are completely determined by the values of the constants a and b.

At space infinity the energy-momentum tensors T_{ab} in Eq. (16) must approach zero sufficiently fast. Further we expect that the metric must be asymptotically (anti–) de Sitter space, depending on the value of Λ. This, with the equations of motion, leads to the asymptotic expansion at large r;

$$u = u_0 + u_1\frac{1}{r} + \cdots, \quad w = w_0 + w_1\frac{1}{2} + \cdots,$$

$$m = M + m_1\frac{1}{r} + \cdots, \quad p = p_0 + p_4\frac{1}{r^4} + \cdots, \quad (27)$$

where u_0, u_1, w_0, w_1, m_1, p_0, and p_4 are constants to be determined and M is the ADM mass, $M = m(\infty) - m(0)$.

III. CONSERVED CHARGES

Solutions to Eqs. (22)–(25) are classified by the ADM mass, $M = m(\infty)$, electric and magnetic charges, Q_E and Q_M. From the Gauss flux theorem

$$\begin{pmatrix} Q_E \\ Q_M \end{pmatrix} = \frac{e}{4\pi}\int dS_k\sqrt{-g}\begin{pmatrix} F^{k0} \\ \tilde{F}^{k0} \end{pmatrix} \quad (28)$$

are conserved. With the ansatz in the singular gauge (11) and the asympotitic behavior (27), the charges are given by

$$\begin{pmatrix} Q_E \\ Q_M \end{pmatrix} = \begin{pmatrix} u_1 p_0 \\ 1-w_0^2 \end{pmatrix}\frac{\tau_3}{2}. \quad (29)$$

Notice that the electric charge Q_E is determined by u_1, whereas the magnetic charge Q_M by w_0. If (u,w,m,p) is a solution, then $(-u,w,m,p)$ is also a solution. Dyon solutions come in a pair with $(\pm Q_E, Q_M, M)$.

The charges (28) are not gauge invariant, however. Under a local gauge transformation $A \rightarrow UAU^{-1} - (i/e)dUU^{-1}$, Q_E and Q_M are transformed to

$$\begin{pmatrix} Q_E{}^U \\ Q_M{}^U \end{pmatrix} = \frac{e}{4\pi} \int dS_k \sqrt{-g}\, U(x) \begin{pmatrix} F^{k0} \\ \tilde{F}^{k0} \end{pmatrix} U^{-1}(x). \quad (30)$$

In non-Abelian gauge theory a set of charges $\{Q_E{}^U, Q_M{}^U\}$ are conserved. In the rest of the paper we use the charges, Eq. (29), defined in the singular gauge.

The effective charge Q_{eff} [1] is defined by the asymptotic behavior of $H(r)$;

$$H(r) = 1 - \frac{2M}{r} + \frac{Q_{\text{eff}}^2}{r^2} - \frac{1}{3}\Lambda r^2. \quad (31)$$

In terms of the coefficients in Eq. (27), $Q_{\text{eff}}^2 = -2m_1$. This requires that $m_1 < 0$ which indeed is the case. After inserting Eq. (27) into Eqs. (22) and (25) we find the relation

$$Q_{\text{eff}}^2 = 2v\,\text{Tr}(Q_E^2 + Q_M^2) - \frac{4\Lambda}{3}\frac{p_4}{p_0}. \quad (32)$$

Equation (24) implies that $p(r)$ is a monotonically decreasing positive function so that $p_0 > 0$ and $p_4 > 0$. The effective charge is smaller (larger) than $2v\,\text{Tr}(Q_E^2 + Q_M^2)$ for $\Lambda > 0$ (<0). The relation (32) incidentally implies that the charges defined in the singular gauge have a physical, gauge-invariant meaning.

IV. SUM RULES

Sum rules are obtained from the equations of motion. First, multiply both sides of Eq. (23) by u and integrate in part:

$$pr^2 uu'\big|_{r_1}^{r_2} = \int_{r_1}^{r_2} dr \left\{ r^2 p(u')^2 + 2\frac{p}{H}u^2 w^2 \right\}. \quad (33)$$

Secondly, multiply both sides of Eq. (22) by w and integrate in part:

$$\frac{H}{p}ww'\big|_{r_1}^{r_2} = \int_{r_1}^{r_2} dr \left\{ \frac{H}{p}(w')^2 - \frac{p}{H}u^2 w^2 - \frac{1}{pr^2}w^2(1-w^2) \right\}. \quad (34)$$

Thirdly, divide both sides of Eq. (22) by w and integrate in part:

$$-\frac{Hw'}{pw}\bigg|_{r_1}^{r_2} = \int_{r_1}^{r_2} dr \left\{ \frac{p}{H}u^2 + \frac{H}{p}\left(\frac{w'}{w}\right)^2 + \frac{1}{pr^2}(1-w^2) \right\}. \quad (35)$$

These relations are valid, provided the integrals on the right-hand sides are defined. Several important conclusions follow from Eqs. (33)–(35).

A. In asymptotically flat space

Consider Eq. (33) with $r_1 = 0$ and $r_2 = \infty$. For regular solutions $u(0) = 0$. Both p and H approach constant as $r \rightarrow \infty$. The finiteness of the ADM mass requires that $uw|_{r=\infty} = 0$. In the expansion (27), $u_0 w_0 = 0$. On the other hand, if $u_0 \neq 0$, $w_0 = 0$ and Eq. (23) implies $w_1 \neq 0$ so that Eq. (22) cannot be satisfied. Hence, $u(\infty) = 0$. Then the left-hand side (lhs) of Eq. (33) vanishes, implying that $u(r)$ must vanish identically. There is no regular electrically charged solution. Furthermore, Eq. (22) can be solved only if $(w_0)^2 = 1$ as $H(\infty) = 1$, therefore the magnetic charge Q_M vanishes.

Suppose that $w(r)$ never vanishes and $w^2 \leqslant 1$ for $0 < r < \infty$. Consider Eq. (35) with $r_1 = 0$ and $r_2 = \infty$. The lhs vanishes, but the integrand on the rhs is positive definite except for the pure gauge configuration $w(r) = \pm 1$. This implies that nontrivial solutions with $w^2 \leqslant 1$ must vanish at least once.

We also note that the singular solution $w(r) = 0$, $u'(r) = r^{-2}$, and $p(r) = 1$ is nothing but the Reissner-Nordström (RN) solution.

B. In asymptotically de Sitter space

In asymptotically de Sitter space $H(r) \rightarrow -\Lambda r^2/3$ as $r \rightarrow \infty$. In this case the finiteness of the ADM mass does not forbid nonvanishing uw at $r = \infty$. However, there arises a cosmological horizon at $r = r_h$, where $H(r_h) = 0$.

It follows from Eqs. (24) and (25) that u or w must vanish at $r = r_h$. Now consider Eq. (35). Suppose that $w(r_h) \neq 0$, or equivalently $w \neq 0$ for $r_1 \leqslant r \leqslant r_2 = r_h$ for some $r_1 > 0$, the left-hand side is finite so that $u(r_h) = 0$. However, Eq. (23) implies that $u(r)$ is a monotonically increasing or decreasing function. With the boundary condition $u(0) = 0$, the only possibility available is $u(r) = 0$. This conclusion remains valid even if $w(r_h) = 0$. In this case the first and second terms on the rhs give positively divergent contributions near r_h, whereas the lhs remains finite. To summarize, there is no solution with nonvanishing $u(r)$.

If $w > 0$ for $0 \leqslant r \leqslant r_h$, then the left-hand side vanishes in the $r_1 \rightarrow 0$ and $r_2 \rightarrow r_h$ limit. If one further assumes that $w^2 \leqslant 1$ in the interval, the integrand on the right-hand side is positive definite so that the only solution is $w(r) = \pm 1$, which is a pure gauge.

This argument also shows that a nontrivial solution $w(r)$, which satisfies $w^2 \leqslant 1$ for $0 \leqslant r \leqslant r_h$, must vanish at least once in this interval. We have numerically looked for solutions in which $w^2 \geqslant 1$ for small r, but have found that no such solution exists.

C. In asymptotically anti−de Sitter space

In asymptotically anti–de Sitter space there exist solutions in which $H(r) > 0$ everywhere. As $H(r) \sim |\Lambda| r^2/3$ for large r, the condition for the finiteness of the ADM mass requires only that $uw \rightarrow$ const. In other words, both u and w

may approach nonvanishing values as $r \rightarrow \infty$. Consequently, with the expansion (27) the lhs of Eq. (33) with $r_1 = 0$ and $r_2 = \infty$ is $-p_0 u_0 u_1$ and can be nonvanishing. Solutions with $u(r) \neq 0$ are allowed.

