Glueballs in a Hamiltonian light-front approach to pure-glue QCD

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We calculate a renormalized Hamiltonian for pure-glue QCD and diagonalize it. The renormalization procedure is designed to produce a Hamiltonian that will yield physical states that rapidly converge in an expansion in free-particle Fock-space sectors. To make this possible, we use light-front field theory to isolate vacuum effects, and we place a smooth cutoff on the Hamiltonian to force its free-state matrix elements to quickly decrease as the difference of the free masses of the states increases. The cutoff violates a number of physical principles of light-front pure-glue QCD, including Lorentz covariance and gauge covariance. This means that the operators in the Hamiltonian are not required to respect these physical principles. However, by requiring the Hamiltonian to produce cutoff-independent physical quantities and by requiring it to respect the unviolated physical principles of pure-glue QCD, we are able to derive recursion relations that define the Hamiltonian to all orders in perturbation theory in terms of the running coupling. We approximate all physical states as two-gluon states (thus they are relatively simple single-glueball states), and use our recursion relations to calculate to second order the part of the Hamiltonian that is required to compute the spectrum. We diagonalize the Hamiltonian using basis-function expansions for the gluons' color, spin, and momentum degrees of freedom. We examine the sensitivity of our results to the cutoff and use them to analyze the nonperturbative scale dependence of the coupling. We investigate the effect of the dynamical rotational symmetry of light-front field theory on the rotational degeneracies of the spectrum and compare the spectrum to recent results from other approaches. Finally, we examine our wave functions and analyze the various sources of error in our calculation.

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I. INTRODUCTION

A solution to a quantum field theory that is close to our physical intuition is possible if we can develop a formalism in which the physical states of the theory rapidly converge in an expansion in free-particle Fock-space sectors (free sectors). Such an expansion is unlikely to be possible in an equal-time approach to many of the more interesting theories, such as QCD. This is because in equal-time approaches to these theories, the physical states must be built on top of a complicated vacuum unless the volume of space is severely limited. For this reason, we work in light-front field theory (LFFT). In LFFT, it is possible to force the vacuum to be empty by removing from the theory all particles that have zero longitudinal momentum.[1] Any physical effects of these particles must be incorporated into the operators of the theory in order to obtain correct physical quantities.

In LFFT, the Hamiltonian is trivially related to the invariant-mass operator[2] (IMO), and it is more natural to work with the IMO because it is manifestly boost invariant. If the IMO satisfies three conditions in the basis of free-particle Fock-space states (free states), then its eigenstates will rapidly converge in an expansion in free sectors. First, the diagonal matrix elements of the IMO must be dominated by the free part of the IMO. Second, the off-diagonal matrix elements of the IMO must quickly decrease as the difference of the free masses of the states increases. If the IMO satisfies these first two conditions, then each of its eigenstates will be dominated by free-state components with free masses that are close to the mass of the eigenstate. The third condition on the IMO is that the free mass of a free state must quickly increase as the number of particles in the state increases. If the IMO satisfies all three conditions, then the number of particles in a free-state component that dominates an eigenstate will be limited from above. This means that the IMO's eigenstates will rapidly converge in an expansion in free sectors.[3]

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[1]This is because there are no negative longitudinal momenta and momentum conservation requires the three-momenta of the constituents of the vacuum to sum to zero.

[2]The invariant-mass operator is given by the square of the momentum operator: $\mathcal{P}^\mu \mathcal{P}_\mu = \mathcal{M}^2$. See Appendix A for more details.

[3]There are three subtleties here. The first subtlety is that the first and third conditions on the IMO will not be satisfied for those free states in which many of the particles have negligible center-of-mass transverse momentum and little or no mass. However, the contributions of these free states to the physical states in which we are interested are typically suppressed. For example, in QCD these free states have very large widths in transverse position space and are thus highly suppressed by confinement. In QED, the particles with negligible center-of-mass transverse momentum and no mass are long-transverse-wavelength photons. These photons decouple from the physical states in which we are typically interested, e.g., charge-singlet states such as hydrogen and positronium. Thus the contributions to these physical states from the free states containing these photons are suppressed. The second subtlety is that exactly how quickly the IMO's off-diagonal matrix elements must decrease and the free mass of a free state must increase are not known. We assume that the rates that we are able to achieve are sufficient. This can be verified by diagonalizing the IMO and examining the rate of convergence of the free-sector expansion of its eigenstates. The third subtlety is that the coefficients of the expansion for highly excited eigenstates may grow for a number of free sectors and then peak before diminishing and becoming rapidly convergent.

