Consistent Batalin-Fradkin quantization of infinitely reducible first class constraints

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We reconsider the problem of Becchi-Rouet-Stora-Tyutin (BRST) quantization of a mechanics with infinitely reducible first class constraints. Following an earlier recipe [Phys. Lett. B **381**, 105 (1996)], the original phase space is extended by purely auxiliary variables, the constraint set in the enlarged space being the first stage of reducibility. The BRST charge involving only a finite number of ghost variables is explicitly constructed.

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The problem of infinitely reducible first class constraints originated from superstring theory where a fully satisfactory covariant quantization seems to be an unsolved problem. Taking a simpler mechanics analogue in four dimensions these look like

$$p^2=0, \quad (p_\theta \sigma^n p_n)_{\dot\alpha}=0, \quad (\sigma^n p_{\bar\theta} p_n)_\alpha=0, \tag{1}$$

where $(p_n, p_{\theta\alpha}, p_{\bar\theta\dot\alpha})$ are momenta conjugate to the variables parametrizing a conventional $R^{4|4}$ superspace $(x^n,\theta^\alpha,\bar\theta^{\dot\alpha})$ and $\sigma^n{}_{\alpha\dot\alpha}$ are the Pauli matrices. Owing to the null vector p_n entering the problem, only half of the fermionic constraints is linearly independent. In particular, the identity

$$(p_\theta \sigma^n p_n)_{\dot\alpha} Z_1{}^{\dot\alpha\alpha}+Z_1{}^{\alpha}p^2\equiv 0, \tag{2}$$

where $Z_1{}^{\dot\alpha\alpha}=(\tilde\sigma^n p_n)^{\dot\alpha\alpha}$, $Z_1{}^\beta=p_\theta^\beta$ holds. On the constraint surface not all of the functions $Z_1{}^{\dot\alpha\alpha}$ prove to be independent:

$$Z_1{}^{\dot\alpha\alpha}Z_{2\alpha\dot\beta}\approx 0, \quad Z_{2\alpha\dot\beta}=(\sigma^n p_n)_{\alpha\dot\beta}. \tag{3}$$

Apparently this process can be continued, the system at hand being an infinite stage of reducibility [1]. It is worth mentioning that, although the correct counting of degrees of freedom can be achieved in the course of Becchi-Rouet-Stora-Tyutin (BRST) quantization by making use of Euler's regularization [2], the expression for the BRST charge involves an infinite ghost tower [3] and, hence, looks formal.

A recipe on how to supplement infinitely reducible first class constraints up to a constraint system of finite stage of reducibility has been proposed recently [4]. It suffices to extend the original phase space by purely auxiliary variables $(\Lambda^n,\ p_{\Lambda m})$, $(\chi^\alpha,\ p_{\chi\alpha})$, $(\bar\chi_{\dot\alpha},\ p_{\bar\chi}^{\dot\alpha})$, with Λ being a real boson and $(\chi,\bar\chi)$ a pair of complex conjugate fermions. These are required to satisfy reducible constraints such as those in Eq. (1) (one can check that the number and the class of the constraints are just enough to suppress dynamics in the sector [4]):

$$p_{\chi\alpha}=0, \quad (\chi\sigma^n\Lambda_n)_{\dot\alpha}=0, \tag{4}$$

$$p_{\bar\chi\dot\alpha}=0, \quad (\sigma^n\bar\chi\Lambda_n)_\alpha=0, \tag{5}$$

$$p_{\Lambda n}=0, \quad \Lambda^2=0, \quad 1-\Lambda p=0. \tag{6}$$

In the extended phase space the reducibility of the original constraints (1) can be compensated by that coming from the sector of additional variables to put the fermionic constraints in the irreducible form

$$\bar\Phi_{\dot\alpha}\equiv(p_\theta\sigma^n p_n+p_\chi\sigma^n\Lambda_n)_{\dot\alpha}=0, \tag{7}$$

$$\Phi_\alpha\equiv(p_n\sigma^n p_{\bar\theta}+\Lambda_n\sigma^n p_{\bar\chi})_\alpha=0, \tag{8}$$