In critical cases a cosmological horizon appears and $H(r_h) = 0$. In this case the argument above for the asymptotically de Sitter case applies and either u or w must vanish at $r = r_h$. We shall find, indeed, $w(r_h) = 0$ below.

V. SOLITON SOLUTIONS

A. The BK solution

Particlelike solutions of the EYM equations in asymptotically Minkowski space were first found by Bartnik and McKinnon (BK) [1] in 1988. When $u = 0$, we can solve Eqs. (22)–(25) numerically with the boundary condition $a = 0$.

As already discussed, $w(r) = 1$ and $H(r) = 1$ corresponds to a pure gauge configuration. Similarly, if $w = 0$ as $r \rightarrow \infty$ we are left with the RN solution which is singular at $r = 1$. The Yang-Mills charge (29) vanishes in asymptotically Minkowski space since $w_0 = \pm 1$. The effective charge $Q_{\text{eff}} = 0$ in all BK solutions, since $Q_M = \Lambda = 0$. All solutions possess a metric that is asymptotically Schwarzschild.

There is a discrete set of BK solutions, labeled by the number of nodes in w, $n \in [1, \infty)$, and the free shooting parameter b. Since all BK solutions have at least one node, $n \geq 1$, the solutions are unstable against spherically symmetric perturbations [2].

B. Solutions with a cosmological horizon

Solutions in asymptotically de Sitter space were obtained by adding a cosmological constant (Λ) term to the Einstein equations. Solutions display the same basic properties as the BK solutions [10]. Just as in the BK solution, these equations are solved numerically, using the shooting method and requiring $a = 0$.

In asymptotically de Sitter space, a cosmological horizon, where $H = 0$, develops at $r = r_h$. At the horizon $H'(r_h) \neq 0$. Near the horizon

$$w(r) = w_0 + w_1 x + w_2 x^2 + \cdots,$$

$$p(r) = p_0 + p_1 x + P_2 x^2 + \cdots,$$

$$m(r) = m_0 + m_1 x + m_2 x^2 + \cdots,$$

$$H(r) = h_1 x + h_2 x^2 + \cdots, \quad (36)$$

where $x = r - r_h$. m_0 and r_h are related by $1 - (2m_0/r_h) - (\Lambda r_h^2/3) = 0$. With given w_0, r_h, and p_0, Eqs. (22), (24), and (25) (with $u = 0$) determine all other coefficients, provided $w_0 \neq 0$, ± 1 and $p_0 \neq 0$;

$$m_1 = \frac{v}{2r_h^2}(w_0^2 - 1)^2,$$

$$h_1 = -\frac{2m_1}{r_h} + \frac{2m_0}{r_h^2} - \frac{2}{3}\Lambda r_h,$$

$$w_1 = -\frac{w_0(1 - w_0^2)}{h_1 r_h^2},$$

$$p_1 = -\frac{2v w_1^2 p_0}{r_h}, \quad (37)$$

and so on. This expansion is valid independent of the value of Λ. The critical case $h_1 = H'(r_h) = 0$ requires a special treatment, and will be analyzed in Sec. V C. Künzle and Masood-ul-Alam [5] have argued that $w(r)$ has $\sqrt{|r - r_h|}$ singularity at $r = r_h$. However, we have found that the regular expansion (36) is valid.

Just like the BK case, there are a discrete set of solutions labeled by the number of nodes in $w(r)$, n and the parameter b. Solutions in w and m, have the same form as the BK solutions except that $w(r)$ no longer approaches 1 at infinity. Equation (32) implies that there is a nonvanishing charge Q_M (or $w \neq 1$) if $\Lambda \neq 0$ and that the solutions are all asymptotically the Reissner-Nordström type. $|w(\infty)|$ is slightly greater than 1 for the $n = 1$ solutions. When $n \geq 2$, $1 > w(\infty) > 0$, where $w(\infty) \rightarrow 0$ as $n \rightarrow \infty$.

The mass $m(r)$ also stays finite. For all indices n, it is small near the origin and does not grow until it approaches the horizon where it quickly climbs to a value near 1. After the horizon it stays almost constant.

The position of the horizon depends on Λ. As long Λ is below some critical value, the geometry approaches Reissner–Nordström–de Sitter space in the asymptotic region as indicated by Eq. (32). Above some critical value the topology changes and a singularity appears. Since the position of the horizon depends on Λ, there is a value for Λ where this singularity and the horizon meet, in which the topology becomes a completely regular manifold. Above this value for Λ, the solutions are no longer regular. More details of the topology dependence on Λ can be found in [10]. All the solutions are unstable [12].

C. Solutions in asymptotically anti−de Sitter space

As already discussed, there are no boundary conditions that forbid a solution to the EYM equations in asymptotically anti–de Sitter space with a nonzero electric component $u(r)$, to the Yang-Mills fields. Solutions to Eqs. (22)–(25) are determined with the cosmological constant Λ fixed at some negative value.

1. Monopole solutions

Monopole solutions are obtained by setting $a = 0$ ($u = 0$). By varying the initial condition parameter b, a continuum of monopole solutions are found which are regular in the entire space. Just as in the BK and dS solutions, w crosses the axis an arbitrary number of times depending on the value of the adjustable shooting parameter b. In contrast to the $\Lambda = 0$ and $\Lambda > 0$ cases which have a discrete set of solutions in b and n, there is a continuum of solutions in b for each n. Typical solutions are displayed in Fig. 1.

The behavior of m and p is similar to that of the asymptotically dS solutions [10]. In contrast, as shown in Fig. 1,
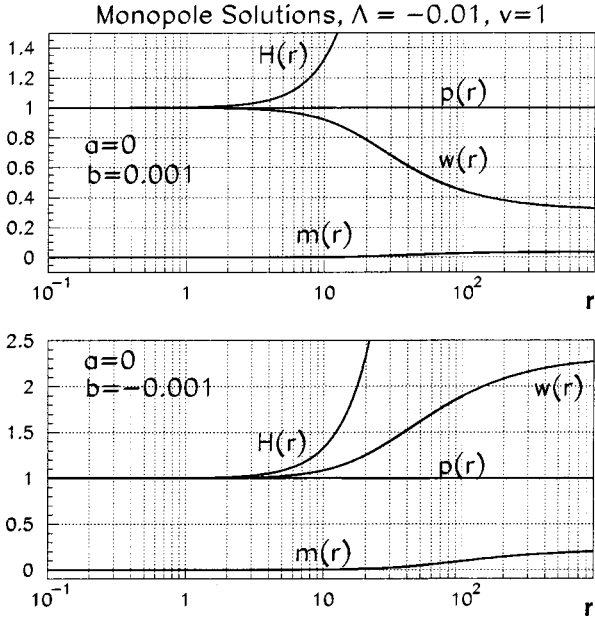
FIG. 1. Monopole solutions for $\Lambda = -0.01$ and $v = 1$. $(a,b) = (0,0.001)$ and $(0, -0.001)$. (w, m) at $r = \infty$ are $(0.318, 0.035)$ and $(2.304, 0.201)$.

there exist solutions where w has no nodes. These solutions are of particular interest because they are shown to be stable against linear perturbations.

2. Dyon solutions

Dyon solutions to the EYM equations are determined if the adjustable shooting parameter a is chosen to be nonzero for a given negative Λ. Just as in the monopole solutions, we find a continuum of solutions where w crosses the axis an arbitrary number of times depending on a and b. Also similar to the monopole solutions is the existence of solutions where w does not cross the axis. As shown in Fig. 2, the electric component u of the EYM equations starts at zero and monotonically increases to some finite value. The behavior of w, m, H, and p is similar to that in the monopole solutions.

Just as for the monopole case, dyon solutions are found for a continuous set of parameters, a and b. For some values of a and b, solutions blow up, or the function $H(r)$ crosses the axis and becomes negative.

