Properties of the massive Thirring model from the *XYZ* spin chain

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We consider here the massive Thirring model regularized with the *XYZ* spin chain. We numerically calculate the mass ratios of particles which lie in the discrete part of the spectrum and obtain results in accordance with the DHN formula and in disagreement with recent calculations in the literature based on the numerical Bethe ansatz and infinite momentum frame methods. We also analyze the short distance behavior of these states and evaluate the conformal dimensions. This paper, taken together with the previous one for the sine-Gordon model, confirms the duality relation between two models formulated by Klassen and Melzer [Int. J. Mod. Phys. A **8**, 4131 (1993)].

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I. INTRODUCTION

The massive Thirring model (MTM) and sine-Gordon model (SGM) are important as a testing laboratory for understanding ideas proposed for other more complicated field theories. In this paper we propose to calculate certain physical quantities for the MTM by performing an explicit diagonalization of its lattice regularization with the *XYZ* spin chain. As the first task we want to calculate the masses of breathers. The previous calculations have been based on the semiclassical method [1], on factorized scattering theory [2], or on the Bethe ansatz method [3–9].

The additional interest in avoiding the previously mentioned assumptions is due to recent criticism [10–12] [the authors claim that there is only one breather in the whole attractive region, and with different mass than the Dashen-Hasslacher-Neveu (DHN) formula predicts]. The same authors challange also the well-known duality relation between the MTM and SGM [13–17]. The precise meaning and extent of this equivalence was formulated by Klassen and Melzer [17] (notice that the models are not equivalent when they have a finite size in space).

One important criticism relates to the use of the so-called string conjecture. Indeed, violations of this conjecture are observed in the literature [18]. Despite the fact that, at least until now, it was not known that these violations affect any relevant results, it would be desirable to have a calculation which does not rely on the string conjecture.

It is for this reason that we want to treat the MTM without using the above-mentioned assumptions. Our approach will be based on direct numerical diagonalization of the *XYZ* spin chain which is a lattice regularization of the MTM [5,6]. This method is suited for analyses of low discrete states in the spectrum, but becomes less and less effective when we go to higher states. Such an approach was used in the literature for other problems, e.g., conformal unitary models perturbed by some relevant operator [19–21].

We also intend to calculate conformal dimensions of operators creating breather states. There are conjectured values for them [17]. By explicit calculation we confirm this conjecture for the first breather but get different results for the second breather.

Recently [22] we have performed a similar calculation for the SGM. The regularization in this case was the *XXZ* spin chain in a transverse field. The results on the masses of breathers and conformal dimensions agree as statements on relation of two models would suggest, so it maybe considered also as an independent check of the SGM-MTM correspondence [17].

II. MTM AS A MASSIVE PERTURBATION OF THE GAUSSIAN MODEL

The MTM is a $(1+1)$-dimensional field theory of a Dirac spinor field ψ, defined classically by the Lagrangian

$$\mathcal{L}_{MTM}=i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi-\lambda\bar{\psi}\psi-\frac{g}{2}(\bar{\psi}\gamma_{\mu}\psi)(\bar{\psi}\gamma^{\mu}\psi). \quad (2.1)$$

Here λ is a dimensionful parameter which sets the mass scale in a theory which is conformaly invariant when $\lambda=0$. However, although λ enters Eq. (2.1) as a (bare) mass, its mass dimension d_λ is not equal to 1, but is determined from the (nontrivial) anomalous dimension of the field $\bar{\psi}\psi$. The dimensionless coupling constant g is scale invariant (vanishing beta function), but it is not uniquely defined due to the existence of different regularizations of the (conserved) current $\bar{\psi}\gamma^{\mu}\psi$. Correspondingly, there is at least a one-parameter family of definitions of g. Our definition will be the same as the one used by Coleman [13] (Schwinger definition). We shall find it more convenient to use the parameter β related to g with

$$\frac{4\pi}{\beta^2}=1+\frac{g}{\pi}. \quad (2.2)$$

Here β is the dimensionless coupling constant from the dually related SGM.

