Brane new world

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We study a Randall-Sundrum cosmological scenario consisting of a domain wall in anti–de Sitter space with a strongly coupled large *N* conformal field theory living on the wall. The AdS-CFT correspondence allows a fully quantum mechanical treatment of this CFT, in contrast with the usual treatment of matter fields in inflationary cosmology. The conformal anomaly of the CFT provides an effective tension which leads to a de Sitter geometry for the domain wall. This is the analogue of Starobinsky's four dimensional model of anomaly driven inflation. Studying this model in a Euclidean setting gives a natural choice of boundary conditions at the horizon. We calculate the graviton correlator using the Hartle-Hawking ''no boundary'' proposal and analytically continue to Lorentzian signature. We find that the CFT strongly suppresses metric perturbations on all but the largest angular scales. This is true independently of how the de Sitter geometry arises, i.e., it is also true for four dimensional Einstein gravity. Since generic matter would be expected to behave like a CFT on small scales, our results suggest that tensor perturbations on small scales are far smaller than predicted by all previous calculations, which have neglected the effects of matter on tensor perturbations.

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I. INTRODUCTION

Randall and Sundrum (RS) have suggested [1] that four dimensional gravity may be recovered in the presence of an infinite fifth dimension provided that we live on a domain wall embedded in anti–de Sitter space (AdS). Their linearized analysis showed that there is a massless bound state of the graviton associated with such a wall as well as a continuum of massive Kaluza-Klein modes. More recently, linearized analyses have examined the spacetime produced by matter on the domain wall and concluded that it is in close agreement with four dimensional Einstein gravity [2,3].

RS used horospherical coordinates based on slicing AdS into flat hypersurfaces. These horospherical coordinates break down at the horizons shown in Fig. 1. An issue that has not received much attention so far is the role of boundary conditions at these Cauchy horizons in AdS. With stationary perturbations, one can impose the boundary conditions that the horizons remain regular. Indeed, without this boundary condition the solution for stationary perturbations is not well defined. Even for non-perturbative departures from the RS solution, like black holes, one can impose the boundary condition that the AdS horizons remain regular [4,5,2,6,7]. Non-stationary perturbations on the domain wall, however, will give rise to gravitational waves that cross the horizons. This will tend to focus the null geodesic generators of the horizon, which will mean that they will intersect each other on some caustic. Beyond the caustic, the null geodesics will not lie in the horizon. However, null geodesic generators of the future event horizon cannot have a future endpoint [8] and so the endpoint must lie to the past. We conclude that if the past and future horizons remain non-singular when perturbed[1] (as required for a well-defined boundary condition) then they must intersect at a finite distance from the wall. By contrast, the past and future horizons do not intersect in the RS ground state but go off to infinity in AdS.

The RS horizons are like the horizons of extreme black holes. When considering perturbations of black holes, one normally assumes that radiation can flow across the future horizon but that nothing comes out of the past horizon. This is because the past horizon isn't really there, and should be replaced by the collapse that formed the black hole. To justify a similar boundary condition on the Randall-Sundrum past horizon, one needs to consider the initial conditions of the universe.

The main contender for a theory of initial conditions is the ''no boundary'' proposal[2] [10] that the quantum state of the universe is given by a Euclidean path integral over compact metrics. The simplest way to implement this proposal for the Randall Sundrum idea is to take the Euclidean version of the wall to be a four sphere at which two balls of AdS_5 are joined together. In other words, take two balls in AdS_5, and glue them together along their four sphere boundaries. The result is topologically a five sphere, with a delta function of curvature on a four dimensional domain wall separating the two hemispheres. If one analytically continues to Lorentzian signature, one obtains a four dimensional de Sitter hyperboloid, embedded in Lorentzian anti–de Sitter space, as shown in Fig. 2. The past and future RS horizons, are replaced by the past and future light cones of the points at the centres of the two balls. Note that the past and future horizons now intersect each other and are non extreme, which means they

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[1] It has been shown that the Kaluza-Klein (KK) modes of RS give rise to singular horizons [9].

[2] Other approaches to quantum cosmology in the RS model have been discussed in [11,12]. Boundary conditions motivated by a Euclidean approach were also used in [3] for a flat domain wall.

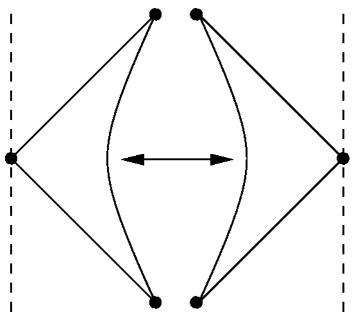

FIG. 1. Carter-Penrose diagram of anti–de Sitter space with a flat domain wall. The dotted line denotes timelike infinity and the arrows denote identifications. The heavy dots denote points at infinity. Note that the Cauchy horizons intersect at infinity.

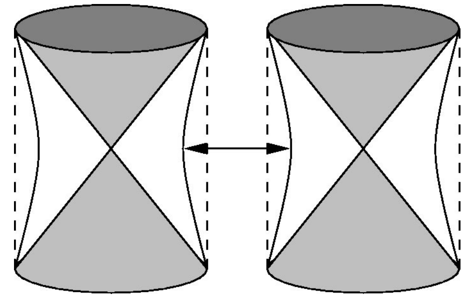

FIG. 2. Anti–de Sitter space with a de Sitter domain wall. AdS is drawn as a solid cylinder, with the boundary of the cylinder (dashed line) representing timelike infinity. The light cone shown is the horizon. The arrows denote identifications.

are stable to small perturbations. A perfectly spherical Euclidean domain wall will give rise to a four dimensional Lorentzian universe that expands forever in an inflationary manner.[3]

In order for a spherical domain wall solution to exist, the tension of the wall must be larger than the value assumed by RS, who had a flat domain wall. We shall assume that matter on the wall increases its effective tension, permitting a spherical solution. In Sec. III, we consider a strongly coupled large N conformal field theory (CFT) on the domain wall. On a spherical domain wall, the conformal anomaly of the CFT increases the effective tension of the domain wall, making the spherical solution possible. The Lorentzian geometry is a de Sitter universe with the conformal anomaly driving inflation,[4] an idea introduced long ago by Starobinsky [19].

The no boundary proposal allows one to calculate unambiguously the graviton correlator on the domain wall. In particular, the Euclidean path integral itself uniquely specifies the allowed fluctuation modes, because perturbations that have infinite Euclidean action are suppressed in the path integral. Therefore, in this framework, there is no need to impose by hand an additional, external prescription for the vacuum state for each perturbation mode. In addition, the AdS-CFT correspondence allows a fully quantum mechanical treatment of the CFT, in contrast with the usual classical treatment of matter fields in inflationary cosmology.

Finally, we analytically continue the Euclidean correlator into the Lorentzian region, where it describes the spectrum of quantum mechanical vacuum fluctuations of the graviton field on an inflating domain wall with conformally invariant matter living on it. We find that the quantum loops of the large N CFT give spacetime a rigidity that strongly suppresses metric fluctuations on small scales. Since any matter would be expected to behave like a CFT at small scales, this result probably extends to any inflationary model with sufficiently many matter fields. It has long been known that matter loops lead to short distance modifications of gravity. Our work shows that these modifications can lead to observable consequences in an inflationary scenario.

Although we have carried out our calculations for the RS model, we shall show that results for four dimensional Einstein gravity coupled to the CFT can be recovered by taking the domain wall to be large compared with the AdS scale. Thus our conclusion that metric fluctuations are suppressed holds independently of the RS scenario.

The spherical domain wall considered in this paper analytically continues to a Lorentzian de Sitter universe that inflates forever. However, Starobinsky [19] showed that the conformal anomaly driven de Sitter phase is unstable to evolution into a matter dominated universe. If such a solution could be obtained from a Euclidean instanton then it would have an $O(4)$ symmetry group, rather than the $O(5)$ symmetry of a spherical instanton. We shall study such models for both the RS model and four dimensional Einstein gravity in a separate paper.

The AdS-CFT correspondence [20–22] provides an explanation of the RS behavior[5] [23]. It relates the RS model to an equivalent four dimensional theory consisting of general relativity coupled to a strongly interacting conformal field theory and a logarithmic correction. Under certain circumstances, the effects of the CFT and logarithmic term are negligible and pure gravity is recovered. We review this correspondence in Sec. II.

In Sec. III we present our calculation of the graviton correlator on the instanton and demonstrate how the result is continued to Lorentzian signature. Section IV contains our conclusions and some speculations. This paper also includes two appendices which contain technical details that we have omitted from the text.

II. RANDALL-SUNDRUM FROM AdS-CFT

The AdS-CFT correspondence [20–22] relates type IIB supergravity theory in $AdS_5 \times S^5$ to a $\mathcal{N}=4$ $U(N)$ supercon-

[3]Such inflationary brane-world solutions have been studied in [13–16,11]. For a discussion of other cosmological aspects of the RS model, see [17] and references therein.

[4]A similar idea was recently discussed within the context of renormalization group flow in the AdS-CFT correspondence [18]. However, in that case the CFT was the CFT dual to the bulk AdS geometry, not a new CFT living on the domain wall.

[5]This was first pointed out in unpublished remarks of Maldacena and Witten.

formal field theory. If g_{YM} is the coupling constant of this theory then the 't Hooft parameter is defined to be $\lambda = g_{YM}^2 N$. The CFT parameters are related to the supergravity parameters by [20]

$$l = \lambda^{1/4} l_s, \tag{2.1}$$

$$\frac{l^3}{G} = \frac{2N^2}{\pi}, \tag{2.2}$$

where l_s is the string length, l the AdS radius and G the *five* dimensional Newton constant. Note that λ and N must be large in order for stringy effects to be small. The CFT lives on the conformal boundary of AdS_5. The correspondence takes the following form:

$$\begin{aligned} Z[\mathbf{h}] &\equiv \int d[\mathbf{g}] \exp(-S_{grav}[\mathbf{g}]) \\ &= \int d[\phi] \exp(-S_{CFT}[\phi;\mathbf{h}]) \\ &\equiv \exp(-W_{CFT}[\mathbf{h}]), \end{aligned} \tag{2.3}$$

here $Z[\mathbf{h}]$ denotes the supergravity partition function in AdS_5. This is given by a path integral over all metrics in AdS_5 which induce a given conformal equivalence class of metrics $\mathbf{h}$ on the conformal boundary of AdS_5. The correspondence relates this to the generating functional W_{CFT} of connected Green's functions for the CFT on this boundary. This functional is given by a path integral over the fields of the CFT, denoted schematically by ϕ. Other fields of the supergravity theory can be included on the left hand side; these act as sources for operators of the CFT on the right hand side.

