Approximated fourth order calculation of the vacuum wave function of (2+1) dimensional SU(3) lattice gauge theory

Ping Hui
Department of Physics, Guangdong Education College, Guangzhou 510303, China

Xi-Yan Fang and Qi-Zhou Chen
CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China
and Department of Physics, Zhongshan University, Guangzhou 510275, China
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We propose a scheme to improve the coupled cluster expansion in lattice gauge theory (LGT), based on the application of the random phase approximation, in order to approximate a wave function in terms of a linear combination of Wilson loops. Using this method, we study the vacuum energy and vacuum wave function in (2+1)D SU(3) LGT up to fourth order. The vacuum energy is lower than that obtained by the unimproved approach. The coefficients μ_0, μ_2 of the vacuum wave function show good scaling behavior and convergence.

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I. INTRODUCTION

QCD has been accepted to be the most successful gauge theory of strongly interacting particles. QCD in the pure gauge sector possesses a nontrivial vacuum structure and bound states called glueballs. In 1980, Greensite [1] argued that the vacuum state of SU(N) lattice gauge theory (LGT) should be $\exp[R(U)]$. Using the Kogut-Susskind Hamiltonian [2]

$$H=\frac{g^2}{2a}\left[\sum_l E_l^2-\frac{2}{g^4}\sum_p \mathrm{Tr}(U_p+U_p^\dagger)\right], \tag{1}$$

he obtained the Schrödinger eigenvalue equation for the vacuum wave function:

$$\sum_l \{[E_l^a,[E_l^a,R(U)]]+[E_l^a,R(U)][E_l^a,R(U)]\}$$
$$-\frac{2}{g^4}\sum_p \mathrm{Tr}(U_p+U_p^\dagger)=\frac{2a}{g^2}\epsilon_\Omega, \tag{2}$$

where g is the coupling constant, ϵ_Ω the vacuum energy, E_l^a the ''chromoelectric'' field on the link l, and $R(U)$ is a single sum over the lattice of clusters consisting of Wilson loops at fixed relative orientation and separation [3].

However, it is difficult to find the exact solution of Eq. (2). As far as we know, no exact solution has been reported. Many numerical [1,4–6] and analytical efforts [3,7–13] have been made. Some significant results and progress were reported. In 1993, Guo *et al.* [10] developed an approximate scheme with the eigenvalue equations truncated according to the continuum limit, and calculated the μ_0, μ_2 of the vacuum wave function and 0^{++} glueball mass of (2+1)-dimensional SU(2) LGT [10,11], where μ_0 and μ_2 are long wavelength expansion coefficients defined by [6,10,12]

$$\Psi_0(U)=N\exp\left[-\frac{\mu_0}{e^2}\int d^2x\,\mathrm{Tr}F^2-\frac{\mu_2}{e^6}\int d^2x\,\mathrm{Tr}(DF)^2\right.$$
$$\left.+\text{higher order terms}\right], \tag{3}$$

with F being the field strength tensor and D the covariant derivative. The low order results of μ_0, μ_2 and the glueball mass show good scaling behavior [10,11], but the high order (the fifth order) results of μ_0, μ_2 and the glueball mass are not satisfactory [14].

There are many difficulties in doing high order calculation. One is the rapid proliferation of clusters which occur as the (strong coupling perturbation theory) order increases. A more difficult problem is how to distinguish the independent graphs (i.e., clusters) in expansions. The system of basis functions used in coupled cluster expansions is overcomplete because the gauge group elements are unimodular. For example, in the case of SU(2) LGT, the unimodular condition is $U_{ij}U_{kl}\epsilon_{ik}\epsilon_{jl}=2$, which can be expressed as a more convenient form in cluster expansions,

$$\mathrm{Tr}(U^+V)=\mathrm{Tr}(U)\mathrm{Tr}(V)-\mathrm{Tr}(UV), \tag{4}$$

where ϵ_{ij} is asymmetric with $\epsilon_{12}=1$, and U, V are any element of the SU(2) group, or any gauge link or connected path on a lattice which comprise the graphs. Using Eq. (4), one may find the relations among the graphs and eliminate redundancies [3,10], but it depends heavily on one's experience. Therefore, it is not reliable when the number of graphs becomes large and the configurations of graphs are complex. For example, the total number of independent bases in the fourth order calculation of (2+1)-dimensional SU(2) LGT is reported to be 70 in Ref. [10], while it is 69 in Ref. [3]. The unimodular condition of the SU(3) group is more complicated. This is the main reason why we have not continued the study of Refs. [12], [13] to obtain higher order (for example, the fourth order) results.

