Static solutions in the U(1) gauged Skyrme model

B. M. A. G. Piette*
Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, United Kingdom

D. H. Tchrakian†
Department of Mathematical Physics, National University of Ireland, Maynooth, Maynooth, Ireland
and School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland
(Received 20 August 1999; revised manuscript received 29 March 2000; published 26 June 2000)

We use a prescription to gauge the SU(2) Skyrme model with a U(1) field, characterized by a conserved baryonic current. This model reverts to the usual Skyrme model in the limit of the gauge coupling constant vanishing. We show that there exist axially symmetric static solutions with zero magnetic charge, which can be electrically either charged or uncharged. The energies of the (uncharged) gauged Skyrmions are less than the energy of the (usual) ungauged Skyrmion. For physical values of the parameters the impact of the U(1) field is very small, so that it can be treated as a perturbation to the (ungauged) spherically symmetric hedgehog. This allows the perturbative calculation of the magnetic moment.

PACS number(s): 12.39.Dc

I. INTRODUCTION

For a long time now, much attention has been paid to the Skyrme [1] model in 3 dimensions. It is believed to be an effective theory for nucleons in the large N limit of QCD at low energies. The classical properties as well as the quantum properties of the model are in relatively good agreement with the observed properties of small nuclei [2–4].

Gauged Skyrme models have been used in the past. The U(1) gauged model [2,5] was used to study the decay of nucleons in the vicinity of a monopole [5], while the $SU(2)_L$ gauged model [6,7] was used to study the decay of nucleons when the Skyrme model is coupled to the weak interactions [6,7]. The Skyrme model has also been used to compute the quantum properties of the Skyrmion [3] where the gauge degrees of freedom are quantized to compute the low energy eigenstates of a Skyrmion. These states were identified as the proton, the neutron and the delta.

The aim of this work is to show that the Skyrme model can be coupled to a self-contained electromagnetic field and that this U(1) gauged model has stable classical solutions. In addition to these solitons with vanishing magnetic and electric flux, we show that this system supports solutions with nonvanishing electric flux which are analogous to the dyon solutions of the Georgi-Glashow model, just as the uncharged solitons are the analogues of the monoploles [8] of that model. The electrically charged lumps have larger energy, or mass, than the uncharged soliton, just like the Julia-Zee dyon [9] has larger energy, or is heavier, than the (electrically uncharged) monopole. We shall refer to these lumps as *charged U(1) Skyrmions*.

In addition to its intrinsic interest as a soliton in the Maxwell gauged Skyrme model, the present work is also an example of a soliton in a d-dimensional $SO(N)$ gauged S^d model with $N<d$ for the case $d=3$, $N=2$, extending the results of Ref. [10] which were restricted to the $N=d$ cases. (The work of Ref. [10] consists of establishing topological lower bounds for the generic case, encompassing earlier examples in two [11] and three [12,13] dimensions respectively.) The gauging prescription used here by us coincides precisely with that used in Ref. [5] and permits the establishing of a topological lower bound which did not feature in Ref. [5] and which is carried out here to establish the stability of the soliton. Such lower bounds are absent in the other prescription of gauging the Skyrme model as in Refs. [6,7]. [Notice that we name the sigma models after the manifold in which the fields take their values rather than using the name of the symmetry group for the model. Thus what is sometime called the $O(d+1)$ model in the literature will be refered to as the S^d model.]

The U(1) gauged SU(2) Skyrme model is described by the Lagrangian [5]

$$\mathcal{L}=\frac{F_\pi^2}{16}\mathrm{Tr}(D_\mu U D_\mu U^\dagger) - \frac{1}{32a^2}\mathrm{Tr}([(D_\mu U)U^\dagger,(D_\mu U)U^\dagger])^2-\frac{1}{4}\mathcal{F}_{\mu\nu}^2$$

where the U(1) gauge covariant derivative is

$$D_\mu U=\partial_\mu U+ie\mathcal{A}_\mu[Q,U], \tag{1}$$

where $\mathcal{F}_{\mu\nu}=\partial_\mu\mathcal{A}_\nu-\partial_\nu\mathcal{A}_\mu$ and where the charge matrix of the quarks is expressed as $Q=\frac{1}{2}(\frac{1}{3}1+\sigma_3)$. This differs from the covariant derivative of Ref. [5] only in the unimportant matter of the sign of i in Eq. (1), which we have chosen for consistency of the convention used in Ref. [10].

In what follows it will be more convenient [10] to parametrize the Skyrme field as an S^3 valued field $\phi^a=(\phi^\alpha,\phi^A)$, $\alpha=1,2$, $A=3,4$ subject to the constraint $|\phi^a|^2=1$. The two fields U and ϕ are related to each other via the following expression:

*Email address: B.M.A.G. Piette@durham.ac.uk
†Email address: tigran@thphys.may.ie

$$U=\phi^a\tau^a, \quad U^{-1}=U^\dagger=\phi^a\tilde{\tau}^a \tag{2}$$

where $\tau^a=(i\sigma^\alpha,i\sigma^3,1)$ and $\tilde{\tau}^a=(-i\sigma^\alpha,-i\sigma^3,1)$, in terms of the Pauli matrices $(\sigma^1,\sigma^2,\sigma^3)$.

The gauge covariant derivative now can be reexpressed as

$$D_\mu\phi^\alpha=\partial_\mu\phi^\alpha+A_\mu\varepsilon^{\alpha\beta}\phi^\beta, \quad D_\mu\phi^A=\partial_\mu\phi^A, \tag{3}$$

where $A_\mu=e\mathcal{A}_\mu$ and $F_{\mu\nu}=e\mathcal{F}_{\mu\nu}$.

The Lagrangian for the U(1) gauged Skyrme model can then be written as

$$\mathcal{L}=-\lambda_0F_{\mu\nu}^2+\lambda_1|D_\mu\phi^a|^2-\lambda_2|D_{[\mu}\phi^aD_{\nu]}\phi^b|^2 \tag{4}$$

where the square brackets on the indices imply (total) antisymmetrisation and where $\lambda_0^{-1}=4e^2$, $\lambda_1=F_\pi^2/8$ and $\lambda_2^{-1}=8a^2$. The late Greek indices μ label the Minkowskian coordinates, while the early Greek indices $\alpha=1,2$ and the upper case Latin indices $A=3,4$ label the fields $\phi^a=(\phi^\alpha,\phi^A)$.

The static Hamiltonian pertaining to the Lagrangian (4) is

$$\mathcal{H}=\lambda_0F_{ij}^2+\lambda_1|D_i\phi^a|^2+\lambda_2|D_{[i}\phi^aD_{j]}\phi^b|^2+i2\lambda_0|\partial_iA_0|^2 +A_0^2\{\lambda_1|\phi^\alpha|^2+16\lambda_2[|\phi^\alpha|^2|\partial_i\phi^A|^2+\tfrac{1}{4}|\partial_i(|\phi^A|^2)|^2]\}, \tag{5}$$

where the indices $i=\alpha,3$ label the spacelike coordinates.

To find the static solutions of the model, one would usualy solve the Euler Lagrange equations which minimize the Hamiltonian (5), but because of the electric potential A_0, one must solve the Euler Lagrange equations derived from the Lagrangian (4). We then look for static solutions, but, as for the Julia-Zee dyon [9], we have to impose the proper asymptotic behavior for the electric potential to obtain static solutions which are electrically charged (in the classical sense, i.e. solutions where the flux of the electric field is nonzero).

