Renormalization group improved black hole spacetimes

Alfio Bonanno*
Osservatorio Astrofisico, Viale Andrea Doria 6, 95125 Catania, Italy
and INFN Sezione di Catania, Corso Italia 57, 95100 Catania, Italy

Martin Reuter†
Institut für Physik, Universität Mainz, Staudingerweg 7, 55099 Mainz, Germany
(Received 22 February 2000; published 27 July 2000)

We study the quantum gravitational effects in spherically symmetric black hole spacetimes. The effective quantum spacetime felt by a pointlike test mass is constructed by ''renormalization group improving'' the Schwarzschild metric. The key ingredient is the running Newton constant which is obtained from the exact evolution equation for the effective average action. The conformal structure of the quantum spacetime depends on its ADM mass M and it is similar to that of the classical Reissner-Nordström black hole. For M larger than, equal to, and smaller than a certain critical mass M_{cr} the spacetime has two, one, and no horizon(s), respectively. Its Hawking temperature, specific heat capacity, and entropy are computed as a function of M. It is argued that the black hole evaporation stops when M approaches M_{cr} which is of the order of the Planck mass. In this manner a ''cold'' soliton-like remnant with the near-horizon geometry of $AdS_2 \times S^2$ is formed. As a consequence of the quantum effects, the classical singularity at $r=0$ is either removed completely or it is at least much milder than classically; in the first case the quantum spacetime has a smooth de Sitter core which would be in accord with the cosmic censorship hypothesis even if $M < M_{cr}$.

PACS number(s): 97.60.Lf, 04.60.−m, 11.10.Hi

I. INTRODUCTION

The Schwarzschild spacetime is the unique spherically symmetric vacuum solution of Einstein's equations. Understanding the dynamics of this spacetime when quantum effects of the geometry are switched on has always been one of the most challenging issues from the theoretical point of view. It is in fact very plausible that those effects will play a key role in the very late stages of the gravitational collapse as well as during the evaporation process of a Planck size black hole.

According to the standard semiclassical scenario, a black hole of mass M emits Hawking radiation at a temperature which is inversely proportional to M. During this process, in addition to the radiation of energy to infinity, a negative-energy flux through the horizon is produced. Thereby the mass of the black hole is lowered and the temperature is increased. It is an open question whether this process continues until the entire mass of the black hole has been converted to radiation or whether it stops when the temperature is close to the Planck temperature where the semiclassical arguments are likely to break down.

In the case of a complete evaporation a number of exotic physical processes such as violations of baryon and lepton number conservation or the ''information paradox'' could occur [23]. Let us consider a quantum field on the black hole spacetime whose initial state is described by a pure density matrix $\hat{\rho}$. If we trace over the field modes which are localized inside the event horizon we are left with an effective mixed state density matrix $\hat{\rho}_{\text{eff}}$ for the physics outside the horizon. Of course this does not mean that a pure state has evolved into a mixed state since the incomplete information provided by $\hat{\rho}_{\text{eff}}$ still can be supplemented by the information contained in $\hat{\rho}$ about the degrees of freedom behind the horizon. However, if the black hole evaporates completely those parts of the spacetime which formerly were interior to the horizon disappear entirely, and there are no field degrees of freedom left which could ''know'' about the information missing in $\hat{\rho}_{\text{eff}}$. As a consequence, the initially pure quantum state $\hat{\rho}$ seems to have evolved into a genuinely mixed state $\hat{\rho}_{\text{eff}}$.

Alternatively one could speculate that the evaporation is incomplete, i.e., that it comes to an end when the Schwarzschild radius is close to the Planck length where the semiclassical results apply no longer. In this case the final state of the Hawking evaporation might be some kind of ''cold'' remnant with a mass close to the Planck mass.

It is clear that the problem of the final state should be addressed within a consistent theory of quantum gravity. The standard semiclassical derivation of the Hawking temperature quantizes only the matter field and treats the spacetime metric as a fixed classical background. However, investigating black holes with a radius not too far above the Planck length we must be prepared that quantum fluctuations of the metric play an important role. The standard perturbative quantization of Einstein gravity is of little help here since it leads to a non-renormalizable theory. Also the more advanced attempts at formulating a fundamental theory of quantum gravity (string theory, loop quantum gravity, etc.) do not provide us with a satisfactory answer yet [2]. As a way out we propose in this paper to use the idea of the

*Email address: abo@sunct.ct.astro.it
†Email address: reuter@thep.physik.uni-mainz.de

Wilsonian renormalization group [1] in order to study quantum effects in the Schwarzschild spacetime.

Our basic tool will be a Wilson-type effective action $\Gamma_k[g_{\mu\nu}]$ where k is a scale parameter with the dimension of a mass. In a nutshell, $\Gamma_k[g_{\mu\nu}]$ is constructed in such a way that, when evaluated at tree level, it correctly describes gravitational phenomena, *with all loop effects included*, whose typical momenta are of the order of k. The basic idea is borrowed from the block spin transformations which are used in statistical mechanics in order to ''coarse grain'' spin configurations of lattice systems. In its simplest formulation, when applied to a continuum field theory [3–5], we are given a field $\phi(x)$ defined on a Euclidean spacetime with metric $g_{\mu\nu}$ and dimension d. The averaged or ''blocked'' field $\phi_k(x)$ is defined by means of

$$\phi_k(x)=\int d^dy\,\sqrt{g}(y)\,\rho_k(x-y)\,\phi(y), \tag{1.1}$$

where $\rho_k(x-y)$ is a smearing function that has support only for $||x-y||<k^{-1}$. The ''average action'' Γ_k governs the dynamics of the coarse-grained or macroscopic field Φ. It is obtained from the classical action by integrating over the microscopic degrees of freedom or ''fast variables'':

$$\exp(-\Gamma_k[\Phi])=\int D[\phi]\,\delta(\phi_k-\Phi)\exp(-S[\phi]). \tag{1.2}$$

The blocked field has a very intuitive physical interpretation: it is the field noticed by an observer who uses an experimental apparatus of resolution

$$l\sim k^{-1}. \tag{1.3}$$

This observer sees the field evolving according to the effective equation of motion $\delta\Gamma_k[\Phi]/\delta\Phi(x)=0$.

For continuum field theories the functional integral (1.2) is not easy to deal with, and so we shall use an alternative construction which leads to a functional Γ_k with similar qualitative properties to the one discussed above. We use the method of the ''effective average action'' Γ_k which has been developed in Refs. [6,7]. It is defined in a similar way to the ordinary effective action Γ but it has the additional feature of a built-in infrared cutoff at the scale k. Quantum fluctuations with momenta $p_\mu^2>k^2$ are integrated out in the usual way while the effect of the large distance fluctuations with $p_\mu^2<k^2$ is not included in Γ_k. Hence Γ_k, regarded as a function of k, describes a renormalization group trajectory in the space of all actions; it connects the classical action $S=\Gamma_{k\to\infty}$ to the ordinary effective action $\Gamma=\Gamma_{k=0}$. This trajectory satisfies an exact functional renormalization group (or flow) equation. If one wants to quantize a fundamental theory with action S one integrates this equation from the initial point $\Gamma_\Lambda=S$ down to $\Gamma=\Gamma_{k=0}$. After appropriate renormalizations one then lets $\Lambda\to\infty$.

The flow equation can also be used in order to further evolve (coarse grain) effective field theory actions from one scale k to another. In this case no limit such as $\Lambda\to\infty$ above needs to be taken, i.e., the ultraviolet cutoff is not removed. Hence the evolution of Γ_k from k_1 to k_2 is always well defined even if the theory under consideration, when regarded as a fundamental theory, is not renormalizable.

In the following we consider Einstein gravity as an effective field theory and we identify the standard Einstein-Hilbert action with the average action $\Gamma_{k_{\rm obs}}$. Here $k_{\rm obs}$ is some typical ''observational scale'' at which the classical tests of general relativity have confirmed the validity of the Einstein-Hilbert action. In order to find an approximate solution to the flow equation we assume that also for $k>k_{\rm obs}$, i.e., at higher momenta, Γ_k is well approximated by an action of the Einstein-Hilbert form. The two parameters in this action, Newton's constant and the cosmological constant, will depend on k, however, and the flow equation will tell us how the running Newton constant $G(k)$ and the running cosmological constant $\lambda(k)$ depend on the cutoff. Their experimentally observed values are $G(k_{\rm obs})=G_0$ and $\lambda(k_{\rm obs})=\lambda_0\approx 0$. Since, at least within our approximation, there is essentially no running of these parameters between $k_{\rm obs}$ (the scale of the solar system, say) and cosmological scales ($k\approx 0$) we may set $k_{\rm obs}=0$ and identify the measured parameters with $G(k=0)$ and $\lambda(k=0)$.

The key idea presented in this paper is to use the running Newton constant $G=G(k)$ in order to ''renormalization group improve'' the Schwarzschild spacetime. This idea is borrowed from particle physics. There it is a standard device in order to add the dominant quantum corrections to the Born approximation of some scattering cross section, say. Our implementation of this scheme is similar to the renormalization group based derivation of the Uehling correction to the Coulomb potential in massless QED [8]. One starts from the classical potential energy $V_{\rm cl}(r)=e^2/4\pi r$ and replaces e^2 by the running gauge coupling in the one-loop approximation:

$$e^2(k)=e^2(k_0)[1-b\,\ln(k/k_0)]^{-1}, \quad b\equiv e^2(k_0)/6\pi^2. \tag{1.4}$$

The crucial step is to identify the renormalization point k with the inverse of the distance r. This is possible because in the massless theory r is the only dimensionful quantity which could define a scale. The result of this substitution reads

$$V(r)=-e^2(r_0^{-1})[1+b\,\ln(r_0/r)+O(e^4)]/4\pi r, \tag{1.5}$$

where the IR reference scale $r_0\equiv 1/k_0$ has to be kept finite in the massless theory. We emphasize that Eq. (1.5) is the correct (one-loop, massless) Uehling potential which is usually derived by more conventional perturbative methods [8]. Obviously the position dependent renormalization group improvement $e^2\to e^2(k)$, $k\propto 1/r$ encapsulates the most important effects which the quantum fluctuations have on the electric field produced by a point charge.

In this paper we propose to ''improve'' the Schwarzschild metric by an analogous substitution. We replace the Newton constant by its running counterpart $G(k)$ with an appropriate position-dependent scale $k=k(r)$, where r is the radial Schwarzschild coordinate. At large distances we shall have $k(r)\propto 1/r$ as in QED, but since G is dimensionful there will be deviations at small distances.

This approach has also been used in Ref. [9] where the impact of quantized gravity on the Cauchy horizon singularity occurring in a realistic gravitational collapse has been studied. In this work a perturbative approximation of the function $G(k)$ has been employed. In the present paper we use instead an exact, non-perturbative solution to the evolution equation for $G(k)$ which follows from the ''Einstein-Hilbert truncation.''

Our main results about the quantum corrected Schwarzschild spacetime are the following. For large masses M the quantum effects are essentially negligible. Lowering the mass we find that the radius of the event horizon becomes smaller and that at the same time a second, inner horizon develops out of the $(r=0)$ singularity which is now timelike. When M equals a certain critical mass $M_{\rm cr}$ which is of the order of the Planck mass the two horizons coincide. For $M < M_{\rm cr}$ there is no horizon at all. The causal structure of these spacetimes is similar to the classical Reissner-Nordström spacetimes. It turns out that while the Hawking temperature is proportional to $1/M$ for very heavy black holes it vanishes as M approaches $M_{\rm cr}$ from above. This leads to a scenario for the evaporation process where the Hawking radiation is ''switched off'' once the mass gets close to $M_{\rm cr}$. This picture suggests that the final state of the evaporation could be a critical (extremal) black hole with $M=M_{\rm cr}$.

The rest of this paper is organized as follows. In Sec. II we derive the running of the Newton constant from the renormalization group equation. In Sec. III the correct identification of the position dependent cutoff $k=k(r)$ is discussed. In Sec. IV we ''renormalization group improve'' the eternal black hole spacetime and discuss its properties in detail. In Sec. V we provide an effective matter interpretation of this spacetime. In Sec. VI the Hawking temperature is derived and our scenario for the evaporation process is presented. In Sec. VII we obtain an expression for the thermodynamic entropy of the quantum black hole, while in Sec. VIII we discuss the fate of the $(r=0)$ singularity. The conclusions are contained in Sec. IX. In the Appendix we discuss some problems related to the statistical mechanical entropy of the quantum black hole.

II. THE RUNNING NEWTON CONSTANT

In Ref. [7] the idea of the effective average action [6,10] has been used in order to formulate the quantization of (d-dimensional, Euclidean) gravity and the evolution of scale-dependent effective gravitational actions $\Gamma_k[g_{\mu\nu}]$ by means of an exact renormalization group equation. Furthermore, in order to find approximate solutions to this equation, the renormalization group flow in the infinite dimensional space of all action functionals has been projected on the 2-dimensional subspace spanned by the operators $\sqrt{g}$ and $\sqrt{g}R$ (''Einstein-Hilbert truncation''). Using the background gauge formalism with a background metric $\bar{g}_{\mu\nu}$, this truncation of the ''theory space'' amounts to considering only actions of the form

$$\Gamma_k[g,\bar{g}] = (16\pi G(k))^{-1}\int d^dx\,\sqrt{g}\{-R(g)+2\bar{\lambda}(k)\} + S_{\rm gf}[g,\bar{g}], \tag{2.1}$$

where $G(k)$ and $\bar{\lambda}(k)$ denote the running Newton constant and cosmological constant, respectively, and $S_{\rm gf}$ is the classical background gauge fixing term. For truncations of this type the flow equation reads

$$\partial_t\Gamma_k[g,\bar{g}] = \frac{1}{2}\mathrm{Tr}[(\kappa^{-2}\Gamma_k^{(2)}[g,\bar{g}]+\mathcal{R}_k^{\rm grav}[\bar{g}])^{-1}\partial_t\mathcal{R}_k^{\rm grav}[\bar{g}]] - \mathrm{Tr}[(-\mathcal{M}[g,\bar{g}]+\mathcal{R}_k^{\rm gh}[\bar{g}])^{-1}\partial_t\mathcal{R}_k^{\rm gh}[\bar{g}]], \tag{2.2}$$

with

$$t\equiv \ln k, \tag{2.3}$$

where $\Gamma_k^{(2)}$ stands for the Hessian of Γ_k with respect to $g_{\mu\nu}$, and $\mathcal{M}$ is the Faddeev-Popov ghost operator. The operators $\mathcal{R}_k^{\rm grav}$ and $\mathcal{R}_k^{\rm gh}$ implement the IR cutoff in the graviton and the ghost sector. They are defined in terms of a, to some extent, arbitrary smooth function $\mathcal{R}_k(p^2)\propto k^2R^{(0)}(p^2/k^2)$ by replacing the momentum square p^2 with the graviton and ghost kinetic operator, respectively. Inside loops, they suppress the contribution of infrared modes with covariant momenta $p<k$. The function $R^{(0)}(z), z\equiv p^2/k^2$, has to satisfy the conditions $R^{(0)}(0)=1$ and $R^{(0)}(z)\to 0$ for $z\to\infty$. For explicit computations we use the exponential cutoff

$$R^{(0)}(z) = z[\exp(z)-1]^{-1}. \tag{2.4}$$

If we insert Eq. (2.1) into Eq. (2.2) and project the flow onto the subspace spanned by the Einstein-Hilbert truncation we obtain a coupled system of differential equations for the dimensionless Newton constant

$$g(k)\equiv k^{d-2}G(k) \tag{2.5}$$

and the dimensionless cosmological constant $\lambda(k) \equiv \bar{\lambda}(k)/k^2$:

$$\partial_t g = [d-2+\eta_N]g \tag{2.6}$$

$$\partial_t\lambda = -(2-\eta_N)\lambda + \frac{1}{2}g(4\pi)^{1-d/2}[2d(d+1)\Phi^1_{d/2}(-2\lambda) - 8d\Phi^1_{d/2}(0) - d(d+1)\eta_N\tilde{\Phi}^1_{d/2}(-2\lambda)]. \tag{2.7}$$