The main purpose of this work is to circumvent the above problems. At first, let us review the idea of random phase approximation (RPA) briefly [15]. In the Hubbard model, the transverse magnetic susceptibility is

$$\chi^{\dagger,-}(\vec{q},t)=\frac{1}{N}\sum_{\vec{K}}\langle\langle\theta_{\vec{K}}(\vec{q},t);S^{\dagger}(-\vec{q},0)\rangle\rangle_{\omega+i\eta}, \quad (5)$$

where $\theta_{\vec{K}}(\vec{q},t)=C^{\dagger}_{\vec{K}+\vec{q},\downarrow}(t)C_{\vec{K},\uparrow}(t)$, $S^{+}(\vec{q},0)=\Sigma_{\vec{K}}C^{\dagger}_{\vec{K},\uparrow}(0)\times C_{\vec{K}+\vec{q},\downarrow}(0)$, $\vec{K},\vec{q}$ are the momenta of electrons, $C^{\dagger}_{\vec{K},\uparrow}$ is a creation operator of electrons with momentum $\vec{K}$ and up spin, $C_{\vec{K},\uparrow}$ is the corresponding annihilation operator, N is the volume of lattice, and

$$\langle\langle\theta_{\vec{K}}(\vec{q},t);S^{\dagger}(-\vec{q},0)\rangle\rangle=i\theta(t)\langle[\theta_{\vec{K}}(\vec{q},t),S^{+}(-\vec{q},0)]\rangle, \quad (6)$$

with angular brackets denoting an average over the vacuum state and $\theta(t)$ the θ function. In momentum representation, the Hamiltonian of Hubbard model is

$$H=H_T+H_I, \quad (7)$$

where

$$H_T=\sum_{\vec{K},\sigma}E_{\vec{K}}C^{\dagger}_{\vec{K},\sigma}C_{\vec{K},\sigma},(\sigma=_{\uparrow,\downarrow}) \quad (8)$$

and

$$H_I=\frac{U}{N}\sum_{\vec{K},\vec{K}',\vec{q}'}C^{\dagger}_{\vec{K}+\vec{q}',\uparrow}C^{\dagger}_{\vec{K}'-\vec{q}',\downarrow}C_{\vec{K}',\downarrow}C_{\vec{K},\uparrow}, \quad (9)$$

with U being the average value of Coulomb potential energy [16]. The time-dependent Green function and its dynamical equation are

$$G(t,0)=\langle\langle\theta_k(\vec{q},t);S^{\dagger}(-\vec{q})\rangle\rangle=\int_{-\infty}^{\infty}e^{i\omega t}\langle\langle\theta_{\vec{k}}(\vec{q},\omega);S^{\dagger}(-\vec{q})\rangle\rangle d\omega, \quad (10)$$

$$i\frac{d}{dt}G(t,0)=-\delta(t)\langle[\theta_{\vec{k}}(\vec{q},t),S^{\dagger}(-\vec{q})]\rangle+\langle\langle[\theta_{\vec{k}}(\vec{q},t),H];S^{\dagger}(-\vec{q})\rangle\rangle, \quad (11)$$

where

$$[\theta_{\vec{k}}(\vec{q},t),H_I]=-\frac{U}{N}\sum_{\vec{K},\vec{q}'}[C^{\dagger}_{\vec{K}+\vec{q}',\uparrow}C^{\dagger}_{\vec{k}+\vec{q}-\vec{q}',\downarrow}C_{\vec{k},\uparrow}C_{\vec{K},\uparrow}-C^{\dagger}_{\vec{k}+\vec{q},\downarrow}C^{\dagger}_{\vec{K}-\vec{q}',\downarrow}C_{\vec{K},\downarrow}C_{\vec{k}-\vec{q}',\uparrow}]. \quad (12)$$