When the full equations of motion are written down, one finds as expected that there are static solutions for which $A_0=0$, i.e., solutions for which the electric field is identically zero. For these solutions in the temporal gauge, the equations of motion reduce to the equations obtained by minimizing the Hamiltonian (5). We study the solutions of *unit Baryon charge* of the U(1) gauged Skyrme model with and without an electric field, for various values of the U(1) coupling constant (or equivalently the Skyrme coupling). For physical values of these parameters in the model, we find that the energy (mass) of the gauged Skyrmion does not differ significantly from that of the ungauged charged-1 Skyrmion, namely the familiar hedgehog [1]. This implies that for these values of the physical parameters, the U(1) gauged Skyrmion can be regarded as a perturbation of the (ungauged) hedgehog, enabling the computation of the magnetic moments of the gauged Skyrmion (i.e. the Neutron) and the shift of the energy of the gauged Skyrmion away from the energy of the hedgehog [1], perturbatively using the method employed by Klinkhamer and Manton [14] for the sphaleron of the Weinberg–Salam model.

In Sec. II, we define the topological charge and establish the corresponding lower bound on the energy functional. In Sec. III we present the solutions which have no electric fields in the first subsection and electrically charged solutions in the second subsection. The perturbation analysis of the gauged Skyrmion around the (ungauged) hedgehog is carried out in Sec. IV, and Sec. V is devoted to a discussion of our results.

II. THE TOPOLOGICAL CHARGE AND LOWER BOUND

The definition of the topological charge is based on the criterion that it be equal to the Baryon number, namely the degree of the map. For the gauged theory however, this quantity must be *gauge invariant* as well. This requirement can be systematically [10] satisfied by arranging the gauge invariant topological charge density to be the sum of the usual, *gauge variant* winding number density

$$\varrho_0=\varepsilon_{ijk}\varepsilon^{abcd}\partial_i\phi^a\partial_j\phi^b\partial_k\phi^c\phi^d, \tag{6}$$

plus a total divergence whose surface integral vanishes due to the finite energy conditions, such that the combined density is gauge invariant. In 3 dimensions, this is given explicitly in Refs. [10,13] for the $SO(3)$ gauged S^3 model, and for the present case of interest, namely the $SO(2)$ gauged S^3 model, the charge density can be derived from that of the $SO(3)$ gauged model by contraction of the gauge group $SO(3)$ down to $SO(2)$. It can also be arrived at directly. To state the definition of the charge, we denote the gauge covariant counterpart of Eq. (6) by

$$\varrho_G=\varepsilon_{ijk}\varepsilon^{abcd}D_i\phi^aD_j\phi^bD_k\phi^c\phi^d, \tag{7}$$

so that using the notations (6) and (7) we have the definition of the gauge invariant topological charge

$$\varrho=\varrho_0+\partial_i\Omega_i, \tag{8}$$

$$=\varrho_G+\frac{3}{2}\varepsilon_{ijk}F_{ij}(\varepsilon^{AB}\phi^BD_k\phi^A). \tag{9}$$

In Eq. (8) the density Ω_i is the following gauge *variant* form:

$$\Omega_i=3\varepsilon_{ijk}\varepsilon^{AB}A_j\partial_k\phi^A\phi^B. \tag{10}$$

The flux of Ω_i vanishes, as can deduced by anticipating the finite energy conditions to be stated later.

Note that the 3-volume integral of ϱ_0 in Eq. (8) is the degree of the map for the ungauged system namely the baryon number.

Identifying ϱ, Eq. (9), with the naught component j^0 of the baryon current, j^μ is defined by

$$j^\mu=\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}\partial_\nu\phi^a\partial_\rho\phi^b\partial_\sigma\phi^c\phi^d +3\varepsilon^{\mu\nu\rho\sigma}\partial_\nu(A_\rho\varepsilon^{AB}\phi^B\partial_\sigma\phi^A) \tag{11}$$

$$= \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd}D_\nu\phi^a D_\rho\phi^b D_\sigma\phi^c\phi^d - \frac{3}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\nu\rho}(\varepsilon_{AB}\phi^B D_\sigma\phi^A). \quad (12)$$

The 4-divergence of Eq. (11) receives a contribution only from its first term, which being locally a total divergence implies that the 3-volume integral of j^0 is a conserved quantity. Alternatively we consider the 4-divergence of Eq. (12),

$$\partial_\mu j^\mu = 6\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta}\varepsilon_{AB}D_\mu\phi^\alpha D_\nu\phi^\beta D_\rho\phi^A D_\sigma\phi^B \quad (13)$$

which is analogous to the corresponding quantity in the work of Goldstone and Wilczek [6]. This contrasts with the expression for the total divergence of the topological current in the work of D'Hoker and Farhi [7], where a different gauging prescription is used leading to that quantity being equal to the local anomaly.

We now proceed to find a model whose Hamiltonian $\mathcal{H}_0$ is bounded from below by the topological charge density defined by Eq. (9). We will then show that the Hamiltonian (5) is given by $\mathcal{H}_0$ plus certain positive definite terms.

First of all, we reproduce the density ϱ_G, Eq. (7), in Eq. (9) by using the following inequality:

$$(\kappa_3 D_i\phi^a - \varepsilon_{ijk}\varepsilon^{abcd}\kappa_2^2 D_j\phi^b D_k\phi^c\phi^d)^2 \geqslant 0 \quad (14)$$

where the two constants κ_3 and κ_2 have the dimensions of length. Expanding the square, we get ϱ_G on the right-hand side of

$$\kappa_3^2(D_i\phi^a)^2 + \kappa_2^4(D_{[i}\phi^a D_{j]}\phi^b)^2 \geqslant 2\kappa_3\kappa_2^2\varrho_G. \quad (15)$$

To reproduce the other term in Eq. (9), $\frac{3}{2}\varepsilon_{ijk}F_{ij}(\varepsilon^{AB}\phi^B\partial_k\phi^A)$, we use the following inequality:

$$\left(\kappa_0^2 F_{ij} - \frac{1}{2}\kappa_4\varepsilon_{ijk}\varepsilon^{AB}\phi^B D_k\phi^A\right)^2 \geqslant 0 \quad (16)$$

yielding

$$\kappa_0^4 F_{ij}^2 + \kappa_4^2\frac{1}{4}(\varepsilon^{AB}\phi^B D_i\phi^A)^2 \geqslant \kappa_0^2\kappa_4\varepsilon_{ijk}F_{ij}(\varepsilon^{AB}\phi^B D_k\phi^A). \quad (17)$$

With the special choice for the relative values of the constants $3\kappa_3\kappa_2^2 = \kappa_4\kappa_0^2$, the sum of Eqs. (13) and (15) yields the following:

$$\kappa_0^4 F_{ij}^2 + \kappa_3^2(D_i\phi^a)^2 + \kappa_2^4(D_{[i}\phi^a D_{j]}\phi^b)^2 + \frac{9\kappa_3^2\kappa_2^4}{4\kappa_0^4}(\varepsilon^{AB}\phi^B D_i\phi^A)^2 \geqslant 2\kappa_3\kappa_2^2\varrho. \quad (18)$$