A problem with Eq. (2.3) as it stands is that the usual gravitational action in AdS is divergent, rendering the path integral ill-defined. A procedure for solving this problem was developed in [22,24–29]. First one brings the boundary into a finite radius. Next one adds a finite number of counterterms to the action in order to render it finite as the boundary is moved back off to infinity. These counterterms can be expressed solely in terms of the geometry of the boundary. The total gravitational action for AdS_{d+1} becomes

$$S_{grav} = S_{EH} + S_{GH} + S_1 + S_2 + \cdots. \tag{2.4}$$

The first term is the usual Einstein-Hilbert action[6] with a negative cosmological constant:

$$S_{EH} = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{g}\left(R + \frac{d(d-1)}{l^2}\right) \tag{2.5}$$

the overall minus sign arises because we are considering a Euclidean theory. The second term in the action is the Gibbons-Hawking boundary term, which is necessary for a well-defined variational problem [30]:

$$S_{GH} = -\frac{1}{8\pi G}\int d^d x\sqrt{h}K, \tag{2.6}$$

where K is the trace of the extrinsic curvature of the boundary[7] and h the determinant of the induced metric. The first two counterterms are given by the following [26–29] (we use the results of [29] rotated to Euclidean signature):

$$S_1 = \frac{d-1}{8\pi G l}\int d^d x\sqrt{h}, \tag{2.7}$$

$$S_2 = \frac{l}{16\pi G(d-2)}\int d^d x\sqrt{h}R, \tag{2.8}$$

where R now refers to the Ricci scalar of the boundary metric. The third counterterm is

$$S_3 = \frac{l^3}{16\pi G(d-2)^2(d-4)}\int d^d x\sqrt{h}\left(R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2\right), \tag{2.9}$$

where R_{ij} is the Ricci tensor of the boundary metric and boundary indices i,j are raised and lowered with the boundary metric h_{ij}. This expression is ill-defined for $d=4$, which is the case of most interest to us. With just the first two counterterms, the gravitational action exhibits logarithmic divergences [24–26] so a third term is needed. This term cannot be written solely in terms of a polynomial in scalar invariants of the induced metric and curvature tensors; it makes explicit reference to the cutoff (i.e. the finite radius to which the boundary is brought before taking the limit in which it tends to infinity). The form of this term is the same as Eq. (2.9) with the divergent factor of $1/(d-4)$ replaced by $\log(R/\rho)$, where R measure the boundary radius and ρ is some finite renormalization length scale.

Following [23], we can now use the AdS-CFT correspondence to explain the behavior discovered by Randall and Sundrum. The (Euclidean) RS model has the following action:

$$S_{RS} = S_{EH} + S_{GH} + 2S_1 + S_m. \tag{2.10}$$

Here $2S_1$ is the action of a domain wall with tension $(d-1)/(4\pi G l)$. The final term is the action for any matter present on the domain wall. The domain wall tension can cancel the effect of the bulk cosmological constant to produce a flat domain wall. However, we are interested in a spherical domain wall so we assume that the matter on the wall gives an extra contribution to the effective tension. We

[6]We use a positive signature metric and a curvature convention for which a sphere has positive Ricci scalar.

[7]Our convention is the following. Let n denote the outward unit normal to the boundary. The extrinsic curvature is defined as $K_{\mu\nu} = h_\mu^\rho h_\nu^\sigma \nabla_\rho n_\sigma$, where $h_\mu^\nu = \delta_\mu^\nu - n_\mu n^\nu$ projects quantities onto the boundary.

shall discuss a specific candidate for the matter on the wall later on. The wall separates two balls B_1 and B_2 of AdS.

We want to study quantum fluctuations of the metric on the domain wall. Let $\mathbf{g}_0$ denote the five dimensional background metric we have just described and $\mathbf{h}_0$ the metric it induces on the wall. Let $\mathbf{h}$ denote a metric perturbation on the wall. If we wish to calculate correlators of $\mathbf{h}$ on the domain wall then we are interested in a path integral of the form[8]

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle = \int d[\mathbf{h}]Z[\mathbf{h}]h_{ij}(x)h_{i'j'}(x'), \tag{2.11}$$

where

$$\begin{aligned} Z[\mathbf{h}] &= \int_{B_1\cup B_2} d[\delta\mathbf{g}]d[\phi]\exp(-S_{RS}[\mathbf{g}_0+\delta\mathbf{g}]) \\ &= \exp(-2S_1[\mathbf{h}_0+\mathbf{h}])\int_{B_1\cup B_2} d[\delta\mathbf{g}]d[\phi] \\ &\quad\times\exp(-S_{EH}[\mathbf{g}_0+\delta\mathbf{g}] \\ &\quad -S_{GH}[\mathbf{g}_0+\delta\mathbf{g}]-S_m[\phi;\mathbf{h}_0+\mathbf{h}]), \end{aligned} \tag{2.12}$$

$\delta\mathbf{g}$ denotes a metric perturbation in the bulk that approaches $\mathbf{h}$ on the boundary and ϕ denotes the matter fields on the domain wall. The integrals in the two balls are independent so we can replace the path integral by

$$\begin{aligned} Z[\mathbf{h}] &= \exp(-2S_1[\mathbf{h}_0+\mathbf{h}]) \\ &\quad\times\left(\int_B d[\delta\mathbf{g}]\exp(-S_{EH}[\mathbf{g}_0+\delta\mathbf{g}]-S_{GH}[\mathbf{g}_0+\delta\mathbf{g}])\right)^2 \\ &\quad\times\int d[\phi]\exp(-S_m[\phi;\mathbf{h}_0+\mathbf{h}]), \end{aligned} \tag{2.13}$$

where B denotes either ball. We now take $d=4$ and use the AdS-CFT correspondence (2.3) to replace the path integral over $\delta\mathbf{g}$ by the generating functional for a conformal field theory:

$$\begin{aligned} &\int_B d[\delta\mathbf{g}]\exp(-S_{EH}[\mathbf{g}_0+\delta\mathbf{g}]-S_{GH}[\mathbf{g}_0+\delta\mathbf{g}]) \\ &\quad=\exp(-W_{RS}[\mathbf{h}_0+\mathbf{h}]+S_1[\mathbf{h}_0+\mathbf{h}] \\ &\qquad+S_2[\mathbf{h}_0+\mathbf{h}]+S_3[\mathbf{h}_0+\mathbf{h}]), \end{aligned} \tag{2.14}$$

we shall refer to this CFT as the RS CFT since it arises as the dual of the RS geometry. It has gauge group $U(N_{RS})$, where N_{RS} is given by Eq. (2.2). Strictly speaking, we are using an extended form of the AdS-CFT conjecture, which asserts that supergravity theory in a finite region of AdS is dual to a CFT on the boundary of that region with an ultraviolet cutoff related to the radius of the boundary.[9] The path integral for the metric perturbation becomes

$$\begin{aligned} Z[\mathbf{h}] &= \exp(-2W_{RS}[\mathbf{h}_0+\mathbf{h}]+2S_2[\mathbf{h}_0+\mathbf{h}] \\ &\quad+2S_3[\mathbf{h}_0+\mathbf{h}])\int d[\phi]\exp(-S_m[\phi;\mathbf{h}_0+\mathbf{h}]). \end{aligned} \tag{2.15}$$

The RS model has been replaced by a CFT and a coupling to matter fields and the domain wall metric given by the action

$$-2S_2[\mathbf{h}_0+\mathbf{h}]-2S_3[\mathbf{h}_0+\mathbf{h}]+S_m[\phi;\mathbf{h}_0+\mathbf{h}]. \tag{2.16}$$

The remarkable feature of this expression is that the term $-2S_2$ is precisely the (Euclidean) Einstein-Hilbert action for *four dimensional* gravity with a Newton constant given by the RS value

$$G_4 = G/l. \tag{2.17}$$

Therefore the RS model is equivalent to four dimensional gravity coupled to a CFT with corrections to gravity coming from the third counterterm. This explains why gravity is trapped to the domain wall.

At first sight this appears rather amazing. We started off with a quite complicated five dimensional system and have argued that it is dual to four dimensional Einstein gravity with some corrections and matter fields. However in order to use this description, we have to know how to calculate with the RS CFT. At present, the only way we know of doing this is via AdS-CFT, i.e., going back to the five dimensional description. The point of the AdS-CFT argument is to explain why the RS ''alternative to compactification'' works and also to explain the origin of the corrections to Einstein gravity in the RS model. Note that if the matter on the domain wall dominates the RS CFT and the third counterterm then these can be neglected and a purely four dimensional description is adequate.

III. CFT ON THE DOMAIN WALL

A. Introduction

Long ago, Starobinsky studied the cosmology of a universe containing conformally coupled matter [19]. CFTs generally exhibit a conformal anomaly when coupled to gravity (for a review, see [32]). Starobinsky gave a de Sitter solution in which the anomaly provides the cosmological constant. By analyzing homogeneous perturbations of this model, he showed that the de Sitter phase is unstable but could be long lived, eventually decaying to a Friedmann-Robertson-Walker (FRW) cosmology.

[8] In principle, we should worry about gauge fixing and ghost contributions to the gravitational action. A convenient gauge to use in the bulk is transverse traceless gauge. We shall only deal with metric perturbations that also appear transverse and traceless on the domain wall. The gauge fixing terms vanish for such perturbations and the ghosts only couple to these perturbations at higher orders.

[9] Evidence in support of this extended version of the duality was given in [31].