Because this term cannot be expressed as a linear combination of $\theta_{\vec{k}}(\vec{q},t)$, Eq. (11) is difficult to solve. Using the RPA, a pair of fermion operators in Eq. (12) is replaced by its average value, and Eq. (12) becomes

$$[\theta_{\vec{k}}(\vec{q},t),H_I]\approx-\frac{U}{N}\sum_{\vec{K}}[(\langle C^{\dagger}_{\vec{K},\uparrow}C_{\vec{K},\uparrow}\rangle-\langle C^{\dagger}_{\vec{K},\downarrow}C_{\vec{K},\downarrow}\rangle)\theta_{\vec{k}}(\vec{q},t)-(\langle C^{\dagger}_{\vec{k},\uparrow}C_{\vec{k},\uparrow}\rangle-\langle C^{\dagger}_{\vec{k}+\vec{q},\downarrow}C_{\vec{k}+\vec{q},\downarrow}\rangle)\theta_{\vec{K}}(\vec{q},t)], \quad (13)$$

where $\langle C^{\dagger}_{\vec{k},\sigma}C_{\vec{k},\sigma}\rangle$ is a fermion distribution function. Using this method, the chains are truncated, and the equations are closed and can be solved.

The RPA may also be used in solving the Schrödinger eigenvalue equation (2). In the coupled cluster expansions of wave functions, the coupled clusters (graphs) consist of products of Wilson loops. When a graph consists of two Wilson loops, we replace one of them by its vacuum average according to the RPA, so each cluster contains only one loop, which makes the distinguishing of independent graphs very simple and also makes the number of clusters at each order decrease very much. Therefore, by adopting the RPA, we can avoid the above two problems simultaneously. Introducing the average value of a Wilson loop in coupled cluster expansions has been proposed in Ref. [3]. In that paper, the authors also pointed out the dominance in the vacuum wave function of clusters involving only a single loop. However, just like the usual coupled cluster method, their shifted coupled cluster method suffers the above two problems too, because the independent bases they used are similar to those used in the usual coupled cluster expansions. In addition, the truncation scheme we use is different from theirs: we will adopt the truncation scheme proposed in Ref. [10], which is more effective and simpler.

In this paper, we use the above approach to calculate the vacuum wave function and vacuum energy of $(2+1)$-dimensional SU(3) LGT up to fourth order. We will give the details of calculating third order wave function as well as the vacuum energy and present the results from second to fourth order.

II. APPROXIMATION OF THE VACUUM WAVE FUNCTION

The wave function of the vacuum state is assumed to be

$$|\Omega\rangle=\exp[R(U)]|0\rangle, \quad (14)$$

where the bare vacuum state $|0\rangle$ is defined to be $E_l|0\rangle=0$. The vacuum state and energy ϵ_Ω satisfy the lattice Schrödinger equation

$$H|\Omega\rangle=\epsilon_\Omega|\Omega\rangle, \quad (15)$$

which results in Eq. (2). $R(U)$ in Eq. (2) is expanded in order of the graphs [10,12,17],

$$R(U)=\sum_i R_i(U), \quad (16)$$

where R_i is a linear combination of the ith order graphs with the coefficients to be determined. Generally, the term $[E_l,R_i][E_l,R_j]$ will produce some new graphs which are

different from the graphs with order from 1 to $(i+j-1)$; we define them as the $(i+j)$th order graphs. Therefore, the truncated Schrödinger equation at order M is

$$\sum_l \left\{ \left[E_l, \left[E_l, \sum_{i=1}^{M} R_i(U) \right] \right] + \sum_{i+j \leqslant M} [E_l, R_i(U)] \times [E_l, R_j(U)] \right\} - \frac{2}{g^4} \sum_p \mathrm{Tr}(U_p + U_p^\dagger) = \frac{2a}{g^2} \epsilon_\Omega , \tag{17}$$

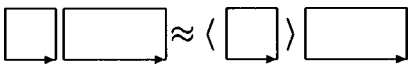
with a being the lattice spacing. The dimensionless coupling constant g is related to the invariant coupling constant e by $g^2 = e^2 a$ in the case of $(2+1)$-dimensional SU(3) LGT.

Suppose R_i and R_j are linear combinations of graphs which contain only one Wilson loop; then, the new graphs produced by $[E_l, R_i][E_l, R_j]$ will contain at most two loops. If there is a connected overlapping graph, we transform it into unconnected overlapping graph with the unimodular condition

$$A_{i_1 j_1} A_{i_2 j_2} A_{i_3 j_3} \epsilon_{j_1 j_2 j_3} = \epsilon_{i_1 i_2 i_3}, \tag{18}$$

where A is any group element of SU(3). Applying the RPA to all those graphs which contain two loops, i.e., replacing one of the two Wilson loops with its vacuum average value, we get all new graphs composed of only one Wilson loop. Now the independent bases are obtained directly, and the number of independent graphs is much smaller than that before applying the RPA, especially at high order.