To satisfy the first condition on the IMO, we assume that we can derive the IMO in perturbation theory. If this is true, then the couplings are small and the diagonal matrix elements of the IMO are dominated by the free part of the IMO. To satisfy the second condition, we place a smooth cutoff on the IMO to force its matrix elements to quickly decrease as the difference of the free masses of the states increases. Once we have removed the particles with zero longitudinal momentum from the theory, it is reasonable to expect that the third and final condition on the IMO will be satisfied automatically. This is because the free-particle dispersion relation of LFFT should force the free mass of a free state to quickly increase as the number of particles in the state increases (see Appendix A of Ref. [1]).

By suppressing the matrix elements of the IMO that have large changes in free mass, the cutoff regulates the ultraviolet divergences of the theory. Unfortunately, it also violates a number of physical principles of LFFT, including Lorentz covariance and gauge covariance. This means that the operators in the IMO are not required to respect these physical principles, and renormalization is *not* simply a matter of adjusting a few canonical parameters. The simplest way to systematically determine the IMO in this case is in perturbation theory. In order for a perturbative computation of the IMO to be strictly valid, the theory must be asymptotically free.[4] If this is the case, then by requiring the IMO to produce cutoff-independent physical quantities and by requiring it to respect the unviolated physical principles of the theory,[5] we can derive recursion relations that define the IMO to all orders in perturbation theory in terms of the fundamental parameters of the field theory. If our cutoff is large enough, then the couplings will be small and the perturbative approximation to the IMO may work well.

The physical principles that we use to determine the IMO form a subset of the full set of physical principles of light-front field theory. This raises the question of how the remaining principles, which are violated by our cutoff, are restored in physical quantities. Since the IMO is uniquely determined by the principles that we use, the remaining principles must be automatically respected by physical quantities derived from our IMO, at least perturbatively. If they are not, then they contradict the principles that we use and no consistent theory can be built upon the complete set of principles. The reason that this process is possible is that there are redundancies among the various physical principles.

It is possible to compute operators other than the IMO in our approach. Although we compute operators perturbatively, we can use these operators to compute nonperturbative quantities. For example, the spectrum can be computed by diagonalizing the IMO (see Sec. VI). However, there are drawbacks to computing operators perturbatively. It is possible that there are intrinsically nonperturbative effects in the theory that require nonperturbative renormalization. Any such effects are neglected in this approach. Another problem is that perturbative renormalization makes nonperturbative physical quantities somewhat cutoff dependent.

In general, field theories have an infinite number of degrees of freedom. However, since our IMO will cause the physical states of the theory to rapidly converge in an expansion in free sectors, we can truncate this expansion. This means that approximate computations of physical quantities will require only a finite number of finite-body matrix elements of operators. In addition, since we assume that we can compute these matrix elements perturbatively, in these computations we only need to consider intermediate states that can be connected perturbatively to the states in the matrix elements. This allows perturbation theory to naturally limit the number of intermediate-state particles that we must consider.

Since we do not truncate the space of intermediate states that can appear when we perturbatively calculate matrix elements of operators, our approach does not use a Tamm-Dancoff truncation[6] [2–4]. (A Tamm-Dancoff truncation arbitrarily limits the number of particles that can appear in any state in the theory. Such a truncation drastically complicates renormalization, i.e., the calculation of the matrix elements of operators.) We also do not completely eliminate any interactions, such as those that change particle number. These strengths of our approach allow us to better describe physical theories. However, the truncation of the free-sector expansion of physical states has drawbacks that are similar to those of perturbative renormalization. It neglects any physical effects that require an infinite number of particles and contributes to the cutoff dependence of nonperturbative physical quantities.

The accuracy of our results and the strength of the cutoff dependence of our nonperturbative physical quantities are determined by the order in perturbation theory to which we calculate the operators of the theory and the number of free sectors that we keep in the expansion of physical states. If we use a cutoff that is too small, then the couplings of the theory will be large, and it will be necessary for us to keep many terms in the expansion of the operators. If we use a cutoff that is too large, then the free-sector expansion of the physical states will converge slowly, and it will be necessary for us to keep many sectors in the states. We assume that if the order of perturbation theory and the number of free sectors are manageable, then there is a range of cutoff values for

[4]Our method may work even if the theory is not asymptotically free. For example, it works in QED because the scale at which the electron charge is large is astronomical.

[5]Some of the physical principles, such as cluster decomposition, are violated in a very specific manner and can still be used to restrict the form of the IMO. However, the restriction in this situation is always weaker than it would have been had there been no violation of the principle.

[6]The only degrees of freedom that we remove from the full theory are the particles with zero longitudinal momentum. We should be able to replace the physical effects of these particles with interactions without compromising the validity of the theory. This is because these particles are vacuum effects or have infinite kinetic energies (or both), and thus are not observable in the laboratory as particles.

which the approximations work well and physical quantities are relatively accurate and cutoff independent.