Here

$$\eta_N(g,\lambda) = \frac{gB_1(\lambda)}{1-gB_2(\lambda)} \tag{2.8}$$

is the anomalous dimension of the operator $\sqrt{g}R$, and the functions $B_1(\lambda)$ and $B_2(\lambda)$ are given by

$$B_1(\lambda)\equiv\frac{1}{3}(4\pi)^{1-d/2}[d(d+1)\Phi^1_{d/2-1}(-2\lambda)-6d \times(d-1)\Phi^2_{d/2}(-2\lambda)-4d\Phi^1_{d/2-1}(0)-24\Phi^2_{d/2}(0)], \tag{2.9}$$

$$B_2(\lambda)\equiv-\frac{1}{6}(4\pi)^{1-d/2}[d(d+1)\tilde{\Phi}^1_{d/2-1}(-2\lambda)-6d \times(d-1)\tilde{\Phi}^2_{d/2}(-2\lambda)],$$

with the cutoff, i.e., $R^{(0)}$-dependent ''threshold'' functions ($p=1,2,\ldots$)

$$\Phi^p_n(w)=\frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}(z)-zR^{(0)\prime}(z)}{[z+R^{(0)}(z)+w]^p}, \tag{2.10}$$

$$\tilde{\Phi}^p_n(w)=\frac{1}{\Gamma(n)}\int_0^\infty dz\, z^{n-1}\frac{R^{(0)}(z)}{[z+R^{(0)}(z)+w]^p}.$$

For further details about the effective average action in gravity and the derivation of the above results we refer to [7].

From now on we shall focus on $d=4$. Furthermore, the cosmological constant plays no role within the scope of our present investigation. We assume that $\overline{\lambda}\ll k^2$ for all scales of interest so that we may approximate $\lambda\approx 0$ in the arguments of $B_1(\lambda)$ and $B_2(\lambda)$. Thus the evolution is governed by the equation

$$\frac{dg(t)}{dt}=[2+\eta_N]g(t)=\beta(g(t)), \tag{2.11}$$

with the anomalous dimension

$$\eta_N(g)=\frac{B_1 g}{1-B_2 g}, \tag{2.12}$$

and the beta function

$$\beta(g)=2g\frac{1-\omega' g}{1-B_2 g}. \tag{2.13}$$

The constants B_1 and B_2 are given by

$$B_1\equiv B_1(0)=-\frac{1}{3\pi}[24\Phi^2_2(0)-\Phi^1_1(0)], \tag{2.14}$$

$$B_2\equiv B_2(0)=\frac{1}{6\pi}[18\tilde{\Phi}^2_2(0)-5\tilde{\Phi}^1_1(0)]. \tag{2.15}$$

We also define

$$\omega\equiv-\frac{1}{2}B_1, \quad \omega'\equiv\omega+B_2. \tag{2.16}$$

For the exponential cutoff (2.4) we have explicitly

$$\Phi^1_1(0)=\frac{\pi^2}{6}, \quad \Phi^2_2(0)=1 \tag{2.17}$$

$$\tilde{\Phi}^1_1(0)=1, \quad \tilde{\Phi}^2_2(0)=\frac{1}{2} \tag{2.18}$$

and

$$\omega=\frac{4}{\pi}\left(1-\frac{\pi^2}{144}\right), \quad B_2=\frac{2}{3\pi}. \tag{2.19}$$

The evolution equation (2.11) displays two fixed points g_*, $\beta(g_*)=0$. There exists an infrared attractive (Gaussian) fixed point at $g_*^{\rm IR}=0$ and an ultraviolet attractive (non-Gaussian) fixed point at

$$g_*^{\rm UV}=\frac{1}{\omega'}. \tag{2.20}$$

This latter fixed point is a higher dimensional analog of the Weinberg fixed point [11] known from $(2+\epsilon)$-dimensional gravity. (Within the present framework it has been studied in [7].)

The UV fixed point separates a weak coupling regime ($g<g_*^{\rm UV}$) from a strong coupling regime where $g>g_*^{\rm UV}$. Since the β function is positive for $g\in[0,g_*^{\rm UV}]$ and negative otherwise, the renormalization group trajectories which result from Eq. (2.11) with Eq. (2.13) fall into the following three classes:

(i) Trajectories with $g(k)<0$ for all k. They are attracted towards $g_*^{\rm IR}$ for $k\to 0$.

(ii) Trajectories with $g(k)>g_*^{\rm UV}$ for all k. They are attracted towards $g_*^{\rm UV}$ for $k\to\infty$.

(iii) Trajectories with $g(k)\in[0,g_*^{\rm UV}]$ for all k. They are attracted towards $g_*^{\rm IR}=0$ for $k\to 0$ and towards $g_*^{\rm UV}$ for $k\to\infty$.

Only the trajectories of type (iii) are relevant for us. We shall not allow for a negative Newton constant, and we also discard solutions of type (ii). They are in the strong coupling region and do not connect to a perturbative large distance regime. (See Ref. [12] for a numerical investigation of the phase diagram.)

The differential equation (2.11) with (2.13) can be integrated analytically to yield

$$\frac{g}{(1-\omega' g)^{\omega/\omega'}}=\frac{g(k_0)}{[1-\omega' g(k_0)]^{\omega/\omega'}}\left(\frac{k}{k_0}\right)^2, \tag{2.21}$$

but this expression cannot be solved for $g=g(k)$ in closed form. However, it is obvious that this solution interpolates between the IR behavior $g(k)\propto k^2$ for $k^2\to 0$ and $g(k)\to 1/\omega'$ for $k\to\infty$.

In order to obtain an approximate analytic expression for the running Newton constant we observe that the ratio ω'/ω is actually very close to unity. Numerically one has $\omega\approx 1.2$, $B_2\approx 0.21$, $\omega'\approx 1.4$, $g_*^{\rm UV}\approx 0.71$ so that $\omega'/\omega\approx 1.18$ is indeed close to 1. Replacing $\omega'/\omega\to 1$ in Eq. (2.21) yields a rather accurate approximation with the same general features as the exact solution. In this case we can easily solve Eq. (2.21):

$$g(k)=\frac{g(k_0)k^2}{\omega g(k_0)k^2+[1-\omega g(k_0)]k_0^2}. \tag{2.22}$$

This function is an *exact* solution to the renormalization group equation with the approximate anomalous dimension $\eta_N=-2\omega g+O(g^2)$ which is the first term in the perturbative expansion of Eq. (2.12):

$$\eta_N=-2\omega g\left[1+\sum_{n=1}^{\infty}(B_2 g)^n\right]. \tag{2.23}$$

Remarkably, for the trajectory (2.22) the quantity $B_2\ g(k)$ remains negligibly small for all values of k. It assumes its largest value at the UV fixed point where $B_2\ g_*^{\rm UV}=0.15$. Thus Eq. (2.22) provides us with a consistent approximation. (This can also be checked by comparing to the numerical solution of Ref. [12].)

In terms of the dimensionful Newton constant $G(k)\equiv g(k)/k^2$ Eq. (2.22) reads

$$G(k)=\frac{G(k_0)}{1+\omega G(k_0)[k^2-k_0^2]}. \tag{2.24}$$

From now on we shall set $k_0=0$ for the reference scale. At least within the Einstein-Hilbert truncation, $G(k)$ does not run any more between scales where the Newton constant was determined experimentally (laboratory scale, scale of the solar system, etc.) and $k\approx 0$ (cosmological scale). Therefore we can identify $G_0\equiv G(k_0=0)$ with the experimentally observed value of the Newton constant. From

$$G(k)=\frac{G_0}{1+\omega G_0 k^2} \tag{2.25}$$

we see that when we go to higher momentum scales k, $G(k)$ decreases monotonically. For small k we have[1]

$$G(k)=G_0-\omega G_0^2 k^2+O(k^4), \tag{2.26}$$

while for $k^2 \gg G_0^{-1}$ the fixed point behavior sets in and $G(k)$ "forgets" its infrared value:

$$G(k)\approx\frac{1}{\omega k^2}. \tag{2.27}$$

In Ref. [13], Polyakov had conjectured an asymptotic running of precisely this form.

III. IDENTIFICATION OF THE INFRARED CUTOFF

In the Introduction we identified the scale k with the inverse distance in order to derive the leading QED correction to the Coulomb potential. In this section we discuss how in the case of a black hole k can be converted to a position-dependent quantity. We write this position-dependent IR cutoff in the form

$$k(P)=\frac{\xi}{d(P)}, \tag{3.1}$$

where ξ is a numerical constant to be fixed later and $d(P)$ is the distance scale which provides the relevant cutoff for the Newton constant when the test particle is located at the point P of the black hole spacetime.

Using Schwarzschild coordinates (t,r,θ,ϕ) and considering spherically symmetric spacetimes, the symmetries of the problem imply that $d(P)$ depends on the r coordinate of P only, $d=d(r)$.

If the test particle is far outside the horizon of the black hole ($r\gg 2G_0M$) where the spacetime is almost flat we expect that $d(r)$ is approximately equal to r. By comparison with the work of Donoghue [14] we shall see that this is actually the case. As a consequence, the function d is normalized such that

$$\lim_{r\to\infty}\frac{d(r)}{r}=1 \tag{3.2}$$

so that the constant ξ fixes the asymptotic behavior

$$k(r)\approx\frac{\xi}{r}\quad \text{for}\ \ r\to\infty. \tag{3.3}$$

Contrary to the situation in QED on flat spacetime, Eq. (3.3) is not a satisfactory identification of $k=k(P)$ for arbitrary points P. The reason is that $d(P)$ should have a coordinate independent meaning, while r is simply one of the local Schwarzschild coordinates. As a way out, we define $d(P)$ to be the proper distance (with respect to the classical Schwarzschild metric) from the point P to the center of the black hole along some curve $\mathcal{C}$:

$$d(P)=\int_{\mathcal{C}}\sqrt{|ds^2|}. \tag{3.4}$$

There is still some ambiguity as for the correct identification of the spacetime curve $\mathcal{C}$. However, at least in the spherically symmetric case, it turns out that all physically plausible candidates lead to cutoffs with the same qualitative features.

We parametrize $\mathcal{C}$ as $x^\mu(\lambda)$ where $x^\mu=(t,r,\theta,\phi)$ are the Schwarzschild coordinates and λ is a (not necessarily affine) parameter along the curve. To start with, let us consider the curve $\mathcal{C}\equiv\mathcal{C}_{(1)}$ defined by $t(\lambda)=t_0$, $r(\lambda)=\lambda$, $\theta(\lambda)=\theta_0$, $\phi(\lambda)=\phi_0$ with $\lambda\in[0,r(P)]$ where $r(P)$ is the r coordinate

[1] In general we would expect that the IR asymptotics might change when we include the running of the cosmological constant. In this case Eq. (2.26) still gives the leading correction in an expansion with respect to $g=k^2G$, but since the cosmological constant is dimensionful this is not necessarily the same as an expansion with respect to k^2. For a first numerical investigation of the impact the cosmological constant has on the running of G, the reader is referred to Souma [12].

of P. This is, even beyond the horizon, a straight ''radial'' line from the origin to P, at fixed values of t, θ and ϕ. If we restrict λ to the interval $[r(P_0), r(P)]$ with P_0 and P both outside the horizon where $ds^2 > 0$ then $\int \sqrt{ds^2}$ is the ordinary spatial proper distance between the points P_0 and P. The definition (3.4) involves the modulus of ds^2 and it generalizes this ''distance'' to the case that at least one of the two points lies within the horizon where r is timelike. (See also [15] for a discussion of this ''distance.'') The explicit result reads for $r < 2G_0M$

$$d_{(1)}(r) = 2G_0M \arctan\sqrt{\frac{r}{2G_0M - r}} - \sqrt{r(2G_0M - r)} \tag{3.5}$$

and for $r > 2G_0M$:

$$d_{(1)}(r) = \pi G_0 M + 2G_0M \ln\left(\sqrt{\frac{r}{2G_0M}} + \sqrt{\frac{r}{2G_0M} - 1}\right) + \sqrt{r(r - 2G_0M)}. \tag{3.6}$$

Note that $d_{(1)}(r)$ is continuous at the horizon. Equation (3.6) shows that indeed $d_1(r) = r + O(\ln r)$ for $r \to \infty$. From Eq. (3.5) we obtain for $r \to 0$

$$d_{(1)}(r) = \frac{2}{3}\frac{1}{\sqrt{2G_0M}} r^{3/2} + O(r^{5/2}), \tag{3.7}$$

which leads to the cutoff

$$k_{(1)}(r) = \frac{3}{2}\xi\sqrt{2G_0M}\left(\frac{1}{r}\right)^{3/2} \quad \text{for} \quad r \to 0. \tag{3.8}$$

This $r^{-3/2}$ behavior has to be contrasted with the r^{-1} dependence of the ''naive'' cutoff $k = \xi/r$.