In the above RPA, when the sizes of two Wilson loops are different, we replace the small one with its vacuum average and let the large one remain unchanged. For example,



(19)

We do so because it is the easiest and simplest way to apply the RPA. Another reason is due to physical considerations. The vacuum or exciting states possess definite correlation lengths. Only when the space occupied by glueball is covered with the Wilson loops is the calculation efficient, so we replace the small loop with its vacuum average and preserve the large loop in Eq. (19).

In the following (the graphs $G_{i,j}$ are shown in Fig. 1), we present the detailed calculation with order $M=3$. Let

$$R_1 = c_0 (G_{1,1} + G_{1,1}^\dagger) = c_0 G_{1,1} + \mathrm{H.c.} \tag{20}$$

Substituting R_1 into Eq. (17), we have

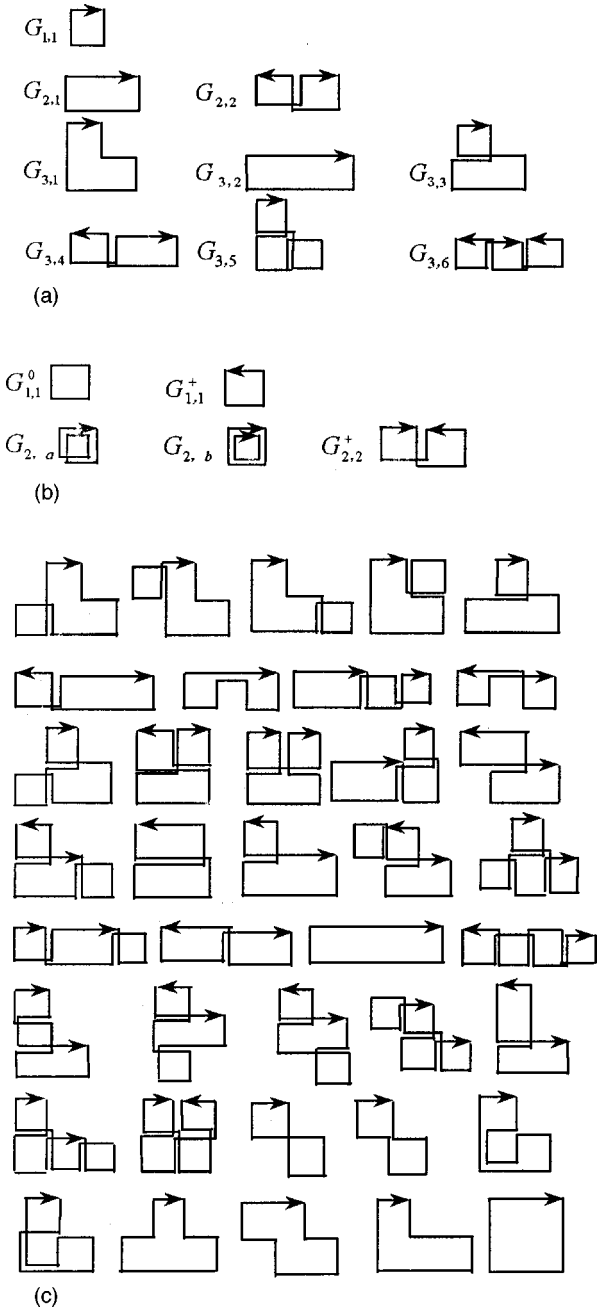


FIG. 1. (a) The first, second, and third order independent graphs used in the approximate third order calculation. (b) The other graphs used in the approximate third order calculation. (c) The fourth order independent graphs used in the approximate fourth order calculation.

$$[E, G_{1,1}][E, G_{1,1}] = -2\left[G_{2,1} - \frac{1}{3} G_{1,1} G_{1,1} \right] + 2\left[G_{2,a} - \frac{1}{3} G_{2,b} \right] \approx -2G_{2,1} + 2\langle G_{1,1}^0 \rangle G_{1,1} - 4G_{1,1}^\dagger , \tag{21}$$

where $\langle G_{1,1}^0 \rangle = \frac{1}{2}[\langle G_{1,1} \rangle + \langle G_{1,1}^\dagger \rangle]$ and $\langle G_{1,1} \rangle = \langle G_{1,1}^\dagger \rangle$, and