The right-hand side of Eq. (18) is now proportional to the topological charge density ϱ defined by Eq. (9) so that the inequality (18) can be interpreted as the topological inequality giving the lower bound on the energy density functional if we define the latter to be the left-hand side of Eq. (18), namely

$$\mathcal{H}_0 = \kappa_0^4 F_{ij}^2 + \kappa_1^2(D_i\phi^a)^2 + \kappa_2^4(D_{[i}\phi^a D_{j]}\phi^b)^2 + \frac{9\kappa_3^2\kappa_2^4}{4\kappa_0^4}(\varepsilon^{AB}\phi^B D_i\phi^A)^2. \quad (19)$$

The Hamiltonian system (19) is almost the Hamiltonian of the gauged Skyrme model (5) (remember that $A_0=0$). It differs from the latter only in its last term. Now we can use the identity

$$(\varepsilon^{AB}\phi^B D_i\phi^A)^2 = (D_i\phi^a)^2 - \left[\frac{1}{2}(\phi^{[\alpha}D_i\phi^{\beta]})^2 + (\phi^{[\alpha}D_i\phi^{A]})^2\right] \quad (20)$$

and add the positive definite term $(\kappa_3^2\kappa_2^4/9\kappa_0^4)[\frac{1}{2}(\phi^{[\alpha}D_i\phi^{\beta]})^2 + (\phi^{[\alpha}D_i\phi^{A]})^2]$ appearing on the right-hand side of Eq. (20) to $\mathcal{H}_0$ in Eq. (19) to end up with the Hamiltonian for the U(1) gauged Skyrme model:

$$\mathcal{H} = \kappa_0^4 F_{ij}^2 + \kappa_1^2(D_i\phi^a)^2 + \kappa_2^4(D_{[i}\phi^a D_{j]}\phi^b)^2 \geqslant 2\kappa_3\kappa_2^2\varrho \quad (21)$$

which is nothing but the static Hamiltonian (5) in the temporal gauge $A_0=0$, and where

$$\lambda_1 = \kappa_3^2\left(1 + \frac{9\kappa_2^4}{4\kappa_0^4}\right), \quad \lambda_0 = \kappa_0^4, \quad \lambda_2 = \kappa_2^4. \quad (22)$$

By virtue of Eq. (18), Eq. (21) is also bounded from below by $2\kappa_3\kappa_2^2\varrho$, namely by a number proportional to the topological charge density ϱ.

We thus see that $\mathcal{H}_0$ can be considered as a minimal [U(1) gauged] model, but from now on, we will restrict our attention to the physically more relevant model (21) and integrate it numerically to find its topologically stable finite energy solitons.

The soliton solutions to the system (21) can only be found by solving the second-order Euler-Lagrange equations, and *not* some first-order Bogomol'nyi equations since saturating the inequalities (14) and (16) would not saturate the lower bound on the energy density functional $\mathcal{H}$. In this context we note that saturating Eqs. (14) and (16) does indeed saturate the topological lower bound on the functional $\mathcal{H}_0$ by virtue of the inequality (18), and should it have turned out that the Bogomol'nyi equations arising from the saturation of Eqs. (14) and (16) supported non-trivial solutions, then $\mathcal{H}_0$ would have been a very interesting system to consider. As it turns out however, these Bogomol'nyi equations have only trivial solutions in exactly the same way as in the case of the (ungauged) Skyrme model [1].

The energy for the static configuration, when the electric field vanishes, is expressed as

$$E(\lambda_0,\lambda_1,\lambda_2) = \int d^3x[\lambda_0 F_{ij}^2 + \lambda_1(D_i\phi^a)^2 + \lambda_2(D_{[i}\phi^a D_{j]}\phi^b)^2] \quad (23)$$

and performing the dilation $x \to \sigma x$, $A_\mu \to \sigma^{-1} A_\mu$, we get

$$E(\lambda_0,\lambda_1,\lambda_2)=\int d^3x\left[\frac{\lambda_0}{\sigma}F_{ij}^2+\sigma\lambda_1(D_i\phi^a)^2 + \frac{\lambda_2}{\sigma}(D_{[i}\phi^a D_{j]}\phi^b)^2\right]. \tag{24}$$

If we choose $\sigma=(\lambda_2/\lambda_1)^{1/2}$ then we have

$$E(\lambda_0,\lambda_1,\lambda_2)=(\lambda_1\lambda_2)^{1/2}E\left(\frac{\lambda_0}{\lambda_2},1,1\right), \tag{25}$$

from which we see that we can set $\lambda_1=\lambda_2=1$ without any loss of generality. By virtue of Eqs. (21) and (25), we can finally state

$$E(\lambda_0,\lambda_1,\lambda_2)\geq 2\left(\frac{\lambda_1\lambda_2}{1+\frac{9\lambda_2}{4\lambda_0}}\right)^{1/2}\int d^3x\varrho. \tag{26}$$

Notice that for the usual Skyrme model we have

$$E_{sk}(\lambda_1,\lambda_2)=\int dx^3[\lambda_1(\partial_i\phi^a)^2+\lambda_2(\partial_{[i}\phi^a\partial_{j]}\phi^b)^2] = (\lambda_1\lambda_2)^{1/2}E_{sk}(1,1)\geq 2(\lambda_1\lambda_2)^{1/2}\int dx^3\varrho_0. \tag{27}$$

We will use Eq. (27) to compare the numerical solutions of the gauged Skyrme model with the solutions of the (ungauged) Skyrme model.

We would like to point out that the topological stability considerations discussed in this section apply only to the solutions with no electric field, i.e., with $A_0=0$.

III. THE SOLITON AND THE CHARGED U(1) SKYRMION

To find the static solutions, we have to look for the largest symmetry group of the functional to be subjected to the variational principle, and look for solutions which are invariant under that symmetry group. For the solutions in the $A_0=0$ gauge this is the static Hamiltonian (21), while for the solutions in the $A_0\neq 0$ it is the Lagrangian (4). For our choice of gauge group the largest symmetry is the $SO(2)$ group corresponding to an axial rotation in space-time and a gauge transformation on the gauge field. Defining the axial variables $r=\sqrt{x_1^2+x_2^2}$ and $z=x_3$ in terms of the coordinates $x_i=(x_\alpha,x_3)$, $\alpha=1,2$, the most general axially symmetric ansatz [15] for the fields $\phi^a=(\phi^\alpha,\phi^A)$ (with $\alpha=1,2$ and $A=3,4$), and, $A_i=(A_\alpha,A_3)$, is

$$\phi^\alpha=\sin f\sin g n^\alpha,\quad \phi^3=\sin f\cos g,\quad \phi^4=\cos f, \tag{28}$$

$$A_\alpha=\frac{a(r,z)-n}{r}\varepsilon_{\alpha\beta}\hat{x}_\beta+\frac{c_2(r,z)}{r}\hat{x}_\alpha,$$

$$A_3=\frac{c_1(r,z)}{r},\quad A_0=\frac{b(r,z)}{r}, \tag{29}$$

with $n^\alpha=(\sin n\phi,\cos n\phi)$ in terms of the azimuthal angle ϕ and $\hat{x}_\alpha=x_\alpha/r$. n in n^α is the *vorticity*, which for the nucleons of interest to us here, equals *one*, $n=1$. The functions a,b,c_1,c_2,f and g both depend on r and z.