$$\bar\Psi_{\dot\alpha}\equiv(\chi\sigma^n\Lambda_n+p_\chi\sigma^n p_n)_{\dot\alpha}=0, \tag{9}$$

$$\Psi_\alpha\equiv(\Lambda_n\sigma^n\bar\chi+p_n\sigma^n p_{\bar\chi})_\alpha=0, \tag{10}$$

while in the bosonic sector one has

$$p^2=0, \tag{11}$$

$$\tilde p_{\Lambda m}\equiv p_{\Lambda m}-(p_\Lambda\Lambda)p_m-(p_\Lambda p)\Lambda_m=0, \tag{12}$$

$$p_\Lambda p=0, \Lambda^2=0, p_\Lambda\Lambda=0, 1-\Lambda p=0. \tag{13}$$

The equivalence to the initial constraint set seems to be more transparent if one makes use of the identity

$$p_\chi{}^\alpha=-\frac{1}{2\Lambda p}p^2p_\theta^\alpha-\frac{1}{2\Lambda p}\Lambda^2\chi^\alpha-\frac{1}{2\Lambda p}\bar\Phi_{\dot\alpha}(\tilde\sigma^m p_m)^{\dot\alpha}-\frac{1}{2\Lambda p}\bar\Psi_{\dot\alpha}(\tilde\sigma^m\Lambda_m)^{\dot\alpha\alpha}, \tag{14}$$

and its complex conjugate. In the new basis the constraints (7), (8), (11), and (12) are first class, whereas Eqs. (9),(10), and (13) involve second class ones. In order to explicitly decouple $\tilde p_\Lambda{}^n=0$ from the second class fermionic constraints it suffices to redefine them as $\tilde p_\Lambda{}^n=0\to\tilde p_\Lambda{}^n-\frac{1}{2}\chi\sigma^n\tilde\sigma^m p_\chi p_m-\frac{1}{2}p_{\bar\chi}\tilde\sigma^m\sigma^n\bar\chi p_m=0$. As the Dirac bracket associated with the second class constraints is introduced, this seems to be inessential here.

Residual reducibility proves to fall in the bosonic sector. Due to the identities (in what follows the symbol "≈" denotes an equality up to a linear combination of *second class* constraints)

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$$\tilde{p}_{\Lambda}\Lambda \approx 0, \quad \tilde{p}_{\Lambda}p \approx 0, \tag{15}$$

there are only two linearly independent components entering Eq. (12), *the system in the extended phase space being first stage of reducibility*.

It is the purpose of this Brief Report to explicitly construct the BRST charge associated with the constraint set (7)–(13), thus giving an efficient way to cure the infinite ghost tower problem intrinsic to the original system (1).

According to the general recipe [1] the nilpotency equation to determine the BRST charge should be solved under the Dirac bracket associated to the second class constraints. Evaluated in specific coordinate sectors this reads (only the brackets to be used below are explicitly given here)

$$\{\chi^{\alpha},p_{\chi\beta}\}=\frac{1}{2}\delta^{\alpha}{}_{\beta}-\frac{2}{\Delta}\Lambda p(\sigma_{nm})_{\beta}{}^{\alpha}\Lambda^{n}p^{m},$$

$$\{\chi^{\alpha},\chi^{\beta}\}=\frac{2}{\Delta}p^{2}(\sigma_{nm})^{\alpha\beta}\Lambda^{n}p^{m},$$

$$\{p_{\chi\alpha},p_{\chi\beta}\}=\frac{2}{\Delta}\Lambda^{2}(\sigma_{nm})_{\alpha\beta}\Lambda^{n}p^{m}; \tag{16}$$

$$\{\Lambda^{n},p_{\Lambda m}\}=\delta^{n}{}_{m}+\frac{2}{\Delta}p^{2}\Lambda^{n}\Lambda_{m}+\frac{2}{\Delta}\Lambda^{2}p^{n}p_{m}-\frac{2}{\Delta}\Lambda p(p^{n}\Lambda_{m}+\Lambda^{n}p_{m}),$$