As we mentioned, we remove from the theory all particles with zero longitudinal momentum. We should replace their physical effects with interactions. However, due to the limitations of our method, we can reproduce only those effects of these particles that can be derived with perturbative renormalization and require only a small number of particles.

There are a number of approaches that are similar to ours [5–12], and some of these methods have been used to calculate the physical states of QCD [12,13]. These calculations are based on nonrelativistic approximations and use sharp step-function cutoffs. Nonrelativistic approximations drastically simplify the diagonalization of the IMO, but are insufficient for states containing light quarks or gluons. Sharp cutoffs prevent the complete cancellation of the infrared divergences that appear in light-front gauge theories.[7]

Our approach is completely relativistic and uses smooth cutoffs to ensure the complete cancellation of the light-front infrared divergences. It is largely based on the renormalization methods of Perry [14], Perry and Wilson [15], Wilson [16], and Glazek and Wilson [5], as well as the Hamiltonian-diagonalization methods of Wegner [17]. In Ref. [1], we developed and tested our method in massless ϕ^3 theory in six dimensions. In Ref. [18], Kylin, Allen, and Perry extended our method to include particle masses. In this paper, we extend our method to pure-glue QCD. This theory is simpler than full QCD due to the reduced number of vertices and the absence of quark masses. For this application, we derive the recursion relations that determine the IMO to all orders in perturbation theory in the running coupling. We approximate all physical states as two-gluon states and use the recursion relations to compute to second order the part of the IMO that is required to compute the spectrum. We diagonalize the IMO in a basis-state expansion and analyze the results.

This paper is organized as follows. Appendix A contains our conventions for light-front pure-glue QCD; so the reader may wish to examine it first. In Sec. II we present our method for computing the free-state matrix elements of the IMO. Some of the details of the method are given in Appendix B. In Section III we define a basis for the expansion of physical states. In Sec. IV we use the recursion relations for the IMO to compute its two-gluon to two-gluon matrix element to second order in the running coupling. This is the only free-state matrix element that we need to solve the eigenvalue equation. In Sec. V we compute the matrix elements of the IMO in the basis that we have defined, in terms of integrals that must be evaluated numerically. Some of the details of these evaluations and other technical issues appear in Appendix C. In Sec. VI we derive our results and analyze the sources of error in the calculation. Finally, in Sec. VII we conclude with a summary and a discussion of the direction of future work. We do not give an introduction to light-front field theory or review the various research efforts in this area. The interested reader should consult the recent reviews and introductions for these purposes [19].

II. THE METHOD FOR COMPUTING FREE-STATE MATRIX ELEMENTS OF THE INVARIANT-MASS OPERATOR

This section summarizes the extension of the results of Secs. II–IV and Appendix D of Ref. [1] to the case of pure-glue QCD. We begin by defining our cutoff, which regulates the free-state matrix elements of the IMO by exponentially suppressing large changes in free mass. We then proceed to place a number of restrictions on the IMO. First we force the IMO at a given cutoff to be unitarily equivalent to itself at a higher cutoff. This implies that the IMO is unitarily equivalent to itself at an infinite cutoff, and will therefore yield cutoff-independent physical quantities. From the statement of unitary equivalence, we develop a perturbative series that relates the interactions at two different cutoffs. We then proceed to use physical principles to restrict the form of the IMO. We require it to conserve momentum and to be invariant under boosts and rotations about the three-axis. Although our cutoff violates exact transverse locality, we are able to require the IMO to respect an approximate transverse locality. In practice this means that the IMO's matrix elements are analytic functions of transverse momenta. The cutoff also violates cluster decomposition, but we show that the implications of this violation are simple enough that we can still use this principle to restrict the form of the IMO.

In order to represent pure-glue QCD, the IMO must become the free IMO in the noninteracting limit, and it must be a function only of the cutoff and the coupling. The final physical restriction on the IMO is that it must reproduce the perturbative scattering amplitudes of pure-glue QCD. This restriction specifies the form of the first-order interaction and part of the second-order interaction.

We use the restrictions from physical principles and the perturbative series that relates the interactions at two different cutoffs to derive the recursion relations that determine the IMO. A crucial step in this process is the removal of the coupling from the perturbative series. This allows us to separate the cutoff dependences of the operators in the interaction from the cutoff dependences of their couplings. Most of the details of this derivation are relegated to Appendix B.