Another plausible spacetime curve $\mathcal{C}$ is the worldline of an observer who falls into the black hole. We define $\mathcal{C} \equiv \mathcal{C}_{(2)}$ to be the radial timelike geodesic of the Schwarzschild metric with vanishing velocity at infinity. For this geodesic, the observer's radial coordinate r and proper time τ are related by [15]

$$\tau - \tau_0 = \frac{2}{3}\frac{1}{\sqrt{2G_0M}}(r_0^{3/2} - r^{3/2}), \tag{3.9}$$

where the constant of integration is chosen such that $r(\tau_0) = r_0$. Equation (3.9) is valid both outside and inside the horizon. Setting $r_0 = 0 = \tau_0$, we see that when the observer has arrived at $r = r(P)$, the remaining proper time it takes him or her to reach the singularity is given by

$$|\tau(P)| = \frac{2}{3}\frac{1}{\sqrt{2G_0M}} r(P)^{3/2}. \tag{3.10}$$

From the point of view of this observer it is meaningless to consider times larger than $|\tau(P)|$ and, as a consequence, frequencies smaller than $|\tau(P)|^{-1}$. This motivates the identification $d_{(2)}(P) = |\tau(P)|$, i.e.,

$$d_{(2)}(r) = \frac{2}{3}\frac{1}{\sqrt{2G_0M}} r^{3/2}, \tag{3.11}$$

which leads to

$$k_{(2)}(r) = \frac{3}{2}\xi\sqrt{2G_0M}\left(\frac{1}{r}\right)^{3/2}. \tag{3.12}$$

Equations (3.11) and (3.12) are exact for all values of r. It is remarkable that $d_{(2)}(r)$ coincides precisely with the approximation for $d_{(1)}(r)$, Eq. (3.7), which is valid for small values of r. This supports our assumption that close to the singularity $(r \to 0)$ the correct cutoff behaves as $k(r) \propto 1/r^{3/2}$.

For large distances, the curves $\mathcal{C}_{(1)}$ and $\mathcal{C}_{(2)}$ lead to different r dependencies of the cutoff: $k_{(1)} \propto 1/r$, $k_{(2)} \propto 1/r^{3/2}$. Quite generally, if a system possesses more than one typical momentum scale, $k_{(1)}, k_{(2)}, k_{(3)}, \ldots$, which can cut off the running of some coupling constant, it is the largest one among those scales which provides the actual cutoff: $k = \text{Max}\{k_{(1)}, k_{(2)}, k_{(3)}, \ldots\}$. In the case at hand we have $k_{(1)} \gg k_{(2)}$ for $r \to \infty$ so that we must set $k = k_{(1)}(r) \propto 1/r$ for large values of r.

The only properties of the function $k(r)$ which we shall use in the following is that it varies as $k(r) \propto 1/r$ for $r \to \infty$ and as $k(r) \propto 1/r^{3/2}$ for $r \to 0$. This behavior can be further confirmed by investigating different choices of $\mathcal{C}$. For instance, a radial timelike geodesic with vanishing velocity at some finite distance from the black hole or a geodesic with non-vanishing velocity at infinity, for small values of r, again reproduces Eq. (3.7).

While we used Schwarzschild coordinates in the above discussion we emphasize that the same results can also be obtained using coordinate systems (such as the Eddington-Finkelstein coordinates) which do not become singular at the horizon.

It turns out that the qualitative features of the quantum corrected black hole spacetimes which we are going to construct in the following are rather insensitive to the precise manner in which $k(r)$ interpolates between the $1/r^{3/2}$ and the $1/r$ behavior. Moreover, most of the general features (horizon structure, etc.) are even independent of the precise form of $k(r)$ for $r \to 0$. Using $k(r) \propto 1/r^{\nu}$ with ν not necessarily equal to 3/2 leads to essentially the same picture. The only issue where the value of ν is of crucial importance is the fate of the singularity at $r = 0$ when quantum effects are switched on.

In concrete calculations we shall use the interpolating function

$$d(r) = \left(\frac{r^3}{r + \gamma G_0 M}\right)^{1/2}, \tag{3.13}$$

with $d(r) = r[1 + O(1/r)]$ and $d(r) = r^{3/2}/\sqrt{\gamma G_0 M} + O(r^{5/2})$ for large and small r's, respectively. From $\mathcal{C}_{(1)}$ and $\mathcal{C}_{(2)}$ we had obtained

$$\gamma = \frac{9}{2}, \tag{3.14}$$

but we shall treat γ as a free parameter. Most of our results turn out to be very robust: qualitatively they are the same for all $\gamma > 0$. By setting $\gamma = 0$, the ansatz (3.13) also allows us to return to the ''naive'' cutoff $k \propto 1/r$, i.e., to $\nu = 1$. Except for questions related to the singularity at $r = 0$, even $\gamma = 0$ will lead to essentially the same qualitative properties of the improved black hole spacetime.

Upon inserting Eq. (3.1) into the running Newton constant (2.25) we obtain the following position-dependent Newton constant $G(r) \equiv G(k(r))$:

$$G(r) = \frac{G_0 d(r)^2}{d(r)^2 + \tilde{\omega} G_0}, \tag{3.15}$$

where

$$\tilde{\omega} \equiv \omega \xi^2. \tag{3.16}$$

For the ansatz (3.13) this yields

$$G(r) = \frac{G_0 r^3}{r^3 + \tilde{\omega} G_0 [r + \gamma G_0 M]}. \tag{3.17}$$

At large distances, the leading correction to Newton's constant is given by

$$G(r) = G_0 - \tilde{\omega} \frac{G_0^2}{r^2} + O\left(\frac{1}{r^3}\right). \tag{3.18}$$

For small distances $r \rightarrow 0$, it vanishes very rapidly:

$$G(r) = \frac{r^3}{\gamma \tilde{\omega} G_0 M} + O(r^4). \tag{3.19}$$

The asymptotic behavior (3.18) can be used in order to fix the numerical value of $\tilde{\omega}$. The idea is to renormalization group improve the classical Newton potential $V(r) = -G_0 m_1 m_2 / r$ of two masses m_1 and m_2 at distance r by replacing the constant G_0 with $G(r)$. Within the approximation (3.18) we obtain

$$V_{\text{imp}}(r) = -G_0 \frac{m_1 m_2}{r} \left[1 - \tilde{\omega} \frac{G_0 \hbar}{r^2 c^3} + \cdots \right], \tag{3.20}$$

where we have reinstated factors of $\hbar$ and c for a moment. We observe that our renormalization group approach predicts a $1/r^3$ correction to the $1/r$ potential. However, the value of the coefficient $\tilde{\omega} = \omega \xi^2$ cannot be obtained by renormalization group arguments alone: the factor ω is a non-universal coefficient of the β function, i.e., it depends on the shape of the function $R^{(0)}$, and also ξ is unknown as long as one does not explicitly identify the specific cutoff for a concrete process.

On the other hand, it was pointed out by Donoghue [14] that the standard perturbative quantization of Einstein gravity leads to a well-defined, finite prediction for the leading large distance correction to Newton's potential. His result reads

$$V(r) = -G_0 \frac{m_1 m_2}{r} \left[1 - \frac{G_0 (m_1 + m_2)}{2 c^2 r} - \hat{\omega} \frac{G_0 \hbar}{r^2 c^3} + \cdots \right], \tag{3.21}$$

where [16] $\hat{\omega} = 118/15\pi$. The correction proportional to $(m_1 + m_2)/r$ is a purely kinematic effect of classical general relativity, while the quantum correction $\propto \hbar$ has precisely the structure we have predicted on the basis of the renormalization group. Comparing Eq. (3.20) to Eq. (3.21) allows us to determine the coefficient $\tilde{\omega}$ by identifying

$$\tilde{\omega} = \hat{\omega} \equiv \frac{118}{15\pi}. \tag{3.22}$$

Contrary to the factors ω and ξ^2, their product $\tilde{\omega} = \omega \xi^2$ has a uniquely determined, measurable value.

A priori the renormalization group analysis yields G as a function of k rather than r, and the function $\mathcal{R}_k$ serves as a mathematical model of an arbitrary, yet unspecified physical mechanism which cuts off the running of G. In the case at hand, this mechanism is the finite distance between the test particle and the black hole; it led to the ansatz $k = \xi / d(r)$. In general the information about the actual physical cutoff mechanism enters at two points:

(a) The function $\mathcal{R}_k$ should be chosen so as to model the actual physics as correctly as possible.

(b) Both the physical cutoff mechanism and the choice for $\mathcal{R}_k$ determine the relation between k and other variables, adapted to the concrete problem, which can parametrize the running of G (r, in our case).

This means that, within our approximation, the $\mathcal{R}_k$ dependence of the correct identification $k = k(r)$ should precisely compensate for the $\mathcal{R}_k$ dependence of $G(k)$. We have seen that this is indeed what happens: ω and ξ appear only in the combination $\tilde{\omega} = \omega \xi^2$. The $\mathcal{R}_k$ dependencies of ω and ξ^2 cancel in this product, and its unambiguous numerical value can be read off from the known asymptotic form of $V_{\text{imp}}(r)$.

IV. IMPROVING THE ETERNAL BLACK HOLE SPACETIME

A. The improved metric

We consider spherically symmetric, Lorentzian metrics of the form

$$ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2, \tag{4.1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$ is the line element on the unit two-sphere and $f(r)$ is an arbitrary ''lapse function.'' The most important example of a metric belonging to this class is the Schwarzschild metric with

$$f(r) = f_{\text{class}}(r) \equiv 1 - \frac{2 G_0 M}{r}. \tag{4.2}$$

While the Schwarzschild spacetime is a solution of the vacuum Einstein equation $R_{\mu\nu} = 0$, we are not going to constrain $f(r)$ by any field equation in the following.

In classical general relativity the metric (4.1) with (4.2) is interpreted as a property of a black hole (or the exterior of a star) *per se*, i.e., the metric is given a meaning even in the absence of a test particle which probes it. Within our approach, we regard $f_{\rm class}$ as a manner of encoding the classical dynamics of a test particle in the vicinity of some ''central body'' of mass M. Because of the actual presence of the test particle, the system defines a physically relevant distance scale $d(r)$ which enters into the cutoff for the running of G. It is our main assumption that, also beyond the Newtonian limit, the leading quantum gravity effects in this system consist of a position-dependent renormalization of the Newton constant in Eq. (4.2). More precisely, we assume that the quantum corrected geometry can be approximated by Eq. (4.1) with

$$f(r)=1-\frac{2G(r)M}{r}, \tag{4.3}$$

where $G(r)$ is given by Eq. (3.17):

$$f(r)=1-\frac{2G_0Mr^2}{r^3+\tilde{\omega}G_0[r+\gamma G_0M]}. \tag{4.4}$$

Let us now analyze the properties of the renormalization group improved spacetime defined by Eq. (4.4). First of all, for $r\rightarrow\infty$ we have

$$f(r)=1-\frac{2G_0M}{r}\left(1-\tilde{\omega}\frac{G_0}{r^2}\right)+O\left(\frac{1}{r^4}\right). \tag{4.5}$$

For large distances, i.e., at order $1/r$, we recover the classical Schwarzschild spacetime. The leading quantum correction appears at order $1/r^3$; since in the Newtonian approximation the potential is given by $[f(r)-1]/2$, this correction is equivalent to the improved potential (3.20) which was independently confirmed by Donoghue's result (3.21). As we discussed in Sec. III already, matching the two results unambiguously fixes the constant $\tilde{\omega}$ to be $\tilde{\omega}=\hat{\omega}=118/15\pi$. Thus our improved lapse function (4.4) does not contain any free parameter. (Recall that the analysis of Sec. III fixes γ to be $\gamma=9/2$. However, to be as general as possible, we shall allow for an arbitrary $\gamma\geqslant 0$ in most of the calculations.)

B. The horizons

Next we determine the structure of the horizons of the improved spacetime. To this end we look for zeros of the function $f(r)$, Eq. (4.4), which is conveniently rewritten as

$$f(r)=\left.\frac{B(x)}{B(x)+2x^2}\right|_{x=r/G_0M} \tag{4.6}$$

with the polynomial B given by

$$B(x)\equiv B_{\gamma,\Omega}(x)=x^3-2x^2+\Omega x+\gamma\Omega, \tag{4.7}$$

where

$$\Omega\equiv\frac{\tilde{\omega}}{G_0M^2}. \tag{4.8}$$

The parameter Ω is a measure for the impact the quantum gravity effects have on the metric. Reinstating factors of $\hbar$ for a moment we have $\tilde{\omega}\propto\hbar$ and $\Omega\propto\hbar$. The classical limit is recovered by setting $\Omega=0$. We see immediately that very heavy black holes ($M\rightarrow\infty$) essentially behave classically. In fact, defining the Planck mass by $m_{\rm Pl}\equiv G_0^{-1/2}$ we have

$$\Omega=\tilde{\omega}\frac{m_{\rm Pl}^2}{M^2}, \tag{4.9}$$

which shows that an Ω of order unity requires M to be not much heavier than $m_{\rm Pl}$.

For $x>0$ the numerator and the denominator of the RHS of Eq. (4.6) have no common zeros; hence r_0 is a zero of $f(r)$ if $x_0=r_0/G_0M$ is a zero of $B(x)$. In the classical case ($\Omega=0$) we have $B_{\gamma,0}(x)=x^2(x-2)$ with its only nontrivial zero $x_0=2$ corresponding to the familiar Schwarzschild horizon at $r_0=2G_0M$.

In the quantum case ($\Omega>0$), $B_{\gamma,\Omega}(x)$ is a generic cubic polynomial which has either one or three simple zeros[2] on the real axis. Since $r\equiv xG_0M$ must be positive, only zeros on the positive real x axis $\mathfrak{R}^+$ can correspond to a horizon. It is easy to see that for any value of Ω and γ, $B(x)$ always has precisely one zero on the *negative* real axis: first we observe that $B(0)=\gamma\Omega>0$ and $B(-\infty)=-\infty<0$ which implies that $B(x)$ has at least one zero on the negative real axis. Furthermore, the derivative $B'(x)=3x^2-4x+\Omega$ is positive for $x<0$, i.e., B is monotonically increasing for $x<0$. As a consequence, B has precisely one zero on the negative real axis. Hence B has either two simple zeros or no zeros at all on the positive real axis $\mathfrak{R}^+$, whereby the two simple zeros might degenerate to form a single double zero.

The three cases are distinguished by the value of the discriminant

$$\mathcal{D}_\gamma(\Omega)=(3\Omega-4)^3+\left(9\Omega+\frac{27}{2}\gamma\Omega-8\right)^2. \tag{4.10}$$

For $\mathcal{D}_\gamma(\Omega)<0$ there are two simple zeros on $\mathfrak{R}^+$, for $\mathcal{D}_\gamma(\Omega)=0$ we have a double zero, and for $\mathcal{D}_\gamma(\Omega)>0$ there exists no zero on $\mathfrak{R}^+$. The discriminant can be factorized as

$$\mathcal{D}_\gamma(\Omega)=27\Omega[\Omega-\Omega_1(\gamma)][\Omega-\Omega_{\rm cr}(\gamma)], \tag{4.11}$$

with

$$\Omega_{\rm cr}(\gamma)=\frac{1}{8}(9\gamma+2)\sqrt{\gamma+2}\sqrt{9\gamma+2}-\frac{27}{8}\gamma^2-\frac{9}{2}\gamma+\frac{1}{2}. \tag{4.12}$$

[2]Here double and triple zeros are counted as two or three simple zeros, respectively.

