Goldberger-Treiman discrepancy

N. F. Nasrallah*
Faculty of Science, Lebanese University, Tripoli, Lebanon
(Received 15 October 1999; published 10 July 2000)

The Golberger-Treiman discrepancy $\Delta_{\rm GT}=1-m_N g_A/f_\pi G_{\pi N}$ is related to the asymptotic behavior of the pionic form factor of the nucleon obtained from baryonic QCD sum rules. The result is $0.015 \lesssim \Delta_{\rm GT} \lesssim 0.022$.

PACS number(s): 11.40.Ha, 11.55.Fv, 11.55.Hx, 13.30.Ce

The Goldberger-Treiman relation (GTR) [1]

$$m_N g_A = f_\pi G_{\pi N}, \tag{1}$$

which relates the nucleon mass m_N, the axial-vector coupling constant in β decay g_A, the π decay constant f_π, and the π-N coupling constant $G_{\pi N}$, is one of the most remarkable relations of hadronic physics. Explicit chiral symmetry breaking by the quark masses leads to small corrections to the GTR, the Goldberger-Treiman discrepancy (GTD) [2]

$$\Delta_{\rm GT} = 1-\frac{m_N g_A}{f_\pi G_{\pi N}}, \tag{2}$$

which arises from the coupling of the divergence of the axial vector current to the $J^p=0^-$ continuum. The evaluation of $\Delta_{\rm GT}$ was recently addressed in the framework of baryon chiral perturbation theory [3]. On the experimental side $g_A = 1.267\pm 0.004$ and $f_\pi = 92.42$ MeV are known to sufficient precision, and most of the uncertainty in $\Delta_{\rm GT}$ results from the uncertainty in $G_{\pi N}$. The most recent determination of $G_{\pi N}$ from NN, $N\bar{N}$, and πN data was by the Nijmegen group [4]:

$$G_{\pi N} = 13.05 \pm 0.08,$$

which corresponds to

$$\Delta_{\rm GT} = 0.014 \pm 0.009. \tag{3}$$

Similar results are obtained by the VPI group [5]. Larger values are given by Bugg and Machleidt [6] and Loiseau *et al.* [7]:

$$G_{\pi N} = 13.65 \pm 0.30,$$

which corresponds to

$$\Delta_{\rm GT} = 0.056 \pm 0.02. \tag{4}$$

The result of theoretical calculations at the loop level [3] does not account even for the smaller value given by Eq. (3) in a parameter free way.

The evaluation of the GTD involves an integral over the imaginary part of the form factor $\Pi(q^2)$, which describes the matrix element of the divergence of the axial current between two nucleon states:

$$\langle P(p')|\partial_n A_n^+|N(p)\rangle = \Pi(q^2)\cdot \overline{\mathcal{U}}(p')\gamma_5 \mathcal{U}(p), \qquad q=p'-p. \tag{5}$$

Access to $\Pi(q^2)$ is provided by the study of the three-point function [8]

$$\Gamma(t,q^2) = -\int\int d^4x\, d^4y\, \exp(-ipx)\exp(iqy) \times \langle 0|T\Psi_\sigma^P(x)\,\partial_\mu A_\mu^+(y)\,\overline{\Psi}_\kappa^N(0)|0\rangle, \tag{6}$$

where $t=p^2$ and $\partial_\mu A_\mu^+ = i(m_u+m_d)(\bar{u}\gamma_5 d)$ express the divergence of the axial currents in terms of quark fields, and

$$\Psi_\sigma^P = \epsilon_{ijk} u_i^T C\gamma_\alpha u_j \gamma_5\gamma_\alpha d_k, \qquad \Psi_\kappa^N = \epsilon_{ijk} d_i^T C\gamma_\alpha d_j \gamma_5\gamma_\alpha u_k \tag{7}$$

are the nucleon currents [9]

Amplitude (6) contains nucleon double- and single-pole contributions as well as a nonsingular contribution of the continuum,

$$\Gamma(t,q^2) = (\gamma_5 \not{q})\left[-\frac{\lambda_N^2 \Pi(q^2) m_N}{(t-m_N^2)^2} + \frac{c}{(t-m_N^2)} + \cdots\right], \tag{8}$$

where c is the unknown coefficient of the single-pole contribution and λ_N represents the coupling of the nucleon to its current,

$$\langle 0|\Psi_\sigma^P|P\rangle = \lambda_N \mathcal{U}_\sigma, \tag{9}$$

and where we have limited ourselves to the tensor structure $\gamma_5\not{q}$ for simplicity. Any other choice is of course *a priori* valid, provided it leads to the stability of the calculation, as will be shown to be the case here.