In this section we will consider the RS analogue of Starobinsky's model by putting a CFT on the domain wall. On a spherical domain wall, the conformal anomaly provides the extra tension required to satisfy the Israel equations. It is appealing to choose the new CFT to be a $\mathcal{N}=4$ superconformal field theory because then the AdS-CFT correspondence makes calculations relatively easy.[10] This requires that the CFT is strongly coupled, in contrast with Starobinsky's analysis.[11]

Our five dimensional (Euclidean) action is the following:

$$S=S_{EH}+S_{GH}+2S_1+W_{CFT}. \tag{3.1}$$

We seek a solution in which two balls of AdS_5 are separated by a spherical domain wall. Inside each ball, the metric can be written

$$ds^2=l^2(dy^2+\sinh^2 y\, d\Omega_d^2), \tag{3.2}$$

with $0\leqslant y\leqslant y_0$. The domain wall is at $y=y_0$ and has radius

$$R=l\sinh y_0. \tag{3.3}$$

The effective tension of the domain wall is given by the Israel equations as

$$\sigma_{eff}=\frac{3}{4\pi Gl}\coth y_0. \tag{3.4}$$

The actual tension of the domain wall is

$$\sigma=\frac{3}{4\pi Gl}. \tag{3.5}$$

We therefore need a contribution to the effective tension from the CFT. This is provided by the conformal anomaly, which takes the value [24–26]

$$\langle T\rangle=-\frac{3N^2}{8\pi^2R^4}. \tag{3.6}$$

This contributes an effective tension $-\langle T\rangle/4$. We can now obtain an equation for the radius of the domain wall:

$$\frac{R^3}{l^3}\sqrt{\frac{R^2}{l^2}+1}=\frac{N^2G}{8\pi l^3}+\frac{R^4}{l^4}. \tag{3.7}$$

It is easy to see that this has a unique positive solution for R. We shall derive this equation directly from the action in Sec. III C.

[10] We emphasize that this use of the AdS-CFT correspondence is independent of the use described above because this new CFT is unrelated to the RS CFT.

[11] Note that the conformal anomaly is the same at strong and weak coupling [25] so any differences arising from strong coupling can only show up when we perturb the system.

We are particularly interested in how perturbations of this model would appear to inhabitants of the domain wall. Thus we are interested in metric perturbations on the sphere

$$ds^2=(R^2\hat{\gamma}_{ij}+h_{ij})dx^i dx^j. \tag{3.8}$$

Here $\hat{\gamma}_{ij}$ is the metric on a unit d-sphere. We shall only consider *tensor* perturbations, for which h_{ij} is transverse and traceless with respect to $\hat{\gamma}_{ij}$. In order to calculate correlators of the metric perturbation, we need to know the action to second order in the perturbation. The most difficult part here is obtaining W_{CFT} to second order. This is the subject of the next subsection.

B. CFT generating function

We want to work out the effect of the perturbation on the CFT on the sphere. To do this we use AdS-CFT. Introduce a fictional AdS region that fills in the sphere. Let $\bar{l},\bar{G}$ be the AdS radius and Newton constant of this region. We emphasize that this region has nothing to do with the regions of AdS that "really" lie inside the sphere in the RS scenario. This new AdS region is bounded by the sphere. If we take $\bar{l}$ to zero then the sphere is effectively at infinity in AdS so we can use AdS-CFT to calculate the generating functional of the CFT on the sphere. In other words, $\bar{l}$ is acting like a cutoff in the CFT and taking it to zero corresponds to removing the cutoff. However the relation

$$\frac{\bar{l}^3}{\bar{G}}=\frac{2N^2}{\pi}, \tag{3.9}$$

implies that if $\bar{l}$ is taken to zero then we must also take $\bar{G}$ to zero since N is fixed (and large).

For the unperturbed sphere, the metric in the new AdS region is

$$ds^2=\bar{l}^2(dy^2+\sinh^2 y\,\hat{\gamma}_{ij}dx^i dx^j), \tag{3.10}$$

and the sphere is at $y=y_0$ given by $R=\bar{l}\sinh y_0$. Note that $y_0\to\infty$ as $\bar{l}\to 0$ since R is fixed. In order to use AdS-CFT for the perturbed sphere, we need to know how the perturbation extends into the bulk. This is done by solving the linearized Einstein equations. It is always possible to choose a gauge in which the bulk metric perturbation takes the form

$$h_{ij}(y,x)dx^i dx^j, \tag{3.11}$$

where h_{ij} is transverse and traceless with respect to the metric on the spherical spatial sections:

$$\hat{\gamma}^{ij}(x)h_{ij}(y,x)=\hat{\nabla}^i h_{ij}(y,x)=0, \tag{3.12}$$

with $\hat{\nabla}$ denoting the covariant derivative defined by the metric $\hat{\gamma}_{ij}$. Since we are only dealing with tensor perturbations, this choice of gauge is consistent with the boundary sitting at constant y. If scalar metric perturbations were included then we would have to take account of a perturbation in the position of the boundary. These issues are discussed in detail in Appendix A.

The linearized Einstein equations in the bulk are (for any dimension)

$$\nabla^2 h_{\mu\nu} = -\frac{2}{l^2} h_{\mu\nu}, \tag{3.13}$$

where μ,ν are $d+1$ dimensional indices. It is convenient to expand the metric perturbation in terms of tensor spherical harmonics $H_{ij}^{(p)}(x)$. These obey

$$\hat{\gamma}^{ij} H_{ij}^{(p)}(x) = \hat{\nabla}^i H_{ij}^{(p)}(x) = 0, \tag{3.14}$$

and they are tensor eigenfunctions of the Laplacian:

$$\hat{\nabla}^2 H_{ij}^{(p)} = (2 - p(p+d-1)) H_{ij}^{(p)}, \tag{3.15}$$

where $p = 2,3,\ldots$. We have suppressed extra labels $k,l,m,\ldots$ on these harmonics. The harmonics are orthonormal with respect to the obvious inner product. See Appendix B and [33] for more details of their properties. The metric perturbation can be written as a sum of separable perturbations of the form

$$h_{ij}(y,x) = f_p(y) H_{ij}^{(p)}(x). \tag{3.16}$$

Substituting this into Eq. (3.13) gives

$$f_p''(y) + (d-4)\coth y\, f_p'(y) - (2(d-2) + [p(p+d-1) + 2(d-3)]\mathrm{cosech}^2 y) f_p(y) = 0. \tag{3.17}$$

The roots of the indicial equation are $p+2$ and $-p-d+3$, yielding two linearly independent solutions for each p. In order to compute the generating functional W_{CFT} we have to calculate the Euclidean action of these solutions. However, because the latter solution goes as $y^{-(p+d-3)}$ at the origin $y=0$ of the instanton, the corresponding fluctuation modes have infinite Euclidean action.[12] Hence they are suppressed in the path integral. Therefore, in contrast to other methods [2,3] where one requires a (rather *ad hoc*) prescription for the vacuum state of each perturbation mode, there is no need to impose boundary conditions by hand in our approach: the Euclidean path integral defines its own boundary conditions, which automatically gives a unique Green function. The path integral unambiguously specifies the allowed fluctuation modes as those which vanish at $y=0$. Note that boundary conditions at the origin in Euclidean space replace the need for boundary conditions at the horizon in Lorentzian space. The solution regular at $y=0$ is given by

$$f_p(y) = \frac{\sinh^{p+2} y}{\cosh^p y} F(p/2,(p+1)/2,p+(d+1)/2,\tanh^2 y). \tag{3.18}$$

This solution can also be written in terms of associated Legendre functions:

$$f_p(y) \propto (\sinh y)^{(5-d)/2} P_{-(d+1)/2}^{-(p+(d-1)/2)}(\cosh y) \propto (\sinh y)^{(4-d)/2} Q_{p+(d-2)/2}^{d/2}(\coth y), \tag{3.19}$$

and the latter can be related to Legendre functions if $d/2$ is an integer, using

$$Q_\nu^m(z) = (z^2-1)^{m/2} \frac{d^m Q_\nu}{dz^m}. \tag{3.20}$$

The full solution for the metric perturbation is

$$h_{ij}(y,x) = \sum_p \frac{f_p(y)}{f_p(y_0)} H_{ij}^{(p)}(x) \int d^d x' \sqrt{\hat{\gamma}} h^{kl}(x') H_{kl}^{(p)}(x'). \tag{3.21}$$

We have a solution for the metric perturbation throughout the bulk region. The AdS-CFT correspondence can now be used to give the generating functional of the CFT on the perturbed sphere:

$$W_{CFT} = S_{EH} + S_{GH} + S_1 + S_2 + \cdots. \tag{3.22}$$

We shall give the terms on the right hand side for $d=4$.

The Einstein-Hilbert action with cosmological constant is

$$S_{EH} = -\frac{1}{16\pi G}\int d^5x \sqrt{g}\left(R + \frac{12}{l^2}\right), \tag{3.23}$$

and perturbing this gives

$$S_{bulk} = -\frac{1}{16\pi G}\int d^5x \sqrt{g} \times \left(-\frac{8}{l^2} + \frac{1}{4} h^{\mu\nu}\nabla^2 h_{\mu\nu} + \frac{1}{2l^2} h^{\mu\nu} h_{\mu\nu}\right) - \frac{1}{16\pi G}\int d^4x \sqrt{\gamma}\left(-\frac{1}{2} n^\mu h^{\nu\rho}\nabla_\nu h_{\mu\rho} + \frac{3}{4} h_{\nu\rho} n^\mu \nabla_\mu h^{\nu\rho}\right), \tag{3.24}$$

where Greek indices are five dimensional and we are raising and lowering with the unperturbed five dimensional metric. $n = l dy$ is the unit normal to the boundary and ∇ is the covariant derivative defined with the unperturbed bulk metric. $\gamma_{ij} = R^2 \hat{\gamma}_{ij}$ is the unperturbed boundary metric. It is im-

[12] This can be seen by surrounding the origin by a small sphere $y=\epsilon$ and calculating the surface terms in the actions that arise on this sphere. They are the same as the surface terms in Eqs. (3.25) and (3.26) below, which are obviously divergent for the modes in question.

portant to keep track of all the boundary terms arising from integration by parts. Evaluating on shell gives

$$S_{EH}=\frac{\bar{l}^3}{2\pi\bar{G}}\int d^4x\sqrt{\hat{\gamma}}\int_0^{y_0}dy\,\sinh^4 y -\frac{\bar{l}^3}{16\pi\bar{G}}\int d^4x\sqrt{\hat{\gamma}}\left(\frac{3}{4\bar{l}^4}h^{ij}\partial_y h_{ij}-\frac{\coth y_0}{\bar{l}^4}h^{ij}h_{ij}\right), \tag{3.25}$$

where we are now raising and lowering with $\hat{\gamma}_{ij}$. The Gibbons-Hawking term is

$$S_{GH}=-\frac{\bar{l}^3}{2\pi\bar{G}}\int d^4x\sqrt{\hat{\gamma}}\left(\sinh^3 y_0\cosh y_0-\frac{1}{8\bar{l}^4}h^{ij}\partial_y h_{ij}\right). \tag{3.26}$$