$$\{p_{\Lambda n},p_{\Lambda m}\}=\frac{2}{\Delta}p^{2}(\Lambda_{n}p_{\Lambda m}-\Lambda_{m}p_{\Lambda n})+\frac{2}{\Delta}pp_{\Lambda}(p_{n}\Lambda_{m}-p_{m}\Lambda_{n})+\frac{2}{\Delta}p\Lambda(p_{\Lambda n}p_{m}-p_{\Lambda m}p_{n})-\frac{i}{\Delta}(\chi^{2}-\bar{\chi}^{2})\epsilon_{nmkl}\Lambda^{k}p^{l},$$

$$\{\Lambda^{n},\Lambda^{m}\}=0; \tag{17}$$

$$\{\theta^{\alpha},p_{\theta\beta}\}=\delta^{\alpha}{}_{\beta}, \quad \{\theta^{\alpha},\theta^{\beta}\}=0,$$

$$\{p_{\theta\alpha},p_{\theta\beta}\}=0; \quad \{p_{n},p_{m}\}=0, \tag{18}$$

plus complex conjugate expressions for the pairs $(\bar{\chi},p_{\bar{\chi}})$, $(\bar{\theta},p_{\bar{\theta}})$.

In the cross sectors the only nonvanishing brackets are (in what follows we will not need the explicit form of the brackets involving the x^{n} variable, these are omitted here)

$$\{p_{\Lambda n},\chi^{\alpha}\}=\frac{1}{\Delta}p^{2}[\Lambda_{n}\chi^{\alpha}+(\chi\sigma_{n}\tilde{\sigma}^{k}\Lambda_{k})^{\alpha}]+\frac{1}{\Delta}p_{n}(\chi\sigma^{k}\Lambda_{k}\tilde{\sigma}^{m}p_{m})^{\alpha}-\frac{1}{\Delta}\Lambda p(\chi\sigma_{n}\tilde{\sigma}^{k}p_{k})^{\alpha}, \tag{19}$$

$$\{p_{\Lambda n},p_{\chi\alpha}\}=\frac{1}{\Delta}\Lambda^{2}(p_{n}\chi_{\alpha}+(\chi\sigma_{n}\tilde{\sigma}^{k}p_{k})_{\alpha})+\frac{1}{\Delta}\Lambda_{n}(\chi\sigma^{k}p_{k}\tilde{\sigma}^{m}\Lambda_{m})_{\alpha}-\frac{1}{\Delta}\Lambda p(\chi\sigma_{n}\tilde{\sigma}^{k}\Lambda_{k})_{\alpha}, \tag{20}$$

plus complex conjugates.

Given the Dirac bracket, the algebra of the first class constraints is easy to evaluate,

$$\{\tilde{p}_{\Lambda n},\tilde{p}_{\Lambda m}\}\approx U_{nm}{}^{k}\tilde{p}_{\Lambda k}+U_{nm}p^{2},$$

$$\{\tilde{p}_{\Lambda n},\Phi_{\alpha}\}\approx U_{n\alpha}^{\beta}\Phi_{\beta}+U_{n\alpha}p^{2},$$

$$\{\tilde{p}_{\Lambda n},\bar{\Phi}_{\dot{\alpha}}\}\approx U_{n\dot{\alpha}}{}^{\dot{\beta}}\bar{\Phi}_{\dot{\beta}}+U_{n\dot{\alpha}}p^{2}, \tag{21}$$

with all other brackets vanishing. The structure functions entering Eq. (21) are given by

$$U_{nm}{}^{k}=\frac{2}{\Delta}[(\Lambda_{n}p^{2}-p_{n})\delta_{m}{}^{k}-(\Lambda_{m}p^{2}-p_{m})\delta_{n}{}^{k}],$$

$$U_{nm}=\frac{i}{\Delta}(p_{\chi}\chi-p_{\bar{\chi}}\bar{\chi})\epsilon_{nmkl}\Lambda^{k}p^{l},$$