In [17] it was shown that the MTM can be viewed as a perturbed conformal field theory (CFT) when the second term in Eq. (2.1) is treated as a (massive) perturbation. We will now repeat here some results of their analyses relevant for our discussion.

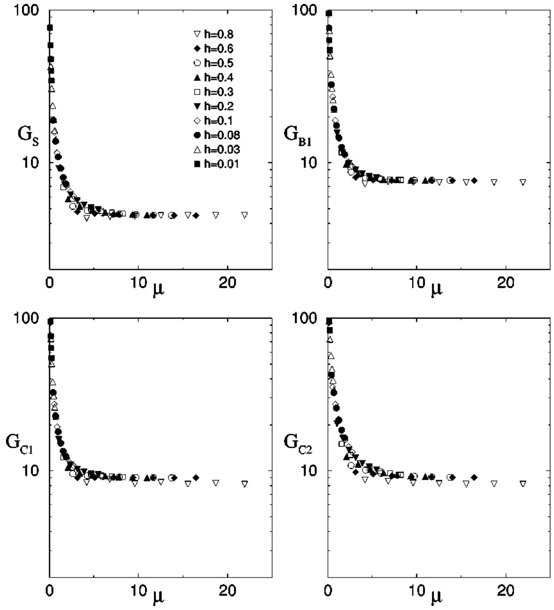

FIG. 1. Scaling functions $\tilde{G}_a(\beta,\mu)$ for the isolated gaps (S, $B1$) plus two lowest ''continuum'' gaps ($C1$, $C2$) of the Hamiltonian (3.1) at $\Delta=0.3$ (or $\beta^2=10.13$, $g=0.76$). For this value of the coupling constant the DHN formula predicts the existence of one breather. The legend in the upper left figure applies to all figures in this article.

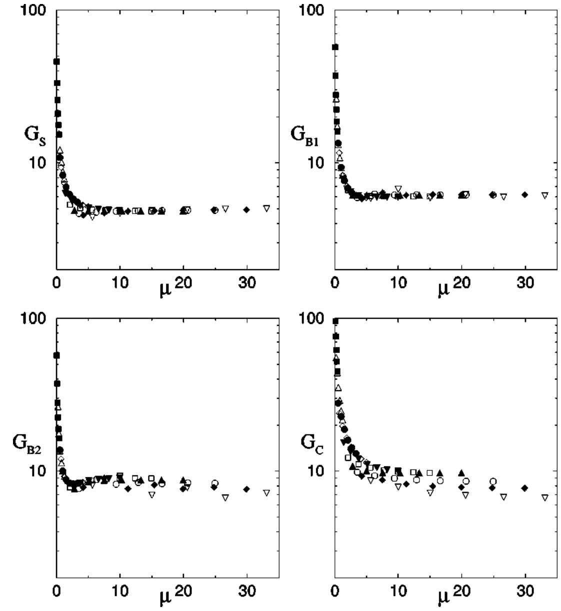

FIG. 2. Scaling functions $\tilde{G}_a(\beta,\mu)$ for the isolated gaps (S, $B1$, $B2$) plus lowest ''continuum'' gap (C) of the Hamiltonian (3.1) at $\Delta=0.6$ (or $\beta^2=7.42$, $g=2.18$). The DHN formula predicts now the existence of two breathers.

An unperturbed theory $\lambda=0$ (approached in the UV limit) is the Thirring model which is a CFT with central charge $c=1$ and an operator algebra generated by

$$L_f=\{V_{m,n}|m\in 2\mathbf{Z},n\in\mathbf{Z}\ \text{ or }\ m\in 2\mathbf{Z}+1,n\in\mathbf{Z}+1/2\},\tag{2.3}$$

where $V_{m,n}(x)$ are primary fields with conformal dimensions

$$(\Delta_{m,n},\bar{\Delta}_{m,n})=\left(\frac{2\pi}{\beta^2}\left(\frac{m\beta^2}{8\pi}+n\right)^2,\frac{2\pi}{\beta^2}\left(\frac{m\beta^2}{8\pi}-n\right)^2\right),\tag{2.4}$$

where we used duality relation (2.2). From Eq. (2.4) we can read off the scaling dimensions and (Lorentz) spin of $V_{m,n}$:

$$d_{m,n}=\Delta_{m,n}+\bar{\Delta}_{m,n}=\frac{m^2\beta^2}{16\pi}+\frac{4\pi n^2}{\beta^2},$$

$$s_{m,n}=\Delta_{m,n}-\bar{\Delta}_{m,n}=mn.\tag{2.5}$$

A whole operator algebra is generated by a quartet of fields $V_{\pm1,\pm1/2}$, which are connected to the fundamental spinor fields ψ, $\bar{\psi}$ by

$$\psi\leftrightarrow\begin{pmatrix}V_{1,1/2}\\ V_{-1,1/2}\end{pmatrix},\qquad \bar{\psi}\leftrightarrow\begin{pmatrix}V_{1,-1/2}\\ V_{-1,-1/2}\end{pmatrix}.\tag{2.6}$$

Now one supposes that Hilbert space of the full (perturbed) theory is isomorphic to that of the unperturbed one. From operator product algebra (OPA) it follows that the (properly normalized) perturbing operator in the MTM (2.1) is

$$\bar{\psi}\psi=V^{(+)}_{2,0}\equiv\frac{1}{2}(V_{2,0}+V_{-2,0}),\tag{2.7}$$

which means that λ has mass dimension $d_\lambda=2-d_{2,0}=2-\beta^2/4\pi$. From the condition of relevancy of the perturbation, i.e., $d_\lambda>0$, we obtain Coleman's bound $\beta^2<8\pi$ ($g>-\pi/2$). Also, from Eqs. (2.3) and (2.7) we can see that the MTM has a $\tilde{U}(1)\times Z_2\times\tilde{Z}_2$ internal symmetry group. The $\tilde{U}(1)$ acts as $V_{m,n}\to e^{i\alpha n}V_{m,n}$, while Z_2 and $\tilde{Z}_2$ are generated by $R:V_{m,n}\to V_{-m,n}$ and $\tilde{R}:V_{m,n}\to V_{m,-n}$, respectively.

III. SPIN CHAIN REGULARIZATION OF THE MTM

It was argued a while ago [5,6] that the MTM on a cylinder with proper (antiperiodic) boundary conditions (B.C.'s) possesses spin chain regularization given by the XYZ spin chain defined by the Hamiltonian

$$H_{XYZ}=H_{XXZ}-h\sum_{n=1}^{N}(\sigma_n^x\sigma_{n+1}^x-\sigma_n^y\sigma_{n+1}^y),\tag{3.1}$$

where $\sigma^{x,y,z}$ are Pauli matrices, N ia an even integer, and H_{XXZ} is the Hamiltonian of the XXZ spin chain:

$$H_{XXZ}=\sum_{n=1}^{N}(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\Delta\sigma_n^z\sigma_{n+1}^z),\tag{3.2}$$

where $-1<\Delta<1$ [we also use standard parametrization $\Delta=-\cos\gamma$, so $\gamma\in(0,\pi)$]. In Eqs. (3.1) and (3.2) sector-dependent B.C.'s should be used:

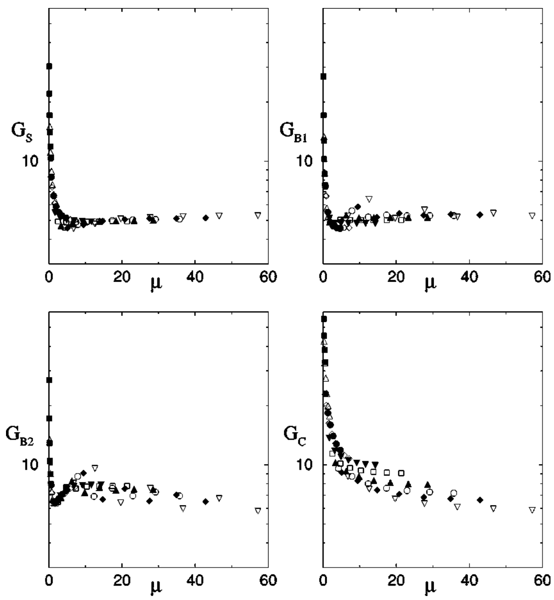

FIG. 3. The same as Fig. 2 but now for $\Delta=0.75$ (or $\beta^2=5.78$, $g=3.69$).