3. Critical solutions

As the parameter b is increased, the minimum of $H(r)$ hits zero from above, i.e., $H(r_h) = H'(r_h) = 0$. This constitutes a special case and needs careful examination. Numerical studies indicate that this happens in a finite range of the parameter a. The critical solution exists for both the $u(r) = 0$ and $u(r) \neq 0$ cases. One example of solutions near the critical value $[(a,b) = (0.01, 0.69)]$ is displayed in Fig. 3. $H(r)$ becomes very close to zero at $r \sim 1$. It has $(Q_E, Q_M, M) \sim (0.015, 0.998, 0.995)$.

When $b = b_c$, w and p vanish at $r = r_h$ as well. As $p(r_h) = 0$, $p(r) = 0$ for $r \geq r_h$. The space ends at $r = r_h$. There is universality at the critical point.
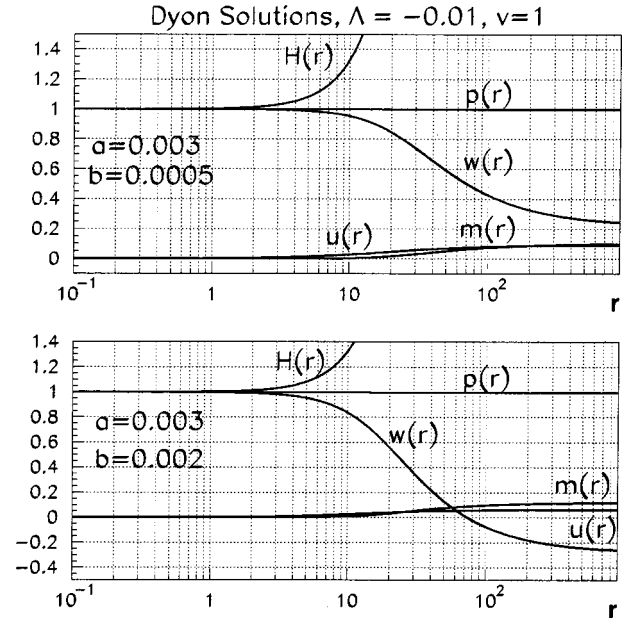

FIG. 2. Dyon solutions for $\Lambda = -0.01$ and $v = 1$. In the top figure $w(r)$ has no node $(n = 0)$, whereas in the bottom figure it has one node $(n = 1)$.

The numerical integration of the differential equations indicates that $m(r)$ and $u(r)$ are regular at $r = r_h$. The appropriate ansatz for the critical solutions with $H(r_h) = H'(r_h) = 0$ is, for $y = r_h - r \geq 0$,

$$u = u_0 + u_1 y + u_2 y^2 + \cdots,$$

$$w = y^\alpha \{ w_0 + w_1 y + w_2 y^2 + \cdots \},$$

$$p = y^\beta \{ p_0 + p_1 y + p_2 y^2 + \cdots \},$$

$$H = y^\gamma \{ h_0 + h_1 y + h_2 y^2 + \cdots \},$$
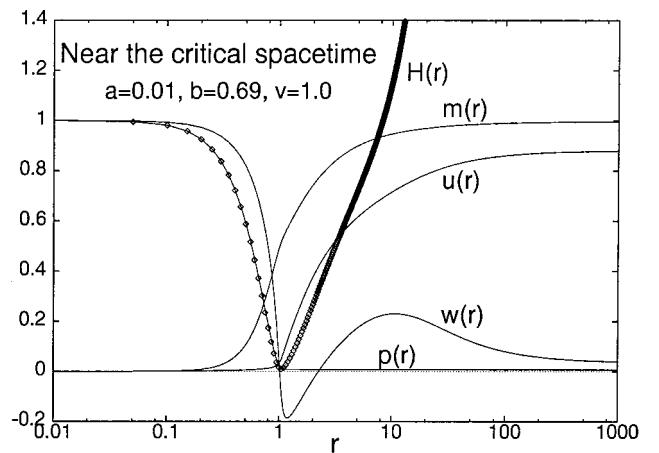
$$m = m_0 + m_1 y + m_2 y^2 + \cdots. \tag{38}$$

FIG. 3. Dyon solution for $\Lambda = -0.01$, $v = 1$, $a = 0.01$, and $b = 0.69$. At $a = 0.01$, the critical value is $b_c = 0.7104$. H almost hits the axis around $r = 1$. At b_c, $p(r) = 0$ for $r \geq r_h$. The space ends at r_h.

Equation (22) implies that $\gamma = 2$. When $u(r) \neq 0$, Eq. (23) leads to $\alpha = \frac{1}{2}$. When $u(r) = 0$, Eq. (24), instead, implies that $\alpha = \frac{1}{2}$. Other relations obtained from Eqs. (22)–(25) are

$$\left(\beta - \frac{3}{2} \right) h_0 = \frac{2}{r_h^2},$$

$$\beta r_h^2 u_1 = \frac{2 w_0^2 u_0}{h_0},$$

$$\beta = \frac{v w_0^2}{2 r_h},$$

$$m_1 = -\frac{v}{2 r_h^2}. \tag{39}$$

The value of the index β is unconstrained when $u = 0$. However, if $u \neq 0$, the consistency of Eq. (22), for instance, demands that 2β be an integer. The smallest value for β which satisfies the first relation in Eq. (39) is $\beta = 2$, as $h_0 > 0$. We have confirmed this by numerical studies. The relation (19) further implies that

$$\frac{2 m_0}{r_h} = 1 - \frac{\Lambda}{3} r_h^2,$$

$$m_1 = \frac{1}{2} (\Lambda r_h^2 - 1). \tag{40}$$

From the two relations for m_1, one in Eq. (39) and the other in Eq. (40), r_h is determined as a function of v and $\Lambda < 0$:

$$r_h^2 = \frac{1}{2|\Lambda|} (\sqrt{1 + 4v|\Lambda|} - 1). \tag{41}$$

To summarize, the indices in Eq. (38) are given by

$$\alpha = \frac{1}{2}, \quad \beta = \gamma = 2. \tag{42}$$

The coefficients m_0, m_1, and

$$w_0^2 = \frac{4 r_h}{v}, \quad h_0 = \frac{4}{r_h^2} \tag{43}$$

are all determined by v and Λ only. We are observing the universality in the behavior of the critical solutions. The coefficients u_0 and p_0 depend on a or b as well.

For small $v|\Lambda| \ll 1$

$$r_h \sim \sqrt{v}, \quad m_0 \sim \frac{1}{2} \sqrt{v}, \quad m_1 \sim -\frac{1}{2},$$

$$w_0 \sim \frac{2}{v^{1/4}}, \quad h_0 \sim \frac{4}{v}. \tag{44}$$

They are all determined by v only. This universal behavior is clearly seen in the solution in Fig. 3 which is very close to the critical one.
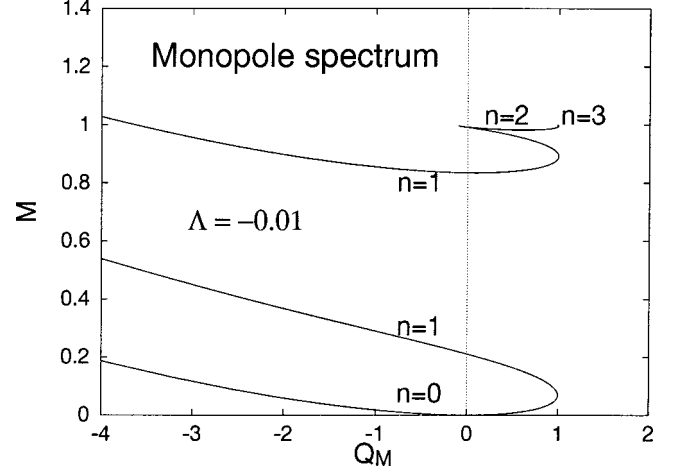

FIG. 4. Mass M is plotted as a function of magnetic charge Q_M for monopole solutions at $\Lambda = -0.01$ and $v = 1$. The number of nodes n in $w(r)$ is also marked. In the lower branch $Q_M = 1$ at $b = 0.00168$.

The meaning of the critical spacetime is yet to be clarified. The space ends at $r = r_h$. It defines a spacetime with a boundary.

4. Spectrum of monopole and dyon solutions

Monopole and dyon solutions permit nonvanishing charges Q_M and Q_E, although there are solutions where $Q_M = 0$ and $Q_E \neq 0$ or where $Q_E = 0$ but $Q_M \neq 0$. Nonzero charges Q_M or Q_E ensures that $Q_{\text{eff}} \neq 0$ [see Eq. (32)] so that solutions are asymptotically of the AdS Reissner-Nordström type.