Our ansatz (29) for the U(1) field consists of decomposing the latter in the most general tensor basis possible. We will find out below, when we compute the Euler-Lagrange equations, that the functions c_1 and c_2 vanish identically. Anticipating this, we suppress them henceforth. In its final form this ansatz agrees with that used in Ref. [15], the latter being arrived at by specializing the Rebbi-Rossi ansatz for the axially symmetric $SO(3)$ field.

The static Hamiltonian, i.e., the T_{00} component of the energy mometum tensor $T_{\mu\nu}$, is then given by

$$H=\int\left\{\frac{\lambda_0}{r^2}\left[a_r^2+a_z^2+\left(b_r-\frac{b}{r}\right)^2+b_z^2\right]+\frac{\lambda_1}{2}\left[f_r^2+f_z^2+\sin^2 f(g_r^2+g_z^2)+\frac{a^2+b^2}{r^2}\sin^2 f\sin^2 g\right] + 2\lambda_2\sin^2 f\left[(f_rg_z-f_zg_r)^2+\sin^2 g\left[\frac{a^2+b^2}{r^2}(f_r^2+f_z^2+(g_r^2+g_z^2)\sin^2 f)\right]\right]\right\}r\,dr\,dz. \tag{30}$$

The boundary conditions for the Skyrmion fields are the same as the boundary conditions for the hedgehog ansatz when expressed in the cylindrical coordinates where $g=\pi/2+\arctan(z/r)$ and defining $R=(z^2+r^2)^{1/2}$, $f\equiv f(R)$ with $f(0)=\pi$ and $\lim_{R\to\infty}f(R)=0$. From this we can deduce that the function f has a fixed value at the origin and at infinity. For smothness along the z axis, each field, that is f, g and A_α must satisfy the condition that the partial derivative with respect to of the field at $r=0$ vanishes. The boundary conditions and the asymptotic behaviors for a and b are chosen so that the gauge fields A_μ are well defined and A_0 looks asymptoticaly like a Coulomb field (i.e., with an electric charge but no magnetic charge). We also require that the total energy be finite. These conditions lead to the following constraints:

$$
\begin{aligned}
& f(0,0)=\pi, \quad f(r\to\infty,z\to\infty)=0, \quad f_r(r=0,z)=0,\\
& g(r=0,z<0)=0, \quad g(r=0,z>0)=\pi, \quad g_R|_{R\to\infty}=0,\\
& a(r=0,z)=1, \quad a_r|_\infty=0, \quad a_z|_\infty=0,\\
& a_r(r=0,z)=0, \quad A_0(r\to\infty,z\to\infty)=V_0+q/r, \quad A_0(r=0,z) \quad =0,
\end{aligned} \tag{31}
$$

where $R=(z^2+r^2)^{1/2}$ and where we have used the notation $\partial a/\partial r = a_r$, etc. Note that the field g is undefined at the origin and the resulting discontinuity of g at that point is an artifact of the coordinate system used. The asymptotic behavior of A_0 at infinity will be discussed in a later section. To solve the equations numerically, it is more conveniant to use the field A_0 rather than b; this is why we have expressed the boundary condition in terms of that field. On the other hand, the equations take a simpler form when written in term of b, so we shall still use it below.

Now the volume integral (with the appropriate normalization of $12\pi^2$) of ϱ_0 given by Eq. (6) is the degree of the map, or, the baryon number. It is straightforward to verify that when the ansatz (28) is substituted in ϱ_0 and the volume integral is computed subject to the boundary conditions given above, the result will equal the integer n defined in Eq. (28). Thus, the baryon number of the field configuration (28) equals the vortex number n. In what follows, we will restrict ourselves to *unit* baryon number, $n=1$, i.e., to the nucleons.

Before we proceed to substitute the anstaz (28), (29) into the field equations, we calculate the Baryon current (11) for the field configurations (28), (29) described by the solutions we seek. We express the spacelike part of this current j_i in the radial direction flowing out of the normal to the surface of the cylinder which we denote by j_r, and in the z direction which we denote by j_z. The result is

$$
j_r=\frac{6}{r}\left[(\dot f g_z-\dot g f_z)a\sin^2 f\sin g+\frac{1}{2}(\dot g a_z-\dot a g_z)\sin f\cos f\sin g+\frac{1}{2}(\dot a f_z-\dot f a_z)\cos g\right], \tag{32}
$$

$$
j_z=-\frac{6}{r}\left[(\dot f g_r-\dot g f_r)a\sin^2 f\sin g+\frac{1}{2}(\dot g a_r-\dot a g_r)\sin f\cos f\sin g+\frac{1}{2}(\dot a f_r-\dot f a_r)\cos g\right], \tag{33}
$$

where we have denoted $\partial f/\partial t=\dot f$, etc. Note that the baryon current (32), (33) are *not* sensitive to the charge of the nucleon, i.e., that the function $b(r,z)$ does not feature in them.

We now turn to the equations to be solved, namely the Euler-Lagrange equations arising from the variational principle applied to the Lagrangian (4), in the *static limit*. Substituting the ansatz (28),(29) into these equations of motion leads to

$$
a_{rr}-\frac{a_r}{r}-a_{zz}-a\sin^2(f)\sin^2(g)\left[\frac{\lambda_1}{2\lambda_0}+\frac{2\lambda_2}{\lambda_0}[f_r^2+f_z^2+(g_r^2+g_z^2)\sin^2(f)]\right]=0,
$$

$$
b_{rr}-\frac{b_r}{r}+\frac{b}{r^2}+b_{zz}-b\sin^2(f)\sin^2(g)\left[\frac{\lambda_1}{2\lambda_0}+\frac{2\lambda_2}{\lambda_0}[f_r^2+f_z^2+(g_r^2+g_z^2)\sin^2(f)]\right]=0,
$$

$$
\begin{aligned}
&\Delta f-\left(g_r^2+g_z^2+\frac{a^2-b^2}{r^2}\sin^2 g\right)\sin f\cos f+4\frac{\lambda_2}{\lambda_1}\sin f[(g_r(f_{zz}g_r-f_r g_{zz}+f_z g_{rz}-f_{rz}g_z-f_z/rg_z)\\
&+g_z(f_{rr}g_z-f_z g_{rr}+f_r g_{rz}-f_{rz}g_r+f_r/rg_z))\sin f+(f_r g_z-g_r f_z)^2\cos f\\
&+\sin g/r^2[(a^2-b^2)(f_r^2+f_z^2-2(g_r^2+g_z^2)\sin^2 f)\cos f\sin g\\
&+((a^2-b^2)(f_{rr}+f_{zz}-f_r/r)+2f_r(aa_r-bb_r)+2f_z(aa_z-bb_z))\sin f\sin g\\
&+2(a^2-b^2)(f_r g_r+f_z g_z)\sin f\cos g]]=0,
\end{aligned}
$$

$$
\begin{aligned}
&\Delta g+2(f_r g_r+f_z g_z)\cot f-\frac{a^2-b^2}{r^2}\sin g\cos g+4\frac{\lambda_2}{\lambda_1}[f_z(f_z g_{rr}-f_{rr}g_z+f_{rz}g_r-f_r g_{rz}+f_z g_r/r)\\
&+f_r(f_r g_{zz}-f_{zz}g_r+f_{rz}g_z-f_z g_{rz}-f_z g_z/r)+\sin g/r^2[(a^2-b^2)((g_r^2+g_z^2)\sin^2 f-f_r^2-f_z^2)\cos g\\
&+((a^2-b^2)(g_{rr}+g_{zz}-g_r/r)+2g_r(aa_r-bb_r)+2g_z(aa_z-bb_z))\sin^2 f\sin g\\
&+(a^2-b^2)(f_r g_r+f_z g_z)\cos f\sin f\sin g]]=0.
\end{aligned} \tag{34}
$$