$$\sigma_{N+1}^{x,y}=\sigma_1^{x,y}(-1)^{N/2}C, \quad \sigma_{N+1}^z=\sigma_1^z, \tag{3.3}$$

where

$$C=\prod_{n=1}^{N}\sigma_n^z. \tag{3.4}$$

From results of Ref. [23] it follows that the XXZ chain (3.2) with B.C. (3.3) gives in the continuum limit a CFT with a space of states equal to that of L_f in Eq. (2.3), where

$$\beta=\sqrt{8(\pi-\gamma)}. \tag{3.5}$$

That leads us to the conjecture that, aside from irrelevant corrections,

$$\sigma_n^x\sigma_{n+1}^x-\sigma_n^y\sigma_{n+1}^y\propto V_{2,0}^{(+)} \tag{3.6}$$

in the continuum limit.

Now, the continuum limit is obtained letting $N\to\infty$ and $h\to 0$, but at the same time keeping fixed the scaling parameter $\tilde{\mu}$:

$$\tilde{\mu}\equiv hN^{d_\lambda}=hN^{2-\beta^2/4\pi}=hN^{2\gamma/\pi}. \tag{3.7}$$

In this limit, the mass gaps of the XYZ chain are expected to satisfy a scaling law

$$\tilde{m}_i=h^{1/d_\lambda}\tilde{G}_i(\gamma,\tilde{\mu})=h^{\pi/2\gamma}\tilde{G}_i(\gamma,\tilde{\mu}). \tag{3.8}$$

The scaling parameter $\tilde{\mu}$ is connected to L (space extension of continuum theory, i.e., MTM). For our purposes it is enough to know that $\tilde{\mu}\to\infty$ ($\tilde{\mu}\to 0$) corresponds to $L\to\infty$ ($L\to 0$), respectively.

IV. MASS SPECTRUM

Our goal here is to calculate the mass ratios of particles in the MTM in the $L\to\infty$ limit using the connection with the XYZ spin chain (3.1). First we must numerically calculate the mass gaps of the spin chain for finite N and h. Then we must make a continuum limit, i.e., take $N\to\infty$ and $h\to 0$, keeping $\tilde{\mu}$ fixed. Finally we should make a $L\to\infty$, i.e., $\tilde{\mu}\to\infty$, limit. In practice, it is preferable to do the following [19–21]: first take $N\to\infty$ with h fixed and afterwards extrapolate $h\to 0$. A difference is that in the latter case one does $\tilde{\mu}\to\infty$ before $h\to 0$. These limits are performed using the BST extrapolation method [24,25].

We numerically diagonalized Hamiltonian (3.1) for up to 16 sites using the Lanczos algorithm. We are interested in the masses, so we only need the zero-momentum sector. We should note here that in Ref. [6] it was shown that true space translations are generated not by an ordinary translation operator on the spin chain, but by its square. From this and the fact that Hamiltonian (3.1) commutes with the operator C, Eq. (3.4), it follows that we can break the Hamiltonian in the momentum-zero sector into four sectors named $0^\pm$, $\pi^\pm$, where $0,\pi$ is macroscopic momentum and $\pm$ denotes eigenvalue of $(-1)^{N/2}C$ (which can only be ± 1 because $C^2=1$). We considered a number of values of coupling constant in the attractive regime ($g>0$, i.e., $\Delta>0$). The structure of the spectrum is in agreement with the DHN prediction; i.e., we obtain vacuum, first breather ($B1$), second breather ($B2$) (when it exists), and ''continuum'' in 0^+; fermion (F) and ''continuum'' in 0^-; antifermion ($\overline{F}$) and ''continuum'' in π^-; ''continuum'' starting with $F\overline{F}$ and $\overline{F}\overline{F}$ in π^+. Names for the particle states and FF, $\overline{F}\overline{F}$ continuum will be confirmed by results for the mass ratios. But even we could not make an extrapolation (because of the poor scaling in the $\tilde{\mu}\to\infty$ limit) for the lowest ''continuum'' state in 0^+ for values of g where the DHN formula predicts that it should be of $B1\,B1$ type; its scaling law in the $\tilde{\mu}\to 0$ limit clearly shows that its scaling dimension is the one we expect for the $B1\,B1$ lowest continuum state, i.e., $d_{4,0}$. We should mention also that spectra in 0^- and π^- are exactly degenerated which means that the F and $\overline{F}$ mass gaps are equal even on the lattice, which was not the case in similar analyses of the SGM in [22].