In Fig. 4 the mass M is plotted as a function of Q_M for monopole solutions at $\Lambda = -0.01$ and $v = 1$. The behavior of the solutions near to $b = b_c = 0.7104$ needs more careful analysis.

Dyon solutions are found in a good portion of the Q_E-Q_M plane. There are solutions with $Q_M = 0$ but $Q_E \neq 0$. Although $Q_M = 0$, i.e., $w(\infty) = \pm 1$, $w(r) \neq 0$. In the shooting parameter space (a, b), these solutions correspond not exactly, but almost to a universal value for $b \sim 0.0054$. See Fig. 5. More surprising is the fact that Q_M takes a quantized value $-(4\pi)^{-1/2}$ at $b = 0.0061$ independent of the value of a within numerical errors. We have not understood why it should be so.

Solutions with no node in $w(r)$ have special importance, as they are stable against small fluctuations. (See Sec. VIII). In Fig. 6 the spectrum of nodeless dyon solutions are presented in the parameter space (a, b). Notice that a must be small enough ($a < 0.005$) even for $b < 0$.

5. Dependence of the coupling v on the solutions

The ADM mass M depends on the value of the coupling $v = 4\pi G/e^2$. As shown in Fig. 7, M increases as v gets larger and decreases when v gets smaller. With fixed (a, b), roughly $M \propto v$. The Yang-Mills fields u and w are roughly independent of v.
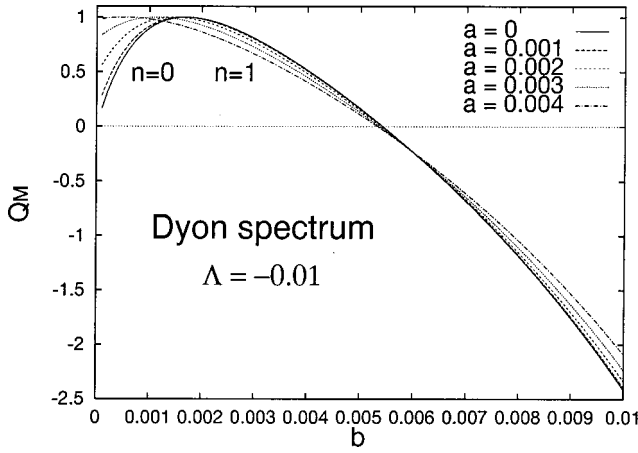
FIG. 5. Magnetic charge Q_M of dyon solutions as a function of the parameter b is plotted with various values of the parameter a. At $b \sim 0.0061$, Q_M is independent of a, taking the quantized value $-(4\pi)^{-1/2} = -0.2821$ within numerical errors.

VI. BLACK HOLE SOLUTIONS

Not long after the BK solutions were discovered, black hole solutions were also found to be contained in the EYM equations [4] if different boundary conditions were used. These solutions generated a large amount of further study, as they apparently violate the no-hair conjecture [6]. Later, EYM black holes were studied in a cosmological context by including a positive cosmological constant [10]. These black hole solutions share most of the properties as the soliton solutions, including their instability.

Recently, purely magnetic black hole solutions were found in asymptotically anti–de Sitter space [13]. These solutions are drastically different from their asymptotically Minkowski or de Sitter counterparts. There are a continuum of solutions in terms of the adjustable shooting parameter that specifies the initial conditions at the horizon. Furthermore, there exist solutions that have no node in w and are in
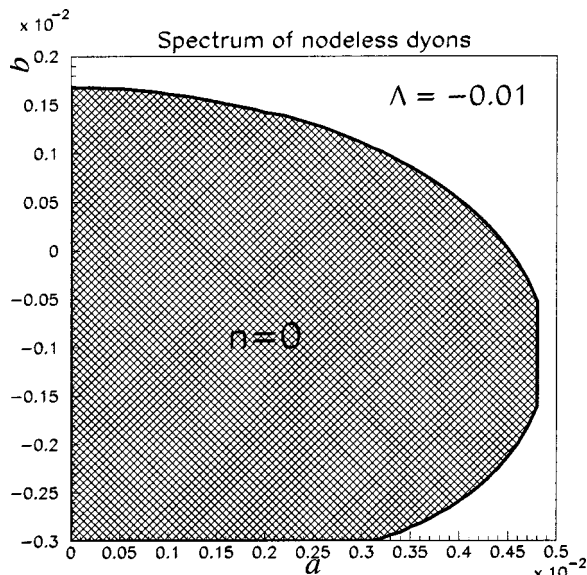

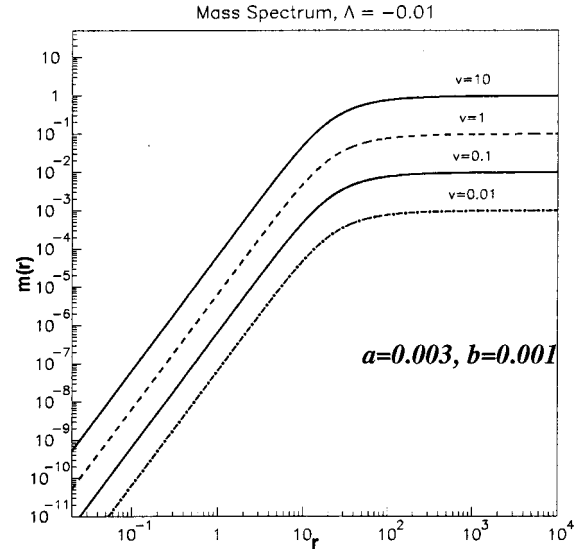
FIG. 6. Spectrum of nodeless dyons.

FIG. 7. A typical solution showing the dependence of v on the mass $m(r)$. $\Lambda = -0.01$, $a = 0.003$, and $b = 0.001$. $w(r)$ and $u(r)$ do not have much dependence on v.

turn stable against spherically symmetric linear perturbations. Here, we discuss the solutions found by Winstanley [13] and also present new dyon black hole solutions. We also discuss the apparent shrinking of the moduli space when the magnitude of Λ is decreased. Similar to the particlelike solutions already discussed, the moduli space becomes discrete in the $\Lambda \to 0$ limit.

A. Boundary conditions at the horizon

Black hole solutions are obtained numerically by specifying the boundary conditions at the horizon and shooting for regular solutions w, u, m, and p for $r_h \leq r < \infty$. The location of the horizon, r_h, and the value $p(r_h) > 0$ can be arbitrarily chosen by scaling of t and r. We look for solutions in which $H(r) > 0$ for $r > r_h$. As $H(r_h) = 0$ but $p(r_h) \neq 0$, Eqs. (22)–(25) require that either u or w vanishes at the horizon. A stronger condition is obtained from the sum rule (35) with $r_1 = r_h$ and $r_2 = \infty$. Its left-hand side (lhs) is finite so that $u(r_h) = 0$ on its rhs. Hence we are led to the expansion

$$w = w_0 + w_1 x + \cdots,$$

$$u = u_1 x + \cdots,$$

$$p = 1 + p_1 x + \cdots,$$

$$H = h_1 x + \cdots,$$

$$m = m_0 + m_1 x + \cdots, \tag{45}$$

where $x = r - r_h$. We have chosen $p(r_h) = 1$ without loss of generality.

There are two adjustable shooting parameters, $(a,b) = (u_1, w_0)$. After inserting the ansatz into Eqs. (19)–(25) we find
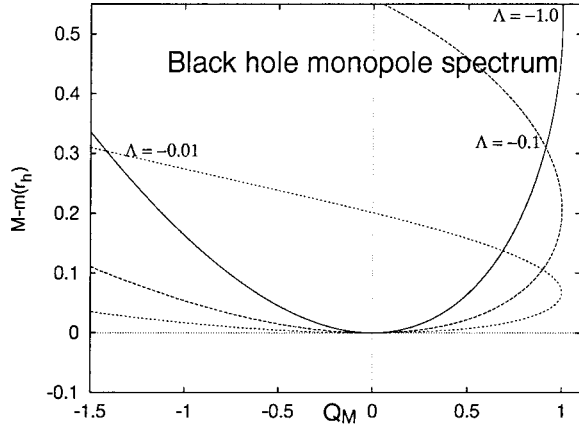
FIG. 8. Black hole mass vs magnetic charge Q_M for the $n=0$ and $n=1$ arms and for different values of Λ.

$$m_0 = \frac{r_h}{2} - \frac{\Lambda r_h^3}{6},$$

$$m_1 = \frac{v}{2}\left\{ \frac{(1-w_0^2)^2}{r_h^2} + u_1^2 \right\},$$

$$h_1 = \frac{1}{r_h}(1 - \Lambda r_h^2 - 2m_1),$$

$$w_1 = -\frac{w_0(1-w_0^2)}{r_h^2 h_1},$$

$$p_1 = -\frac{2v}{r_h}\left\{ (w_1)^2 + \frac{w_0^2 u_1^2}{h_1^2} \right\},$$

$$u_2 = -\left\{ \frac{w_0^2}{r_h^2 h_1} + \frac{1}{r_h} + \frac{p_1}{2} \right\}. \tag{46}$$

The asymptotic expansion at large r is the same as in Eq. (27). The ADM mass is given by $M = m(\infty)$.