The first counterterm is

$$S_1=\frac{3}{8\pi\bar{G}\bar{l}}\int d^4x\sqrt{\gamma} =\frac{3\bar{l}^3}{8\pi\bar{G}}\int d^4x\sqrt{\hat{\gamma}}\left(\sinh^4 y_0-\frac{1}{4\bar{l}^4}h^{ij}h_{ij}\right). \tag{3.27}$$

The second counterterm is

$$S_2=\frac{\bar{l}}{32\pi\bar{G}}\int d^4x\sqrt{\gamma}R =\frac{\bar{l}^3}{32\pi\bar{G}}\int d^4x\sqrt{\hat{\gamma}}\left(12\sinh^2 y_0-\frac{2}{\bar{l}^4\sinh^2 y_0}h^{ij}h_{ij} +\frac{1}{4\bar{l}^4\sinh^2 y_0}h^{ij}\hat{\nabla}^2 h_{ij}\right). \tag{3.28}$$

Thus with only two counterterms we would have

$$W_{CFT}=\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R}{\bar{l}}-\frac{\bar{l}^3}{16\pi\bar{G}}\int d^4x\sqrt{\hat{\gamma}}\left(-\frac{1}{4\bar{l}^4}h^{ij}\partial_y h_{ij} +\frac{1}{\bar{l}^4}h^{ij}h_{ij}\left(\frac{3}{2}-\sqrt{1+\frac{\bar{l}^2}{R^2}}\right)+\frac{1}{\bar{l}^2R^2}h^{ij}h_{ij} -\frac{1}{8\bar{l}^2R^2}h^{ij}\hat{\nabla}^2 h_{ij}\right). \tag{3.29}$$

Ω_4 is the area of a unit four-sphere and we have used Eq. (3.9). The expansion of $\partial_y h_{ij}$ at $y=y_0$ is obtained from

$$\partial_y h_{ij}=\sum_p\frac{f_p'(y_0)}{f_p(y_0)}H_{ij}^{(p)}(x)\int d^4x'\sqrt{\hat{\gamma}}h^{kl}(x')H_{kl}^{(p)}(x') \tag{3.30}$$

and

$$\frac{f_p'(y_0)}{f_p(y_0)}=2+\frac{\bar{l}^2}{2R^2}(p+1)(p+2)+p(p+1)(p+2)(p+3) \times\frac{\bar{l}^4}{4R^4}\log(\bar{l}/R)+\frac{\bar{l}^4}{8R^4}[p^4+2p^3-5p^2-10p-2 -p(p+1)(p+2)(p+3)(\psi(1)+\psi(2)-\psi(p/2 +2)-\psi(p/2+5/2))]+\mathcal{O}\left(\frac{\bar{l}^6}{R^6}\log(\bar{l}/R)\right). \tag{3.31}$$

The psi function is defined by $\psi(z)=\Gamma'(z)/\Gamma(z)$. Substituting into the action we find that the divergences as $\bar{l}\to 0$ cancel at order $R^4/\bar{l}^4$ and $R^2/\bar{l}^2$. The term of order $\bar{l}^4/R^4$ in the above expansion makes a contribution to the finite part of the action [along with a term from the square root in Eq. (3.29)]:

$$W_{CFT}=\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R}{\bar{l}} +\frac{N^2}{256\pi^2R^4}\sum_p\left(\int d^4x'\sqrt{\hat{\gamma}}h^{kl}(x')H_{kl}^{(p)}(x')\right)^2 \times(2p(p+1)(p+2)(p+3)\log(\bar{l}/R)+\Psi(p)), \tag{3.32}$$

where

$$\Psi(p)=p(p+1)(p+2)(p+3)[\psi(p/2+5/2) +\psi(p/2+2)-\psi(2)-\psi(1)] +p^4+2p^3-5p^2-10p-6. \tag{3.33}$$

To cancel the logarithmic divergences as $\bar{l}\to 0$, we have to introduce a length scale ρ defined by $\bar{l}=\epsilon\rho$ and add a counterterm proportional to $\log\epsilon$ to cancel the divergence as ϵ tends to zero. The counterterm is

$$S_3=-\frac{\bar{l}^3}{64\pi\bar{G}}\log\epsilon\int d^4x\sqrt{\gamma}\left(\gamma^{ik}\gamma^{jl}R_{ij}R_{kl}-\frac{1}{3}R^2\right) =-\frac{\bar{l}^3}{64\pi\bar{G}}\log\epsilon\int d^4x\sqrt{\hat{\gamma}}\left(-12+\frac{1}{R^4}\left[2h^{ij}h_{ij} -\frac{3}{2}h^{ij}\hat{\nabla}^2h_{ij}+\frac{1}{4}h^{ij}\hat{\nabla}^4h_{ij}\right]\right). \tag{3.34}$$

This term does indeed cancel the logarithmic divergence, leaving us with

$$W_{CFT}=\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R}{\rho}+\frac{N^2}{256\pi^2R^4}\times\sum_p\left(\int d^4x'\sqrt{\hat{\gamma}}h^{kl}(x')H^{(p)}_{kl}(x')\right)^2\times(2p(p+1)(p+2)(p+3)\log(\rho/R)+\Psi(p)). \tag{3.35}$$

Note that varying W_{CFT} twice with respect to h_{ij} yields the expression for the transverse traceless part of the correlator $\langle T_{ij}(x)T_{i'j'}(x')\rangle$ on a round four sphere. At large p, this behaves like $p^4\log p$, as expected from the flat space result [21]. In fact this correlator can be determined in closed form solely from the trace anomaly and symmetry considerations.[13] However, we shall be be interested in calculating cosmologically observable effects, for which our mode expansion is more useful.

C. The total action

Recall that our five dimensional action is

$$S=S_{EH}+S_{GH}+2S_1+W_{CFT}. \tag{3.36}$$

In order to calculate correlators of the metric, we need to evaluate the path integral

$$Z[\mathbf{h}]=\int_{B_1\cup B_2}d[\delta\mathbf{g}]\exp(-S)=\exp(-2S_1[\mathbf{h}_0+\mathbf{h}]-W_{CFT}[\mathbf{h}_0+\mathbf{h}])\times\left(\int_B d[\delta\mathbf{g}]\exp(-S_{EH}[\mathbf{g}_0+\delta\mathbf{g}]-S_{GH}[\mathbf{g}_0+\delta\mathbf{g}])\right)^2. \tag{3.37}$$

Here $\mathbf{g}_0$ and $\mathbf{h}_0$ refer to the unperturbed background metrics in the bulk and on the wall respectively and $\mathbf{h}$ denotes the metric perturbation on the wall. Many of the terms required here can be obtained from results in the previous section by simply replacing $\bar{l}$ and $\bar{G}$ with l and G. For example, from Eq. (3.27) we obtain

$$S_1[\mathbf{h}_0+\mathbf{h}]=\frac{3l^3}{8\pi G}\int d^4x\sqrt{\hat{g}}\left(\sinh^4 y_0-\frac{1}{4l^4}h^{ij}h_{ij}\right), \tag{3.38}$$

where y_0 is defined by $R=l\sinh y_0$. The path integral over $\delta\mathbf{g}$ is performed by splitting it into a classical and quantum part:

$$\delta\mathbf{g}=\mathbf{h}+\mathbf{h}', \tag{3.39}$$

where the boundary perturbation $\mathbf{h}$ is extended into the bulk using the linearized Einstein equations and the requirement of finite Euclidean action, i.e., $\mathbf{h}$ is given in the bulk by Eq. (3.21). $\mathbf{h}'$ denotes a quantum fluctuation that vanishes at the domain wall. The gravitational action splits into separate contributions from the classical and quantum parts:

$$S_{EH}+S_{GH}=S_0[\mathbf{h}]+S'[\mathbf{h}'], \tag{3.40}$$

where S_0 can be read off from Eqs. (3.25) and (3.26) as

$$S_0=-\frac{3l^3\Omega_4}{2\pi G}\int_0^{y_0}dy\sinh^2 y_0\cosh^2 y_0+\frac{l^3}{16\pi G}\int d^4x\sqrt{\hat{\gamma}}\left(\frac{1}{4l^4}h^{ij}\partial_y h_{ij}+\frac{\coth y_0}{l^4}h^{ij}h_{ij}\right). \tag{3.41}$$

Note that S' cannot be converted to a surface term since $\mathbf{h}'$ does not satisfy the Einstein equations. We shall not need the explicit form for S' since the path integral over $\mathbf{h}'$ just contributes a factor of some determinant Z_0 to $Z[\mathbf{h}]$. We obtain

$$Z[\mathbf{h}]=Z_0\exp(-2S_0[\mathbf{h}_0+\mathbf{h}]-2S_1[\mathbf{h}_0+\mathbf{h}]-W_{CFT}[\mathbf{h}_0+\mathbf{h}]). \tag{3.42}$$

The exponent is given by

$$2S_0+2S_1+W_{CFT}=-\frac{3l^3\Omega_4}{\pi G}\int_0^{y_0}dy\sinh^2 y\cosh^2 y+\frac{3\Omega_4R^4}{4\pi Gl}+\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R}{\rho}+\frac{1}{l^4}\sum_p\left(\int d^4x'\sqrt{\hat{\gamma}}h^{kl}(x')H^{(p)}_{kl}(x')\right)^2\times\left[\frac{l^3}{32\pi G}\left(\frac{f_p'(y_0)}{f_p(y_0)}+4\coth y_0-6\right)+\frac{N^2}{256\pi^2\sinh^4 y_0}\times(2p(p+1)(p+2)(p+3)\log(\rho/R)+\Psi(p))\right]. \tag{3.43}$$

We have kept the unperturbed action in order to demonstrate how the conformal anomaly arises: it is simply the coefficient of the $\log(R/\rho)$ term divided by the area Ω_4R^4 of the sphere. If we set the metric perturbation to zero and vary R in Eq. (3.43) (using $R=l\sinh y_0$) then we reproduce Eq. (3.7).

Having calculated R, we can now choose a convenient value for the renormalization scale ρ. If we were dealing purely with the CFT then we could keep ρ arbitrary. However, since the third counterterm [Eq. (3.34)] involves the square of the Weyl tensor (the integrand is proportional to the difference of the Euler density and the square of the Weyl tensor), we can expect pathologies to arise if this term is present when we couple the CFT to gravity. In other

[13]See [34] for a general discussion of such correlators on maximally symmetric spaces.

words, when coupled to gravity, different choices of ρ lead to different theories. We shall choose the value $\rho=R$ so that the third counter term exactly cancels the divergence in the CFT, with no finite remainder and hence no residual curvature squared terms in the action.