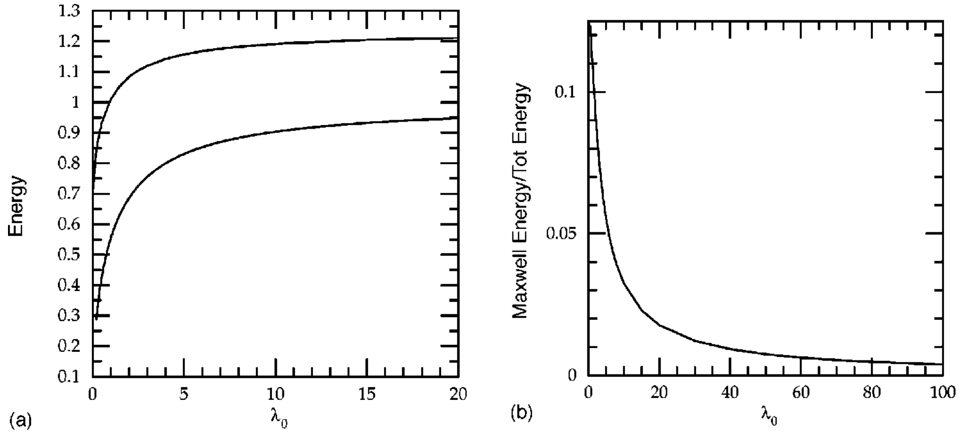

FIG. 1. (a) Energy and topological bound of the gauged Skyrmion in units of $24\pi^2$. (b) Ratio of the electromagnetic and the total energy as a function of λ_0.

In the case of the $A_0=0$ gauge, Eqs. (34) coincide with the Euler-Lagrange equations derived from the positive definite Hamiltonian density (5). Moreover, in that case, those equations also coincide with the Euler-Lagrange equations of the reduced two dimensional Hamiltonian obtained by subjecting Eq. (5) to axial symmetry by substituting the ansatz (28),(29) into it. This is expected due to the strict imposition of symmetry. In the $A_0\neq 0$ gauge, the Euler-Lagrange equations are derived from the Lagrangian (4) which is not positive definite. Nonetheless these equations coincide with those arising from the reduced two dimensional Lagrangian obtained by subjecting the Lagrangian (4) to axial symmetry. (This happens also for the Julia Zee dyon [9].)

A. $A_0=0$: U(1) Skyrme soliton

It is easy to see from Eq. (34) that there are solutions for which $b=0$ (i.e., $A_0=0$). As mentioned before, in that case, Eq. (34) can be obtained by minimizing the Hamiltonian (30). Notice also that setting $a=0$ is not compatible with our boundary conditions (A_i would not be well defined at the origin). We thus expect our gauged solution to carry a nonzero magnetic field.

To show this we have to solve Eq. (34) numerically for the non-vanishing functions $f(r,z),g(r,z)$ and $a(r,z)$.

We have restricted our numerical integrations to the case where the vortex number n appearing in the axially symmetric ansatz (28) is equal to 1, i.e., our soliton carries unit baryon number.

Using Eq. (30), we have found numerically that $E(1,1,1)=24\pi^2 1.01$ whereas for those values of $\lambda_0,\lambda_1,\lambda_2$ the lower bound for the energy given by Eq. (26) is $24\pi^2 0.555$. In Fig. 1, we present the total energy for the gauged Skyrmion as a function of λ_0, together with the lower bound given by Eq. (26). Note that the asymptotic value of $E(\lambda_0,1,1)$ is $24\pi^2 1.232$ as $\lambda_0\to\infty$. As a comparison, the energy (27) for the ungauged Skyrmion is $E_{sk}(1,1)=24\pi^2 1.232$ with a lower bound set at $24\pi^2$. We see that $E(\lambda_0=\infty,1,1)=E_{sk}(1,1)$ which means that as $\lambda_0\to\infty$, the gauge coupling $1/\lambda_0^{1/2}$ goes to zero and the gauged Skyrmion becomes in this limit the ungauged Skyrmion.

It is interesting to note that the energy of gauged Skyrmion is smaller than the energy of the ungauged Skyrmion, as expected, but that on the other hand, the amount by which the energy of the gauged Skyrmion exceeds its topological lower bound is larger than the excess of the energy of the ungauged Skyrmion above its respective topological lower bound. For example we can clearly see from Fig. 1(a) that at $\lambda_0=20$, the energy of the gauged Skyrmion 1.22 (in units of $24\pi^2$) exceeds the lower bound 0.95 by 0.27. This is larger than 0.232, the excess of the ungauged Skyrmion energy over its lower bound. For smaller values of λ_0 Fig. 1(a) shows that the excess of the energy of the gauged Skyrmion over its lower bound is even larger, hence this is a general feature.

In Fig. 1(b), we also see that the Maxwell energy, i.e., the term proportional to λ_0 in Eq. (30), is decreasing as λ_0 increases. Notice that we could have used for the Maxwell energy the sum of all the terms involving the gauge field functions a and b in Eq. (30), but this would lead to a figure similar to Fig. 1(b).

In Fig. 2, we show the profile and the level curve for the energy density of the Skyrmion in the r,z plane for $\lambda_0=1$. One sees clearly that the effect of the gauged field is to make the Skyrmion elongated along the z axis. The magnetic field vectors of the Skyrmion are parallel to the r,z plane. In Fig. 3, we show the configuration of magnetic field using arrows to represent the magnetic field vector at each point on the grid. Notice that there is a vortex around the point $r=2,z=0$. The magnetic field is thus generated by a current flowing on a ring centered around the z axis.

In terms of the usual physical constants [3], we have $\lambda_0^{-1}=4e^2$, $\lambda_1=F_\pi^2/8$ and $\lambda_2^{-1}=8a^2$ where we use a instead of the traditional e for the Skyrme coefficient to avoid confusion with the electric charge.

In our units, $c=\hbar=1$, we have $e=(4\pi\alpha)^{1/2}$ where $\alpha=1/137$ is the fine structure constant. Choosing $F_\pi=186$ MeV, we can find the value for a by requiring that the energy of the neutron $M_n=939$ MeV matches the energy of the Skyrmion:

$$M_n=\frac{F_\pi}{8a}E\left(\frac{2a^2}{e^2},1,1\right).$$

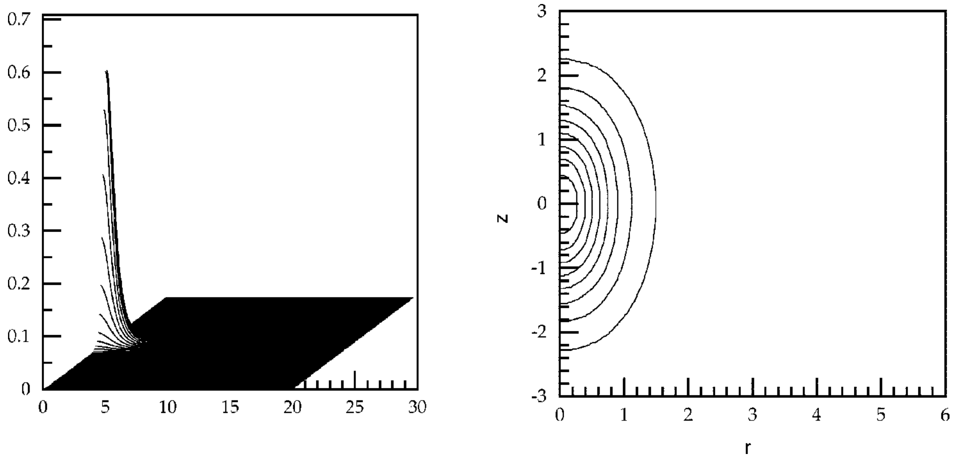

FIG. 2. (a) Energy density for the gauged Skyrmion in the (r,z) plane $(\lambda_0=\lambda_1=\lambda_2=1)$. (b) Energy density level curve for the gauged Skyrmion $(\lambda_0=\lambda_1=\lambda_2=1)$.