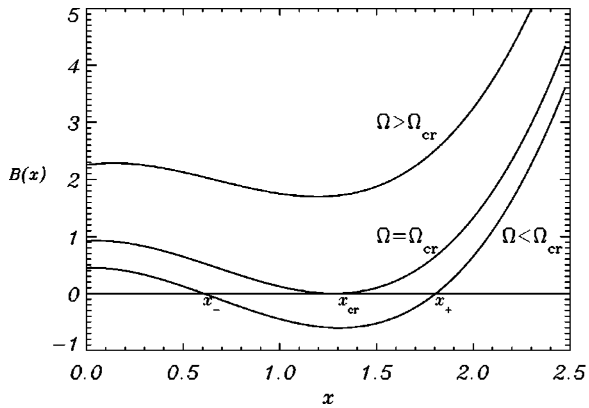

FIG. 1. The function $B_{\gamma,\Omega}(x)$ with $\gamma=9/2$ for different values of Ω. The regime $\Omega<\Omega_{\rm cr}$ ($\Omega>\Omega_{\rm cr}$) corresponds to very heavy (light) black holes.

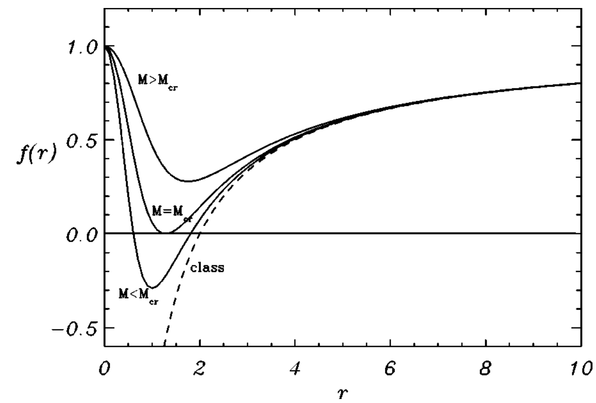

FIG. 2. The lapse function $f(r)$ for various mass values. The dashed line shows $f_{\rm class}(r)$ of the classical Schwarzschild metric.

The function $\Omega_1(\gamma)$ is not important except that it is negative for all $\gamma>0$. As a consequence, the sign of $\mathcal{D}_\gamma(\Omega)$ depends only on whether Ω is smaller or larger than the critical value $\Omega_{\rm cr}$: For $\Omega<\Omega_{\rm cr}(\gamma)$ the polynomial $B_{\gamma,\Omega}$ has two simple zeros x_+ and x_- on $\mathfrak{R}^+$ $(x_+>x_->0)$, for $\Omega>\Omega_{\rm cr}(\gamma)$ it has no zero on $\mathfrak{R}^+$, and for $\Omega=\Omega_{\rm cr}(\gamma)$ the two simple zeros merge into a single double zero at $x_+=x_-\equiv x_{\rm cr}$. This situation is illustrated in Fig. 1.

By virtue of Eq. (4.8), the critical value for Ω defines a critical value for the mass of the black hole:

$$M_{\rm cr}(\gamma)=\left[\frac{\tilde{\omega}}{\Omega_{\rm cr}(\gamma)G_0}\right]^{1/2}. \tag{4.13}$$

For the preferred value $\gamma=9/2$ we have

$$\Omega_{\rm cr}(9/2)=\frac{1}{32}[85\sqrt{85}\sqrt{13}-2819]\approx 0.20 \tag{4.14}$$

while for $\gamma=0$ (''naive'' cutoff $k\propto 1/r$),

$$\Omega_{\rm cr}(0)=1. \tag{4.15}$$

In any case $M_{\rm cr}$ is a number of order unity times $m_{\rm Pl}$.

The zeros $x_\pm$ or $x_{\rm cr}$ of $B(x)$ are equivalent to zeros of $f(r)$ located at

$$r_\pm=x_\pm G_0 M, \quad r_{\rm cr}=x_{\rm cr}G_0 M_{\rm cr}. \tag{4.16}$$

They correspond to horizons of the quantum corrected black hole spacetime. For heavy black holes ($M>M_{\rm cr},\Omega<\Omega_{\rm cr}$) we have an outer horizon at r_+ and an inner horizon at r_-. The function $f(r)$ is positive, i.e., the vector field $\partial/\partial t$ is timelike outside the outer (event) horizon ($r>r_+$) and inside the inner horizon ($r<r_-$); in the region between the horizons ($r_-<r<r_+$) we have $f(r)<0$ so that $\partial/\partial t$ is spacelike. For $M\gg M_{\rm cr}$ the outer horizon coincides essentially with the classical Schwarzschild horizon ($r_+\approx 2G_0M$) while r_- is very close to zero. When we decrease M and approach $M_{\rm cr}$ from above, the outer horizon shrinks and the inner horizon expands. Finally, for $M=M_{\rm cr}$, the two horizons coalesce at $r_+=r_-\equiv r_{\rm cr}$ which corresponds to a double zero of f. For very light black holes with $M<M_{\rm cr}$ the spacetime has no horizon at all.

In Fig. 2 we plot $f(r)$ for various masses M. The values of x_+ and x_- could be written down in closed form as a function of Ω and γ, but the formulas are not very illuminating. Instead, in Fig. 3, we represent them graphically.

C. The critical (extremal) black hole and the Reissner-Nordström analogy

Let us look in more detail at the ''critical'' black hole with $M=M_{\rm cr}$. We know that for $\Omega=\Omega_{\rm cr}(\gamma)$ the polynomial $B_{\gamma,\Omega}(x)$ has a double zero at some $x_{\rm cr}\equiv x_{\rm cr}(\gamma)>0$. Upon inserting Eq. (4.12) into $B_{\gamma,\Omega_{\rm cr}}(x)$ and factorizing the resulting expression with respect to x one finds the following explicit result:

$$x_{\rm cr}(\gamma)=\frac{1}{4}\sqrt{\gamma+2}\sqrt{9\gamma+2}-\frac{3}{4}\gamma+\frac{1}{2}. \tag{4.17}$$

In particular,

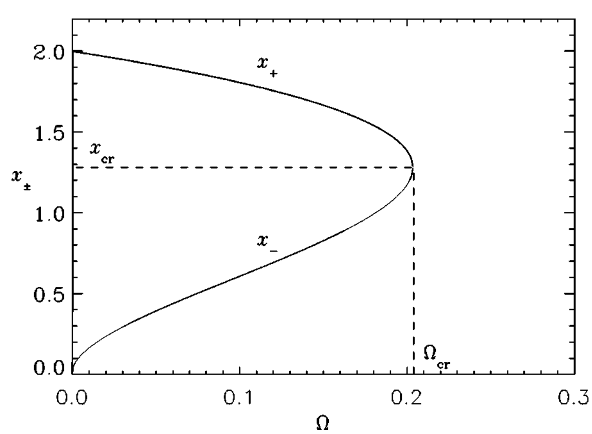

FIG. 3. The zeros x_+ and x_- for $\gamma=9/2$ as a function of Ω. Lowering M from infinity to $M_{\rm cr}$, Ω increases from zero to $\Omega_{\rm cr}$, x_- increases from zero to $x_{\rm cr}$, and x_+ decreases from 2 towards $x_{\rm cr}$. The outer horizon shrinks and the inner horizon expands until they meet at $r_{\rm cr}$.

$$x_{\rm cr}(0)=1,$$

$$x_{\rm cr}(9/2)=\frac{1}{8}[\sqrt{13}\sqrt{85}-23]\approx 1.28.$$

Using Eq. (4.16) the critical radius reads

$$r_{\rm cr}(\gamma)=x_{\rm cr}(\gamma)\left(\frac{\tilde{\omega}G_0}{\Omega_{\rm cr}(\gamma)}\right)^{1/2}, \tag{4.18}$$

with $x_{\rm cr}$ and $\Omega_{\rm cr}$ given by Eq. (4.17) and Eq. (4.12), respectively.

Some of the qualitative features of the quantum black hole are remarkably similar to those of a Reissner-Nordström spacetime (black hole with charge e). Its lapse function reads (in appropriate units)

$$f_{\rm RN}(r)=1-\frac{2G_0M}{r}+\frac{G_0e^2}{r^2}. \tag{4.19}$$

In analogy with Eq. (4.8) we introduce the parameter

$$\Omega_{\rm RN}\equiv\frac{e^2}{G_0M^2}. \tag{4.20}$$

The Reissner-Nordström spacetime has no horizon for $\Omega_{\rm RN}>1$, two horizons with

$$x_{\pm}^{\rm RN}=r_{\pm}^{\rm RN}/G_0M=1\pm\sqrt{1-\Omega_{\rm RN}} \tag{4.21}$$

if $\Omega_{\rm RN}<1$, and a single degenerate horizon at

$$r_{\rm cr}^{\rm RN}=G_0M, \tag{4.22}$$

if $\Omega_{\rm RN}$ equals its critical value $\Omega_{\rm RN}=1$. We observe that, in a sense, the renormalization group improved Schwarzschild spacetime is similar to a Reissner-Nordström black hole whose charge is given by $e=\tilde{\omega}^{1/2}$. In particular, the ''critical'' quantum black hole with $M=M_{\rm cr}$ corresponds to the extremal charged black hole ($\Omega_{\rm RN}=1$).

Let us look more closely at the near-horizon geometry of the critical quantum black hole. If we expand about $r=r_{\rm cr}$ and introduce the new coordinate $\bar{r}\equiv r-r_{\rm cr}$ we have at leading order

$$ds^2=-\left(\frac{\bar{r}}{G_0M_{\rm AdS}}\right)^2dt^2+\left(\frac{G_0M_{\rm AdS}}{\bar{r}}\right)^2d\bar{r}^2+r_{\rm cr}^2d\Omega^2, \tag{4.23}$$

where the mass parameter $M_{\rm AdS}$ is defined by

$$(G_0M_{\rm AdS})^{-2}=\frac{1}{2}f''(r_{\rm cr}(\gamma))|_{\Omega=\Omega_{\rm cr}(\gamma)} \tag{4.24}$$

The metric (4.23) is the Robinson-Bertotti metric for the product of a two-dimensional anti–de Sitter space AdS_2 with a two-sphere, $AdS_2\times S^2$. The parameter $M_{\rm AdS}$ determines the curvature of AdS_2. Using Eqs. (4.4), (4.12), and (4.17) it is obtained in the form

$$M_{\rm AdS}(\gamma)=\sqrt{2/b(\gamma)}M_{\rm cr}, \tag{4.25}$$

where $b(\gamma)$ is a complicated function which we shall not write down here. In particular,

$$b(\gamma=0)=1, \tag{4.26}$$

$$b(\gamma=9/2)=1.123\ldots. \tag{4.27}$$

If we put $M_{\rm AdS}=M$ and $r_{\rm cr}=G_0M$ the metric (4.23) describes also the near-horizon geometry of an extremal Reissner-Nordström black hole of mass M. However, in our case the relative magnitude of the AdS_2 and the S^2 curvature is different. For $\gamma=0$, say, the S^2 curvature is given by the radius $r_{\rm cr}=G_0M_{\rm cr}$ as above, but the AdS_2 curvature is determined by $M_{\rm AdS}=\sqrt{2}M_{\rm cr}$.

D. Large mass expansion of $r_\pm$

It is instructive to look at the location of the horizons in the limit of very heavy black holes. Since $\Omega\propto 1/M^2$, the large-mass expansion in $1/M$ corresponds to an expansion in powers of $\Omega^{1/2}$. Let us start by looking at the leading quantum correction of the outer horizon. Classically we have $r_+=2G_0M$ or $x_+=2$. By inserting an ansatz of the form $x_+=2+c_1\Omega+c_2\Omega^2+\cdots$ into $B(x_+)=0$ and combining equal powers of Ω we can easily determine the coefficients c_j. In leading order one finds $x_+=2-\frac{1}{4}(2+\gamma)\Omega+O(\Omega^2)$ and

$$r_+=2G_0M-\frac{(2+\gamma)\tilde{\omega}}{4M}+O\left(\frac{1}{M^3}\right). \tag{4.28}$$

We see that the quantum corrected r_+ is indeed smaller than its classical value. The leading correction is proportional to $1/M$ and it is independent of the value of Newton's constant. The prefactor of the $1/M$ term is uniquely determined: the arguments of Sec. III yield $\gamma=9/2$ and $\tilde{\omega}$ is fixed by the matching condition (3.22). We believe that Eq. (4.28) is a particularly accurate prediction of our approach.

Let us now look at r_- for $M\to\infty$. Classically, for $\Omega=0$, we have $B(x)=x^2(x-2)$. When we switch on Ω, the double zero at $x=0$ develops into 2 simple zeros, one on the negative and the other on the positive real axis. The latter is the (approximate) x_- we are looking for. As long as $\Omega\ll 1$ we have $x_-\ll 1$ and therefore we may neglect the cubic term in $B(x_-)=0$ relative to the quadratic one. The resulting equation is easily solved:

$$x_-=\frac{1}{4}\sqrt{\Omega}[\sqrt{\Omega}+\sqrt{8\gamma+\Omega}]. \tag{4.29}$$

The asymptotics of this result depends on whether $\gamma>0$ or $\gamma=0$. For $\gamma>0$ we have

$$x_-=\frac{1}{2}\sqrt{2\gamma\Omega}+O(\Omega)=\sqrt{\frac{\gamma\tilde{\omega}}{2G_0}}\frac{1}{M}+O\left(\frac{1}{M^2}\right). \tag{4.30}$$

Obviously x_- vanishes for $M\to\infty$ but because of its $1/M$ behavior the actual radius $r_-=x_-G_0M$ approaches a universal, nonzero limit:

$$r_-=\frac{1}{2}\sqrt{2\gamma\tilde{\omega}G_0}+O\left(\frac{1}{M}\right). \qquad (4.31)$$

Thus the inner horizon does not disappear even for infinitely massive black holes. The situation would be different for $\gamma=0$. There $x_-=\frac{1}{2}\Omega+O(\Omega^{3/2})$ and

$$r_-=\frac{\tilde{\omega}}{2M}+O\left(\frac{1}{M^2}\right) \quad (\gamma=0), \qquad (4.32)$$

which vanishes for $M\to\infty$.

E. The de Sitter core

We expect the improved $f(r)$ to be reliable as long as r is not too close to $r=0$ where the renormalization effects become strong and the quantum corrected geometry differs significantly from the classical one. Therefore Eqs. (4.31) and (4.32) should be taken with a grain of salt, of course. However, if one takes Eq. (4.4) at face value even for $r\to0$, the horizons (4.31), (4.32) acquire a very intriguing interpretation.