The (Euclidean) graviton correlator can be read off from the action as

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle=\frac{128\pi^2R^4}{N^2}\sum_{p=2}^{\infty}W^{(p)}_{iji'j'}(x,x')F(p,y_0)^{-1} \tag{3.44}$$

where we have eliminated l^3/G using Eq. (3.7). The function $F(p,y_0)$ is given by

$$F(p,y_0)=e^{y_0}\sinh y_0\left(\frac{f_p'(y_0)}{f_p(y_0)}+4\coth y_0-6\right)+\Psi(p), \tag{3.45}$$

and the bitensor $W^{(p)}_{iji'j'}(x,x')$ is defined as

$$W^{(p)}_{iji'j'}(x,x')=\sum_{k,l,m,\ldots}H^{(p)}_{ij}(x)H^{(p)}_{i'j'}(x'), \tag{3.46}$$

with the sum running over all the suppressed labels $k,l,m,\ldots$ of the tensor harmonics.

The appearance of N^2 in the denominator in Eq. (3.44) suggests that the CFT suppresses metric perturbations on all scales. This is misleading because R also depends on N. The function $F(p,y_0)$ has the following limiting forms for large and small radius:

$$\lim_{y_0\to\infty}F(p,y_0)=\Psi(p)+p^2+3p+6, \tag{3.47}$$

$$\lim_{y_0\to 0}F(p,y_0)=\Psi(p)+p+6. \tag{3.48}$$

$F(p,y_0)$ has poles at $p=-4,-5,-6,\ldots$ with zeros between each pair of negative integers starting at $-3,-4$. When we analytically continue to Lorentzian signature, we shall be particularly interested in zeros lying in the range $p\geqslant-3/2$. There is one such zero exactly at $p=0$, another near $p=0$ and a third near $p=-3/2$. For large radius, these extra zeros are at $p\approx-0.054$ and $p\approx-1.48$ while for small radius they are at $p\approx0.094$ and $p\approx-1.60$. For intermediate radius they lie between these values, with the zeros crossing through $-3/2$ and 0 at $y_0\approx0.632$ and $y_0\approx1.32$ respectively.

D. Comparison with four dimensional gravity

We discussed in Sec. II how the RS scenario reproduces the predictions of four dimensional gravity when the effects of matter on the domain wall dominates the effects of the RS CFT. In our case we have a CFT on the domain wall. This has action proportional to N^2. The RS CFT is a similar CFT (but with a cutoff) and therefore has action proportional to N_{RS}^2. Hence we can neglect it when $N\gg N_{RS}$. The logarithmic counterterm is also proportional to N_{RS}^2 and therefore also negligible. We therefore expect the predictions of four dimensional gravity to be recovered when $N\gg N_{RS}$. We shall now demonstrate this explicitly.

First consider the radius R of the domain wall given by Eq. (3.7). It is convenient to write this in terms of the rank N_{RS} of the RS CFT (given by $l^3/G=2N_{RS}^2/\pi$)

$$\frac{R^3}{l^3}\sqrt{\frac{R^2}{l^2}+1}=\frac{N^2}{16N_{RS}^2}+\frac{R^4}{l^4}. \tag{3.49}$$

If we assume $N\gg N_{RS}\gg1$ then the solution is

$$\frac{R}{l}=\frac{N}{2\sqrt{2}N_{RS}}\left[1+\frac{N_{RS}^2}{N^2}+\mathcal{O}(N_{RS}^4/N^4)\right]. \tag{3.50}$$

Note that this implies $R\gg l$, i.e., the domain wall is large compared with the anti–de Sitter length scale.

Now let's turn to a four dimensional description in which we are considering a four sphere with no interior. The only matter present is the CFT. The metric is simply

$$ds^2=R_4^2\hat{\gamma}_{ij}dx^idx^j, \tag{3.51}$$

where R_4 remains to be determined. The action is the four dimensional Einstein-Hilbert action (without cosmological constant) together with W_{CFT}. There is no Gibbons-Hawking term because there is no boundary. Without a metric perturbation, the action is simply

$$\begin{aligned}S&=-\frac{1}{16\pi G_4}\int d^4x\sqrt{\gamma}R+W_{CFT}\\&=-\frac{3\Omega_4R_4^2}{4\pi G_4}+\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R_4}{\rho},\end{aligned} \tag{3.52}$$

where G_4 is the four dimensional Newton constant. We want to calculate the value of R_4 so we cannot choose $\rho=R_4$ yet. Varying R_4 gives

$$R_4^2=\frac{N^2G_4}{4\pi}, \tag{3.53}$$

and N is large hence R_4 is much greater than the four dimensional Planck length. Substituting $G_4=G_5/l$, this reproduces the leading order value for R found above from the five dimensional calculation.

We can now go further and include the metric perturbation. The perturbed four dimensional Einstein-Hilbert action is

$$S_{EH}^{(4)}=-\frac{1}{16\pi G_4}\int d^4x\sqrt{\hat{\gamma}}\times\left(12R_4^2-\frac{2}{R_4^2}h^{ij}h_{ij}+\frac{1}{4R_4^2}h^{ij}\hat{\nabla}^2h_{ij}\right). \tag{3.54}$$

Adding the perturbed CFT gives

$$S=-\frac{3N^2\Omega_4}{16\pi^2}+\frac{3N^2\Omega_4}{8\pi^2}\log\frac{R_4}{\rho}+\sum_p\left(\int d^4x'\sqrt{\hat{\gamma}}h^{kl}(x')H_{kl}^{(p)}(x')\right)^2\times\left[\frac{1}{64\pi G_4R_4^2}(p^2+3p+6)+\frac{N^2}{256\pi^2R_4^4}(2p(p+1)\times(p+2)(p+3)\log(\rho/R_4)+\Psi(p))\right]. \tag{3.55}$$

Setting $\rho=R_4$, we find that the graviton correlator for a four dimensional universe containing the CFT is

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle=8N^2G_4^2\sum_{p=2}^{\infty}W_{iji'j'}^{(p)}(x,x')[p^2+3p+6+\Psi(p)]^{-1}. \tag{3.56}$$

This can be compared with the expression obtained from the five dimensional calculation, which can be written

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle=\frac{8N^2G^2}{l^2}[1+\mathcal{O}(N_{RS}^2/N^2)]\sum_{p=2}^{\infty}W_{iji'j'}^{(p)}(x,x')\times[p^2+3p+6+\Psi(p)+4p(p+1)(p+2)(p+3)\times(N_{RS}^2/N^2)\log(N_{RS}/N)+\mathcal{O}(N_{RS}^2/N^2)]^{-1}. \tag{3.57}$$

We have expanded in terms of

$$\frac{N_{RS}^2}{N^2}=\frac{\pi l^3}{2N^2G}. \tag{3.58}$$

The four and five dimensional expressions clearly agree (for $G_4=G/l$) when $N\gg N_{RS}$, i.e., $R\gg l$. There are corrections of order $(N_{RS}^2/N^2)\log(N_{RS}/N)$ coming from the RS CFT and the logarithmic counterterm. In fact, these corrections can be absorbed into the renormalization of the CFT on the domain wall if, instead of choosing $\rho=R$, we choose

$$\rho=R\left(1-\frac{2N_{RS}^2}{N^2}\log(N_{RS}/N)\right). \tag{3.59}$$

The corrections to the four dimensional expression are then of order N_{RS}^2/N^2. We shall not give these correction terms explicitly although they are easily obtained from the exact result (3.44).

E. Lorentzian correlator

In this subsection we shall show how the Euclidean correlator calculated above is analytically continued to give a correlator for Lorentzian signature. We have put many of the details in Appendix B but the analysis is still rather technical so the reader may wish to skip to the final result, which is given in Eq. (3.66). The techniques used here were developed in [35–37].

Let us first introduce a new label $p'=i(p+3/2)$, so that on the four sphere

$$\hat{\nabla}^2H_{ij}^{(p')}=\lambda_{p'}H_{ij}^{(p')}, \tag{3.60}$$

where $p'=7i/2,9i/2,\dots$ and

$$\lambda_{p'}=(p'^2+17/4). \tag{3.61}$$

Recall that there are extra labels on the tensor harmonics that we have suppressed. The set of rank-two tensor eigenmodes on S^4 forms a representation of the symmetry group of the manifold. Hence the sum [Eq. (B2)] of the degenerate eigenfunctions with eigenvalue $\lambda_{p'}$ defines a maximally symmetric bitensor $W_{(p')i'j'}^{ij}(\mu(\Omega,\Omega'))$, where $\mu(\Omega,\Omega')$ is the distance along the shortest geodesic between the points with polar angles Ω and Ω'. The expression of the bitensor in terms of a set of fundamental bitensors with μ-dependent coefficient functions together with the relation between the bitensors on S^4 and Lorentzian de Sitter space are obtained in Appendix B.

The motivation for the unusual labelling is that, as demonstrated in Appendix B, in terms of the label p' the bitensor on S^4 has exactly the same formal expression as the corresponding bitensor on Lorentzian de Sitter space. This property will enable us to analytically continue the Euclidean correlator into the Lorentzian region without Fourier decomposing it. In other words, instead of imposing by hand a prescription for the vacuum state of the graviton on each mode separately and propagating the individual modes into the Lorentzian region, we compute the two-point tensor correlator in real space, directly from the no boundary path integral. Since the path integral unambiguously specifies the allowed fluctuation modes as those which vanish at the origin of the instanton (see discussion in Sec. III B), this automatically gives a unique Euclidean correlator. The technical advantage of our method is that dealing directly with the real space correlator makes the derivation independent of the gauge ambiguities involved in the mode decomposition [37].

We begin by continuing the graviton correlator [Eq. (3.44)] obtained via the five dimensional calculation. The analytic continuation of the correlator for four dimensional gravity [Eq. (3.56)] is completely analogous. In terms of the new label p', the Euclidean correlator (3.44) between two points on the wall is given by

$$\langle h_{ij}(\Omega)h_{i'j'}(\Omega')\rangle=\frac{128\pi^2R^4}{N^2}\sum_{p'=7i/2}^{i\infty}W_{iji'j'}^{(p')}(\mu)G(p',y_0)^{-1} \tag{3.62}$$

where

$$
\begin{aligned}
G(p',y_0) &= F(-ip'-3/2,y_0)\\
&= e^{y_0}\sinh y_0\left(\frac{g'_{p'}(y_0)}{g_{p'}(y_0)}+4\coth y_0-6\right)\\
&\quad +(p'^4-4ip'^3+p'^2/2-5ip'-63/16\\
&\quad +(p'^2+1/4)(p'^2+9/4)[\psi(-ip'/2+5/4)\\
&\quad +\psi(-ip'/2+7/4)-\psi(1)-\psi(2)]), \quad (3.63)
\end{aligned}
$$

with $g_{p'}(y)=Q^2_{-ip'-1/2}(\coth y)$, which follows from Eq. (3.19). The function $G(p',y_0)$ is real and positive for all values of p' in the sum and for arbitrary $y_0\geqslant 0$.