In Fig. 1(a), we can read the value of $E(\lambda_0,1,1)$ (given in units of $24\pi^2$ MeV) with $\lambda_0=2a^2/e^2$. We now have to find the value of λ_0 for which $24\pi^2E(\lambda_0,1,1)=2(2\lambda_0)^{1/2}eM_n/F_\pi$. The intersection between the curve $(eM_n/6\pi^2F_\pi)(2\lambda_0)^{1/2}$ and the curve $E(\lambda_0,1,1)$ in Fig. 1(a) is located in the region where the energy is virtually equal to the asymptotic value $E(\lambda_0,1,1)=1.232$. This means that $a\approx3\pi^2 1.232F_\pi/M_n\approx7.2$ and that $\lambda_0\approx1138$. We can thus conclude that the effective impact of the Maxwell term we have added to the Skyrme model is relatively small.

This justifies the procedure used in [5] where the Skyrmion was coupled with an external magnetic field of a magnetic monopole. Indeed, as the Maxwell field generated by a Skyrmion is very small (for the parameters fitting the actual mass of the nucleons) the external field is much larger than the Skyrmion's magnetic field.

It would be interesting to find the differences between the electromagnetic quantities obtained from the ungauged model, as in [3], and our U(1) gauged model. We are not able to compute the solutions of the U(1) gauged model for the physical value of the parameter λ_0 as this is too large, but as we now know that the influence of the gauge field is very small, we can compute the latter *perturbatively* around the (ungauged) Hedgehog as an induced field. This enables the evalution of the energy correction and the induced magnetic moment. This perturbative analysis will be be carried out in Sec. IV.

It can also be concluded that if the U(1) gauged Skyrme model were quantized as in [3] (by quantizing the zero modes corresponding to the global gauge transformation) but taking into account the electromagnetic field generated classically by the Skyrmion, the result would not differ very markedly from what was obtained in [3].

B. $A_0\neq0$: charged U(1) Skyrmion

We can now look for solutions with a non-zero electric charge by requiring that the field b in our ansatz (29) does not vanish. To do this we follow the same procedure as Julia and Zee [9] and require that the electric field be asymptotically of the form $A_0=V_0+q/(r^2+z^2)^{1/2}$ where V_0 and q are two constants. In practice, one computes solutions for different values of V_0 and evaluates q by computing the electric flux. We sought only those solutions, for which the electric flux equals 4π times the charge of the electron.

It is important to realize that in this case, Eqs. (34) are obtained after minimizing the action and thus they do not minimize the Hamiltonian (30).

In our units, the charge of the electron is 0.303. In Fig. 4 we show the energy as a function of λ_0, as well as V_0 as a function of λ_0, so that $q=0.303$.

One can see that, for a fixed value of λ_0, the energy of the *charged gauged* Skyrmion is smaller than the energy of the *ungauged* Skyrmion when $\lambda_0<7$ but it is always larger than the energy of the *uncharged gauged* Skyrmion. If the energies of the electrically charged and uncharged gauged Skyrmions were interpreted as the the masses of the proton and the neutron m_P and m_N, then on this purely classical level we would have to conclude that $(m_P-m_N)>0$ which is not correct. This is expected on the basis of its analogy with the dyon [9]. Clearly, to calculate this mass difference correctly one would have to perform the collective coordinate quantization as in Ref. [3], which we do not do here.

The energy of the charged Skyrmion increases with λ_0. It is unfortunately very difficult to carry out the numerical computations accurately when λ_0 is very large.

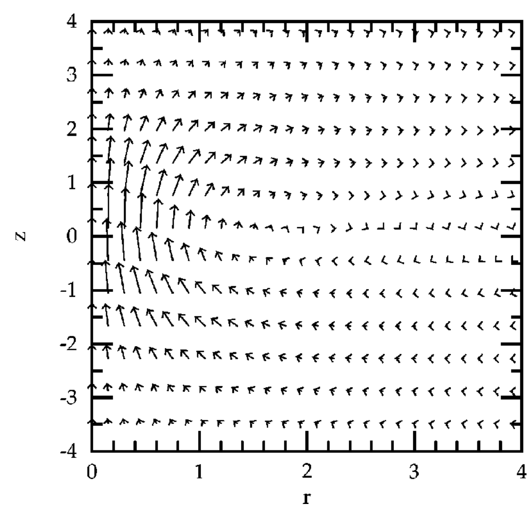

FIG. 3. Magnetic field of the gauged Skyrmion $(\lambda_0=\lambda_1=\lambda_2=1)$.

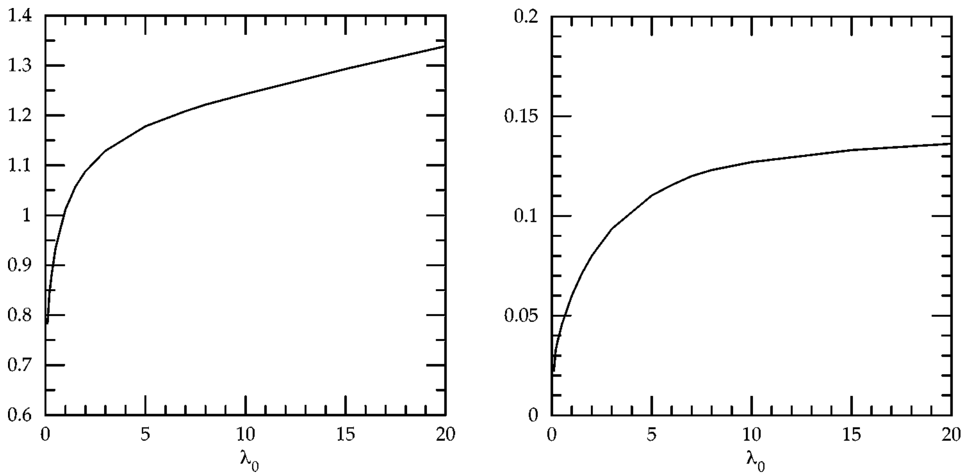

FIG. 4. (a) Energy of the charged Skyrmion. (b) V_0 as a function of λ_0.