Expanding $f(r)$ about $r=0$ one finds for $\gamma>0$

$$f(r)=1-2(\gamma\tilde{\omega}G_0)^{-1}r^2+O(r^3). \qquad (4.33)$$

Recalling that (4.1) with $f_{\rm dS}(r)=1-\Lambda r^2/3$ is the metric of de Sitter space we see that, at very small distances, the quantum corrected Schwarzschild spacetime looks like a de Sitter space with an effective cosmological constant

$$\Lambda_{\rm eff}=6(\gamma\tilde{\omega}G_0)^{-1} \qquad (4.34)$$

(For $\gamma=9/2$, $\Lambda_{\rm eff}\approx0.06m_{\rm Pl}^2$.) This result is quite remarkable since there exist longstanding speculations in the literature about a possible de Sitter core of realistic black holes [17]. Those speculations were based upon purely phenomenological considerations and no derivation from first principles has been given so far.[3] Instead, if the renormalization group improved metric is reliable also at very short distances, the de Sitter core and in particular the regularity of the metric at $r=0$ is an automatic consequence. The validity of the improved $f(r)$ for $r\to0$ will be discussed in detail in Sec. VIII.

The de Sitter metric (4.33) has a ''cosmological'' horizon at $r_{\rm dS}=\sqrt{3/\Lambda_{\rm eff}}$. This value coincides precisely with the approximate r_- of Eq. (4.31). The asymptotic de Sitter form (4.33) is obtained only if $\gamma>0$. For $\gamma=0$ the expansion starts with a term linear in r:

$$f(r)=1-\frac{2M}{\tilde{\omega}}r+O(r^2) \quad (\gamma=0). \qquad (4.35)$$

This spacetime is *not* regular at $r=0$; there remains a curvature singularity at the origin. We shall come back to this point in Sec. VIII.

F. The special case $\gamma=0$

While close to $r=0$ [where the use of our improved $f(r)$ is anyhow questionable], the physics implied by the quantum corrected metric strongly depends on the parameter γ; the essential features of the spacetime related to larger distances are fairly insensitive to the value of γ. In particular, it is easy to see that the general pattern of horizons (two, one, or no horizon, their M dependence, etc.) is qualitatively the same for all values of γ. Even $\gamma=0$ gives the same general picture as the preferred value $\gamma=9/2$. Therefore some of the calculations in the following sections will be performed for $\gamma=0$ which simplifies the algebra and leads to much more transparent results. For $\gamma=0$ we have, for instance,

$$x_\pm=r_\pm/G_0M=1\pm\sqrt{1-\Omega}, \qquad (4.36)$$

$$\Omega_{\rm cr}=1, \quad x_{\rm cr}=1 \qquad (4.37)$$

$$M_{\rm cr}=\sqrt{\tilde{\omega}/G_0}, \qquad (4.38)$$

$$r_{\rm cr}=\sqrt{\tilde{\omega}G_0}. \qquad (4.39)$$

It is amusing to see that the explicit formula for the location of the horizons, Eq. (4.36), coincides exactly with the corresponding expression for the Reissner-Nordström black hole, Eq. (4.21). Note also that because of Eq. (4.37) the parameter Ω can be interpreted as the ratio

$$\Omega=\frac{M_{\rm cr}^2}{M^2} \quad (\gamma=0). \qquad (4.40)$$

G. Geodesics and causal structure

The global structure of our black hole spacetime is quite similar to the one of the Reissner-Nordström charged black hole. In particular, the $r=0$ hypersurface is timelike now. The Penrose diagram of the spacetime is shown in Fig. 4 for $M>M_{\rm cr}$. It is clear from the location of the horizons that we can distinguish the following main regions:

I and V : $r_+<r<\infty$

II and IV: $r_-<r<r_+$

III and III′: $0<r<r_-$

The features of the motion in such a spacetime are particularly evident if we consider a test particle which moves radially on a timelike geodesic. The equations of motion are given by

$$\frac{dr}{d\tau}=\pm(\mathcal{E}^2-f(r))^{1/2}, \qquad (4.41)$$

[3]However, two-dimensional dilaton gravity has been shown [19] to contain nonsingular quantum black holes asymptotic to de Sitter space.

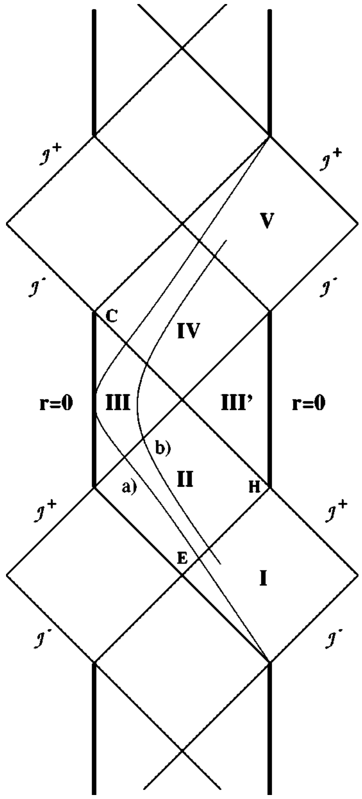

FIG. 4. Penrose conformal diagram of the quantum black hole spacetime.

$$\frac{dv}{d\tau}=f(r)^{-1}(\mathcal{E}\pm[\mathcal{E}^2-f(r)]^{1/2}), \tag{4.42}$$

where we have used Eddington-Finkelstein coordinates with v being the advanced time coordinate. Furthermore $\mathcal{E}$ denotes the constant of motion associated with the timelike Killing vector field $\xi^\mu=\delta^\mu_v$,

$$\mathcal{E}=-\xi_\mu u^\mu, \tag{4.43}$$

where u^μ is the four-velocity of a static observer. The choice of the sign in Eq. (4.41) and Eq. (4.42) depends upon whether the test particle is travelling on a path of decreasing $(-)$ or increasing $(+)$ radius r. From Eq. (4.41) we deduce that the proper acceleration is

$$\frac{d^2r}{d\tau^2}=-\frac{1}{2}\frac{\partial f(r)}{\partial r}=-\frac{MG_0 r(r^3-\tilde{\omega}G_0 r-2\tilde{\omega}G_0^2\gamma M)}{(r^3+\tilde{\omega}G_0[r+\gamma G_0 M])^2}, \tag{4.44}$$

wherefrom one sees that the radial motion is ruled by a Newton-type equation of motion with respect to the proper time τ. It contains the potential function $\Phi(r)=\frac{1}{2}f(r)$ with the properties $\Phi(0)=\Phi(\infty)=\frac{1}{2}\equiv\Phi_{\max}$ and $\Phi_{\min}<0$. If we identify the ''energy'' of the motion with $\overline{\mathcal{E}}=\mathcal{E}^2/2$ we have from Eq. (4.41)

$$\frac{1}{2}\dot{r}^2+\Phi(r)=\overline{\mathcal{E}}. \tag{4.45}$$

It is thus possible to discuss the radial motion by the help of simple mechanical arguments referring to Fig. 2 . In particular we use Eq. (4.45) in order to determine the inflection points, i.e., zero-velocity configurations where $\Phi(r)=\overline{\mathcal{E}}$, and where the sign in Eqs. (4.41) and (4.42) has to be changed.

There are basically three types of motions depending on the value of $\overline{\mathcal{E}}$:

(i) $\overline{\mathcal{E}}>\Phi_{\max}$. The motion is unbounded. It is a free falling test particle that starts its motion in region I, crosses the event horizon (EH in Fig. 4) and eventually reaches $r=0$ in region III in a finite amount of proper time, with non-zero velocity and finite proper acceleration. It would thus cross the inner horizon (CH in Fig. 4) and continue its journey in regions IV and V. This is for instance the path a) in Fig. 4. It should be noted that this behavior is unlike the Reissner-Nordström one. In that case $\Phi(0)=\infty$ and there is always an inflection point in region III. The particle is thus bounced away from the Reissner-Nordström central singularity at some non-zero value of the radius, before continuing its motion in region IV.

(ii) $\overline{\mathcal{E}}\in[0,\Phi_{\max}]$. The motion is bounded. Starting in regions I it evolves into regions II and III and it eventually continues in regions IV and V. Let us first consider the case $\overline{\mathcal{E}}<\Phi_{\max}$. Then there is an inflection point in region III and $r=0$ is avoided. A further inflection point is present in region V where the trajectory reaches the same initial conditions as in region I. The situation is shown in Fig. 4 with the path b). If $\overline{\mathcal{E}}=\Phi_{\max}$ the inflection point is at $r=0$. If $\gamma\neq 0$ this is also an equilibrium point since, as it follows from Eq. (4.45), the proper acceleration is zero, $\Phi'(0)=0$. The particle reaches the center in an infinite amount of proper time. If $\gamma=0$ the proper acceleration at $r=0$ is not zero, $\Phi'(0)=-\tilde{\omega}/M$, and $r=0$ is not an equilibrium configuration. Close to the origin the particle feels a repulsive force of strength $\tilde{\omega}/M$.

(iii) $\overline{\mathcal{E}}\in[\Phi_{\min},0]$. The motion is bounded. It starts in region II where it has two inflection points and it continues indefinitely in this region.

It would be possible to study along similar lines spacelike and null geodesics as well as the $M\leqslant M_{\rm cr}$ cases, but we shall not present this analysis here.

The Penrose diagram of Fig. 4 was obtained within the approximation of a vanishing running cosmological constant. If, at the UV scale, the bare cosmological constant is set to a value much smaller than $m_{\rm Pl}^2$, a running $\overline{\lambda}$ does not change the qualitative structure of the horizons which we derived above. It might, however, modify the spacetime at large, cosmological distances. Since in the present paper we are only interested in the physics at Planckian distances we shall not investigate this issue here.

V. EFFECTIVE MATTER INTERPRETATION AND ENERGY CONDITIONS

Let us suppose that our quantum black hole has been generated by an ''effective'' matter fluid that simulates the effect of the quantum fluctuations of the metric. We assume that this coupled gravity-matter system satisfies the conventional Einstein equations $G_{\mu\nu}=8\pi G_0 T_{\mu\nu}$. The stress-energy tensor of a (not necessarily classical) perfect fluid with the symmetries of our spacetime reads

$$T^{\alpha}{}_{\beta}=\mathrm{diag}(-\rho,P_r,P_\perp,P_\perp). \tag{5.1}$$

It is thus possible to use the Einstein equations in order to *define* the components of $T^{\alpha}{}_{\beta}$ in the following way

$$-8\pi G_0\rho=G^t_t=G^r_r, \tag{5.2}$$

$$8\pi G_0 P_\perp=G^\theta_\theta. \tag{5.3}$$

Here $G_{\mu\nu}$ is the Einstein tensor of the metric (4.1) with (4.4). A straightforward calculation shows that

$$\rho=-P_r=\frac{1}{4\pi}\frac{MG_0\tilde{\omega}(2r+3\gamma G_0 M)}{(r^3+\tilde{\omega}G_0[r+\gamma G_0 M])^2} \tag{5.4}$$

$$P_\perp=\frac{1}{4\pi}\frac{MG_0\tilde{\omega}(3r^4+6r^3G_0\gamma M-3\tilde{\omega}G_0^3\gamma^2M^2-r^2\tilde{\omega}G_0-3\tilde{\omega}G^2r\gamma M)}{(r^3+\tilde{\omega}G_0[r+\gamma G_0 M])^3} \tag{5.5}$$

The total energy outside a given radius r,

$$E(r)=4\pi\int_r^\infty \rho(r')r'^2dr' \tag{5.6}$$

is given by

$$E(r)=\frac{\tilde{\omega}G_0M(r+\gamma G_0 M)}{r^3+\tilde{\omega}G_0[r+\gamma G_0 M]}. \tag{5.7}$$

This quantity is finite and positive definite for any value of γ. In particular we find the surprisingly simple result

$$E_{\mathrm{tot}}\equiv E(0)=M, \tag{5.8}$$

which identifies the total energy of the fluid with the mass of the black hole.

It is not difficult to realize that this ''magic'' equality is a consequence of the boundary conditions set at spatial infinity, $f=1-2G_0M/r+O(1/r^2)$, and of the behavior of f at $r=0$. For *every* metric of the form (4.1), with an arbitrary function $f(r)$, the definitions (5.2) and (5.6) lead to the expression

$$\begin{aligned}E(r)&=-\frac{1}{G_0}\lim_{\hat{r}\to\infty}\int_r^{\hat{r}}ds[(sf(s))'-1]\\&=M+\frac{r}{2G_0}(f(r)-1).\end{aligned} \tag{5.9}$$

Obviously $E(0)$ equals M if $rf(r)\to 0$ when $r\to 0$, and this is always satisfied in our model, for any value of γ. Note that in ordinary Schwarzschild spacetimes with ADM mass M, since $r-rf(r)=2G_0M$, it follows that $E_{\mathrm{tot}}=0$. In our picture the quantum effects can be interpreted as a non-zero ρ and $P_\perp$. It is thus remarkable that their global contribution is exactly equal to the total mass of the spacetime.

For $M<M_{\mathrm{cr}}$ we have seen from the discussion of Sec. III that no horizon is present and that, contrary to what happens in classical Reissner-Nordström spacetimes for $\Omega_{\mathrm{RN}}>1$, no naked singularity occurs (for $\gamma>0$). The spacetime, in this case, resembles a soliton-like particle with a Planckian rest mass given by Eq. (5.8). The energy of the fluid is then localized in a cloud around $r=0$ with $\partial_r E(r)<0$ always, and $E(\infty)=0$.

It is possible to show that our ''effective'' fluid does not meet all the requirements in order to be considered a classical fluid. In fact, it violates the dominant energy condition in some regions, depending on the values of γ. In particular, the condition $P_\perp-\rho\leqslant 0$ is not always satisfied since it amounts to

$$r^4+3r^3\gamma G_0M-3r^2\tilde{\omega}G_0-8r\tilde{\omega}\gamma G_0^2M\leqslant 0 \tag{5.10}$$

which does not hold for some interval of values of the radial coordinate. For instance, for $\gamma=0$ the left hand side of (5.10) is positive when $r\leqslant\sqrt{3\tilde{\omega}G_0}$.

In the improved black hole spacetimes there are also ''zero gravity'' hypersurfaces, where the Weyl and Ricci curvatures are zero, in analogy to what has been found in [18] in a phenomenological model with a de Sitter core. In fact the ''Coulombian'' component of the Weyl tensor can be shown to be

$$\Psi_2=-\frac{1}{3}\frac{MG_0r(3r^5-r^3\tilde{\omega}G_0-6r^2\tilde{\omega}\gamma G_0^2M-\tilde{\omega}^2\gamma G_0^3M)}{(r^3+\tilde{\omega}G_0[r+\gamma G_0 M])^3} \tag{5.11}$$

It should be noted that the Weyl curvature is regular at $r=0$ where it is always zero. Similarly, the Ricci scalar reads

$$R=-\frac{4\tilde{\omega}G_0M(r^4+3r^3\gamma G_0M-3r^2\tilde{\omega}G_0-8r\tilde{\omega}\gamma G_0^2M-6\tilde{\omega}\gamma^2G_0^3M^3)}{(r^3+\tilde{\omega}G_0[r+\gamma G_0M])^3}. \tag{5.12}$$

In general the location of zero-curvature hypersurfaces depends on the value of γ and of the black hole mass. In the case of $\gamma=0$ one sees from the above expressions that at the radii $r=\sqrt{\tilde{\omega}G_0/3}$ and $r=\sqrt{3\tilde{\omega}G_0}$ one or the other of the two main scalar curvature invariants of our spacetime is zero. In particular, for $M<M_{\rm cr}$ and $\gamma=0$ the radius

$$r=\sqrt{3\tilde{\omega}G_0} \tag{5.13}$$

can be thought of as the characteristic length of the particle-like soliton structure arising from the renormalization group improvement of the spacetime of a nearly Planckian black hole.