The next step is to evaluate $\Gamma(t,q^2)$ in QCD. To this end use is made of the operator product expansion of the currents entering in Eq. (6). The lowest dimensional operators, which provide the dominant contributions at short distances, are the unit operator and the operators $\bar{q}q$ and $G_{\mu\nu}G^{\mu\nu}(=GG)$. As we shall only use the coefficient of $\gamma_5\not{q}$ in the expansion of the currents entering into Eq. (6), the even dimensional operators 1 and GG will be multiplied by the small quark mass m_q, and their contribution will be greatly reduced as compared to the one of the odd dimensional operator $\bar{q}q$. The contribution of the latter was evaluated in the third of Refs. [8]:

*Email address: nsrallh@cyberia.net.lb

$$\Gamma^{\rm QCD}(t,q^2)=\frac{-2m_q\langle\bar{q}q\rangle}{2\pi^2}\left[\frac{t}{q^2}\ln(-t)+\frac{1}{8}\ln(-t)-\frac{1}{4}tI_0(t,q^2)+\frac{1}{4}\ln(-q^2)+\cdots\right](\gamma_5\not{q}),\tag{10}$$

with

$$I_0(t,q^2)=\int_0^1\frac{dx}{t-x(1-x)q^2}\ln\left(\frac{-x(1-x)q^2}{-t}\right),\tag{11}$$

and $-2m_q\langle\bar{q}q\rangle$ given by the Gell-Mann-Oakes-Renner relation

$$-2m_q\langle\bar{q}q\rangle=f_\pi^2m_\pi^2,\quad f_\pi=0.0924\ \text{GeV}.\tag{12}$$

Expression (10) holds for both t and q^2 in the deep Euclidean region. The next step is to extrapolate to the nucleon mass shell, i.e., to obtain $\Pi^{\rm QCD}(q^2)$ from expressions (8) and (10) and the analytic properties of $\Gamma(t,q^2)$. This extrapolation to the mass shell is done over a large interval of the variables t and q^2. The method of QCD sum rules provides a tool for such an extrapolation, where the approximations are well defined and where the numerical stability of the result provides a useful check of their validity. Note that the smallness of $\Delta_{\rm GT}$ results from the smallness of the quark masses, and is not calculated as the difference of large numbers. If this were the case the method of QCD sum rules would not be accurate enough to be reliable.

For q^2 fixed at a large negative value, $\Gamma(t,q^2)$ has a cut on the positive t axis starting at $t=(m_N+m_\pi)^2$, in addition to the nucleon pole structure exhibited in Eq. (8). Consider now the Laplace type integral [10] $(1/2\pi i)\int_c dt \exp(-t/M^2)\Gamma(t,q^2)$ in the complex t plane over a closed contour c consisting of a circle of large radius R and two straight lines above and below the cut which run from threshold to R, M^2 is the usual ''Borel mass'' parameter. The exponential provides a convenient damping of the contribution of the integral over the continuum. This contribution is of course unknown, and provides the main uncertainty in the QCD sum-rule approach. It could be greatly damped by decreasing the parameter M^2, but, as is well known, this enhances the contribution of the higher-order unknown terms in the operator product expansion. We hope to obtain an intermediate range of M^2 for which the contribution of the continuum and that of the higher-order terms are both negligible. If these approximations are adequate, this will show up in the stability of the result. We shall find out that this is the case. On the circle, Γ is well approximated by $\Gamma^{\rm QCD}$, except possibly for a small region near the real axis.

We then obtain

$$\Pi^{\rm QCD}(q^2)+c'(q^2)M^2=\frac{M^6f_\pi^2m_\pi^2}{2\pi^2\lambda_N^2\exp\left(\frac{-m_N^2}{M^2}\right)m_N}\times\left[E_1\left(\frac{R}{M^2}\right)\frac{1}{q^2}+\frac{3}{8M^2}E_0\left(\frac{R}{M^2}\right)-\frac{q^2}{4M^4}\int_0^1dx\int_0^R\frac{dt\exp\left(\frac{-t}{M^2}\right)}{q^2-\frac{t}{x(1-x)}}\right],\tag{13}$$

with

$$E_i\left(\frac{R}{M^2}\right)=\int_0^{R/M^2}x^i\exp(-x)dx.\tag{14}$$

$\Pi(s=q^2)$ is an analytic function in the complex s plane, except for a simple pole at $s=m_\pi^2$ and a right-hand cut running along the positive real axis from $s=9m_\pi^2$ to ∞:

$$\Pi(s)=\frac{-2f_\pi m_\pi^2G_{\pi N}}{s-m_\pi^2}+\cdots;\tag{15}$$

furthermore,

$$\Pi(s=0)=2m_Ng_A.\tag{16}$$

Next consider the integral $(1/2\pi i)\int_{c'}(ds/s)(s-m'^2)\Pi(s)$, where m' is a mass parameter and c' is a closed contour consisting of a circle of large radius R' and two straight lines above and below the cut which run from threshold to R'.[1] Cauchy's theorem implies

$$-m'^22m_Ng_A-2f_\pi G_{\pi N}(m_\pi^2-m'^2)=\frac{1}{\pi}\int_{9m_\pi^2}^R\frac{ds}{s}(s-m'^2)\text{Im}\,\pi(s)+\frac{1}{2\pi i}\oint\frac{ds}{s}(s-m'^2)\pi^{\rm QCD}(s),\tag{17}$$

where we have used $\Pi(s)=\Pi^{\rm QCD}(s)$ on the circle.