At this stage, it is worth saying a few words about the numerical methods we have used. To compute the static solutions, we have employed a relaxation method using finite differences on a regular grid $(dr=dz)$. This discretization method is similar to the one employed in the numerical computation of the solutions of Skyrme models in $2+1$ and $3+1$ dimensions [16–18]. To compute the electrically charged solutions, we have imposed the boundary condition $b(\infty,\infty)=V_0$ for different V_0 and using a dichotomic method, we have determined the values that give a solution with the same electric flux as the proton. Most of the simulations where done on 200×400 or 300×600 grids. By computing the same solution for various lattice sizes, we have empirically obtained the following relation for the expression of the energy of a solution: $E=E_0+Kdr^2$ where E_0 is the exact solution, $dr=dz$ the lattice spacing and K is a constant which depends on λ_0 but takes values between 1 and 0.5. We see thus that the energies we have obtained are accurate to within one or one half of a percent. This inaccuracy in the value of the energy is comparable to that of many other similar works on 2 dimensional systems [17,18], and though it might look large, it does not affect any of the conclusions we have drawn.

IV. PERTURBATION AROUND THE HEDGEHOG

We have seen from the work of Sec. III A above that the energy of the U(1) gauged Skyrmion for the physical values of the parameters, namely of the pion decay constant and the U(1) coupling, does not differ significantly from that of the ungauged Hedgehog. It is therefore justified, for these values of the parameters, to treat the U(1) field as a perturbation to the Hedgehog in the same way as Klinhamer and Manton [14] treat the U(1) field as a perturbation to the SU(2) sphaleron. We will then be able to compute the magnetic moment of the neutron, as well as the (small) deviation of its mass from that of the Hedgehog.

The equations for the fields (ϕ^a,A_μ) are derived from the Lagrangian (1). The equation for A_μ will be of the form

$$\lambda_0\partial_\nu F_{\mu\nu}=j_\mu. \tag{35}$$

The method consists of setting the gauge field A_μ to zero in the current j_μ in Eq. (35) (and in the equation for ϕ^a) and calculating the resulting induced electromagnetic field A_μ. The gauge field computed this way can then be interpeted as the U(1) field generated by the (ungauged) Hedgehog Skyrme field. With this perturbative procedure, it is possible to calculate the induced static magnetic potential A_i $(i=1,2,3)$, but not the static electric potential A_0, which in this scheme vanishes and can only be calculated nonperturbatively. The reason simply is that restricting to the use of the static Hedgehog, the zeroth component of the current j_0 at $A_\mu=0$ vanishes, resulting in turn in vanishing induced potential A_0 according to Eq. (35).

As a consequence the electric field will be identically zero, which implies that we can derive the equation from the static Hamiltonian rather than from the Lagrangian. Notice also that we could try to compute perturbatively a solution for the electrically charges skyrmion by keeping in j_0 the terms proportional to A_0, instead of setting A_0 to 0, and impose the condition that the electric field is assymptoticaly like that of the proton. This perturbation method would not make much sense though as one would expect the electric field to be quite large close to the skyrmion.

The relevant energy functional is Eq. (23), and the resulting equation arising from the variation of the gauge field A_i is

$$\lambda_0\partial_j F_{ij}=j_i \tag{36}$$

$$j_i=-\varepsilon^{\alpha\beta}\bigg(\frac{1}{2}\lambda_1\phi^\beta D_i\phi^\alpha+\lambda_2[(\partial_j|\phi_\gamma|^2)\times D_i\phi^\alpha D_j\phi^\beta+2\phi^\beta D_{[i}\phi^\alpha D_{j]}\phi^A D_j\phi^A]\bigg). \tag{37}$$

We are concerned here with the case where $A_i=0$ Eq. (37) and the chiral field $\phi^a=(\phi^\alpha,\phi^3,\phi^4)$ in Eq. (37) describes the Hedgehog, i.e.,

$$\phi^\alpha=\sin F(R)\hat{x}^\alpha,\quad \phi^3=\sin F(R)\hat{x}^3,\quad \phi^4=\cos F(R), \tag{38}$$

where now $\hat{x}^a = x^a/R$, with $R=\sqrt{r^2+z^2}$. By virtue of Eq. (36) the current (37), given by Eq. (38) and $A_i=0$, will now induce a (small) U(1) field A_i, with curvature $\mathrm{F}_{ij}=\partial_i \mathrm{A}_j - \partial_j \mathrm{A}_i$.

The shift in the energy of the Hedgehog due to the induced U(1) field A_i is

$$\Delta E = \int d^3x (\lambda_0 \mathrm{F}_{ij}\mathrm{F}_{ij} + 4\mathrm{A}_i \mathrm{j}_i), \tag{39}$$

in which $\mathrm{j}_i = j_i(0)$ is the current (37) for $A_i=0$. [Note that all quantities evaluated at $A_i=0$ are denoted by Roman script, e.g., $\mathrm{j}_i=j_i(0)$, as well as the induced connection and curvature A_i and F_{ij}.] When Eq. (36) is satified for the induced U(1) field,

$$\Delta E = -\lambda_0 \int d^3x \mathrm{F}_{ij}\mathrm{F}_{ij} \tag{40}$$

$$= 2\int d^3x \mathrm{A}_i \mathrm{j}_i, \tag{41}$$

which is, as expected, a strictly negative quantity.

The current $\mathrm{j}_i = (\mathrm{j}_\alpha, \mathrm{j}_3)$ in Eqs. (39) and (41) is given, for the Hedghog field configuration (38), by

$$\mathrm{j}_\alpha = \frac{\sin^2 F}{R}\left(\frac{\lambda_1}{2} + 2\lambda_2\left(F'^2 + \frac{\sin^2 F}{R^2}\right)\right)\varepsilon_{\alpha\beta}\hat{x}_\beta \tag{42}$$

$$\mathrm{j}_3 = 0. \tag{43}$$

We now note that $\partial_i \mathrm{j}_i = 0$, which means that Eq. (36), for $A_i=0$, takes the following form in terms of the induced U(1) connection $\mathrm{A}_i = (\mathrm{A}_\alpha, 0)$:

$$\lambda_0 \Delta \mathrm{A}_\alpha = -\mathrm{j}_\alpha. \tag{44}$$

The solution is well known and can be expressed, using the obvious notation $\mathrm{j}_\alpha(\mathbf{x}) = \mathrm{j}(R)\varepsilon_{\alpha\beta}\hat{x}_\beta$ in terms of Eq. (42), as

$$\mathrm{A}_\alpha(\mathbf{x}) = -\frac{1}{4\pi\lambda_0}\varepsilon_{\alpha\beta}\int \frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathrm{j}(R')\hat{x}_\beta d\mathbf{x}', \tag{45}$$

with [19]

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{R_<^l}{R_>^{l+1}}\overline{Y_m^{(l)}}(\theta',\phi')Y_m^{(l)}(\theta,\phi).$$

After performing the angular integrations we have

$$\mathrm{A}_\alpha(\mathbf{x}) = -I(R)\varepsilon_{\alpha\beta}\hat{x}_\beta, \tag{46}$$

with $I_{(\not{})}(R)$ given by the integral

$$I(R) = \frac{1}{3\lambda_0}\left(\int_0^R \frac{R'}{R^2}\mathrm{j}(R')R'^2 dR' + \int_R^\infty \frac{R}{R'^2}\mathrm{j}(R')R'^2 dR'\right). \tag{47}$$

Finally, in the $R \gg 1$ region of interest, the induced U(1) potential is

$$\mathrm{A}_\alpha(\mathbf{x}) = -\frac{\hat{I}}{R^2}\varepsilon_{\alpha\beta}\hat{x}_\beta, \quad \text{with} \quad \hat{I} = \frac{1}{3\lambda_0}\int_0^\infty s^3 \mathrm{j}(s)ds, \tag{48}$$

to be evaluated numerically using the numerically constructed hedgehog profile function $F(x)$ (38).