A. The cutoff

Our goal is to derive recursion relations that uniquely determine the IMO to all orders in perturbation theory in the running coupling. The IMO is a function of the cutoff, and can be split into the canonical free IMO and an interaction (see Appendix A for our light-front pure-glue QCD conventions):

$$\mathcal{M}^2(\Lambda) = \mathcal{M}^2_{\text{free}} + \mathcal{M}^2_{\text{int}}(\Lambda). \tag{1}$$

The cutoff is implemented on the matrix elements of the IMO:

[7]These appear due to the exchange of massless gauge particles with arbitrarily small longitudinal momentum.

$$\langle F|\mathcal{M}^2(\Lambda)|I\rangle = \langle F|\mathcal{M}^2_{\rm free}|I\rangle + \langle F|\mathcal{M}^2_{\rm int}(\Lambda)|I\rangle$$
$$= M_F^2\langle F|I\rangle + e^{-\Delta_{FI}^2/\Lambda^4}\langle F|V(\Lambda)|I\rangle, \quad (2)$$

where $|F\rangle$ and $|I\rangle$ are eigenstates of the free IMO with eigenvalues M_F^2 and M_I^2, and Δ_{FI} is the difference of these eigenvalues:

$$\Delta_{FI} = M_F^2 - M_I^2. \quad (3)$$

$V(\Lambda)$ is the interaction with the Gaussian cutoff factor removed, and we refer to it as the ''reduced interaction.'' To determine the IMO, we must determine the reduced interaction.

We will see that $\langle F|V(\Lambda)|I\rangle$ does not grow exponentially as Δ_{FI}^2 gets large; so the exponential in Eq. (2) forces the off-diagonal matrix elements of the IMO to rapidly diminish as Δ_{FI}^2 grows. This satisfies the second of our three conditions on the IMO and regulates it.

B. The restriction to produce cutoff-independent physical quantities

Our cutoff violates a number of physical principles of LFFT, including Lorentz covariance and gauge covariance.[8] This means that the operators in the IMO are not required to respect these physical principles. In addition, since there is no locality in the longitudinal direction in Hamiltonian LFFT,[9] these operators can contain arbitrary functions of longitudinal momenta. To uniquely determine the IMO in this case, we have to place some restrictions on it. The first restriction is that it must produce cutoff-independent physical quantities. To enforce this, we require $\mathcal{M}^2(\Lambda)$ to satisfy

$$\mathcal{M}^2(\Lambda) = U(\Lambda,\Lambda')\mathcal{M}^2(\Lambda')U^\dagger(\Lambda,\Lambda'), \quad (4)$$

where U is a unitary transformation that changes the IMO's cutoff and

$$\Lambda < \Lambda' < 2\Lambda. \quad (5)$$

We have placed an upper limit on Λ' because Eq. (4) is perturbatively valid only if Λ' is not too much larger than Λ [20]. Note that we are considering $\mathcal{M}^2(\Lambda)$ to be a function of its argument; i.e., $\mathcal{M}^2(\Lambda')$ has the same functional dependence on Λ' that $\mathcal{M}^2(\Lambda)$ has on Λ. In Ref. [1], we proved that Eq. (4) forces $\mathcal{M}^2(\Lambda)$ to produce cutoff-independent physical quantities.

[8]Our regulator breaks these symmetries because the mass of a free state is neither gauge-invariant nor rotationally invariant (except for rotations about the three-axis).

[9]That there is no longitudinal locality in Hamiltonian LFFT is evident from the fact that the longitudinal momentum of a free particle appears in the denominator of its dispersion relation, $(\vec{p}_\perp^2 + m^2)/p^+$.

The unitary transformation that we use is designed to alter the cutoff implemented in Eq. (2), and is a simplified version of a transformation introduced by Wegner [17], modified for implementation with the IMO. It is uniquely defined by a linear first-order differential equation:

$$\frac{dU(\Lambda,\Lambda')}{d(\Lambda^{-4})} = T(\Lambda)U(\Lambda,\Lambda'), \quad (6)$$

with one boundary condition:

$$U(\Lambda,\Lambda) = \mathbf{1}. \quad (7)$$

$U(\Lambda,\Lambda')$ is unitary as long as $T(\Lambda)$ is anti-Hermitian and linear [1]. We define

$$T(\Lambda) = [\mathcal{M}^2_{\rm free}, \mathcal{M}^2(\Lambda)], \quad (8)$$

which is anti-Hermitian and linear.