We have the Euclidean correlator defined as an infinite sum. However, the eigenspace of the Laplacian on de Sitter space suggests that the Lorentzian propagator is most naturally expressed as an integral over real p'. We must therefore first analytically continue our result from imaginary to real p'. The coefficient $G(p',y_0)^{-1}$ of the bitensor is analytic in the upper half complex p'-plane, apart from three simple poles on the imaginary axis. One of them is always at $p'=3i/2$, regardless of the radius of the sphere. Let the position of the remaining two poles be written $p'_k=i\Lambda_k(y_0)$. If we take the radius of the domain wall to be large compared with the AdS scale (which is necessary for corrections to four dimensional Einstein gravity to be small) then[14] $0<\Lambda_k\leqslant 3/2$, with $\Lambda_1\sim 0$ and $\Lambda_2\sim 3/2$. Since $G(p',y_0)$ is real on the imaginary p'-axis, the residues at these poles are purely imaginary. In order to extend the correlator into the complex p'-plane, we must also understand the continuation of the bitensor itself. As shown in Appendix B, the condition of regularity at opposite points on the four sphere imposed by the completeness relation [Eq. (B4)] is sufficient to uniquely specify the analytic continuation of $W^{(p')}_{iji'j'}(\mu)$ into the complex p'-plane. The extended bitensor is defined by Eqs. (B5), (B8), and (B11).

Now we are able to write the sum in Eq. (3.62) as an integral along a contour $\mathcal{C}_1$ encircling the points $p'=7i/2,9i/2,\ldots ni/2$, where n tends to infinity. This yields

$$
\langle h_{ij}(\Omega)h_{i'j'}(\Omega')\rangle = \frac{-i64\pi^2R^4}{N^2}\int_{\mathcal{C}_1}dp'\tanh p'\pi \times W^{(p')}_{iji'j'}(\mu)G(p',y_0)^{-1}. \quad (3.64)
$$

Since we know the analytic properties of the integrand in the upper half complex p'-plane, we can distort the contour for the p' integral to run along the real axis. At large imaginary p' the integrand decays and the contribution vanishes in the large n limit. However as we deform the contour towards the real axis, we encounter three extra poles in the $\cosh p'\pi$ factor, the pole at $p'=3i/2$ becoming a double pole due to the simple zero of $G(p',y_0)$. In addition, we have to take in account the two poles of $G(p',y_0)^{-1}$ at $p'=i\Lambda_k$.

For the $p'=5i/2$ pole, it follows from the normalization of the tensor harmonics that $W^{(5i/2)}_{iji'j'}=0$. Indirectly, this is a consequence of the fact that spin-2 perturbations do not have a dipole or monopole component. The meaning of the remaining two poles of the $\tanh p'\pi$ factor has been extensively discussed in [37], where the continuation is described of the two-point tensor fluctuation correlator from a four dimensional $O(5)$ instanton into open de Sitter space. They represent non-physical contributions to the graviton propagator, arising from the different nature of tensor harmonics on S^4 and on Lorentzian de Sitter space. In fact, a degeneracy appears between $p'_t=3i/2$ and $p'_t=i/2$ tensor harmonics and respectively $p'_v=5i/2$ vector harmonics and $p'_s=5i/2$ scalar harmonics on S^4. More precisely, the tensor harmonics that constitute the bitensors $W^{iji'j'}_{(3i/2)}$ and $W^{iji'j'}_{(i/2)}$ can be constructed from a vector (scalar) quantity. Consequently, the contribution to the correlator from the former pole is pure gauge, while the latter eigenmode should really be treated as a scalar perturbation, using the perturbed scalar action. Henceforth we shall exclude them from the tensor spectrum. This leaves us with the poles of $G(p',y_0)$ at $p'=i\Lambda_k$. If we deform the contour towards the real axis, we must compensate for them by subtracting their residues from the integral. We will see that these residues correspond to discrete "supercurvature" modes in the Lorentzian tensor correlator.

The contribution from the closing of the contour in the upper half p'-plane vanishes. Hence our final result for the Euclidean correlator reads

$$
\begin{aligned}
&\langle h_{ij}(\Omega)h_{i'j'}(\Omega')\rangle\\
&= \frac{-i64\pi^2R^4}{N^2}\Bigg[\int_{-\infty}^{+\infty}dp'\tanh p'\pi W^{(p')}_{iji'j'}(\mu)\\
&\quad \times G(p',y_0)^{-1}+2\pi\sum_{k=1}^{2}\tan\Lambda_k\pi W^{(i\Lambda_k)}_{iji'j'}(\mu)\\
&\quad \times \mathrm{Res}(G(p',y_0)^{-1};i\Lambda_k)\Bigg]. \quad (3.65)
\end{aligned}
$$

The analytic continuation from a four sphere into Lorentzian closed de Sitter space is given by setting the polar angle $\Omega=\pi/2-it$. Without loss of generality we may take $\mu=\Omega$, and μ then continues to $\pi/2-it$. We then obtain the correlator in de Sitter space where one point has been chosen as the origin of the time coordinate.

The continuation of the bitensor $W^{(p')}_{iji'j'}(\mu)$ is given in Appendix B. An extra subtlety arises if one wants to identify the continued bitensor with the usual sum of tensor harmonics on de Sitter space. It turns out that in order to do so, one must extract a factor $ie^{p\pi}/\sinh p'\pi$ from its

[14] If we decrease the radius of the domain wall, then the poles move away from each other. Their behavior follows from the discussion below Eqs. (3.47) and (3.48). For $y_0\leqslant 0.632$, Λ_1 becomes slightly smaller than zero while for $y_0\leqslant 1.32$, Λ_2 becomes slightly greater than 3/2.

coefficient functions.[15] We denote the final form of the bitensor by $W^{L(p')}_{iji'j'}(\mu(x,x'))$, which is defined in the Appendix, Eqs. (B5), (B8), and (B16).

The extra factor $ie^{p\pi}/\sinh p'\pi$ combines with the factor $-i\tanh p'\pi$ in the integrand to give $e^{p'\pi}/\cosh p'\pi$. Furthermore, since $G(-p',y_0)=\overline{G}(p',y_0)$, we can rewrite the correlator as an integral from 0 to ∞. We finally obtain the Lorentzian tensor Feynman (time-ordered) correlator,

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle = \frac{128\pi^2R^4}{N^2}\left[\int_0^{+\infty}dp'\tanh p'\pi W^{L(p')}_{iji'j'}(\mu)\Re(G(p',y_0)^{-1}) + \pi\sum_{k=1}^{2}\tan\Lambda_k\pi W^{L(i\Lambda_k)}_{iji'j'}(\mu)\mathrm{Res}(G(p',y_0)^{-1};i\Lambda_k)\right] + i\frac{128\pi^2R^4}{N^2}\left[\int_0^{+\infty}dp' W^{L(p')}_{iji'j'}(\mu)\Re(G(p',y_0)^{-1}) - \pi\sum_{k=1}^{2}W^{L(i\Lambda_k)}_{iji'j'}(\mu)\mathrm{Res}(G(p',y_0)^{-1};i\Lambda_k)\right]. \tag{3.66}$$

In this integral the bitensor $W^{L(p')}_{iji'j'}(\mu(x,x'))$ may be written as the sum of the degenerate rank-two tensor harmonics on closed de Sitter space with eigenvalue $\lambda_{p'}=(p'^2+17/4)$ of the Laplacian. Note that the normalization factor $\tilde{Q}_{p'}=p'(4p'^2+25)/48\pi^2$ of the bitensor is imaginary at $p'=i\Lambda_k$ and the residues of G^{-1} are also imaginary, so the quantities in square brackets are all real. Both integrands in Eq. (3.66) vanish as $p'\to 0$, so the correlator is well-behaved in the infrared.

For cosmological applications, one is usually interested in the expectation of some quantity squared, like the microwave background multipole moments. For this purpose, all that matters is the symmetrized correlator, which is just the real part of the Feynman correlator.

Gravitational waves provide an extra source of time-dependence in the background in which the cosmic microwave background (CMB) photons propagate. In particular, the contribution of gravitational waves to the CMB anisotropy is given by the integral in the Sachs-Wolfe formula, which is basically the integral along the photon trajectory of the time derivative of the tensor perturbation. Hence the resulting microwave multipole moments $\mathcal{C}_l$ can be directly determined from the graviton correlator.

We can therefore understand the effect of the strongly coupled CFT on the microwave fluctuation spectrum by comparing our result (3.66) with the transverse traceless part of the graviton propagator in four-dimensional de Sitter spacetime [41]. On the four-sphere, this is easily obtained by varying the Einstein-Hilbert action with a cosmological constant. In terms of the bitensor, this yields

$$\langle h_{ij}(\Omega)h_{i'j'}(\Omega')\rangle = 32\pi G_4R^2\sum_{p'=7i/2}^{i\infty}\frac{W^{(p')}_{iji'j'}(\mu(\Omega,\Omega'))}{\lambda_{p'}-2}, \tag{3.67}$$

which continues to

$$\langle h_{ij}(x)h_{i'j'}(x')\rangle = 32\pi G_4R^2\int_0^{+\infty}\frac{dp'}{\lambda_{p'}-2}W^{L(p')}_{iji'j'}(\mu(x,x')). \tag{3.68}$$

This can be compared with Eq. (3.66). Note that (apart from the pole at $p'=3i/2$ corresponding to the gauge mode mentioned before) there are no supercurvature modes. We defer a detailed discussion of the effect of the CFT on the tensor perturbation spectrum in de Sitter space to the next section.