$$U_{n\alpha}{}^{\beta}=\frac{1}{2}(\sigma_{n}\tilde{\sigma}^{k}p_{k})_{\alpha}{}^{\beta}+\frac{1}{\Delta}\Lambda_{n}p^{2}\delta_{\alpha}{}^{\beta}+\frac{1}{\Delta}(\Lambda_{n}p^{2}-p_{n})\times(\Lambda^{k}\sigma_{k}\tilde{\sigma}^{l}p_{l})_{\alpha}{}^{\beta},$$

$$U_{n\alpha}=\frac{1}{2}(\sigma_{n}p_{\bar{\theta}})_{\alpha}-\frac{1}{\Delta}\Lambda_{n}(p^{k}\sigma_{k}p_{\bar{\theta}})_{\alpha}+\frac{1}{\Delta}(\Lambda_{n}p^{2}-p_{n})\times(\Lambda^{k}\sigma_{k}p_{\bar{\theta}})_{\alpha}, \tag{22}$$

and $U_{n\dot{\alpha}}{}^{\dot{\beta}}=(U_{n\alpha}{}^{\beta})^{*}$, $U_{n\dot{\alpha}}=(U_{n\alpha})^{*}$. Worth noting also is the orthogonality of the structure functions obtained to the vectors p_{n}, Λ^{n} which holds on the second class constraints surface.

Having evaluated the structure functions, we are now in a position to construct the BRST charge. Associated with the first class constraints (7), (8), (11), and (12) are the primary ghosts (minimal sector) $(C^{\dot{\alpha}},\bar{\mathcal{P}}_{\dot{\alpha}})$, $(C^{\alpha},\bar{\mathcal{P}}_{\alpha})$, $(C,\bar{\mathcal{P}})$, $(C^{n},\bar{\mathcal{P}}_{n})$. These have the standard properties

$$\epsilon(C^{A})=\epsilon(\bar{\mathcal{P}}^{A})=\epsilon_{A}+1,$$

$$gh(C^{A})=-gh(\bar{\mathcal{P}}^{A})=1. \tag{23}$$

To compensate the overcounting in the sector $(C^{n},\bar{\mathcal{P}}_{n})$ [only two components entering Eq. (12) are linearly independent] one further introduces the secondary ghosts [1] $(C^{1},\bar{\mathcal{P}}^{1})$, $(C^{2},\bar{\mathcal{P}}^{2})$, these obeying

$$\epsilon(C^{1,2})=\epsilon(\overline{\mathcal{P}}^{1,2})=0,$$
$$gh(C^{1,2})=-gh(\overline{\mathcal{P}}^{1,2})=2. \tag{24}$$

The nilpotency equation on the BRST charge

$$\{\Omega_{min},\Omega_{min}\}\approx 0, \tag{25}$$

should then be solved under the boundary condition

$$\Omega_{min}=\Phi_\alpha C^\alpha+\bar{\Phi}_{\dot{\alpha}}C^{\dot{\alpha}}+\tilde{p}_{\Lambda n}C^n+p^2C+\overline{\mathcal{P}}_n\Lambda^nC^1+\overline{\mathcal{P}}_np^nC^2+\cdots, \tag{26}$$

which, through Eq. (25), automatically generates both the algebra (21) and the identities (15).

Calculating the contribution of the boundary terms into Eq. (25),

$$\{\Omega_{min},\Omega_{min}\}\approx 2\overline{\mathcal{P}}_m\{\Lambda^m,\tilde{p}_{\Lambda n}\}C^1C^n-2(U_{n\alpha}{}^{\beta}\Phi_\beta+U_{n\alpha}p^2)C^\alpha C^n-2(U_{n\dot{\alpha}}{}^{\dot{\beta}}\bar{\Phi}_{\dot{\beta}}+U_{n\dot{\alpha}}p^2)C^{\dot{\alpha}}C^n-(U_{nm}{}^{k}\tilde{p}_{\Lambda k}+U_{nm}p^2)C^mC^n+\cdots, \tag{27}$$

one can partially clarify the structure of the terms lacking in Eq. (26). In particular, extending the ansatz (26) by the three new contributions