Comparing Eq. (48) with the usual Maxwell potential of a magnetic dipole $\boldsymbol{\mu}$

$$\mathbf{A}(\mathbf{x}) = \frac{\boldsymbol{\mu}\times\mathbf{x}}{4\pi R^3},$$

we find that $\boldsymbol{\mu} = (0,0,\mu)$ is

$$\mu = 4\pi\hat{I}. \tag{49}$$

We can evaluate the magnetic moment (49) and the energy correction (41) induced by the electromagnetic field by evaluating the integral (47) and (48) numerically. If we take the experimental values $F_\pi = 186$ MeV and $a=7.2$ we obtain

$$\mu = 0.01393 \ \text{fm} = 0.43 \ \text{nm}, \tag{50}$$

$$\Delta E = -0.1 \ \text{keV}. \tag{51}$$

The experimental value for the magnetic moment of the proton and the neutron are respectively $\mu_p = 0.0902$ fm $=2.79$ nm and $\mu_n = 0.0617$ fm$=-1.91$ nm. If, on the other hand, we take the values of the parameters derived in [3], $F_\pi = 129$ MeV and $a=5.45$, we have

$$\mu = 0.0468 \ \text{fm} = 1.449 \ \text{nm}, \tag{52}$$

$$\Delta E = -0.32 \ \text{keV}. \tag{53}$$

The magnetic moment of a particle is strictly speaking a quantum property and it should be computed by quantizing the SU(2) gauge degree of freedom as in [3]. Nevertheless, we see that if we take the parameters derived in [3] the classical magnetic moment is of the correct order of magnitude. The sign is of course undetermined as the classical magnetic moment is a vector. We can thus conclude that our model offers a reasonable classical description of nucleons and affords a method for computing the electromagnetic field generated by the Skyrmion, classically. It is quite surprising to see that a quantum property like the magnetic moment can be reasonably predicted by a purely classical procedure.

V. SUMMARY AND DISCUSSION

We have shown that the SU(2) Skyrme model gauged with U(1) has two types of finite energy static solutions, electrically uncharged and charged respectively. Both of these solutions are axially symmetric and carry no magnetic charge but support a magnetic field shaped like a torus centred around the axis of symmetry, albeit resulting in zero magnetic flux. The uncharged solutions, like the ungauged Skyrmion, have a topological lower bound. The electrically charged solutions are the analogues of the Julia-Zee dyons [9] of the Georgi-Glashow model.

Concerning the stability of the electrically neutral solution, which is expected to be stable by virtue of the lower bound on the energy, we have not made any quantitative effort to test it. We expect however that the solitons of this gauged Skyrme model are stable, or that at least they have stable branches for all values of the parameters in the model. This expectation is based on our knowledge of the corresponding situation when the Skyrme model is gauged instead with SO(3) [20,21], in which case the equations arising from the imposition of spherical symmetry were one dimensional, and hence technically much more amenable to the numerical integration. In that case it was found that in addition to stable branches of solutions, there were also some unstable branches bifurcating from the former, the important matter being that there were indeed stable branches of solutions, characterized by the (ranges of the) parameters of the model. It would be very interesting to carry out the analysis corresponding to that of [20,21], for the considerably more complex case of the axially symmetric equations at hand. This however is technically beyond the scope of the present work.

The energies of the gauged uncharged Skyrmions are smaller than the energy of the usual ungauged Skyrmion. When the gauge coupling $1/\lambda_0^{1/2}$ goes to 0, the uncharged gauged Skyrmion tends to the ungauged Skyrmion. We also note that the energy of the electrically charged Skyrmion is higher than the uncharged one, just as the mass of the dyon is higher than that of the monopole of the Georgi-Glashow model.

Perhaps the most interesting physical result of the present work is that when parameters in the model are fitted to reproduce physical quantities, it turns out that the effect of the Maxwell term in the Skyrme Lagrangian is very small. This is because for the physical value of the constant $\lambda_0 = 1138$, the energy of the gauged uncharged Skyrmion differs little from that of the ungauged Skyrmion, as seen from Fig. 1(b). The gauged Skyrmion field itself is thus nearly radially symmetric (though the gauge field is not).

Having found that the influence of the electromagnetic field on the Skyrmion is small, we were pointed in the direction of treating the magnetic potential as an induced field perturbatively around the (ungauged) Hedgehog. We have been able thus, to compute the classical magnetic moment of the (uncharged) Skyrmion of *unit* baryon charge, namely of the neutron. The result is that the classical magnetic moment of the Skyrmion matches surprisingly well to the experimental values of the magnetic moments of the nucleons.

ACKNOWLEDGMENTS

We are grateful to Y. Brihaye and B. Kleihaus for enlightening discussions. We are particularly indebted to J. Kunz for suggesting to us the perturbative analysis of Sec. IV. This work was supported under projects BCA 96/024 of Forbairt/British Council, as well as IC/99/070 and IC/98/035 of Enterprise–Ireland.

[1] T. H. R. Skyrme, Proc. R. Soc. London **A260**, 127 (1961); Nucl. Phys. **31**, 556 (1962).
[2] E. Witten, Nucl. Phys. **B223**, 422 (1983).
[3] G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).
[4] C. Barnes, W. K. Baskerville, and N. Turok, Phys. Lett. B **411**, 180 (1997).
[5] C. G. Callan, Jr. and E. Witten, Nucl. Phys. **B239**, 161 (1984).
[6] J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47**, 986 (1981).
[7] E. D'Hoker and E. Farhi, Nucl. Phys. **B241**, 109 (1984).
[8] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974); A. M. Polyakov, Pis'ma Zh. Éksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].
[9] B. Julia and A. Zee, Phys. Rev. D **11**, 2227 (1975).
[10] D. H. Tchrakian, Lett. Math. Phys. **40**, 891 (1997).
[11] B. J. Schroers, Phys. Lett. B **356**, 291 (1995); J. Gladikowski, B. M. A. G. Piette, and B. J. Schroers, Phys. Rev. D **53**, 844 (1995).
[12] L. D. Fadde'ev, Lett. Math. Phys. **1**, 289 (1976).
[13] K. Arthur and D. H. Tchrakian, Phys. Lett. B **378**, 187 (1996); Y. Brihaye and D. H. Tchrakian, Nonlinearity **11**, 891 (1998).
[14] F. Klinkhamer and N. S. Manton, Phys. Rev. D **30**, 2212 (1984).
[15] B. Kleihaus, J. Kunz, and Y. Brihaye, Phys. Lett. B **273**, 100 (1991); Phys. Rev. D **46**, 3587 (1992).
[16] R. A. Battye and P. M. Sutcliffe, Phys. Rev. Lett. **79**, 363 (1997).
[17] B. Piette and W. J. Zakrzewski, J. Comput. Phys. **145**, 359 (1998).
[18] J. Gladikowski and M. Hellmund, Phys. Rev. D **56**, 5194 (1997).
[19] See, for example, J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[20] Y. Brihaye and D. H. Tchrakian, Nonlinearity **11**, 891 (1998).
[21] Y. Brihaye, B. Kleihaus, and D. H. Tchrakian, J. Math. Phys. **40**, 1136 (1999).