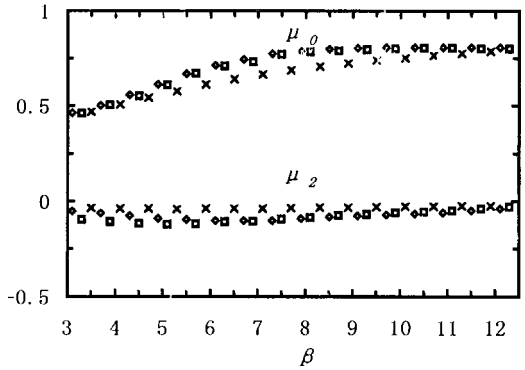


FIG. 2. μ_0 and μ_2 as a function of $\beta=6/g^2$. The crosses stand for the second order results of μ_0 and μ_2, the diamonds for the third order results of μ_0 and μ_2, and the squares for the fourth order of results μ_0 and μ_2,

$$[E,G_{1,1}][E,G_{1,1}^{\dagger}]\approx G_{2,2}+G_{2,2}^{\dagger}-6. \tag{22}$$

Therefore, we obtain the second order approximation of R:

$$R_2=c_1G_{2,1}+c_2G_{2,2}+\text{H.c.} \tag{23}$$

Similarly, we obtain the third order approximation of R:

$$R_3=b_1G_{3,1}+b_2G_{3,2}+b_3G_{3,3}+b_4G_{3,4}+b_5G_{3,5}+b_6G_{3,6}+\text{H.c.} \tag{24}$$

There are six independent graphs in R_3 [see Fig. 1(a)]. The coefficients of each independent graph on both sides of Eq. (17) should be equal to each other, which leads to

$$(G_{3,1}+\text{H.c.})\left[\frac{32}{3}b_1-4c_0c_1\right]=0,$$

$$(G_{3,2}+\text{H.c.})\left[\frac{32}{3}b_2-2c_0c_1\right]=0,$$

$$(G_{3,3}+\text{H.c.})[13b_3+4c_0(c_1-c_2)]=0,$$

$$(G_{3,4}+\text{H.c.})[13b_4+2c_0(c_1-c_2)]=0,$$

$$(G_{3,5}+\text{H.c.})\left[\frac{46}{3}b_5+4c_0c_2\right]=0,$$

$$(G_{3,6}+\text{H.c.})\left[\frac{46}{3}b_6+2c_0c_2\right]=0,$$

$$(G_{2,1}+\text{H.c.})[8c_1-2c_0^2+2c_0(3c_1\langle G_{1,1}^0\rangle+5c_2)+2\langle G_{1,1}^0\rangle(b_3+b_4)]=0,$$

$$(G_{2,2}+\text{H.c.})\left[\frac{31}{3}c_2+2c_0^2+2c_0(3c_1+5c_2\langle G_{1,1}^0\rangle)+2\langle G_{1,1}^0\rangle(b_5+b_6)\right]=0,$$

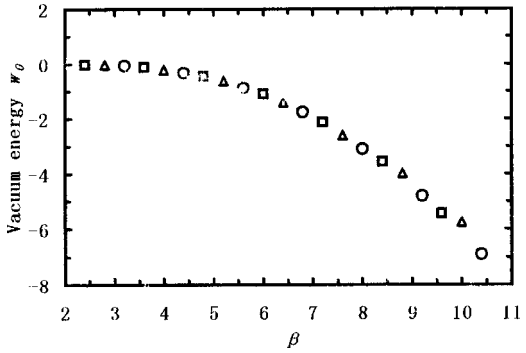


FIG. 3. The vacuum energy against β. The squares stand for the second order results of w_0, the triangles for the third order results of w_0, and the circles for the fourth order of results of w_0.

$$(G_{1,1}+\text{H.c.})\left[\frac{16}{3}c_0-\frac{2}{g^4}+2c_0^2(\langle G_{1,1}^0\rangle-2)-c_0(1+\langle G_{1,1}^0\rangle)(6c_1+10c_2)+c_2\langle G_{1,1}^0\rangle\right]=0,$$

$$w_0=-12c_0^2, \tag{25}$$

where w_0 is defined by $W|\Omega\rangle=w_0|\Omega\rangle$ with $W=(2a/g^2)H$, and thereby $w_0=(2a/g^2)\epsilon_\Omega$.