B. New electrically and magnetically charged black hole solutions

Just as the soliton solutions, purely magnetically charged black hole solutions are obtained by setting the adjustable parameter a to zero. The behavior of the solutions are similar to that of the solitons (see Ref. [13] for more information). The number of nodes n in w can be 0, 1, 2, ... The black hole monopole spectrum of mass versus charge is displayed in Fig. 8. It shows the spectrum for the $n=0$ and $n=1$ arms.

Solutions with both magnetic and electric charge are obtained by giving a a finite value. Dyon black hole solutions are similar to the monopole solutions except that u is non-zero. At the horizon u starts at zero and monotonically increases asymptotically to a finite value. H starts at one and quickly diverges. p starts at one and remains almost constant. Typical black hole dyon solutions are shown in Fig. 9.

Again black hole dyon solutions with no node in $w(r)$ are stable against small spherically symmetric perturbations. The
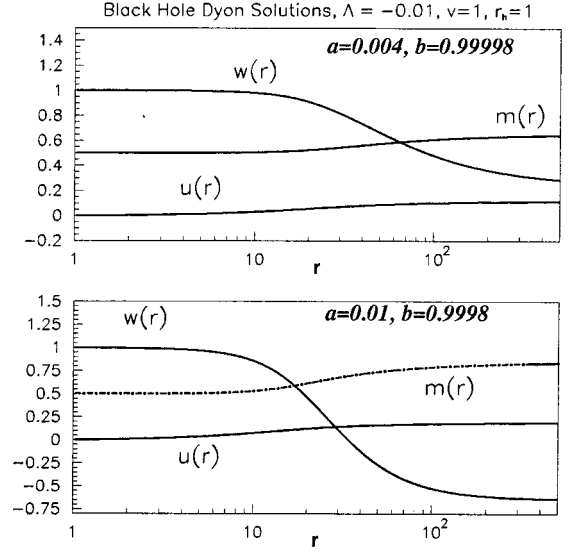

FIG. 9. Typical dyon black hole solutions with no node and one node. $H(r)$ and $p(r)$ were not plotted for the sake of clarity. The top figure corresponds to $(a,b)=(0.004, 0.99998)$, whereas the bottom corresponds to $(0.01, 0.9998)$.

spectrum of those nodeless black hole dyons in the parameter space (a,b) is plotted in Fig. 10. Notice the similarity between Figs. 6 and 10. The nodeless solutions exist only for small $a=u_1 < 0.0055$. $b=w_0$ must be around 1.

VII. DEPENDENCE ON Λ-FRACTAL STRUCTURE

The soliton and black hole solutions depend nontrivially on the value of the cosmological constant Λ. It has not been well understood why the continuum of solutions for negative Λ become a discrete set of solutions in the $\Lambda \to 0$ limit, and remain discrete for all $\Lambda > 0$. Just as Fig. 8 shows for the black hole solutions, Figs. 4 and 11 shows the spectrum in mass vs magnetic charge Q_M plane for a give Λ. The width
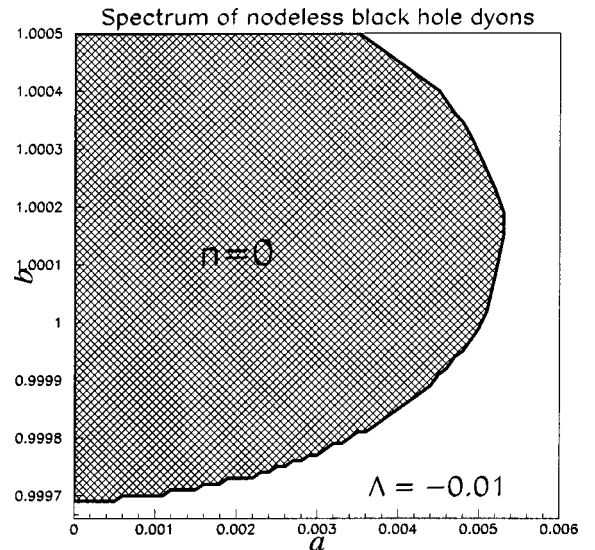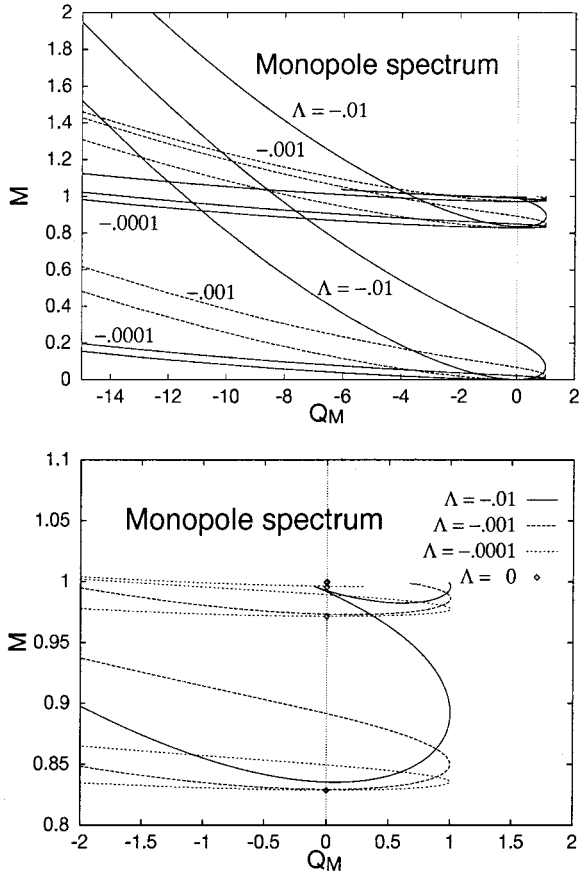

FIG. 10. Spectrum of black hole solutions.

FIG. 11. The appearance and collapse of soliton solutions in $\Lambda \to 0$. The bottom figure is the blow-up of a part of the top figure near $(Q_M, M) = (0,1)$, in which the first four Bartnik-McKinnon solutions are also marked. It is seen that as $\Lambda \to 0$, solutions on a branch collapse to a point. New branches emerge as $|\Lambda|$ becomes smaller, too.

of each branch for a given Λ gets smaller as Λ approaches zero. Figure 11 indicates that as $\Lambda \to 0$, the branches collapse to one point, the BK solution, as the continuum of solutions vanishes. It is still unknown mathematically why and how this occurs.

We would like to point out that there is a fractal structure in the moduli space of the solutions. This is most clearly seen in the parameter b vs mass M plot as displayed in Fig. 12. As Λ becomes smaller, a new branch appears. The shape of branches has approximate self-similarity. Similarly, in Fig. 13 the magnetic charge Q_M is plotted against b. Delicate structure is observed near the critical $b = b_c$ which signifies the critical solution discussed in Section V C 3. There may be some connection between the limiting point in the monopole spectrum and the critical solution.