In Figs 1–3 we present numerical results for the scaled gaps $\tilde{G}_i$ for four states: fermion (F), first breather ($B1$), second breather ($B2$), and lowest state in the FF continuum (C). This is of course a check of the scaling relation (3.8). Finally, partially extrapolated mass ratios

$$\tilde{r}_a(\Delta,h)=\lim_{\substack{N\to\infty\\ h\ \text{fixed}}}\frac{\tilde{m}_a}{\tilde{m}_F}=\lim_{\substack{N\to\infty\\ h\ \text{fixed}}}\frac{\tilde{G}_a}{\tilde{G}_F}, \quad a\in\{B1,B2,C\}, \tag{4.1}$$

and fully extrapolated mass ratios of the first breather

$$\tilde{r}_{B1}(\Delta)=\lim_{h\to 0}\tilde{r}_a(\Delta,h), \tag{4.2}$$

TABLE I. Estimates for the mass gap ratios $\tilde{r}_a$ as a function of h at $\Delta=0.3$ ($\beta^2=10.13$, $g=0.76$). In this regularization soliton and antisoliton gaps are exactly degenerated. We also added the DHN prediction (only one breather for this value of the coupling constant) and the prediction of Fujita *et al.* (only one breather for all $g>0$). The numbers in parentheses give the estimated uncertainty in the last given digit.

	h									
$\tilde{r}_a$	0.8	0.6	0.5	0.4	0.3	0.2	0.1	$h\to 0$	DHN	Fujita *et al.*
$B1$	1.6341 (3)	1.7007 (4)	1.718 (1)	1.730 (3)	1.734 (8)	1.74 (2)	1.67 (6)	1.747 (6)	1.745	1.777
$C1$	1.786 (7)	2.0013 (5)	2.000 (2)	2.001 (3)	1.98 (1)	2.00 (2)	2.07 (5)		2.000	2.000
$C2$	1.797 (2)	2.0011 (8)	1.999 (2)	2.001 (6)	2.00 (1)	2.00 (3)	1.93 (8)		2.000	2.000

TABLE II. The same as Table I but now for $\Delta=0.6$ ($\beta^2=7.42$, $g=2.18$).

	h									
$\tilde{r}_a$	0.8	0.6	0.5	0.4	0.3	0.2	0.1	$h\to 0$	DHN	Fujita *et al.*
$B1$	1.187 (6)	1.2443 (6)	1.2587 (3)	1.26491105 (3)	1.2638 (5)	1.254 (2)	1.240 (9)	1.24 (2)	1.223	1.337
$B2$	1.2 (1)	-	1.694 (2)	1.807020 (8)	1.8753 (8)	1.913 (4)	1.89 (2)		1.935	2.000
C	1.29 (1)	1.536 (4)	1.734 (2)	1.99998 (2)	2.003 (2)	2.00 (1)	1.99 (4)		2.000	2.000

TABLE III. The same as Table I but now for $\Delta=0.75$ ($\beta^2=5.78$, $g=3.69$).

	h									
$\tilde{r}_a$	0.8	0.6	0.5	0.4	0.3	0.2	0.1	$h\to 0$	DHN	Fujita *et al.*
$B1$	-	1.027 (4)	1.030 (1)	1.0255 (4)	1.0112 (2)	0.9870 (4)	0.948 (2)	0.91 (2)	0.905	1.052
$B2$	1.0 (4)	1.21 (8)	1.35 (2)	1.485 (6)	1.5716 (3)	1.6219 (9)	1.641 (7)		1.614	2.000
C	1.528 (3)	1.25 (1)	1.360 (4)	1.553 (4)	1.803 (2)	2.005 (4)	1.97 (2)		2.000	2.000