VI. HAWKING TEMPERATURE AND BLACK HOLE EVAPORATION

A. The Euclidean manifold

Let us consider Lorentzian black hole metrics of the type (4.1) with an essentially arbitrary function $f(r)$. For the time being we only assume that f has a simple zero at some r_+, $(f(r_+)=0, f'(r_+)\neq 0)$ and that it increases monotonically from zero to $f(\infty)=1$ for $r>r_+$. The behavior of $f(r)$ for $r<r_+$ will not matter in the following. There exists a standard method for associating a Euclidean black hole spacetime to metrics of this type [21]. The first step is to perform a ''Wick rotation'' by setting $t=-i\tau$ and taking τ real:

$$ds_E^2=f(r)d\tau^2+f(r)^{-1}dr^2+r^2d\Omega^2. \tag{6.1}$$

This line element defines a Euclidean metric on the manifold coordinatized by (τ,r,θ,ϕ) with $r>r_+$ where ds_E^2 is positive definite. In order to investigate the properties of this manifold we trade r for a new coordinate ρ defined by

$$\rho=\frac{1}{2\pi}\beta_{\rm BH}\sqrt{f(r)}, \tag{6.2}$$

with the constant

$$\beta_{\rm BH}\equiv\frac{4\pi}{f'(r_+)}. \tag{6.3}$$

The new coordinate ranges from $\rho=0$ to $\rho=\beta_{\rm BH}/2\pi$ corresponding to $r=r_+$ and $r\rightarrow\infty$, respectively. Thus the line element becomes

$$ds_E^2=\rho^2\left(\frac{2\pi}{\beta_{\rm BH}}\right)^2d\tau^2+\left[\frac{f'(r_+)}{f'(r(\rho))}\right]^2d\rho^2+r(\rho)^2d\Omega^2, \tag{6.4}$$

where r is a function of ρ now. Close to the horizon ($r=r_+$ or $\rho=0$) this metric simplifies considerably:

$$ds_E^2\approx\rho^2d\hat{\tau}^2+d\rho^2+r_+^2d\Omega^2. \tag{6.5}$$

In writing down (6.5) we introduced the rescaled Euclidean time

$$\hat{\tau}\equiv\frac{2\pi}{\beta_{\rm BH}}\tau. \tag{6.6}$$

Leaving aside the $r_+^2d\Omega^2$ term for a moment, we see that (6.5) looks like the metric of a 2-dimensional Euclidean plane written in polar coordinates $\hat{\tau}$ and ρ. For this to be actually true, $\hat{\tau}$ must be considered an angular variable with period 2π. If $\hat{\tau}$ is periodic with a period different from 2π, the space has a conical singularity at $\rho=0$. In order to avoid this singularity we require the unrescaled time τ to be an angle-like variable with period $\beta_{\rm BH}$, $\tau\in[0,\beta_{\rm BH}]$. With this periodic identification, Eq. (6.4) defines a Euclidean metric on what is referred to as the Euclidean black hole manifold. It has the topology of $R^2\times S^2$.

If we put quantized matter fields on the Euclidean black hole spacetime, their Green's functions inherit the periodicity in the time direction. Thus they appear to be thermal Green's functions with the temperature given by

$$T_{\rm BH}=\beta_{\rm BH}^{-1}=\frac{1}{4\pi}f'(r_+). \tag{6.7}$$

This is the Bekenstein-Hawking temperature for the general class of black holes with metrics of the type (4.1).

B. Temperature and specific heat

Equation (6.7) is an essentially ''kinematic'' statement and its derivation does not assume any specific form of the field equations for the metric. Hence we may apply it to the renormalization group improved Schwarzschild metric and investigate how the quantum gravity effects modify the Hawking temperature. By differentiating Eq. (4.4) we find

$$T_{\rm BH}(M)=\frac{1}{8\pi G_0M}\left[1-\frac{\Omega}{x_+^2}-\frac{2\gamma\Omega}{x_+^3}\right], \tag{6.8}$$

where x_+ and Ω are considered functions of M. When we switch off quantum gravity ($\tilde{\omega}=0$) or look at very heavy black holes ($M\rightarrow\infty$) we have $\Omega=0$ and recover the classical result

$$T_{\rm BH}^{\rm class}(M)=\frac{1}{8\pi G_0M}. \tag{6.9}$$

(In the present context, the term ''classical'' refers to the usual ''semiclassical'' treatment with quantized matter fields on a classical geometry.) Obviously the quantum corrected Hawking temperature is always smaller than the classical one: $T_{\rm BH} < T_{\rm BH}^{\rm class}$.

In order to be more explicit we continue our investigation for the special value $\gamma=0$. Using Eqs. (4.36) and (4.38) we obtain

$$T_{\rm BH}(M) = \frac{1}{4\pi G_0 M}\,\frac{\sqrt{1-\Omega}}{1+\sqrt{1-\Omega}}, \tag{6.10}$$

$$= \frac{1}{4\pi G_0 M_{\rm cr}}\,\frac{\sqrt{\Omega(1-\Omega)}}{1+\sqrt{1-\Omega}}, \tag{6.11}$$

with $\Omega = M_{\rm cr}^2/M^2$. Equation (6.11) is quite similar, though not identical to the corresponding expression for the temperature of the Reissner-Nordström black hole:

$$T_{\rm BH}^{\rm RN} = \frac{1}{2\pi G_0 M}\,\frac{\sqrt{1-\Omega_{\rm RN}}}{(1+\sqrt{1-\Omega_{\rm RN}})^2}. \tag{6.12}$$

The large mass expansion of $T_{\rm BH}$ reads

$$T_{\rm BH}(M) = \frac{1}{8\pi G_0 M}\left[1 - \frac{1}{4}\left(\frac{M_{\rm cr}}{M}\right)^2 - \frac{1}{8}\left(\frac{M_{\rm cr}}{M}\right)^4 + O(M^{-6})\right], \tag{6.13}$$

with $M_{\rm cr}^2 = \tilde{\omega}/G_0$. Probably the first few terms of this series are a rather precise prediction of our method because they correspond to a spacetime which is only very weakly distorted by quantum effects.

Let us look at what happens when M approaches $M_{\rm cr}$ from above. We set

$$\Omega = \Omega_{\rm cr} - \epsilon = 1 - \epsilon \tag{6.14}$$

and study the limit $\epsilon \to 0^+$. Note that because $\Omega = M_{\rm cr}^2/M^2$ for $\gamma = 0$,

$$\epsilon = 1 - \frac{M_{\rm cr}^2}{M}. \tag{6.15}$$

Expanding Eq. (6.11) yields

$$T_{\rm BH}(M) = \frac{\sqrt{\epsilon}}{4\pi G_0 M_{\rm cr}}\left[1 - \sqrt{\epsilon} + \frac{1}{2}\epsilon + O(\epsilon^{3/2})\right] \tag{6.16}$$

or, to lowest order,

$$T_{\rm BH}(M) = \frac{1}{4\pi\tilde{\omega}}\sqrt{M^2 - M_{\rm cr}} + O(M^2 - M_{\rm cr}). \tag{6.17}$$

We see that $T_{\rm BH}$ vanishes as M approaches its critical value $M_{\rm cr}$. This conclusion is true for any value of γ. In fact, as $M \searrow M_{\rm cr}$ the simple zero of f at r_+ tends to become a double zero, i.e., $f'(r_+) \to 0$, and therefore $T_{\rm BH} \to 0$ by Eq. (6.7). We emphasize, however, that the statement

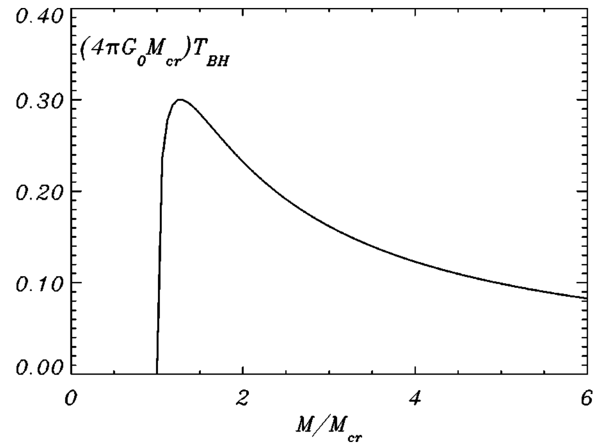

FIG. 5. The Hawking temperature of the quantum black hole (multiplied by $4\pi G_0 M_{\rm cr}$) as a function of $M/M_{\rm cr}$. The maximum temperature is reached for $\tilde{M}_{\rm cr} \approx 1.27 M_{\rm cr}$.

$$T_{\rm BH}(M_{\rm cr}) = 0 \tag{6.18}$$

should always be understood in the sense of a limit $M \searrow M_{\rm cr}$ because strictly speaking the above derivation of the Hawking temperature does not apply for the critical (extremal) black hole with M exactly equal to $M_{\rm cr}$.

In Fig. 5 the Hawking temperature is plotted for the full range of mass values. For large values of M we recover the classical $1/M$ decay, and for $M > M_{\rm cr}$ the temperature increases with M. The Hawking temperature reaches its maximum for a certain mass $\tilde{M}_{\rm cr} > M_{\rm cr}$ which plays the role of another ''critical'' temperature (see below). By definition,

$$\frac{dT_{\rm BH}}{dM}(\tilde{M}_{\rm cr}) = 0. \tag{6.19}$$

The Ω value related to $\tilde{M}_{\rm cr}$ will be denoted $\tilde{\Omega}_{\rm cr}$; for $\gamma = 0$ it reads

$$\tilde{\Omega}_{\rm cr} = \left(\frac{M_{\rm cr}}{\tilde{M}_{\rm cr}}\right)^2. \tag{6.20}$$

Differentiating (6.11) leads to $\tilde{\Omega}_{\rm cr} = (\sqrt{5}-1)/2 \approx 0.62$, which yields $\tilde{M}_{\rm cr} = M_{\rm cr}\tilde{\Omega}_{\rm cr}^{-1/2} \approx 1.27 M_{\rm cr}$. The maximum at $\tilde{M}_{\rm cr}$ is surprisingly close to $M_{\rm cr}$ so that the drop from the peak value of $T_{\rm BH}$ down to zero is rather steep.

Even though we do not have a full statistical mechanical formalism with a partition function and a free energy functional at our disposal, Eq. (5.8) suggests to identify the internal energy U of the black hole with its total mass M. Then the standard thermodynamical relation $C_V = (\partial U/\partial T)_V$ amounts to the following definition for the specific heat capacity of the black hole:

$$C_{\rm BH} = \frac{dM}{dT_{\rm BH}} = \left(\frac{dT_{\rm BH}}{dM}\right)^{-1}. \tag{6.21}$$

From Eq. (6.11) we obtain

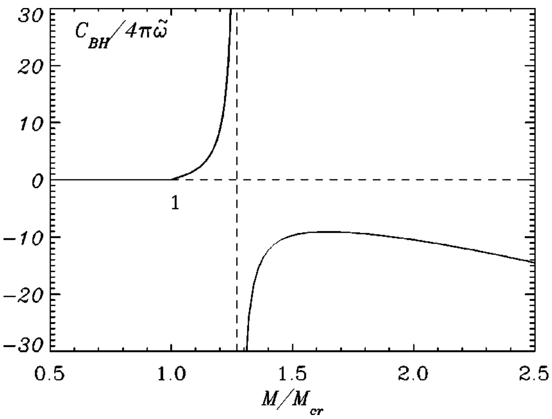

FIG. 6. The specific heat $C_{\rm BH}$ in units of $4\pi\tilde{\omega}$ as a function of $M/M_{\rm cr}$. The singularity occurs at $\tilde{M}_{\rm cr}/M_{\rm cr}\approx 1.27$.

$$C_{\rm BH}=-4\pi\tilde{\omega}\frac{(1-\Omega)[1+\sqrt{1-\Omega}]^2}{\Omega[\Omega^2+(1-2\Omega)(1+\sqrt{1-\Omega})]}. \tag{6.22}$$

The specific heat $C_{\rm BH}$ is negative for $M>\tilde{M}_{\rm cr}$ and becomes positive for $M_{\rm cr}<M<\tilde{M}_{\rm cr}$. It has a singularity at $M=\tilde{M}_{\rm cr}$ which signals a kind of phase transition at this value of the mass. In Fig. 6, $C_{\rm BH}$ is shown as a function of M. For very heavy black holes one has

$$C_{\rm BH}=-8\pi G_0 M^2\left[1+\frac{3}{4}\left(\frac{M_{\rm cr}}{M}\right)^2+\frac{19}{16}\left(\frac{M_{\rm cr}}{M}\right)^4+O(M^{-6})\right] \tag{6.23}$$

In the limit $M\to\infty$ we recover the classical value $C_{\rm BH}^{\rm class}=-8\pi G_0 M^2$, and we observe that the leading quantum corrections make the already negative specific heat even more negative.[4] In the limit $M\searrow M_{\rm cr}$ the specific heat vanishes according to

$$C_{\rm BH}=4\pi\tilde{\omega}\sqrt{\epsilon}[1+2\sqrt{\epsilon}+4\epsilon+O(\epsilon^{3/2})]$$
$$=4\pi\tilde{\omega}\sqrt{1-\frac{M_{\rm cr}^2}{M^2}}+\cdots. \tag{6.24}$$

C. Stopping the evaporation process

From our result for the mass dependence of the Bekenstein-Hawking temperature the following scenario for the black hole evaporation with the leading quantum correction included emerges. As long as the black hole is very heavy the classical relation $T_{\rm BH}\propto 1/M$ is approximately valid. The black hole radiates off energy, thereby lowering its mass and increasing its temperature. This tendency is counteracted by the quantum effects. The actual temperature stays always below $T_{\rm BH}^{\rm class}$. Once the mass is as small as $\tilde{M}_{\rm cr}$, the temperature reaches its maximum value $T_{\rm BH}(\tilde{M}_{\rm cr})$. For even smaller masses it drops very rapidly and it vanishes once M has reached its critical mass $M_{\rm cr}$, which is of the order of $m_{\rm Pl}$.