To solve for $\mathcal{M}^2(\Lambda)$ perturbatively, we need to turn Eq. (4) into a perturbative restriction on the reduced interaction $V(\Lambda)$. We outlined how to do this in Ref. [1], and here we simply state the results that we need. The perturbative version of Eq. (4) in terms of the reduced interaction is

$$V(\Lambda) - V(\Lambda') = \delta V, \quad (9)$$

where δV is the change to the reduced interaction and is a function of both Λ and Λ':

$$\langle F|\delta V|I\rangle = \frac{1}{2}\sum_K \langle F|V(\Lambda')|K\rangle\langle K|V(\Lambda')|I\rangle T_2^{(\Lambda,\Lambda')}(F,K,I)$$
$$+ \frac{1}{4}\sum_{K,L}\langle F|V(\Lambda')|K\rangle\langle K|V(\Lambda')|L\rangle$$
$$\times\langle L|V(\Lambda')|I\rangle T_3^{(\Lambda,\Lambda')}(F,K,L,I)$$
$$+ \mathcal{O}([V(\Lambda')]^4). \quad (10)$$

In this equation, the sums are over complete sets of free states and the cutoff functions are defined by

$$T_2^{(\Lambda,\Lambda')}(F,K,I) = \left(\frac{1}{\Delta_{FK}} - \frac{1}{\Delta_{KI}}\right)\left(e^{2\Lambda'^{-4}\Delta_{FK}\Delta_{KI}} - e^{2\Lambda^{-4}\Delta_{FK}\Delta_{KI}}\right) \quad (11)$$

and

$$T_3^{(\Lambda,\Lambda')}(F,K,L,I)=\left(\frac{1}{\Delta_{KL}}-\frac{1}{\Delta_{LI}}\right)\left(\frac{1}{\Delta_{KI}}-\frac{1}{\Delta_{FK}}\right)e^{2\Lambda'^{-4}\Delta_{KL}\Delta_{LI}}(e^{2\Lambda^{-4}\Delta_{FK}\Delta_{KI}}-e^{2\Lambda'^{-4}\Delta_{FK}\Delta_{KI}})$$
$$+\left(\frac{1}{\Delta_{KL}}-\frac{1}{\Delta_{LI}}\right)\frac{\Delta_{FK}+\Delta_{IK}}{\Delta_{KL}\Delta_{LI}+\Delta_{FK}\Delta_{KI}}(e^{2\Lambda'^{-4}(\Delta_{FK}\Delta_{KI}+\Delta_{KL}\Delta_{LI})}-e^{2\Lambda^{-4}(\Delta_{FK}\Delta_{KI}+\Delta_{KL}\Delta_{LI})})$$
$$+\left(\frac{1}{\Delta_{FK}}-\frac{1}{\Delta_{KL}}\right)\left(\frac{1}{\Delta_{LI}}-\frac{1}{\Delta_{FL}}\right)e^{2\Lambda'^{-4}\Delta_{FK}\Delta_{KL}}(e^{2\Lambda^{-4}\Delta_{FL}\Delta_{LI}}-e^{2\Lambda'^{-4}\Delta_{FL}\Delta_{LI}})$$
$$+\left(\frac{1}{\Delta_{FK}}-\frac{1}{\Delta_{KL}}\right)\frac{\Delta_{FL}+\Delta_{IL}}{\Delta_{FK}\Delta_{KL}+\Delta_{FL}\Delta_{LI}}(e^{2\Lambda'^{-4}(\Delta_{FK}\Delta_{KL}+\Delta_{FL}\Delta_{LI})}-e^{2\Lambda^{-4}(\Delta_{FK}\Delta_{KL}+\Delta_{FL}\Delta_{LI})}). \quad (12)$$

The above definitions for the cutoff functions assume that none of the Δ's that appear in the denominators is zero. In the event one of them is zero, the appropriate cutoff function is defined by

$$T_i^{(\Lambda,\Lambda')}(\Delta=0)=\lim_{\Delta\to 0} T_i^{(\Lambda,\Lambda')}(\Delta). \quad (13)$$

C. Restrictions from physical principles

Equation (9) is the first restriction on the IMO. To uniquely determine the IMO, we need to place additional restrictions on it, and we do this using the physical principles of the theory that are not violated by the cutoff. (See footnote 5.)

1. Symmetry principles

Any LFFT should exhibit manifest momentum conservation, boost covariance, and covariance under rotations about the three-axis. Our cutoff does not violate any of these principles; so we restrict the IMO to conserve momentum and to be invariant under boosts and rotations about the three-axis.

2. Transverse locality

Ideally, the IMO should be local in the transverse directions, and thus each of its matrix elements should be expressible as a finite series of powers of transverse momenta with expansion coefficients that are functions of longitudinal momenta. In our case, the cutoff suppresses interactions that have large transverse-momentum transfers and replaces them with interactions that have smaller transverse-momentum transfers. This is equivalent to suppressing interactions that occur over small transverse separations and replacing them with interactions that occur over larger transverse separations; so we do not expect our interactions to be perfectly transverse-local. Nonetheless, we expect that interactions in $\mathcal{M}^2(\Lambda)$ should appear local relative to transverse separations larger than Λ^{-1} or, equivalently, to transverse momenta less than Λ. This means that for transverse momenta less than Λ we should be able to approximate each matrix element of $\mathcal{M}^2(\Lambda)$ as a finite power series in $\vec{p}_\perp/\Lambda$. We enforce this by assuming that transverse locality is violated in the weakest manner possible, i.e., that any matrix element of the IMO can be expressed as an *infinite* series of powers of transverse momenta with an infinite radius of convergence. In other words, we assume that the matrix elements of the IMO are analytic functions of transverse momenta.