$$\tfrac{1}{2}\overline{\mathcal{P}}_k\tilde{U}_{nm}{}^{k}C^mC^n+\overline{\mathcal{P}}_\alpha U^\alpha_{n\beta}C^\beta C^n+\overline{\mathcal{P}}_{\dot{\alpha}}U_{n\dot{\beta}}{}^{\dot{\alpha}}C^{\dot{\beta}}C^n, \tag{28}$$

with

$$\tilde{U}_{nm}{}^{k}=U^k_{nm}-\frac{2}{\Delta}p^k(\Lambda_np_m-\Lambda_mp_n),$$

$$\tilde{U}_{nm}{}^{k}\Lambda^m\approx\frac{2}{\Delta}\{\Lambda^k,p_{\Lambda n}\},\qquad \tilde{U}^k_{nm}p^m\approx 0, \tag{29}$$

one can get rid of the first term (which is a manifestation of reducibility of the constraints) and those involving $\tilde{p}_\Lambda$, Φ, $\bar{\Phi}$

$$\{\Omega_{min},\Omega_{min}\}\approx -U_{nm}p^2C^mC^n-2U_{n\alpha}p^2C^\alpha C^n-2U_{n\dot{\alpha}}p^2C^{\dot{\alpha}}C^n-2\overline{\mathcal{P}}_\alpha U^\alpha_{n\gamma}U^\gamma_{m\beta}C^mC^nC^\beta-2\overline{\mathcal{P}}_{\dot{\alpha}}U_{n\dot{\gamma}}{}^{\dot{\alpha}}U_{m\dot{\beta}}{}^{\dot{\gamma}}C^mC^nC^{\dot{\beta}}+\cdots. \tag{30}$$

In order to verify Eq. (30) a number of Jacobi identities associated to the constraint algebra (21) should be used. These are omitted here.

It is instructive then to give the explicit form of the terms quadratic in the structure functions which enter Eq. (30) ($U_{m\dot{\alpha}}{}^{\dot{\beta}}U_{n\dot{\beta}}{}^{\dot{\gamma}}-U_{n\dot{\alpha}}{}^{\dot{\beta}}U_{m\dot{\beta}}{}^{\dot{\gamma}}$ is obtained by complex conjugation)

$$U_{m\alpha}{}^{\beta}U_{n\beta}{}^{\gamma}-U_{n\alpha}{}^{\beta}U^\gamma_{m\beta}=\Big\{(\sigma_{nm})_\alpha{}^{\beta}+\frac{1}{\Delta}(\Lambda_np_m-\Lambda_mp_n)(\Lambda_l\sigma^l\tilde{\sigma}^kp_k)_\alpha{}^{\gamma}+\frac{1}{\Delta}\Lambda_m(\sigma_n\tilde{\sigma}^kp_k)_\alpha{}^{\gamma}-\frac{1}{\Delta}\Lambda_n(\sigma_m\tilde{\sigma}^kp_k)_\alpha{}^{\gamma}-\frac{1}{\Delta}(\Lambda_mp^2-p_m)(\sigma_n\tilde{\sigma}^k\Lambda_k)_\alpha{}^{\gamma}+\frac{1}{\Delta}(\Lambda_np^2-p_n)(\sigma_m\tilde{\sigma}^k\Lambda_k)_\alpha{}^{\gamma}+\frac{1}{\Delta}(\Lambda_np_m-\Lambda_mp_n)\delta^\gamma_\alpha\Big\}p^2\equiv\Pi^\gamma_{mn\alpha}p^2. \tag{31}$$

Being factors of p^2 these suggest a further amendment:

$$\overline{\mathcal{P}}U_{n\alpha}C^\alpha C^n+\overline{\mathcal{P}}U_{n\dot{\alpha}}C^{\dot{\alpha}}C^n+\tfrac{1}{2}\overline{\mathcal{P}}U_{nm}C^mC^n-\tfrac{1}{2}\overline{\mathcal{P}\mathcal{P}}_\alpha\Pi^\alpha_{nm\beta}C^mC^nC^\beta-\tfrac{1}{2}\overline{\mathcal{P}\mathcal{P}}_{\dot{\alpha}}\Pi_{nm\dot{\beta}}{}^{\dot{\alpha}}C^mC^nC^{\dot{\beta}}. \tag{32}$$