In order to solve Eq. (25), we must determine $\langle G_{1,1}\rangle$ first. According to the Feynman-Hellman theorem, we make the following change [3]:

$$W\rightarrow W+c_0'(G_{1,1}+G_{1,1}^{\dagger}). \tag{26}$$

Using $\langle G_{1,1}\rangle=\langle G_{1,1}^{\dagger}\rangle$, we have

$$2\langle G_{1,1}^0\rangle=\left.\frac{\partial w_0}{\partial c_0'}\right|_{c_0'=0}=-24c_0\left.\frac{\partial c_0}{\partial c_0'}\right|_{c_0'=0}. \tag{27}$$

Solving the above algebraic equations, we can obtain the wave function of the vacuum state on the lattice. In the continuum limit, the wave function has the form of Eq. (3) [6,10,12]. Expanding clusters in order of the spacing a, we get the coefficient μ_0 (or μ_2) as a linear combination of the coefficients c_i and b_i, consequently, and obtain the function of μ_0 (or μ_2) against $1/g^2$.

III. RESULTS AND DISCUSSIONS

In Fig. 2, we present the results of μ_0,μ_2 vs $\beta=6/g^2$ from second order to fourth order. The crosses stand for the second order results of μ_0 and μ_2, the diamonds for the third order results, and the squares for the fourth order results. From Fig. 2, we see that the three curves of μ_0 (or μ_2) show very good scaling behavior and a convergent trend in the weak coupling region $\beta=6.5-12.5$. It is interesting that the curves of the third and fourth order results are overlapped in the scaling region, which proves that the approach is effective and convergent. From the fourth order results, we obtain

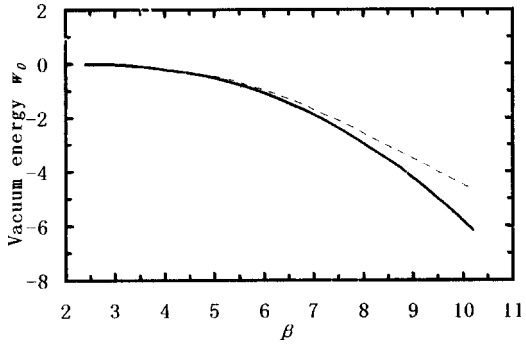


FIG. 4. The vacuum energy vs β. The dashed curve stands for the third order vacuum energy obtained without using the RPA [12], while the solid curve stands for that obtained with the RPA method.

$$\mu_0 = 0.8025 \pm 0.0051,$$

$$\mu_2 = -0.0749 \pm 0.0054,$$

where the errors are the standard errors resulted from the fourth order data in the scaling region. In Ref. [12], the estimated values resulting from the third order calculation are $\mu_0 \approx 0.5411$ and $\mu_2 \approx -0.0781$, but there is no proof to show that the results are convergent, and the scaling behavior of the results is not as good as that in this paper.

As far as we know, such results as the above which show good convergent behavior at relatively low order have not been reported before [3,11,13,17]. It is perhaps because that when we do the Mth order calculation, the unoverlapping loop graphs with order M have no contribution to the calculation of the former $(M-1)$ order wave function when having not used the RPA [11,12] while they have in our present scheme [we see the coefficients of the third unoverlapping loop graphs $b_1 - b_6$ emerge in the third order equations (25)].

The curves of w_0 against β from second order to fourth order are plotted in Fig. 3. The three curves are close to each other and also show good convergent behavior. For comparison, we give the results of the third order vacuum energy obtained through the approach with (solid curve) and without (dashed curve) the RPA, respectively, in Fig. 4. We see that the energy obtained with the RPA is lower than that without the RPA. This show the reasonableness of our present scheme.

In summary, using the RPA, we succeed in overcoming the two problems, i.e., the problem of the rapid proliferation of clusters as the order of approximated calculation rises and the problem of how to distinguish independent bases in the coupled cluster expansions. The computed results of SU(3) LGT in $2+1$ dimensions show that our scheme is reasonable [18].

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