3. Cluster decomposition

Since the matrix elements of the IMO conserve momentum, they can be written as a sum of terms, with each term containing a unique product of momentum-conserving delta functions [21]. We require the IMO to satisfy approximate cluster decomposition [1]; i.e. when any of its matrix elements is written as an expansion in the possible products of momentum-conserving delta functions, the coefficient of any term in the expansion is restricted as follows. (If there is more than one possible set of spectators for a given product of momentum-conserving delta functions, then the coefficient has to be broken into a distinct part for each possible set, and these restrictions hold for each part separately.) It can depend on the cutoff, the quantum numbers of the interacting particles, and the total longitudinal momentum. It must be proportional to a quantum-number-conserving Kronecker delta for each discrete quantum number for each spectator. It can have no other dependence on the quantum numbers of spectators, and it cannot contain delta functions that fix momenta.

The reason that the IMO does not respect exact cluster decomposition and that the coefficients in the delta-function expansion can depend on the momenta of spectators (through a dependence on the total longitudinal momentum) is that our cutoff on free-mass differences violates cluster decomposition. To see this, note that the change in free mass for some process is given by $P^+(\Sigma_j p_j'^- - \Sigma_i p_i^-)$, where the p_i's are the momenta of the particles in the initial state, the p_j''s are the momenta of the particles in the final state, and P^+ is the total longitudinal momentum of each state. The minus momenta of any spectators cancel in this difference, but their longitudinal momenta still contribute to the overall factor of P^+.

4. Representation of the theory of interest

The preceding restrictions on $\mathcal{M}^2(\Lambda)$ are valid for any LFFT in more than two dimensions. In order to represent a particular theory, we must place additional restrictions on $\mathcal{M}^2(\Lambda)$.

We assume that we can compute $\mathcal{M}^2(\Lambda)$ perturbatively, which means that we can expand $V(\Lambda)$ in powers of the coupling at the scale Λ. Our cutoff has no effect in the noninteracting limit; so our IMO must reproduce free pure-glue QCD in this limit. According to Eq. (2), this means that $V(\Lambda)$ vanishes in the noninteracting limit.

In pure-glue QCD, the only fundamental parameter is the coupling; so we require the IMO to depend only on it and the scale. (For an example of the application of our method to a theory with more than one parameter, see Ref. [18].) In this case, the expansion of $V(\Lambda)$ takes the form

$$V(\Lambda)=\sum_{r=1}^{\infty} g_\Lambda^r V^{(r)}(\Lambda), \tag{14}$$

where g_Λ is the coupling at the scale Λ. We refer to $V^{(r)}(\Lambda)$ as the $\mathcal{O}(g_\Lambda^r)$ reduced interaction, although for convenience the coupling is factored out.

g_Λ is the correct fundamental parameter for pure-glue QCD if and only if its definition is consistent with the canonical definition of the coupling. The canonical definition is

$$g=[16\pi^3 p_1^+\delta^{(3)}(p_1-p_2-p_3)\langle\!\langle g_2g_3|v|g_1\rangle\!\rangle]^{-1}\times\langle g_2g_3|\mathcal{M}^2|g_1\rangle, \tag{15}$$

where v is the canonical interaction and $\langle\!\langle j|v|i\rangle\!\rangle$ denotes a modified matrix element of v. (See Appendix A for our light-front pure-glue QCD conventions.) The denominator removes all dependence on momentum, spin, and color in the canonical matrix element for gluon emission, and thus isolates the coupling. We define the coupling g_Λ by

$$\begin{aligned} g_\Lambda&=\{[16\pi^3 p_1^+\delta^{(3)}(p_1-p_2-p_3)\langle\!\langle g_2g_3|v|g_1\rangle\!\rangle]^{-1}\times\langle g_2g_3|\mathcal{M}^2(\Lambda)|g_1\rangle\}^{s_n=1;c_n=n}_{\vec p_{2\perp}=\vec p_{3\perp};p_2^+=p_3^+;\epsilon=0}\\ &=\{[16\pi^3 p_1^+\delta^{(3)}(p_1-p_2-p_3)\langle\!\langle g_2g_3|v|g_1\rangle\!\rangle]^{-1}\times\langle g_2g_3|V(\Lambda)|g_1\rangle\}^{s_n=1;c_n=n}_{\vec p_{2\perp}=\vec p_{3\perp};p_2^+=p_3^+;\epsilon=0}, \end{aligned} \tag{16}$$

where $n=1$, 2, 3, and ϵ is a longitudinal-momentum cutoff that we define below. Momentum conservation and boost invariance imply that the matrix element that defines the coupling can depend only on the transverse momentum of particle 2 in the center-of-mass frame and the ratio p_2^+/p_1^+. The restrictions $\vec p_{2\perp}=\vec p_{3\perp}$ and $p_2^+=p_3^+$ fix these quantities to be 0 and $\frac{1}{2}$, respectively.