VIII. STABILITY

It has been shown that the soliton and black hole solutions in asymptotically Minkowski and de Sitter space, which necessarily have at least one node in $w(r)$, are unstable [2,11,10,3]. In contrast, the monopole and black hole solutions in the asymptotically anti–de Sitter space with no node
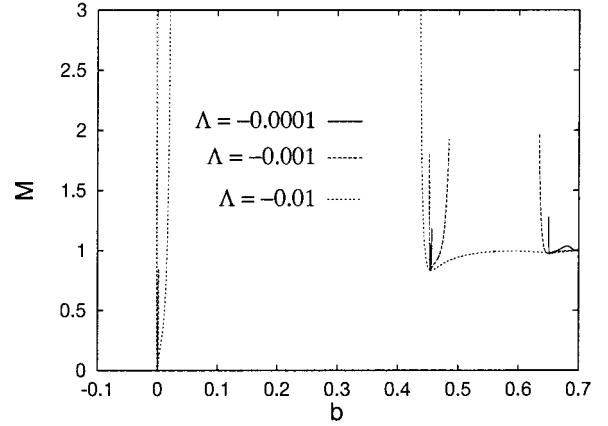

FIG. 12. The monopole spectrum. b vs M (mass) is plotted with various values of Λ.

in $w(r)$ are stable for $u = 0$. One expects the presence of the electric field not to change the stability of the solitons and black hole configurations.

In this section we give a detailed discussion for establishing the stability. We find that in the asymptotically anti–de Sitter space the boundary condition for the resultant Schrödinger problem becomes subtle, and that the previous argument given in the asymptotically Minkowski space needs elaboration.

A. Perturbation equations

We consider small time-dependent perturbations to the static solutions to the coupled EYM equations. In the static solutions $v(r) = \tilde{w}(r) = 0$. In the general ansatz, Eqs. (11) and (14), we set

$$u(r,t) = u(r) + \delta u(r,t),$$

$$w(r,t) = w(r) + \delta w(r,t),$$

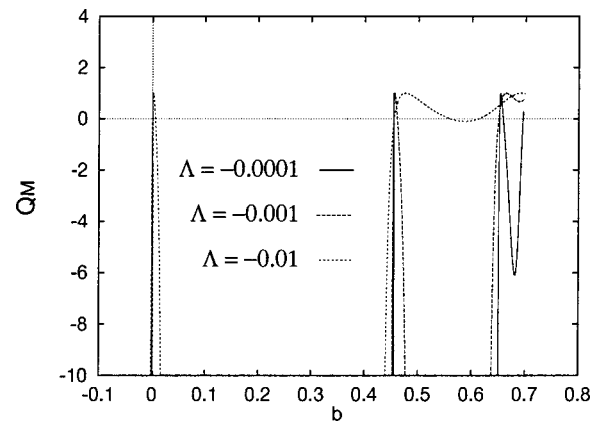$$\tilde{w}(r,t) = \delta \tilde{w}(r,t),$$

$$\nu(r,t) = \delta \nu(r,t),$$

FIG. 13. The monopole spectrum. b vs Q_M (magnetic charge) is plotted with various values of Λ.

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$$p(r,t) = p(r) + \delta p(r,t),$$

$$H(r,t) = H(r) + \delta H(r,t), \tag{47}$$

and $m(r,t) = m(r) + \delta m(r,t)$. Substituting Eq. (47) into the Yang-Mills equation (15) and retaining only terms linear in perturbations, one finds

$$[r^2 u' \, \delta p + r^2 p (\delta u' - \delta \dot{v})]' - \frac{2uw^2 p}{H}\left(\frac{\delta p}{p} - \frac{\delta H}{H}\right)$$

$$- \frac{2pw}{H}(w\,\delta u + 2u\,\delta w + \delta \dot{w}) = 0, \tag{48}$$

$$r^2 p(\delta \dot{u}' - \delta \ddot{v}) + r^2 u' \, \delta \dot{p} - \frac{2H}{p}[w\,\delta \tilde{w}' - w'\,\delta \tilde{w} + w^2 \delta v] = 0, \tag{49}$$

$$\left[\frac{w'H}{p}\left(\frac{\delta H}{H} - \frac{\delta p}{p}\right) + \frac{H}{p}\,\delta w'\right]' + \frac{pu}{H}(u\,\delta w + 2w\,\delta u + 2\,\delta \dot{w})$$

$$- \frac{p}{H}\,\delta \ddot{w} + \frac{wu^2 p}{H}\left(\frac{\delta p}{p} - \frac{\delta H}{H}\right) - \frac{w(1-w^2)}{r^2 p^2}\,\delta p$$

$$+ \frac{1-3w^2}{r^2 p}\,\delta w = 0, \tag{50}$$

$$\left[\frac{H}{p}(\delta \tilde{w}' + w\,\delta v)\right]' + \frac{p}{H}(u^2\,\delta \tilde{w} - 2u\,\delta \dot{w} - w\,\delta \dot{u} - \delta \ddot{\tilde{w}})$$

$$- \frac{uwp}{H}\left(\frac{\delta \dot{p}}{p} - \frac{\delta \dot{H}}{H}\right) + \frac{(1-w^2)}{r^2 p}\,\delta \tilde{w} + \frac{H}{p}w'\,\delta v = 0. \tag{51}$$

The Einstein equations (18) and (20) yield

$$\delta m' = v\left\{r^2 p^2 u'(\delta u' - \delta \dot{v}) + \left(r^2 u'^2 + \frac{2u^2 w^2}{H}\right)p\,\delta p\right.$$

$$+ \left(w'^2 - \frac{p^2 u^2 w^2}{H^2}\right)\delta H + 2Hw'\,\delta w'$$

$$+ \frac{2uwp^2}{H}(w\,\delta u + u\,\delta w + \delta \dot{w}) - \left.\frac{2w(1-w^2)}{r^2}\,\delta w\right\}, \tag{52}$$

$$\left(\frac{\delta p}{p}\right)' = -\frac{4v}{r}\left\{w'\,\delta w' + \frac{p^2 u^2 w^2}{H^2}\left(\frac{\delta p}{p} - \frac{\delta H}{H}\right)\right.$$

$$+ \left.\frac{p^2 uw}{H^2}(w\,\delta u + u\,\delta w + \delta \dot{w})\right\}, \tag{53}$$

$$\delta \dot{H} = -\frac{4vH}{r}\{w'\,\delta \dot{w} - u(w'\,\delta \tilde{w} - w\,\delta \tilde{w}') + uw^2\,\delta v\}. \tag{54}$$

There is residual gauge invariance specified by a gauge function $\Omega(r,t)$ in Eqs. (9) and (12). Making use of this freedom, one can always set either $\delta u(r,t) = 0$ or $\delta v(r,t) = 0$.

B. Stability analysis

In examining time-dependent fluctuations around monopole solutions for which $u(r) = 0$, it is convenient to work in the $\delta u(t,r) = 0$ gauge. Equations (48), (49), and (51) become

$$(r^2 p\,\delta v)' = -\frac{2pw}{H}\,\delta \tilde{w}, \tag{55}$$

$$r^2 p\,\delta \ddot{v} + \frac{2H}{p}(w\,\delta \tilde{w}' - w'\,\delta \tilde{w} + w^2\,\delta v) = 0, \tag{56}$$

$$\left[\frac{H}{p}(\delta \tilde{w}' + w\,\delta v)\right]' - \frac{p}{H}\,\delta \ddot{\tilde{w}} + \frac{1-w^2}{r^2 p}\,\delta \tilde{w} + \frac{H}{p}w'\,\delta v = 0, \tag{57}$$

whereas Eqs. (50), (52), (53), and (54) become

$$\left[\frac{H}{p}\left\{w'\left(\frac{\delta H}{H} - \frac{\delta p}{p}\right) + \delta w'\right\}\right]' - \frac{p}{H}\,\delta \ddot{w} - \frac{w(1-w^2)}{r^2 p^2}\,\delta p$$

$$+ \frac{1-3w^2}{r^2 p}\,\delta w = 0, \tag{58}$$

$$\delta m' = v\left\{w'^2\,\delta H + 2Hw'\,\delta w' - \frac{2w(1-w^2)}{r^2}\,\delta w\right\}, \tag{59}$$

$$\left(\frac{\delta p}{p}\right)' = -\frac{4v}{r}w'\,\delta w', \tag{60}$$

$$\delta H = -\frac{4v}{r}Hw'\,\delta w. \tag{61}$$

Notice that Eqs. (55)–(57) involve only δv and $\delta \tilde{w}$, defining the odd-parity group, whereas Eqs. (58)–(61) involve only δw, δH, and δp, defining the even-parity group. The number of the equations is larger than the number of the unknown functions. Indeed, one equation in each group follows from the others.