TABLE IV. The same as Table I but now for $\Delta=0.9$ ($\beta^2=3.61$, $g=7.79$).

	h											
$\tilde{r}_a$	0.6	0.5	0.4	0.35	0.3	0.25	0.2	0.15	0.1	$h\to 0$	DHN	Fujita *et al.*
$B1$	0.82 (1)	0.821 (8)	0.795 (2)	0.779 (2)	0.758 (1)	0.7356 (5)	0.7083 (4)	0.6763 (4)	0.63250 (7)	0.52 (4)	0.521	0.668
$B2$	0.9 (2)	1.0 (2)	1.13 (6)	1.18 (4)	1.18 (1)	1.202 (8)	1.1964 (9)	1.187 (2)	1.163 (2)		1.005	1.336
C	0.99 (1)	1.010 (8)	1.166 (8)	1.218 (8)	1.285 (8)	1.366 (6)	1.487 (8)	1.70 (1)	1.987 (8)		2.000	2.000

TABLE V. Scaling dimensions of particle states in the MTM as conjectured from our numerical results.

State	Operator	Scaling dimension
Fermion	$V_{1,1/2}$	$\dfrac{\beta^2}{16\pi}+\dfrac{\pi}{\beta^2}$
Antifermion	$V_{1,-1/2}$	$\dfrac{\beta^2}{16\pi}+\dfrac{\pi}{\beta^2}$
First breather	$V_{2,0}^{(-)}$	$\dfrac{\beta^2}{4\pi}$
Second breather	$V_{2,0}^{(+)}$	$\dfrac{\beta^2}{4\pi}$
$\psi_1\psi_2$ continuum	$V_{0,1}$	$\dfrac{4\pi}{\beta^2}$

are given in Tables I–IV, together with the DHN predictions [1] and predictions of Fujita *et al.* [10,11]. Finally, the extrapolation $h\to 0$ was possible only for the first breather because scaling of the second breather is worse and asks for a larger N (probably $N\geqslant 24$). One can see that our results strongly confirm DHN and reject Fujita *et al*.

V. UV (CONFORMAL) LIMIT OF PARTICLE STATES

Let us now turn our attention to the opposite, i.e., UV, limit of our results for the XYZ spin chain. We mentioned in Sec. III that it obtained when $\tilde{\mu}\to 0$. From conformal perturbation theory we expect the scaling relation

$$\tilde{m}_a = h^{\pi/2\gamma}[2\pi\zeta d_a \tilde{\mu}^{-\pi/2\gamma} + \tilde{H}_a(\gamma,\tilde{\mu})], \tag{5.1}$$

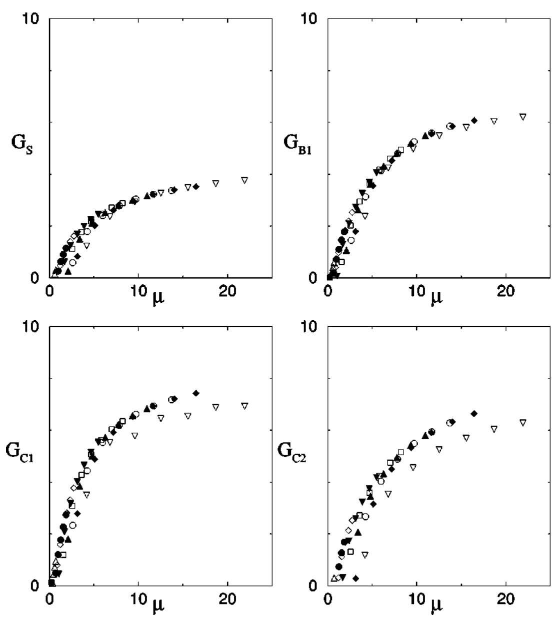

FIG. 4. Reduced scaling functions $\tilde{H}_a(\beta,\mu)$ at $\Delta=0.3$ (or $\beta^2=10.13$). The legend is the same as in Fig. 1.