IV. CONCLUSION

We have studied a Randall-Sundrum cosmological scenario consisting of a domain wall in anti–de Sitter space with a large N conformal field theory living on the wall. The conformal anomaly of the CFT provides an effective tension which leads to a de Sitter geometry for the domain wall. We have computed the spectrum of quantum mechanical vacuum fluctuations of the graviton field on the domain wall, according to Euclidean no boundary initial conditions. The Euclidean path integral unambiguously specifies the tensor correlator with no additional assumptions. This is the first calculation of quantum fluctuations for RS cosmology.

In the usual inflationary models, one considers the classical action for a single scalar field. In that context, it is con-

[15]The underlying reason is that there exist two independent bitensors of the form defined by Eqs. (B5) and (B8). Under the integral in the Lorentzian correlator, they are related by the factor $ie^{p\pi}/\sinh p'\pi$. It follows from the continuation of the completeness relation [Eq. (B4)] that the sum of degenerate tensor harmonics on de Sitter space equals the second independent bitensor, rather than the bitensor that we obtain by continuation from S^4. Therefore, in order to express the Lorentzian two-point tensor correlator in terms of tensor harmonics, we must extract this factor from the bitensor. We refer the interested reader to the Appendix for the details.

sistent to neglect quantum matter loops, on the grounds that they are small. On the other hand, in this paper we have studied a strongly coupled large N CFT living on the domain wall, for which quantum loops of matter are important. By using the AdS-CFT correspondence, we have performed a fully quantum mechanical treatment of this CFT. The most notable effect of the large N CFT on the tensor spectrum is that it suppresses small scale fluctuations on the microwave sky. It can be seen from Eq. (3.66) that the CFT yields a $(p'^4 \ln p')^{-1}$ behavior for the graviton propagator at large p' (in agreement with the flat space results of [40]), instead of the usual p'^{-2} falloff [Eq. (3.68)]. In other words, quantum loops of the CFT give spacetime a rigidity that strongly suppresses metric fluctuations on small scales. Note that this is true independently of how the de Sitter geometry arises, i.e. it is also true for four dimensional Einstein gravity. In addition, the coupling of the CFT to tensor perturbations gives rise to two additional discrete modes in the tensor spectrum. Although this is a novel feature in the context of inflationary tensor perturbations, it is not surprising. In conventional open inflationary scenarios for instance, the coupling of scalar field fluctuations with scalar metric perturbations introduces a supercurvature mode with an eigenvalue of the Laplacian close to the discrete de Sitter gauge mode [42,35]. The former discrete mode at $p'=i\Lambda_1 \sim 3i/2$ in Eq. (3.66) is nothing else than the analogue of this well known supercurvature mode in the scalar fluctuation spectrum. The second mode has an eigenvalue $p'=i\Lambda_2 \sim 0$. Its interpretation is less clear, but it is clearly an effect of the matter on the domain wall. However it hardly contributes to the correlator because $\tan \Lambda_2 \pi$ is very small.

The effect of the CFT on large scales is more difficult to quantify because of the complicated p'-dependence of the tensor correlator [Eq. (3.66)] in the low-p' regime. Generally speaking, however, long-wavelength tensor correlations in closed (or open) models for inflation are very sensitive to the details of the underlying theory, as well as to the boundary conditions at the instanton. Since tensor fluctuations do give a substantial contribution to the large scale CMB anisotropies, this may provide an additional way to observationally distinguish different inflationary scenarios [38].

Most matter fields can be expected to behave like a CFT at small scales. Furthermore, fundamental theories such as string theory predict the existence of a large number of matter fields. Therefore, our results based on a quantum treatment of a large N CFT may be accurate at small scales for any matter. If this is the case then our result shows that tensor perturbations at small angular scales are much smaller than predicted by calculations that neglect quantum effects of matter fields.

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APPENDIX A: CHOICE OF GAUGE

This appendix demonstrates how a metric perturbation on the boundary of a ball of AdS is decomposed into vector, scalar and tensor components.

Consider a ball of perturbed AdS with a spherical boundary. Let $\overline{l}$ be the AdS length scale. Gaussian normal coordinates are introduced by defining $\overline{l}y$ to be the geodesic distance of a point from the origin. The surfaces of constant y are spheres on which we introduce coordinates x^i. In these coordinates the metric takes the form

$$ds^2 = \overline{l}^2 (dy^2 + \sinh^2 y\, \hat{\gamma}_{ij}(x) dx^i dx^j) + h_{ij}(y,x) dx^i dx^j. \tag{A1}$$

The ball of AdS has been perturbed, so the boundary will be at a position $y = y_0 + \xi(x)$.

Let the induced metric perturbation on the boundary be $\hat{h}_{ij}(x)$. This can be decomposed into scalar, vector and tensor perturbations with respect to the round metric on the sphere [39]:

$$\hat{h}_{ij}(x) = \hat{\theta}_{ij} + 2\hat{\nabla}_{(i}\hat{\chi}_{j)} + \hat{\nabla}_i \hat{\nabla}_j \hat{\phi} + \hat{\gamma}_{ij}\hat{\psi}, \tag{A2}$$

where we use hats to denote quantities defined on the sphere (i.e. quantities that depend only on x). $\hat{\theta}_{ij}$ is a transverse traceless tensor on the sphere and $\hat{\chi}_i$ is a transverse vector on the sphere. $\hat{\phi}$ and $\hat{\psi}$ are scalars on the sphere. $\hat{\chi}_i$ and $\hat{\phi}$ can be gauged away by infinitesimal coordinate transformations on the sphere of the form $x^i = \tilde{x}^i - \eta^i(\tilde{x}) - \partial^i \eta(\tilde{x})$ where η^i is transverse. Therefore we shall assume that $\hat{\chi}$ and $\hat{\phi}$ vanish. Note that it is not possible to gauge away $\hat{\psi}$ or ξ. This paper only deals with tensor perturbations so we shall assume that the scalars $\hat{\psi}$ and ξ are vanishing. The induced metric perturbation is then transverse and traceless and can be extended into the bulk as described in Sec. III. The scalars will be discussed in our next paper.

APPENDIX B: MAXIMALLY SYMMETRIC BITENSORS

A maximally symmetric bitensor T is one for which $\sigma^* T = 0$ for any isometry σ of the maximally symmetric manifold. Any maximally symmetric bitensor may be expanded in terms of a complete set of fundamental maximally symmetric bitensors with the correct index symmetries. For instance

$$\begin{aligned} T_{iji'j'} = {} & t_1(\mu) g_{ij} g_{i'j'} + t_2(\mu) n_{(i} g_{j)(i'} n_{j')} \\ & + t_3(\mu)[g_{ii'} g_{jj'} + g_{ji'} g_{ij'}] + t_4(\mu) n_i n_j n_{i'} n_{j'} \\ & + t_5(\mu)[g_{ij} n_{i'} n_{j'} + n_i n_j g_{i'j'}]. \end{aligned} \tag{B1}$$

The coefficient functions $t_j(\mu)$ depend only on the distance $\mu(\Omega,\Omega')$ along the shortest geodesic from the point Ω to the point Ω'. $n_{i'}(\Omega,\Omega')$ and $n_i(\Omega,\Omega')$ are unit tangent vectors to the geodesics joining Ω and Ω' and $g_{ij'}(\Omega,\Omega')$ is the parallel propagator along the geodesic, i.e., $V^i g_i^{j'}$ is the vector at Ω' obtained by parallel transport of V^i along the geodesic from Ω to Ω' [43].

The set of tensor eigenmodes on S^4 (or on de Sitter space) forms a representation of the symmetry group of the manifold. It follows in particular that their sum over the parity states $\mathcal{P}=\{e,o\}$ and the quantum numbers k, l, and m on the three sphere defines a maximally symmetric bitensor on S^4 (or dS space) [43]:

$$W^{ij}_{(p')i'j'}(\mu)=\sum_{\mathcal{P}klm} q^{(p')ij}_{\mathcal{P}klm}(\Omega)q^{(p')\mathcal{P}klm}_{i'j'}(\Omega')^{*}. \quad \text{(B2)}$$

On S^4 the label p' takes the value $7i/2, 9i/2, \ldots$. It is related to a real label p by $p'=i(p+3/2)$. The ranges of the other labels are then $0\leqslant k\leqslant p$, $0\leqslant l\leqslant k$ and $-l\leqslant m\leqslant l$. On de Sitter space there is a continuum of eigenvalues $p'\in[0,\infty)$. We will assume from now on that the eigenmodes are normalized by the condition

$$\int \sqrt{\gamma}d^4\Omega\, q^{(p')ij}_{\mathcal{P}klm}q^{(p'')*}_{\mathcal{P}'k'l'm'ij}=\delta^{p'p''}\delta_{\mathcal{P}\mathcal{P}'}\delta_{ll'}\delta_{mm'}. \quad \text{(B3)}$$

The completeness relation on the four sphere may then be written as

$$\gamma^{-(1/2)}\delta^{ij}{}_{i'j'}(\Omega-\Omega')=\sum_{p'=7i/2}^{+i\infty} W^{ij}_{(p')i'j'}(\mu(\Omega,\Omega')). \quad \text{(B4)}$$

Explicit formulas for the components of these tensors may be found in [33]. In this appendix we will determine $W^{(p')}_{iji'j'}(\mu)$ simultaneously on the four sphere and de Sitter space. The construction of the analogous bitensor on S^3 and H^3 is given in [44] and their relation is described in [37].