In the classical picture based upon $T_{\rm BH}^{\rm class}\propto 1/M$ the black hole becomes continuously hotter during the evaporation process. In the above scenario, on the other hand, its temperature never exceeds $T_{\rm BH}(\tilde{M}_{\rm cr})$, and the evaporation process comes to a complete halt when the mass has reached $M_{\rm cr}$. This suggests that the critical (or extremal) black hole with $M=M_{\rm cr}$ could be the final state of the evaporation of a Schwarzschild black hole. If stable, the critical black hole would indeed constitute a Planck-size remnant of burnt-out macroscopic black holes. It is ''cold'' in the sense that $\lim_{M\searrow M_{\rm cr}}T_{\rm BH}(M)=0$, so that it is stable at least against the classical Hawking radiation mechanism as we know it.

It is interesting to see how long it takes a black hole with the initial mass $M_{\rm i}$ to reduce its mass to some final value $M_{\rm f}$ via Hawking radiation. Stefan's law provides us with a rough estimate of the radiation power. The mass loss per unit proper time of an infinitely far away, static observer is approximately given by

$$-\frac{dM}{dt}=\sigma\mathcal{A}(M)T_{\rm BH}(M)^4. \tag{6.25}$$

Here σ is a constant and $\mathcal{A}=4\pi r_+^2$ is the area of the outer horizon:

$$\mathcal{A}(M)=8\pi G_0^2 M^2\left[1-\frac{1}{2}\Omega+\sqrt{1-\Omega}\right]. \tag{6.26}$$

In the classical case the above differential equation becomes $-dM/dt\propto M^{-2}$. It is easily integrated with the result that only a *finite* amount of time $t(M_{\rm i}\to 0)\propto M_{\rm i}^3$ is needed in order to completely radiate away the initial mass. The problems such as the information paradox mentioned in the Introduction are particularly severe because the catastrophic end point of the evolution ($T_{\rm BH}\to\infty$) is reached within a finite time.

Looking at the quantum black hole now, we assume that the initial mass $M_{\rm i}$ is already close to $M_{\rm cr}$ so that we may use the approximation (6.17) on the RHS of Eq. (6.25):

$$-\frac{dM}{dt}=\frac{\sigma G_0}{(4\pi\tilde{\omega})^3}(M^2-M_{\rm cr}^2)+\cdots. \tag{6.27}$$

Obviously the radiation power decreases quickly as $M\searrow M_{\rm cr}$. Integrating Eq. (6.27) yields for the time to go from $M_{\rm i}$ to $M_{\rm f}$:

$$t(M_{\rm i}\to M_{\rm f})=16\pi^3\tilde{\omega}^2\sigma^{-1}\left(\frac{1}{M_{\rm f}-M_{\rm cr}}-\frac{1}{M_{\rm i}-M_{\rm cr}}\right). \tag{6.28}$$

We see that this time diverges for $M_{\rm f}=M_{\rm cr}$, i.e., it takes an *infinitely long* time to reduce the mass from any given $M_{\rm i}$ down to the critical mass. Clearly the reason is that, because of the T^4 behavior, the radiation power becomes very small

[4]This is the same tendency as in the Weyl-gravity model of Ref. [20], for instance.

when we approach the ''cold'' critical black hole. In a certain sense, this result is a reflection of the third law of black hole thermodynamics which states that it is impossible to achieve an exactly vanishing surface gravity, i.e., $T=0$, by any physical process.

The back reaction of the Hawking radiation on the metric is neglected in the above arguments. We believe that most probably (contrary to the case of the classical Hawking black hole) its inclusion would not lead to qualitative changes of the picture. The reason is that dM/dt is very small in both the early *and the late* stage of the evaporation process, and that in between its value is bounded above.

VII. ENTROPY OF THE QUANTUM BLACK HOLE

One of the most intriguing aspects of black hole thermodynamics is the entropy associated with the horizon of a black hole. It is one of the central but as yet unresolved questions if and how this entropy can be interpreted within a ''microscopic'' statistical mechanics by counting the number of micro states which are inaccessible to our observation [23]. Another important question is how the classical relation between the entropy and the surface area of the horizon

$$S_{\text{class}}=\frac{\mathcal{A}_{\text{class}}}{4G_0} \tag{7.1}$$

changes if quantum (gravity) effects are taken into account.

Our approach of renormalization group improving the Schwarzschild spacetime makes a definite prediction for the quantum correction of the entropy. The key ingredient is the function $T_{\text{BH}}=T_{\text{BH}}(M)$ which we obtained in Sec. VI. From general thermodynamics we know that the entropy $S=S(U,V,\ldots)$ satisfies $(\partial S/\partial U)_V=1/T$. In the present context we identify the energy U with the mass M, and since the volume dependence plays no role $S=S(M)$ satisfies $dS/dM=1/T_{\text{BH}}(M)$. Upon integration we have

$$S(M)-S(M_{\text{cr}})=\int_{M_{\text{cr}}}^{M}\frac{dM'}{T_{\text{BH}}(M')}, \tag{7.2}$$

where the reference point was chosen to be the critical mass. For simplicity we continue the analysis for $\gamma=0$; inserting the corresponding Hawking temperature (6.11) into (7.2) we obtain

$$S(M)-S(M_{\text{cr}})=2\pi\tilde{\omega}\int_{M_{\text{cr}}^2/M^2}^{1}\frac{d\Omega}{\Omega^2}\left[1+\frac{1}{\sqrt{1-\Omega}}\right]. \tag{7.3}$$

The integral yields for $M\geqslant M_{\text{cr}}$

$$S(M)-S(M_{\text{cr}})=2\pi\tilde{\omega}[\Omega^{-1}\sqrt{1-\Omega}(1+\sqrt{1-\Omega})+\text{artanh}\sqrt{1-\Omega}], \tag{7.4}$$

with $\Omega\equiv M_{\text{cr}}^2/M^2$ on the RHS of Eq. (7.4). Equation (7.4) is our prediction for the quantum corrected entropy of the black hole geometry. Its large-M expansion reads

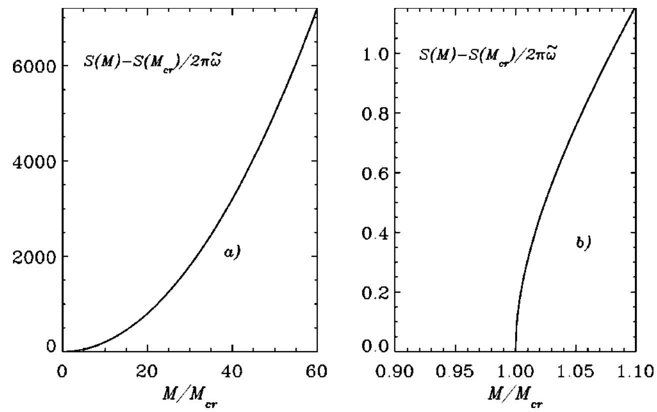

FIG. 7. (a) The entropy $S(M)-S(M_{\text{cr}})$ in units of $2\pi\tilde{\omega}$ as a function of M/M_{cr}. (b) The same function for M near M_{cr}.

$$S(M)-S(M_{\text{cr}})=\frac{\mathcal{A}_{\text{class}}}{4G_0}+2\pi\tilde{\omega}\left[\ln\left(\frac{2M}{M_{\text{cr}}}\right)-\frac{3}{2}-\frac{3}{8}\left(\frac{M_{\text{cr}}}{M}\right)^2-\frac{5}{32}\left(\frac{M_{\text{cr}}}{M}\right)^4+O(M^{-6})\right], \tag{7.5}$$

with the classical area $\mathcal{A}_{\text{class}}=4\pi(2G_0M)^2$. For very heavy black holes we recover the classical entropy S_{class} as the difference of $S(M)$ and the integration constant $S(M_{\text{cr}})$ whose value remains undetermined here. The leading quantum correction is proportional to ln(M). Remarkably, very similar ln(M) terms had been found with rather different methods [24]. While some of the earlier results were plagued by the presence of numerically undefined cutoffs, Eq. (7.4) is perfectly finite. When M approaches M_{cr} from above, the entropy difference displays a square-root behavior:

$$S(M)-S(M_{\text{cr}})=4\pi\tilde{\omega}\sqrt{\epsilon}\left[1+\frac{1}{2}\sqrt{\epsilon}+O(\epsilon)\right]. \tag{7.6}$$

In Fig. 7 the entropy is shown as a function of M.

The above calculation of $S(M)$ was within the framework of ''phenomenological'' thermodynamics. For an attempt at interpreting it within an underlying statistical mechanics we refer to the Appendix.

VIII. IS THERE A CURVATURE SINGULARITY AT $r=0$

We saw already that for $r\to 0$ the renormalization group improved black hole metric approaches to that of de Sitter space. The quantum black hole seems to have a ''de Sitter core'' of a similar type to the regular black holes which were introduced in Ref. [18] on a phenomenological basis. This means in particular that the quantum corrected spacetime is completely regular, i.e., contrary to the ordinary Schwarzschild black hole it is free from any curvature singularity. However, because the classical and the quantum geometries are very different for $r\to 0$ and the quantum effects play a dominant role there, it seems problematic to describe a possibly regular core as an ''improvement'' of the singular Schwarzschild spacetime. Therefore some comments concerning the applicability of our approximation at very small distances are appropriate.

The regularity of the improved metric comes about because the $1/r$ behavior of $f_{\rm class}=1-2G_0M/r$ is tamed by a very fast vanishing of the Newton constant at small distances. Close to the core of the black hole we are in the regime where the running of $G(k)$ is governed by the UV fixed point, $G(k)\approx 1/\omega k^2$, so that the position-dependent Newton constant is approximately given by

$$G(r)\approx \tilde{\omega}^{-1}d(r)^2. \tag{8.1}$$

It is important to keep in mind that the distance function $d(r)$ depends on the classical metric which we are going to improve. In Sec. III we started from the Schwarzschild background and found that for all sensible curves $\mathcal{C}$,

$$d_{\rm Sch}(r)\propto r^{3/2}, \tag{8.2}$$

so that

$$G_{\rm Sch}(r)\propto r^3. \tag{8.3}$$

Taking (8.3) literally means that the improved $f=1-2G(r)M/r$ is of the de Sitter form $1-(\text{const})r^2$ for $r\rightarrow 0$.

However, if the actual quantum geometry really was de Sitter, there is no point in evaluating $d(r)$ for the Schwarzschild background. In fact, if we calculate $d(r)$ for the de Sitter metric the asymptotic behavior is different:

$$d_{\rm dS}(r)\approx r. \tag{8.4}$$

Incidentally, this is precisely the d function which obtains by setting $\gamma=0$ in Eq. (3.13). Equation (8.4) entails that the Newton constant vanishes more slowly than in (8.3):

$$G_{\rm dS}(r)\propto r^2. \tag{8.5}$$

Inserting (8.5) into $f_{\rm class}$ we obtain a lapse function which approaches $f=1$ only *linearly*,

$$f(r)=1-cr+O(r^2). \tag{8.6}$$

(Here c is a constant.)

The metric with an f function of the general form

$$f(r)=1-cr^{\nu}, \tag{8.7}$$

where c and ν are constant has the exact curvature invariants

$$R=c(\nu+1)(\nu+2)r^{\nu-2}, \tag{8.8}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}=c^2(\nu^4-2\nu^3+5\nu^2+4)r^{2\nu-4}, \tag{8.9}$$

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}=\frac{c^2}{144}(\nu-1)^2(\nu-2)^2r^{2\nu-4}. \tag{8.10}$$

This means that the "$G_{\rm dS}$-improved" black hole of Eq. (8.6) has a curvature singularity at its center:

$$R=\frac{6c}{r}+\cdots, \tag{8.11}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}=8\frac{c^2}{r^2}+\cdots. \tag{8.12}$$

Equation (8.10) shows that the square of the Weyl tensor is regular for this metric. Even if, contrary to the "$G_{\rm Sch}$-improved" spacetime, the "$G_{\rm dS}$-improved" geometry is singular at the origin, it is much less singular than it was classically. For the Schwarzschild metric one has

$$(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})_{\rm Sch}=48\frac{G_0^2M^2}{r^6}, \tag{8.13}$$

with an additional factor of $1/r^4$ compared to Eq. (8.12).

Within the present framework, we have no criterion for deciding whether the improvement $G_0\rightarrow G(r)$ should be done with $d_{\rm Sch}$, $d_{\rm dS}$, or the d function of some unknown metric interpolating between Schwarzschild and de Sitter. This is a principal limitation of our approach. It appears plausible that $f(r)\approx 1-cr^{\nu}$ for $r\rightarrow 0$ with the exponent ν somewhere in between the values resulting from $d_{\rm Sch}$ improvement ($\nu=2$) and $d_{\rm dS}$ improvement ($\nu=1$). Except for $\nu=2$, the quantum black hole would have a curvature singularity at its center then. A reliable calculation of the exponent ν seems to be extremely difficult, though. Nevertheless it is probably a safe prediction that the central singularity is much weaker than its classical counterpart. The reason is that we found quantum gravity to be asymptotically free and that near the UV fixed point $G(k)\propto 1/k^2$. In one way or another, this k dependence must translate into a "switching off" of the gravitational interaction at small distances.

The improvement with $d_{\rm dS}$ is equivalent to setting $\gamma=0$ in the formulas of the previous sections. While the cases $\gamma=0$ and $\gamma>0$ are qualitatively different for $r\rightarrow 0$, we saw already that the other features of the quantum black holes (horizons, Hawking radiation, entropy, etc.) are essentially the same in both cases.

IX. SUMMARY AND CONCLUSIONS

In this paper we used the method of the renormalization group improvement in order to obtain a qualitative understanding of the quantum gravitational effects in spherically symmetric black hole spacetimes.

As far as the structure of horizons is concerned, the quantum effects are small for very heavy black holes ($M\gg m_{\rm Pl}$). They have an event horizon at a radius r_+ which is close to, but always smaller than, the Schwarzschild radius $2G_0M$. Decreasing the mass of the black hole the event horizon shrinks. There is also an inner (Cauchy) horizon whose radius r_- increases as M decreases. For $M\rightarrow\infty$ it assumes its nonzero (if $\gamma\neq 0$) minimal value. When M equals the critical mass $M_{\rm cr}$ which is of the order of the Planck mass the two horizons coincide. The near-horizon geometry of this critical black hole is that of $AdS_2\times S^2$. For $M<M_{\rm cr}$ the spacetime has no horizon at all.

While the exact fate of the singularity at $r=0$ cannot be decided within our present approach, we argued that either it is not present at all or it is at least much weaker than its classical counterpart. In the first case the quantum spacetime has a smooth de Sitter core so that we are in accord with the cosmic censorship hypothesis even if $M<M_{\rm cr}$.

The conformal structure of the quantum black hole is very similar to that of the classical Reissner-Nordström spacetime. In particular its $(r=0)$ hypersurface is timelike, in contradistinction to the Schwarzschild case where it is spacelike. In this respect the classical limit $\hbar\to 0$ is discontinuous, as is the limit $e\to 0$ of the Reissner-Nordström black hole.