Our definition of the coupling is consistent with the canonical definition because the conditions on the matrix elements in Eq. (16) have no effect on the right-hand side (RHS) of Eq. (15) and do not force $\langle g_2g_3|\mathcal{M}^2|g_1\rangle$ to vanish. According to Eq. (14), the IMO is *coupling coherent* [14,15,22] because the couplings of its noncanonical operators are functions only of the fundamental parameters of the theory and they vanish in the noninteracting limit.

We have assumed that the IMO obeys approximate transverse locality, which means that we can expand any matrix element $\langle F|V^{(r)}(\Lambda)|I\rangle$ in powers of transverse momenta. Each term in this expansion is either cutoff dependent or cutoff independent. We define $V^{(r)}_{\rm CD}(\Lambda)$ to be the cutoff-dependent part of $V^{(r)}(\Lambda)$, i.e., the part that produces the cutoff-dependent terms in transverse-momentum expansions of matrix elements of $V^{(r)}(\Lambda)$. We define $V^{(r)}_{\rm CI}$ to be the cutoff-independent part of $V^{(r)}(\Lambda)$, i.e., the part that produces the cutoff-independent terms in transverse-momentum expansions of matrix elements of $V^{(r)}(\Lambda)$. Then

$$V^{(r)}(\Lambda)=V^{(r)}_{\rm CD}(\Lambda)+V^{(r)}_{\rm CI}. \tag{17}$$

This separation is necessary because the procedures for computing $V^{(r)}_{\rm CD}(\Lambda)$ and $V^{(r)}_{\rm CI}$ differ.

We believe that if $\mathcal{M}^2(\Lambda)$ is to reproduce the perturbative scattering amplitudes of pure-glue QCD, then it is necessary and sufficient for the reduced interaction to satisfy the following conditions.[10] We prohibit $V^{(r)}(\Lambda)$ from having a three-point interaction unless r is odd, and we prohibit $V^{(r)}(\Lambda)$ from having a four-point interaction unless r is even. We require

$$\begin{aligned} V^{(1)}=&\mathcal{P}^+\frac{16\pi^3}{2!}\int D_1D_2D_3 a_2^\dagger a_3^\dagger a_1\delta^{(3)}(p_1-p_2-p_3)\times\langle\!\langle g_2g_3|v|g_1\rangle\!\rangle\\ &+\mathcal{P}^+\frac{16\pi^3}{2!}\int D_1D_2D_3 a_3^\dagger a_1a_2\delta^{(3)}(p_1+p_2-p_3)\times\langle\!\langle g_3|v|g_1g_2\rangle\!\rangle \end{aligned} \tag{18}$$

and

$$\begin{aligned} V^{(2)}_{\rm CI}=&\mathcal{P}^+\frac{16\pi^3}{2!2!}\int D_1D_2D_3D_4 a_3^\dagger a_4^\dagger a_1a_2\times\delta^{(3)}(p_1+p_2-p_3-p_4)\sum_{i=1}^4\theta^{(i)}_{1,2,;3,4}\langle\!\langle g_3g_4|v|g_1g_2\rangle\!\rangle_i\\ &+\mathcal{P}^+\frac{16\pi^3}{3!}\int D_1D_2D_3D_4 a_2^\dagger a_3^\dagger a_4^\dagger a_1\times\delta^{(3)}(p_1-p_2-p_3-p_4)\sum_{i=1}^4\theta^{(i)}_{1;2,3,4}\langle\!\langle g_2g_3g_4|v|g_1\rangle\!\rangle_i\\ &+\mathcal{P}^+\frac{16\pi^3}{3!}\int D_1D_2D_3D_4 a_4^\dagger a_1a_2a_3\times\delta^{(3)}(p_1+p_2+p_3-p_4)\sum_{i=1}^4\theta^{(i)}_{1,2,3;4}\langle\!\langle g_4|v|g_1g_2g_3\rangle\!\rangle_i, \end{aligned} \tag{19}$$

where

[10] Any longitudinal regulator that is consistent with the physical principles that we use to restrict the IMO is sufficient. It is not necessary to use our ϵ cutoff in order to reproduce the perturbative scattering amplitudes of pure-glue QCD.