The first term on right-hand side of Eq. (17) represents an integral over the unknown continuum. As m'^2 is varied between threshold and R, this integral changes sign, which implies that it vanishes for some value of m'^2 which we adopt. Because m' is an unknown parameter that we shall vary

[1] R' need of course not be equal to R, but they are of the same order and any reasonable difference between them results only in negligible numerical effects; thus we take $R'=R$ to simplify the notation.

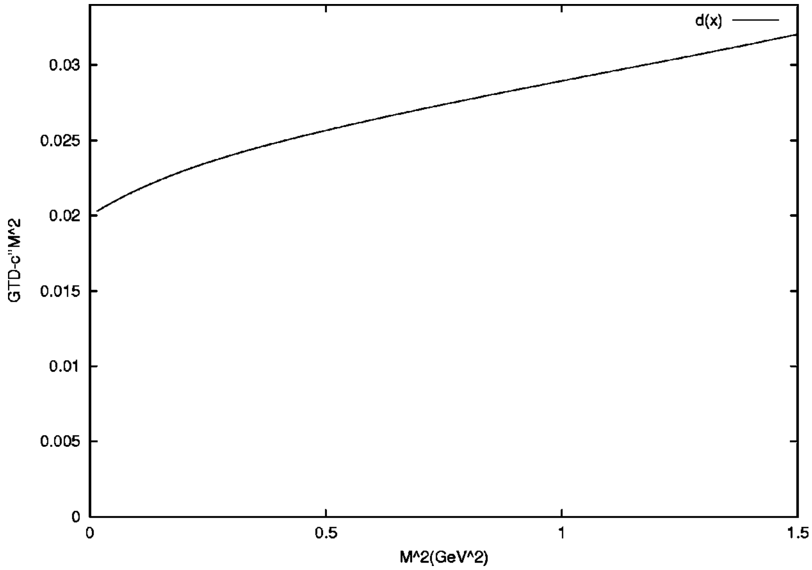

FIG. 1. The first term on the right-hand side of Eq. (19) plotted against M^2. Its linear variation for $M^2 \gtrsim 0.4\,\mathrm{GeV}^2$ compensates for the second term $c'' M^2$.

within reasonable limits it is superfluous to include any contribution of the continuum near threshold. The GTD then follows from Eq. (17):

$$\Delta_{\mathrm{GT}} = \frac{m_\pi^2}{m'^2} + \frac{1}{2 f_\pi G_{\pi N} m'^2} \frac{1}{2\pi i} \oint \frac{ds}{s} (s - m'^2) \Pi^{\mathrm{QCD}}(s), \tag{18}$$

and, when expression (13) for Π^{QCD} is used,

$$\Delta_{\mathrm{GT}} = \frac{m_\pi^2}{m'^2} \Bigg\{ 1 + \frac{1}{4\pi^2 G_{\pi N}} \left(\frac{f_\pi}{m_N}\right) \frac{1}{\lambda_N^2 \exp\left(-\frac{m_N^2}{M^2}\right)} \Bigg[E_1\left(\frac{R}{M^2}\right) - \frac{3}{8} \frac{m'^2}{M^2} E_0\left(\frac{R}{M^2}\right) - \frac{1}{4} \int_0^1 dx \int_0^{x(1-x)(R/M^2)} dy \times \exp(-y) \left(\frac{y}{x(1-x)} - \frac{m'^2}{M^2} \right) \Bigg] \Bigg\} + c'' M^2. \tag{19}$$

λ_N^2 is obtained in a similar fashion from a study of the nucleonic two-point function $\int d^4x \exp(iqx) \langle 0 | T \Psi(x) \Psi(0) | 0 \rangle$ [9], with the result

$$(2\pi)^4 \lambda_N^2 \exp\left(\frac{-m_N^2}{M^2}\right) = \frac{M^6}{4} E_2\left(\frac{R}{M^2}\right) - \frac{\pi^2}{2} \left\langle \frac{\alpha_s GG}{\pi} \right\rangle M^2 E_0\left(\frac{R}{M^2}\right) + \frac{32}{3} \pi^4 \langle (\bar{q} q)^2 \rangle. \tag{20}$$