The Hawking temperature of very heavy quantum black holes is given by the semiclassical $1/M$ law. As M decreases, $T_{\rm BH}$ reaches a maximum at $\tilde{M}_{\rm cr}\approx 1.27 M_{\rm cr}$ and then drops to $T_{\rm BH}=0$ at $M=M_{\rm cr}$. The specific heat capacity has a singularity at $\tilde{M}_{\rm cr}$. It is negative for $M>\tilde{M}_{\rm cr}$, but positive for $\tilde{M}_{\rm cr}>M>M_{\rm cr}$. We argued that the vanishing temperature of the critical black hole leads to a termination of the evaporation process once the black hole has reduced its mass to $M=M_{\rm cr}$. This supports the idea of a cold, Planck-size remnant as the final state of the evaporation. For an infinitely far away static observer this final state is reached after an *infinite* time only.

For $M>M_{\rm cr}$, the entropy of the quantum black hole is a well defined, monotonically increasing function of the mass. For heavy black holes we recover the classical expression $\mathcal{A}/4G_0$. The leading quantum corrections are proportional to $\ln(M/M_{\rm cr})$.

In conclusion we believe that the idea of the renormalization group improvement which, in elementary particle physics, is already well known is a promising new tool in order to study the influence of quantized gravity on the structure of spacetime. In the present work we focused on black holes, but it is clear that this approach has many more potential applications such as the very early universe, for instance.

ACKNOWLEDGMENTS

One of us (M.R.) would like to thank the Department of Physics of Catania University for the cordial hospitality extended to him while this work was in progress. He also acknowledges a NATO traveling grant. A.B. would like to thank the Department of Physics of Mainz University for financial support and for the cordial hospitality extended to him when this work was in progress. We are also grateful to INFN, Sezione di Catania for financial support.

APPENDIX A: THE STATISTICAL MECHANICAL ENTROPY

Our previous computation of $S(M)$ in Sec. VII is within the framework of ''phenomenological'' thermodynamics. Ultimately one would like to derive this thermodynamics from the statistical mechanics based upon a fundamental Hamiltonian $\hat{H}$ which describes the microscopic degrees of freedom of both gravity and matter. The aim would be to compute a partition function such as $Z(\beta)=\mathrm{Tr}[\exp(-\beta\hat{H})]$ and then to derive the free energy F, the internal energy U, the entropy S and similar thermodynamic quantities from it $(T\equiv 1/\beta)$:

$$F=-T\ln Z, \tag{A1}$$

$$U=T^2\frac{\partial}{\partial T}\ln Z, \tag{A2}$$

$$S=-\frac{\partial F}{\partial T}. \tag{A3}$$

In the original work of Gibbons and Hawking [22] the partition function was taken to be the functional integral of the pure Euclidean quantum gravity,

$$Z(\beta)=\int\mathcal{D}g_{\mu\nu}\exp(-I[g]),$$

where the integration is over all Euclidean metrics which are time periodic with period β. (Here $I[g]$ denotes the Einstein-Hilbert action with the Gibbons-Hawking surface term included.) The saddle point approximation of the integral yields, to leading order,

$$Z(\beta)\approx\sum_{g_0^{\rm class}}e^{-I[g_0^{\rm class}]}, \tag{A4}$$

where the ''sum'' is over all saddle points $g_0^{\rm class}$ of I with period β. Considering only saddle points of the Schwarzschild black hole type, the latter requirement means that only the hole of mass $M=\beta/8\pi G_0$ is relevant. For this ''Gibbons-Hawking instanton,'' $\beta_{\rm BH}$ equals the externally prescribed value of β (''on-shell'' approach). By using its action $I=4\pi G_0 M^2$ in $-\ln Z=\beta F\approx I$ one can derive the entire classical black hole thermodynamics.

It seems plausible to assume that the exact quantum gravity partition function in the Schwarzschild black hole sector is of the form

$$Z(\beta)=e^{-\Gamma[g_0]}, \tag{A5}$$

where $\Gamma[g]$ is some effective action functional, and g_0 is a stationary point of Γ with the same topology as $g_0^{\rm class}$:

$$\frac{\delta\Gamma}{\delta g_{\mu\nu}}[g_0]=0. \tag{A6}$$

Γ and g_0 are the quantum corrected versions of I and $g_0^{\rm class}$, respectively. We set

$$\Gamma=I+\Gamma_{\rm quant} \tag{A7}$$

so that $\Gamma_{\rm quant}$ encapsulates the quantum effects. (The statistical mechanics based upon the one-loop approximation $\Gamma_{\rm quant}=\frac{1}{2}\ln\det(\delta^2 I/\delta g^2)$ has already been developed to some extent [24].) The partition function (A5) and the thermodynamics derived from it contain quantum corrections of two types:

(i) The saddle point g_0, the metric of the ''quantum black hole,'' differs from the classical instanton $g_0^{\rm class}$.

(ii) In order to obtain βF, the metric g_0 is inserted into Γ rather than I.

Coming back to the renormalization group approach, it is natural to identify the saddle point g_0 with the Euclidean version of the renormalization group improved Schwarzschild metric, Eq. (6.4) with (4.4), which is denoted $g_{\rm imp}$ from now on. In this manner the quantum effects of (i) are approximately taken into account. However, $g_{\rm imp}$ was obtained by a direct improvement of a classical *solution* rather than of the classical *action*. Thus, within the framework used in the present paper, we do not know the functional Γ for which $g_{\rm imp}$ is an (approximate) saddle point and which would determine the partition function via (A5). The best we can do in this situation is to tentatively neglect the quantum effects of (ii), i.e., to assume that $\Gamma_{\rm quant}[g_{\rm imp}]$ is much less important than $I[g_{\rm imp}]$ and to approximate (A5) by

$$Z(\beta)\approx e^{-I[g_{\rm imp}]}. \tag{A8}$$

In the following we investigate if (A8) can give rise to an acceptable thermodynamics. We shall employ the ''off-shell'' formalism (conical singularity method) developed in Ref. [25] to which we refer for further details.

We evaluate the action I for a general Euclidean metric of the type (6.1) or (6.4) where $f(r)$ is arbitrary to a large extent. We only assume that it has a simple zero at some r_+. Its asymptotic behavior is required to be

$$f(r)=1-\frac{2G_0M}{r}+O\left(\frac{1}{r^2}\right) \tag{A9}$$

for some fixed constant M. Furthermore, we assume that the Euclidean time τ in (6.4) is an angle-like, periodic variable with period β. Here β is the argument of the partition function. It has a prescribed value which in general does not coincide with $\beta_{\rm BH}\equiv 4\pi/f'(r_+)$. The corresponding Euclidean manifold is denoted $\mathcal{M}_\beta$.

If we introduce the 2π-periodic rescaled time variable

$$\hat\tau\equiv\frac{2\pi}{\beta}\tau \tag{A10}$$

then, near the horizon, the metric (6.4) becomes

$$ds_E^2\approx\rho^2\left(\frac{\beta}{\beta_{\rm BH}}\right)^2 d\hat\tau^2+d\rho^2+r_+^2 d\Omega^2, \tag{A11}$$

which coincides with (6.5) only for the ''on-shell'' value $\beta=\beta_{\rm BH}$. For $\beta\neq\beta_{\rm BH}$ the space $\mathcal{M}_\beta$ has a conical singularity at $\rho=0$, the angular deficit being $\delta=2\pi(1-\beta/\beta_{\rm BH})$. As a consequence, the curvature scalar on $\mathcal{M}_\beta$ has a delta-function singularity at $\rho=0$.

The Einstein-Hilbert action $I\equiv I_{\rm reg}+I_{\rm sing}$ on $\mathcal{M}_\beta$ consists of a regular part and a singular part containing the contribution from the delta-function singularity. The regular part $I_{\rm reg}\equiv I_V+I_S$ has a volume and a surface contribution,

$$I_V=-\frac{1}{16\pi G_0}\int_{\mathcal{M}_\beta} d^4x\sqrt{g}R, \tag{A12}$$

$$I_S=-\frac{1}{8\pi G_0}\int_{\partial\mathcal{M}_\beta} d^3x\sqrt{\gamma}(K-K_0), \tag{A13}$$

where γ and K are the metric and the extrinsic curvature on the boundary $\partial\mathcal{M}_\beta$ at infinity ($r\to\infty$). (K_0 is the corresponding value for a flat metric.)

The volume contribution is evaluated most easily by returning to the original r coordinate. Then, after performing the trivial angle and τ integrations,

$$I_V=-\frac{\beta}{4G_0}\int_{r_+}^{\infty} dr\, r^2R(r), \tag{A14}$$

where R is the curvature scalar for the metric (6.1). It reads

$$R(r)=-\frac{1}{r^2}\left[\frac{d^2}{dr^2}(r^2f(r))-2\right] \tag{A15}$$

and therefore the integral (A14) feels the behavior of f only at the horizon and at infinity:

$$I_V=\beta\frac{r_+}{2G_0}-\frac{\beta}{4G_0}r_+^2f'(r_+)-\frac{1}{2}\beta M. \tag{A16}$$

The evaluation of (A13) with (A9) proceeds as in the standard case [22]:

$$I_S=\frac{1}{2}\beta M. \tag{A17}$$

Adding (A17) to (A16) cancels precisely the last term of (A16) which originated from the upper limit of the integral (A14). Using (6.3) the sum contains only data related to the horizon,

$$I_{\rm reg}=\beta\frac{r_+}{2G_0}-\frac{\beta}{\beta_{\rm BH}}\frac{\mathcal{A}}{4G_0}, \tag{A18}$$

where $\mathcal{A}\equiv 4\pi^2r_+^2$ is its area.

For the metric (A11), the singular contribution

$$I_{\rm sing}=-\frac{1}{16\pi G_0}\int_{\mathcal{M}_\beta} d^4x\sqrt{g}R_{\rm sing}, \tag{A19}$$

with $R_{\rm sing}\propto\delta(\rho)$ has already been evaluated in Ref. [26]. The result is

$$I_{\rm sing}=-\left(1-\frac{\beta}{\beta_{\rm BH}}\right)\frac{\mathcal{A}}{4G_0}. \tag{A20}$$

Adding (A20) to (A18) we obtain the complete action evaluated on $\mathcal{M}_\beta$:

$$I=\beta\frac{r_+}{2G_0}-\frac{\mathcal{A}}{4G_0}. \tag{A21}$$

This is the result we wanted to derive. We emphasize that it is valid for black holes with an essentially arbitrary $f(r)$ and, as a consequence, arbitrary ADM mass M and Hawking temperature $T_{\rm BH}=\beta_{\rm BH}^{-1}$.

If we specialize for the renormalization group improved Schwarzschild black hole of a given mass M, the action becomes

$$I[g_{\rm imp}]=\beta\frac{r_+(M)}{2G_0}-\frac{\mathcal{A}(M)}{4G_0}, \tag{A22}$$

with $r_+(M)$ and $\mathcal{A}(M)$ given by (4.36) and (6.26), respectively.

If we tentatively insert the action (A22) into (A8) and use (A3) to calculate the entropy from $F\approx\beta^{-1}I[g_{\rm imp}]$ we obtain

$$S=\frac{\mathcal{A}(M)}{4G_0}. \tag{A23}$$

Apart from the modified relation between $\mathcal{A}$ and M, this is precisely the classical entropy. It is clear from Eq. (6.26) that (A23) differs from the correct result (7.4) already at the leading order of the large-M corrections.

Thus we must conclude that the ''statistical mechanics entropy'' (A23) fails to reproduce the quantum corrections contained in the ''thermodynamical entropy'' (7.4). The lesson to be learned from this failure is that, at least as far as the entropy is concerned, the quantum mechanical modification of the action from I to Γ is essential. Improving only the saddle point ($g_0^{\rm class}\rightarrow g_{\rm imp}$) but neglecting $\Gamma_{\rm quant}$ is not sufficient in order to obtain a meaningful partition function.

[1] K. Wilson and J. Kogut, Phys. Rep., Phys. Lett. **12C**, 75 (1974).
[2] For a recent review see C. Rovelli, hep-th/9910131.
[3] J. Polchinski, Nucl. Phys. **B231**, 269 (1984).
[4] A. Ringwald and C. Wetterich, Nucl. Phys. **B334**, 506 (1990); C. Wetterich, *ibid.* **B352**, 529 (1991); S. Liao and J. Polonyi, Ann. Phys. (N.Y.) **222**, 122 (1993); M. Reuter and C. Wetterich, Nucl. Phys. **B391**, 147 (1993); **B408**, 91 (1993).
[5] A. Bonanno, Phys. Rev. D **52**, 969 (1995); A. Bonanno and D. Zappalà, *ibid.* **55**, 6135 (1997).
[6] C. Wetterich, Phys. Lett. B **301**, 90 (1993); M. Reuter and C. Wetterich, Nucl. Phys. **B417**, 181 (1994); **B427**, 291 (1994); **B506**, 483 (1997).
[7] M. Reuter, Phys. Rev. D **57**, 971 (1998).
[8] W. Dittrich and M. Reuter, *Effective Lagrangians in Quantum Electrodynamics* (Springer, Berlin, 1985).
[9] A. Bonanno and M. Reuter, Phys. Rev. D **60**, 084011 (1999).
[10] For an introduction, see M. Reuter, in *Proceedings of the 5th Hellenic School and Workshop on Elementary Particle Physics*, Corfu, Greece (1995), hep-th/9602012.
[11] S. Weinberg, in *General Relativity: an Einstein Centenary Survey*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[12] W. Souma, Prog. Theor. Phys. **102**, 181 (1999).
[13] A. Polyakov, in *Gravitation and Quantization*, Proceedings of the Les Houches Summer School, edited by J. Zinn-Justin and B. Julià, Session 57 (North-Holland, Amsterdam, 1995).
[14] J.F. Donoghue, Phys. Rev. Lett. **72**, 2996 (1994); Phys. Rev. D **50**, 3874 (1994).
[15] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[16] H.W. Hamber and S. Liu, Phys. Lett. B **357**, 51 (1995).
[17] V.P. Frolov, M.A. Markov, and V.F. Mukhanov, Phys. Rev. D **41**, 383 (1990).
[18] I. Dymnikova, Gen. Relativ. Gravit. **24**, 235 (1992).
[19] A. Strominger, Phys. Rev. D **46**, 4396 (1992).
[20] B. Hasslacher and E. Mottola, Phys. Lett. **99B**, 221 (1981).
[21] S.W. Hawking, in *General Relativity: an Einstein Centenary Survey* (Ref. [11]).
[22] G. Gibbons and S. Hawking, Phys. Rev. D **15**, 2752 (1977).
[23] For an introduction, see R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[24] V.P. Frolov, W. Israel, and S.N. Solodukhin, Phys. Rev. D **54**, 2732 (1996).
[25] S.N. Solodukhin, Phys. Rev. D **54**, 3900 (1996).
[26] R.B. Mann and S.N. Solodukhin, Phys. Rev. D **54**, 3932 (1996).