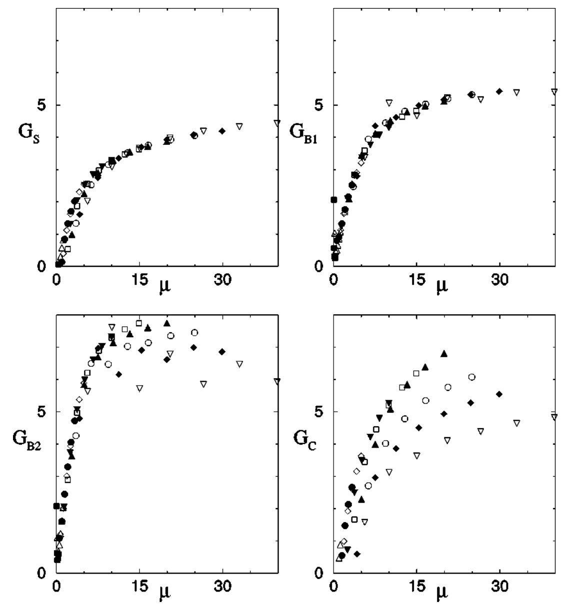

FIG. 5. The same as Fig. 4 but now for $\Delta=0.6$ (or $\beta^2=7.42$).

where d_a is the scaling dimension of the state a, and ζ is a well-known normalization factor,

$$\zeta=\frac{2\pi\sin\gamma}{\gamma}.$$

From Eq. (5.1) we can obtain the scaling dimensions of the particle states F, $B1$, and $B2$ from the condition that $\tilde{H}_a$ should be less singular than $\tilde{G}_a$. Our results are given in Table V. They differ from those conjectured in [17] only for the second breather, which has scaling dimension equal to that of the first breather. These results are in agreement with those in [22] for the SGM. In Figs. 4–6 we show the numeric

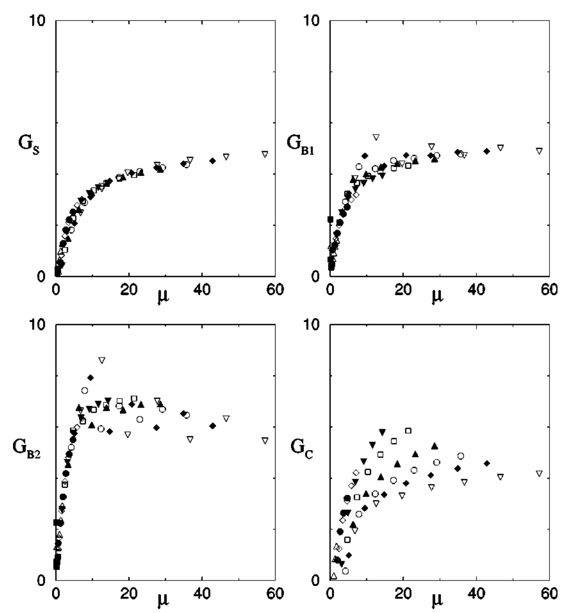

FIG. 6. The same as Fig. 4 but now for $\Delta=0.75$ (or $\beta^2=5.78$).

results for reduced scaling functions, where we used values from Table V for the scaling dimensions.

VI. CONCLUSION

We have calculated in this paper the masses of breather states and the anomalous dimensions of related operators for the MTM using spin chain regularization. This is a direct numerical calculation independent of assumptions such as the semiclassical approximation [1], factorized scattering theory [2], or Bethe ansatz method [3,4,6]. On the other hand, in a series of papers based on numerical calculation within the Bethe ansatz method [11] or using the infinite momentum frame technique [10], different results have been claimed. Our calculation confirms the conventional spectrum. In addition we calculate the anomalous dimensions of operators creating breather states. It agrees with conjecture in [17] for the first breather but disagrees for the second breather. This result is consistent with the previous calculation for the sine-Gordon model, i.e., consistent with equivalence relation between the two models [17].

Note added in proof. In recent work Fevarati *et al.* [26] analytically confirmed our result for the scaling dimension of the second breather using an extension of the nonlinear integral equation (NLIE) method.

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