After tedious calculations with the extensive use of Jacobi identities one can verify that the complete BRST charge

$$\Omega_{min}=\Phi_\alpha C^\alpha+\bar{\Phi}_{\dot{\alpha}}C^{\dot{\alpha}}+\tilde{p}_{\Lambda n}C^n+p^2C+\overline{\mathcal{P}}_n\Lambda^nC^1+\overline{\mathcal{P}}_np^nC^2+\tfrac{1}{2}\overline{\mathcal{P}}_k\tilde{U}_{nm}{}^{k}C^mC^n+\overline{\mathcal{P}}_\alpha U^\alpha_{n\beta}C^\beta C^n+\overline{\mathcal{P}}_{\dot{\alpha}}U_{n\dot{\beta}}{}^{\dot{\alpha}}C^{\dot{\beta}}C^n+\overline{\mathcal{P}}U_{n\alpha}C^\alpha C^n+\overline{\mathcal{P}}U_{n\dot{\alpha}}C^{\dot{\alpha}}C^n+\tfrac{1}{2}\overline{\mathcal{P}}U_{nm}C^mC^n-\tfrac{1}{2}\overline{\mathcal{P}\mathcal{P}}_\alpha\Pi^\alpha_{nm\beta}C^mC^nC^\beta-\tfrac{1}{2}\overline{\mathcal{P}\mathcal{P}}_{\dot{\alpha}}\Pi_{nm\dot{\beta}}{}^{\dot{\alpha}}C^mC^nC^{\dot{\beta}}, \tag{33}$$

is nilpotent. Only a finite number of ghost generations proved to be needed in the extended phase space.

Finally, it is worth mentioning that a formal consideration of the present paper can be directly applied to specific models. In particular, the superparticle due to Siegel [5], after a proper Hamiltonian treatment, leads precisely to Eq. (1) that we started with. The latter theory has been previously considered in the alternative harmonic superspace approach [6] . This makes use of Lorentz harmonics [6] in order to extract linearly independent components from the fermionic constraints (1) in a covariant way. Having obtained a system of rank two, our result here is in perfect agreement with that of

Ref. [6]. The present formulation, however, has the advantage that all the variables involved obey the standard spin–statistics relations. Furthermore, the scheme outlined in this article proves to admit a Lagrangian formulation [4,7], the latter seems to be problematic in the approach [6].

Another related approach to be mentioned is that by Diaz and Zanelli [8] who improved an earlier (noncovariant) quantization proposal by Kallosh [9] (see also related work [10]). The infinite proliferation of ghosts has been truncated there by imposing appropriate conditions on the ghosts variables, the latter involving specific (covariant) projectors. In this respect, it would be interesting to consider the truncation of the infinite ghost tower already at the second step, following the approach by Diaz and Zanelli. We expect that the result will agree with the outcome of our technique. This and other questions related to possible applications to superparticle, superstring will be considered in a forthcoming publication [7].

[1] I.A. Batalin and E.S. Fradkin, Phys. Lett. **122B**, 157 (1983).
[2] R. Kallosh, W. Troost, and A. Van Proeyen, Phys. Lett. B **212**, 212 (1988).
[3] F. Eßler, M. Hatsuda, E. Laenen, W. Siegel, J.P. Yamron, T. Kimura, and A. Miković, Nucl. Phys. **B364**, 67 (1991).
[4] A.A. Deriglazov and A.V. Galajinsky, Phys. Lett. B **381**, 105 (1996).
[5] W. Siegel, Class. Quantum Grav. **2**, L95 (1985).
[6] E.R. Nissimov and S.J. Pacheva, Phys. Lett. B **189**, 57 (1987).
[7] S. Bellucci and A. Galajinsky, hep-th/0002071.
[8] A.H. Diaz and J. Zanelli, Phys. Lett. B **202**, 347 (1988).
[9] R. Kallosh, Phys. Lett. B **195**, 369 (1987).
[10] S. Bellucci, Mod. Phys. Lett. A **5**, 2253 (1990).