To derive the equation for each unknown function in a closed form, we introduce the tortoise radial coordinate ρ by

$$\frac{d\rho}{dr} = \frac{p}{H} \tag{62}$$

with which the equations for w, p, and m become

$$\frac{d^2 w}{d\rho^2} = -\frac{H}{r^2 p^2}w(1-w^2), \tag{63}$$

$$\frac{dp}{d\rho} = -\frac{2vp^2}{rH}\left(\frac{dw}{d\rho}\right)^2, \tag{64}$$

$$\frac{dm}{d\rho} = v\left\{ p\left(\frac{dw}{d\rho}\right)^2 + \frac{H}{2r^2p}(1-w^2)^2 \right\}.$$

(65)

The range of ρ is finite, $0 \leq \rho \leq \rho_{\max}$, since $p \to p_0$ and $H \to |\Lambda| r^2/3$ as $r \to \infty$:

$$\rho = \begin{cases} r & \text{for } r \sim 0, \\ \rho_{\max} - \dfrac{3\rho_0}{|\Lambda|r} & \text{for } r \sim \infty, \end{cases}$$

(66)

where $p = p_0 + O(1/r)$.

In the odd parity group Eq. (55) expresses δw in terms of $\delta\nu$. Substituting it into Eq. (56) and making use of Eq. (63), one finds

$$\left\{ -\frac{d^2}{d\rho^2} + U_\beta(\rho) \right\}\beta = \omega^2\beta,$$

$$U_\beta = \frac{H}{r^2p^2}(1+w^2) + \frac{2}{w^2}\left(\frac{dw}{d\rho}\right)^2,$$

$$\delta\nu = \frac{w}{r^2p}\beta, \quad \delta\widetilde{w} = -\frac{1}{2w}\frac{d}{d\rho}(w\beta).$$

(67)

Here we have supposed fluctuations to be harmonic: $\delta\nu(r,t) = e^{-i\omega t}\delta\nu(\rho)$ and $\delta\widetilde{w}(r,t) = e^{-i\omega t}\delta\widetilde{w}(\rho)$. Equation (57) follows from Eqs. (55), (56), and (63).

In the even-parity group, Eqs. (60) and (61) express δp and δH (δm) in terms of δw. Equation (59) automatically follows from Eqs. (60), (61), (63), and (64). Equation (58) becomes, with the use of Eqs. (60), (61), and (63).

$$\left\{ -\frac{d^2}{d\rho^2} + U_w(\rho) \right\}\delta w = \omega^2\delta w,$$

$$U_w = \frac{H}{r^2p^2}(3w^2 - 1) + 4v\frac{d}{d\rho}\left[\frac{p}{rH}\left(\frac{dw}{d\rho}\right)^2\right].$$

(68)

Again harmonic fluctuations $\delta w(r,t) = e^{-i\omega t}\delta w(\rho)$ are supposed.

Equations (67) and (68) have the same form as the Schrödinger equation on a one-dimensional interval. Both of the potentials U_β and U_w are singular at $\rho = 0$, behaving as $+2/\rho^2$. U_β has an additional singularity if w has a zero at ρ_k; $U_\beta \sim +2/(\rho - \rho_k)^2$.

The integrated energy-momentum density $T^{ab}\sqrt{-g}d^3x$ due to fluctuations must remain finite. At the origin $r = 0$ it implies that $\delta w = \delta\widetilde{w} = 0$, whereas $\delta\nu = O(1)$. Taking advantage of the general coordinate invariance, one can impose $\delta p = 0$ and $\delta H = -2\delta m/r = 0$. At $r \sim \infty$, $\delta w'$, $\delta\widetilde{w}'$, $\delta\nu = O(r^{-2})$. These are mild boundary conditions. One can impose more strict conditions such as the regularity at $r = 0$ and vanishing at $r = \infty$. As physical perturbations we demand that all δw, $\delta\widetilde{w}$, $\delta\nu$, δH, and δp vanish at $r = \infty$.

In Eq. (67) the potential $U_\beta(\rho)$ is positive definite. However, this does not necessarily mean that the eigenvalue ω^2 is positive definite. It depends on the boundary condition.

Clearly $\beta(0) = 0$. At $\rho = \rho_{\max}$, $\delta\nu = \delta\widetilde{w} = 0$ so that $\beta' + h\beta = 0$ where $h = w'/w$. Note that

$$h = \frac{1}{w}\frac{dw}{d\rho}\bigg|_{\rho_{\max}} = \frac{|\Lambda|}{3p_0}\frac{r^2}{w}\frac{dw}{dr}\bigg|_{r=\infty} = \frac{|\Lambda|w_1}{3p_0w_0}$$

(69)

where w_j's are defined in Eq. (27). For the monopole configurations with no nodes in w, $h < 0$ ($h > 0$) when w is monotonically decreasing (increasing).

Following Courant and Hilbert [28], we define

$$\mathcal{D}(\varphi; h) = \int_0^{\rho_{\max}} d\rho\{\varphi'(\rho)^2 + U_\beta(\rho)\varphi(\rho)^2\} + h\varphi(\rho_{\max})^2,$$

$$\mathcal{N}(\varphi) = \int_0^{\rho_{\max}} d\rho\,\varphi(\rho)^2.$$

(70)

If $w(r)$ is nodeless, then $U_\beta(\rho)$ is regular on the interval except at $\rho = 0$. The equation implies that $\beta = O(\rho^2)$ near the origin. In this case, an eigenfunction $\beta(\rho)$ in Eq. (67) satisfies

$$\omega^2 = \frac{\mathcal{D}(\beta)}{\mathcal{N}(\beta)}.$$

(71)

It follows immediately that all eigenvalues ω^2 are positive definite if $h \geq 0$ so that the solution is stable against small odd-parity perturbations.

For $h < 0$ more careful analysis is necessary. The lowest eigenvalue $\omega^2 \equiv \lambda_1$ in the eigenvalue equation (67) is exactly the lower bound of the set of values assumed by the functional $\mathcal{D}(\varphi, h)$, where φ is any function continuous on the interval $[0, \rho_{\max}]$ with piecewise continuous derivatives satisfying $\varphi(0) = 1$ and $\mathcal{N}(\varphi) = 1$:

$$\lambda_1(h) = \min_\varphi \mathcal{D}(\varphi; h).$$

(72)

If $\lambda_1 > 0$, then the solution is stable against odd-parity perturbations. Suppose that $\varphi_1(\rho)$ saturates the lower bound for h_1: $\lambda_1(h_1) = \mathcal{D}(\varphi_1; h_1)$. As

$$\lambda_1(h_1) = \mathcal{D}(\varphi_1; h_2) + (h_1 - h_2)\varphi_1(\rho_{\max})^2$$

$$\geq \lambda_1(h_2) + (h_1 - h_2)\varphi_1(\rho_{\max})^2,$$

(73)

where $\lambda_1(h)$ is a monotonically increasing function of h. Hence, if $\lambda_1(h_1) > 0$, then $\lambda_1(h) > 0$ for $h \geq h_1$.

To establish the stability we utilize the residual gauge invariance. There is a zero-mode (with $\omega^2 = 0$) for Eq. (67) with an appropriate boundary condition h_0. In the $\Lambda = 0$ case the existence of the zero mode was utilized to prove the instability of the BK and black hole solutions which has at least one node in $w(r)$ [26,27]. Consider the time-independent gauge function $\Omega(r)$ in Eq. (9). For $|\Omega| \ll 1$, $\delta\nu = d\Omega/dr$ and $\delta\widetilde{w} = -w\Omega$. Equation (55) is satisfied if

$$\frac{d}{d\rho}\left(\frac{r^2p^2}{H}\frac{d\Omega}{d\rho}\right) = 2w^2\Omega.$$

(74)

As $\Omega(0)=0$, $\Omega \sim a\rho + O(\rho^3)$ for $\rho \sim 0$. Hence, Eq. (74) determines $\Omega(r)$ up to an overall constant. $\beta_0 = (r^2 p^2 / wH)(d\Omega/d\rho)$ is the zero mode of Eq. (67) and $U_\beta = \beta_0''/\beta_0$, where $\Omega' \equiv d\Omega/d\rho$, etc. In this case $\Omega(\rho_{\max}) \neq 0$ and $h_0 = -\beta_0'/\beta_0|_{\rho_{\max}}$ differs from h in the eigenvalue problem under consideration. If $h > h_0$, then $\lambda_1(h) > 0$, establishing the stability. As $d(w\beta_0)/d\rho = 2w^2\Omega$,

$$h = h_0 + \frac{2w^2 H\Omega}{r^2 p^2 \Omega'}. \tag{75}$$

Nodeless solutions ($w>0$) are stable if $\Omega/\Omega' > 0$ at $\rho_{\max}$.