The choice of M^2 in Eq. (19), as well as the consistency of the method, is dictated by stability considerations. If there are values of M^2 small enough to provide an adequate damping of the continuum and large enough to justify the neglect of the contributions of higher-order condensates in the operator product expansion, this should show up in the stability of expression (19). This means that the first term on the right-hand side (RHS) of Eq. (19) should show a linear behavior which compensates for the linear variation of $c'' M^2$ in some intermediate range of M^2 (note that the curve need show no horizontal plateau; this happens only if $c''=0$). The value of m'^2 is expected to be close to (albeit smaller because of the weight factor $1/s$) the maximum of the $\pi'(1.7\ \mathrm{GeV}^2)$ bump. It seems reasonable to vary it in the range $1\ \mathrm{GeV}^2 \lesssim m'^2 \lesssim 1.5\,\mathrm{GeV}^2$. For the gluon condensate we use the standard value $\langle \alpha_s GG/\pi \rangle = 0.012\,\mathrm{GeV}^2$. For $\langle (\bar{q} q)^2 \rangle$ the choice $\langle \bar{q} q \rangle^2$ (vacuum saturation hypothesis) is usually made, but as this seems to be too stringent an assumption [10], we take $\langle qq^2 \rangle = \beta \langle qq \rangle^2$. Varying β between 1 and 3 has no noticeable effect on the result.

In Fig. 1 the first term on the RHS of Eq. (19) is plotted against M^2 for $m'^2 = 1\ \mathrm{GeV}^2$ and $\beta = 1$. It clearly exhibits a slow linear variation in the range $0.5\,\mathrm{GeV}^2 < M^2 < 1.5\,\mathrm{GeV}^2$, which gives

$$\Delta_{\mathrm{GT}} = 0.022.$$

Varying m'^2 as discussed above yields, finally,

$$0.015 \lesssim \Delta_{\mathrm{GT}} \lesssim 0.022, \tag{21}$$

which is consistent with the value given by Eq. (3), and clearly favors the smaller value of $G_{\pi N}$.

It is finally worth investigating the possibility that the value of the quark condensate $\langle \bar{q} q \rangle$ is much smaller than what results from the GOR relation [Eq. (12)]. This is the case, for example, in ''generalized chiral perturbation theory'' [11]. We would then have

$$\Delta_{\mathrm{GT}} \simeq \frac{m_\pi^2}{m'} \quad \text{or} \quad 0.010 \lesssim \Delta_{\mathrm{GT}} \lesssim 0.014. \tag{22}$$

[1] M. L. Golderger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).
[2] C. A. Dominguez, Riv. Nuovo Cimento **8**, 1 (1985), and references therein.
[3] J. L. Goity, R. Lewis, M. Schvellinger, and L. Zhang, Phys. Lett. B **454**, 115 (1999); U.-G. Meissner, in *Themes in Strong Interaction*, Proceedings of the 12th HUGS et CEBAF, edited by J. L. Goity (World Scientific, Singapore, 1998), p. 139, and references therein; J. Gasser, M. E. Sainio, and A. Švarc, Nucl. Phys. B **307**, 779 (1988); E. Jenkins and A. V. Manohar, Phys. Lett. B **255**, 558 (1991); N. H. Fuchs, H. Sazdjian, and J. Stern, *ibid.* **238**, 380 (1990).
[4] J. J. de Swart, M. C. M. Rentmeester, and R. G. E. Timmermans, Proceedings of MENU97, TRIUMPH Report No. TRI-97-1 (1997), p. 96; M. C. M. Rentmeester, R. G. E. Timmermans, J. L. Friar and J. J. de Swart, Phys. Rev. Lett. **82**, 4992 (1999).
[5] R. A. Arndt, J. I. Strokowsky, and R. L. Workman, Phys. Rev. C **52**, 2246 (1995).
[6] D. V. Bugg and R. Machleidt, Phys. Rev. C **52**, 1203 (1995).
[7] B. Loiseau, T. E. V. Ericson, T. Rahm, J. Blomgren, and N. Olson, πN-Newsletter **13**, 117 (1997).
[8] L. J. Reinders, H. R. Rubinstein, and S. Yazaki, Nucl. Phys. B **213**, 109 (1983); K. G. Chetyrkin, S. G. Gorishny, A. B. Krasulin, S. A. Larin, and V. A. Matveev, Institute of Nuclear Research of the Academy of Sciences of the USSR, Report P-0337 (1984); N. F. Nasrallah, Phys. Lett. B **178**, 409 (1986).
[9] B. Ioffe, Z. Phys. C **18**, 67 (1983).
[10] M. Davier, L. Girlanda, A. Höcker, and J. Stern, Phys. Rev. D **58**, 096014 (1998).
[11] N. H. Fuchs, H. Sazdjian, and J. Stern, Phys. Lett. B **269**, 183 (1991); J. Stern, H. Sazdjian, and N. H. Fuchs, Phys. Rev. D **47**, 3814 (1993).