The bitensor $W^{ij}_{(p')i'j'}(\mu)$ has some additional properties arising from its construction in terms of the transverse and traceless tensor harmonics $q^{(p)\mathcal{P}klm}_{ij}$. The tracelessness of $W^{(p')}_{iji'j'}$ allows one to eliminate two of the coefficient functions in Eq. (B1). It may then be written as

$$\begin{aligned}W^{(p')}_{iji'j'}(\mu)=&w^{(p')}_1[g_{ij}-4n_in_j][g_{i'j'}-4n_{i'}n_{j'}]\\&+w^{(p')}_2[4n_{(i}g_{j)(i'}n_{j')}+4n_in_jn_{i'}n_{j'}]\\&+w^{(p')}_3[g_{ii'}g_{jj'}+g_{ji'}g_{ij'}-2n_ig_{i'j'}n_j\\&-2n_{i'}g_{ij}n_{j'}+8n_in_jn_{i'}n_{j'}]. \quad \text{(B5)}\end{aligned}$$

This expression is traceless on either the index pair ij or $i'j'$. The requirement that the bitensor be transverse $\nabla^iW^{(p')}_{iji'j'}=0$ and the eigenvalue condition $(\nabla^2-\lambda_{p'})W^{iji'j'}_{(p')}=0$ impose additional constraints on the remaining coefficient functions $w^{(p')}_j(\mu)$. To solve these constraint equations it is convenient to introduce the new variables on S^4 (in de Sitter space, μ is replaced by $\pi/2-i\tilde{\mu}$)

$$\alpha(\mu)=w^{(p)}_1(\mu)+\frac{2}{3}w^{(p)}_3(\mu)$$

$$\beta(\mu)=\frac{8}{(\lambda_p+8)\sin\mu}\frac{d\alpha(\mu)}{d\mu}. \quad \text{(B6)}$$

In terms of a new argument $z=\cos^2(\mu/2)$ (or its continuation on de Sitter space) the transversality and eigenvalue conditions imply for $\alpha(z)$

$$z(1-z)\frac{d^2\alpha(z)}{d^2z}+[4-8z]\frac{d\alpha(z)}{dz}=(\lambda_{p'}+8)\alpha(z) \quad \text{(B7)}$$

and then for the coefficient functions

$$\begin{aligned}w_1=&-\frac{6}{5}[(\lambda_{p'}+28)z(1-z)-45/6]\alpha(z)\\&+\frac{6}{20}[(\lambda_{p'}+8)z(1-z)(1-2z)]\beta(z)\\w_2=&\frac{9}{5}\left[(\lambda_{p'}+28)z(1-z)+\frac{20}{3}(1-z)-\frac{20}{6}\right]\alpha(z)\\&-\frac{6}{20}[(\lambda_{p'}+8)z(1-z)(4-3z)]\beta(z) \quad \text{(B8)}\\w_3=&\frac{9}{5}[(\lambda_{p'}+28)z(1-z)-40/6]\alpha(z)\\&-\frac{9}{20}[(\lambda_{p'}+8)z(1-z)(1-2z)]\beta(z)\end{aligned}$$

with $\lambda_{p'}=(p'^2+17/4)$.

Notice that Eq. (B7) is precisely the hypergeometric differential equation, which has a pair of independent solutions $\alpha(z)$ and $\alpha(1-z)$ where

$$\alpha(z)=Q_{p'}\ {}_2F_1(7/2+ip',7/2-ip',4,z). \quad \text{(B9)}$$

$Q_{p'}$ is a constant. The solution for $\beta(z)$ follows from Eq. (B6) and is given by

$$\beta(z)=Q_{p'}\ {}_2F_1(9/2-ip',9/2+ip',5,z). \quad \text{(B10)}$$

We will determine below which solution corresponds to the bitensor defined by Eq. (B2).

Our discussion so far applies to either S^4 or de Sitter space. We now specialize to the case of S^4 and will later obtain results for de Sitter space by analytic continuation. The hypergeometric functions on S^4 may be expressed in terms of Legendre polynomials in $\cos\mu$ (Eq. [15.4.19] in [45]),

$$\alpha(\mu)=Q_{p'}\Gamma(4)2^3(\sin\mu)^{-3}P^{-3}_{-1/2+ip'}(-\cos\mu),$$

$$\beta(\mu)=Q_{p'}\Gamma(5)2^4(\sin\mu)^{-4}P^{-4}_{-1/2+ip'}(-\cos\mu). \quad \text{(B11)}$$

The solutions for $\alpha(z)$ and $\beta(z)$ are singular at $z=1$ (i.e. for coincident points on S^4) for generic values of p'. However, for the values of p' corresponding to the eigenvalues of the

Laplacian on S^4, they are regular everywhere on S^4. Similarly, $\alpha(1-z)$ and $\beta(1-z)$ are generically singular for antipodal points on S^4 and regular for these special values of p'. For these special values, $\alpha(z)$ and $\alpha(1-z)$ are no longer linearly independent but related by a factor of $(-1)^{(n+1)/2}$ where $n=-2ip'=7,9,11,\ldots$. This follows from the relation (Eq. [8.2.3] in [45])

$$P_\nu^\mu(-z)=e^{i\nu\pi}P_\nu^\mu(z)-\frac{2}{\pi}e^{-i\mu\pi}\sin[\pi(\nu+\mu)]Q_\nu^\mu(z), \tag{B12}$$

where the second term vanishes for $p'=7i/2,9i/2,\ldots$. In fact, the hypergeometric series terminates for these values of p' and the hypergeometric functions reduce to Gegenbauer polynomials $C_{n-7/2}^{(7/2)}(1-2z)$. We have a choice between using $\alpha(z)$ and $\alpha(1-z)$ in the bitensor for these values of p'. However, to obtain the Lorentzian correlator, we had to express the discrete sum (3.62) as a contour integral. Since the Euclidean correlator obeys a differential equation with a delta function source at $\mu=0$, we must maintain regularity of the integrand at $\mu=\pi$ when extending the bitensor in the complex p'-plane. In other words, for generic p', we need to work with the solution $\alpha(z)$, rather then $\alpha(1-z)$. We shall therefore choose $\alpha(z)$, since this is the solution that we will analytically continue.

The above conditions leave the overall normalization of the bitensor undetermined. To fix the normalization constant $Q_{p'}$, consider the biscalar quantity

$$g^{ii'}g^{jj'}W_{iji'j'}^{(p')}(\mu)=12w_1^{(p')}-6w_2^{(p')}+24w_3^{(p')}. \tag{B13}$$

In the coincident limit $\Omega\to\Omega'$ and $z\to 1$ this yields

$$W_{ij}^{(p')ij}(\Omega,\Omega)=\sum_{\mathcal{P}klm}q_{ij}^{(p')\mathcal{P}klm}(\Omega)q^{(p')\mathcal{P}lmij}(\Omega)^*$$
$$=-72\alpha(1). \tag{B14}$$

Since $F(0)=1$ we have $\alpha(1)=Q_{p'}(-1)^{(1+n)/2}$. By integrating over the four-sphere and using the normalization condition (B3) on the tensor harmonics one obtains, for $n=-2ip'=7,9,11,\ldots$

$$Q_{p'}=\frac{ip'(4p'^2+25)}{48\pi^2(-1)^{(1+n)/2}}=\frac{p'(4p'^2+25)}{48\pi^2\sinh p'\pi}. \tag{B15}$$

We conclude that the properties of the bitensor appearing in the tensor correlator completely determine its form. Notice that in terms of the label p' we have obtained a unified functional description of the bitensor on S^4 and de Sitter space. However, its explicit form is very different in the two cases because the label p' takes on different values. It is precisely this description that has enabled us in Sec. III to analytically continue the correlator from the Euclidean instanton into de Sitter space without Fourier decomposing it. We shall conclude this Appendix by describing in detail the subtleties of this analytic continuation at the level of the bitensor.

To perform the continuation to de Sitter space we note that the geodesic separation μ on S^4 continues to $\pi/2-it$, so $z=\frac{1}{2}(1+i\sinh t)$ on de Sitter space. The continuation of the hypergeometric functions (B11) yields

$$\alpha(z)=\Gamma(4)2^3(\cosh t)^{-3}P_{-1/2+ip'}^{-3}(-i\sinh t),$$
$$\beta(z)=\Gamma(5)2^4(\cosh t)^{-4}P_{-1/2+ip'}^{-4}(-i\sinh t). \tag{B16}$$

However, an extra subtlety arises if one wants to identify the continued bitensor with the usual sum of tensor harmonics on de Sitter space. In particular, in order for the bitensor to correspond to the usual sum of rank-two tensor harmonics on the real p'-axis, one must choose the second solution $\alpha(1-z)$ to the hypergeometric equation, rather than $\alpha(z)$ that enters in the continued bitensor. This is easily seen by performing the continuation on the completeness relation [Eq. (B4)], which should continue to an integral over p' from 0 to ∞ of the Lorentzian bitensor, defined as the sum (B2) over the degenerate tensor harmonics on de Sitter space. Writing Eq. (B4) as a contour integral and continuing to Lorentzian de Sitter space yields

$$g^{-(1/2)}\delta^{ij}{}_{i'j'}(x-x')$$
$$=\int_{-\infty}^{+\infty}dp'\tanh p'\pi W_{(p')i'j'}^{ij}(\mu(x,x')). \tag{B17}$$

Clearly this is not the correct completeness relation according to the equivalent definition (B2) of the bitensor on de Sitter space. But let us relate the continued bitensor in Eq. (B17) to the independent bitensor in which the solutions $\alpha(1-z)$ enter. This can be done by applying Eq. (B12) to the Legendre polynomials in Eq. (B16). By closing the contour in the upper half p'-plane, one sees there is no contribution to the integral (and indeed to the tensor correlator) from the second term in Eq. (B12), because its prefactor cancels the $\cosh^{-1}(p'\pi)$-factor in Eq. (B17), making the integrand analytic in the upper half p'-plane (up to gauge modes). Hence, under the integral both solutions are simply related by the factor $ie^{p\pi}$. In addition one needs to extract the $\sinh^{-1}p'\pi$-factor[16] from $Q_{p'}$. The completeness relation then becomes

$$g^{-(1/2)}\delta^{ij}{}_{i'j'}(x-x')=\int_0^{+\infty}dp'W_{L(p')i'j'}^{ij}(\mu(x,x')), \tag{B18}$$

and the hypergeometric functions $\alpha(1-z)$ and $\beta(1-z)$ that constitute the final bitensor $W_{L(p')i'j'}^{ij}(\mu(x,x'))$ are given by

[16]Remember that $Q_{p'}$ gained the factor $\sinh^{-1}p'\pi$ because we have chosen the solution $\alpha(z)$ on the four sphere. The correct normalization constant for the independent bitensor, obtained from the normalization condition on the tensor harmonics, is then $\tilde{Q}_{p'}=\sinh p'\pi Q_{p'}$.

$$\alpha(1-z)=\tilde{Q}_{p'}\Gamma(4)2^3(\cosh t)^{-3}P^{-3}_{-1/2+ip'}(i\sinh t),$$

$$\beta(1-z)=\tilde{Q}_{p'}\Gamma(5)2^4(\cosh t)^{-4}P^{-4}_{-1/2+ip'}(i\sinh t), \tag{B19}$$

with $\tilde{Q}_{p'}=p'(4p'^2+25)/48\pi^2$.

On the real p'-axis, $W^{L(p')}_{iji'j'}(\mu)$ equals the sum (B2) of the degenerate rank-two tensor harmonics on closed de Sitter space with eigenvalue $\lambda_{p'}=(p'^2+17/4)$ of the Laplacian.

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