Solving Eq. (74) numerically, we have determined $\Omega(\rho)$ to find that indeed $h > h_0$ for nodeless solutions. This analysis also shows that h becomes exactly h_0 for the configuration with $w(r=\infty)=0$. In this limiting case the zero mode is not normalizable; it diverges as $(\rho - \rho_{\max})^{-1}$.

This is a general behavior. When w has a node at $\rho_k < \rho_{\max}$, there appears a negative ω^2 mode which behaves as $(\rho - \rho_k)^{-1}$ near ρ_k.

If $w(r)$ has n nodes, i.e., $w(r_j)=0$ ($j=1,...,n$), the potential U_β develops $(\rho - \rho_j)^{-2}$ singularities. Volkov *et al.* have shown for the BK solutions in the $\Lambda=0$ case that there appear exactly n negative eigenmodes ($\omega^2 < 0$) if w has n nodes [26]. A similar conclusion has been obtained for black hole solutions as well [27]. Their argument needs elaboration in the $\Lambda < 0$ case, however.

To investigate the eigenvalue spectrum of Eq. (67) in this case, it is convenient to consider the dual equation as was done in [26]. One can write the Schrödinger equation in Eq. (67) as

$$Q_+ Q_- \varphi_n = \lambda_n \varphi_n,$$

$$Q_\pm = \pm \frac{d}{d\rho} + \frac{\beta_0'}{\beta_0}, \tag{76}$$

where $\beta_0(\rho)$ is the zero mode described above, $\varphi_n = O(\rho^2)$ near $\rho=0$ and $\varphi_n' + h\varphi_n = 0$ at $\rho_{\max}$. The dual equation is given by

$$Q_- Q_+ \tilde{\varphi}_n = \left\{ -\frac{d^2}{d\rho^2} + \tilde{U}_\beta \right\} \tilde{\varphi}_n = \lambda_n \tilde{\varphi}_n,$$

$$\tilde{U}_\beta = -\frac{H}{r^2 p^2}(1+w^2) - \frac{8ww'H\Omega}{r^2 p^2 \Omega'} + 8\left(\frac{w^2 H\Omega}{r^2 p^2 \Omega'}\right)^2. \tag{77}$$

φ_n and $\tilde{\varphi}_n$ are related to each other by

$$\tilde{\varphi}_n = \begin{cases} Q_- \varphi_n & \text{for } \lambda_n \neq 0, \\ \varphi_n^{-1} = \beta_0^{-1} & \text{for } \lambda_n = 0. \end{cases} \tag{78}$$

However, the boundary condition for $\tilde{\varphi}_n(\rho)$ depends on λ_n:

$$\tilde{\varphi}_n(0) = 0,$$

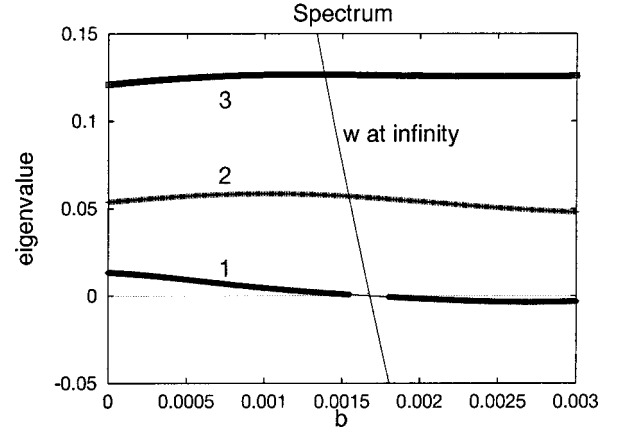$$\tilde{\varphi}_n'(\rho_{\max}) + \tilde{h}_n \tilde{\varphi}_n(\rho_{\max}) = 0,$$

FIG. 14. Eigenvalues λ_n ($n=1,2,3$) in the dual Schrödinger equation (77) with the boundary condition (79) are displayed for monopole configurations with varying values of b. $\Lambda = -0.01$ and $v=1$. When $w(\infty)$ develops a node, the negative eigenvalue mode arises, signaling the instability of the solution. $w(\infty)=0$ at $b = 0.001\,68$, where the numerical evaluation of λ_1 becomes difficult.

$$\tilde{h}_n = -h_0 - \frac{\lambda_n}{h - h_0}. \tag{79}$$

The advantage of considering the dual equation is that the dual potential $\tilde{U}_\beta(\rho)$ is regular except at $\rho=0$ where it behaves as $+6/\rho^2$. However, the eigenvalue λ_n has to be determined self-consistently such that the boundary condition (79) is satisfied. We have determined λ_n's numerically for the monopole configurations in the lower branch in Fig. 4. The first, second, and third eigenvalues are displayed in Fig. 14. One sees that $\lambda_1 > 0$ for the nodeless configurations, but the unstable mode develops when w has a node.

The wave function of the unstable mode in the original equation, not in the dual equation, diverges at the zeroes of $w(r)$. In other words, the instability sets in around the zeros
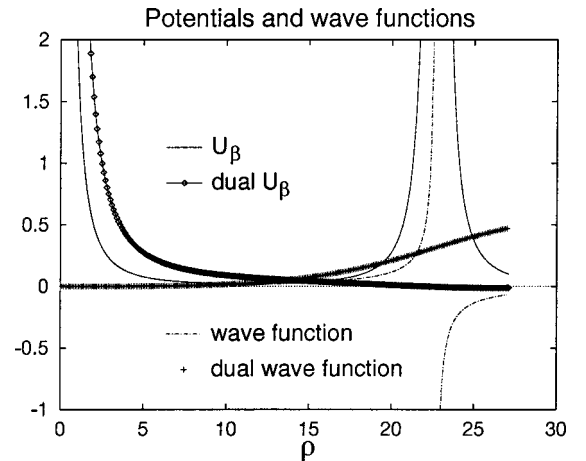

FIG. 15. $U_\beta(\rho)$ and $\tilde{U}_\beta(\rho)$ for the monopole solution at $b = 0.0025$, $\Lambda = -0.01$, and $v=1$. The wave function of the unstable mode ($\lambda = -0.0033$) are also plotted. $\tilde{\varphi}_1(\rho)$ in the dual equation (77) is a regular smooth function, but $\varphi_1(\rho)$ in the original equation (76) diverges at the zero of w.

of $w(r)$. The potential $U_\beta(\rho)$ and $\tilde{U}_\beta(\rho)$ for the solution at $b=0.0025$ are plotted in Fig. 15. At the node of w, U_β diverges, but $\tilde{U}_\beta$ remains finite. The wave function $\tilde{\varphi}_1(\rho)$ of the lowest eigenvalue ($\lambda_1=-0.0033$) and the corresponding $\varphi_1(\rho)$ also have been plotted in Fig. 15.

For the even-parity perturbation the potential $U_w(\rho)$ in Eq. (68) is not positive definite. The first term in U_w becomes negative for $w^2<1/3$. The second term also can become negative when w' vanishes at finite r. We have solved the Schrödinger equation (68) numerically for typical monopole solutions, and found that for the solutions with no node in $w(r)$, the eigenvalues ω^2 are always positive even if $w(r=\infty)<1/\sqrt{3}$. Hence we have established the stability of the monopole solutions with no node in $w(r)$.

IX. SUMMARY

New monopole, dyon, and black hole solutions to the Einstein-Yang-Mills equations have been found in the asymptotically anti–de Sitter space. The solutions with no node in the non-Abelian field strengths are shown to be stable against spherically symmetric perturbations. The non-

trivial boundary condition plays a crucial role in developing the instability for solutions with nodes. The stability of nodeless dyon solutions needs to be established.

Though electric and magnetic charges of monopole and dyon solutions are not quantized in classical theory, they are expected to be quantized in quantum theory. If this is the case, then at least solutions with the smallest charge would become absolutely stable.

We have also found the critical spacetime solutions which end at finite r. These solutions may have connections to black hole solutions, though more detailed study is necessary.

The solutions found in the present paper may have profound consequences in the evolution of the early universe which may have gone through the anti–de Sitter phase. We hope to report on these subjects in future publications.

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