$$\theta^{(1)}_{1,2;3,4}=1,$$

$$\theta^{(2)}_{1,2;3,4}=\theta(|p_1^+-p_3^+|-\epsilon\mathcal{P}^+),$$

$$\theta^{(3)}_{1,2;3,4}=\theta(|p_1^+-p_4^+|-\epsilon\mathcal{P}^+),$$

$$\theta^{(4)}_{1,2;3,4}=1, \tag{20}$$

and

$$\theta^{(1)}_{1;2,3,4}=1,$$

$$\theta^{(2)}_{1;2,3,4}=\theta(|p_1^+-p_2^+|-\epsilon\mathcal{P}^+),$$

$$\theta^{(3)}_{1;2,3,4}=\theta(|p_1^+-p_3^+|-\epsilon\mathcal{P}^+),$$

$$\theta^{(4)}_{1;2,3,4}=\theta(|p_1^+-p_4^+|-\epsilon\mathcal{P}^+), \tag{21}$$

and

$$\theta^{(1)}_{1,2,3;4}=1,$$

$$\theta^{(2)}_{1,2,3;4}=\theta(|p_1^+-p_4^+|-\epsilon\mathcal{P}^+),$$

$$\theta^{(3)}_{1,2,3;4}=\theta(|p_2^+-p_4^+|-\epsilon\mathcal{P}^+),$$

$$\theta^{(4)}_{1,2,3;4}=\theta(|p_3^+-p_4^+|-\epsilon\mathcal{P}^+). \tag{22}$$

The presence of ϵ in these step-function cutoffs ensures that we will avoid divergences from exchanged gluons (either instantaneous or real) with infinitesimal longitudinal momentum. In Sec. V we show how we can take ϵ to zero before we diagonalize $\mathcal{M}^2(\Lambda)$.

We have not yet proved that the above conditions on the reduced interaction are necessary and sufficient to reproduce the perturbative scattering amplitudes of pure-glue QCD. We can, however, show in ϕ^3 theory [1] that analogous conditions are necessary and sufficient to reproduce all second-order scattering amplitudes.

D. The recursion relations for the invariant-mass operator

The restrictions that we have placed on the IMO are sufficient to uniquely determine it order-by-order in perturbation theory. In this subsection, we present the recursion relations that define $\mathcal{M}^2(\Lambda)$ in terms of $V^{(1)}$ and $V^{(2)}_{\rm CI}$, which we have defined above. To begin, we consider the restriction that forces the IMO to produce cutoff-independent physical quantities:

$$V(\Lambda)-V(\Lambda')=\delta V. \tag{23}$$

This restriction is in terms of the reduced interaction and the change to the reduced interaction.

δV is defined in Eq. (10), which makes it clear that since $V(\Lambda')$ can be expanded in powers of $g_{\Lambda'}$, so can δV:

$$\delta V=\sum_{t=2}^{\infty} g^t_{\Lambda'}\delta V^{(t)}. \tag{24}$$

We refer to $\delta V^{(t)}$ as the $\mathcal{O}(g^t_{\Lambda'})$ change to the reduced interaction, although for convenience the coupling is factored out. Note that $\delta V^{(t)}$ is a function of Λ and Λ'.

Now Eq. (23) can be expanded in powers of g_Λ and $g_{\Lambda'}$:

$$\sum_{t=1}^{\infty} g^t_\Lambda V^{(t)}(\Lambda)-\sum_{t=1}^{\infty} g^t_{\Lambda'}V^{(t)}(\Lambda')=\sum_{t=2}^{\infty} g^t_{\Lambda'}\delta V^{(t)}. \tag{25}$$

This equation is a bit tricky to use because it involves the couplings at two different scales. To see how they are related, consider the matrix element of Eq. (23) for $g_1\to g_2g_3$:

$$\langle g_2g_3|V(\Lambda)|g_1\rangle-\langle g_2g_3|V(\Lambda')|g_1\rangle=\langle g_2g_3|\delta V|g_1\rangle. \tag{26}$$

According to the definition of the coupling, this equation implies that

$$g_\Lambda-g_{\Lambda'}=\{[16\pi^3p_1^+\delta^{(3)}(p_1-p_2-p_3)\langle\!\langle g_2g_3|v|g_1\rangle\!\rangle]^{-1}\times\langle g_2g_3|\delta V|g_1\rangle\}^{s_n=1;c_n=n}_{\vec p_{2\perp}=\vec p_{3\perp};p_2^+=p_3^+;\epsilon=0}. \tag{27}$$

Since $V^{(1)}$ changes particle number by 1, inspection of Eq. (10) reveals that $\langle g_2g_3|\delta V|g_1\rangle$ is $\mathcal{O}(g^3_{\Lambda'})$; so

$$g_\Lambda=g_{\Lambda'}+\mathcal{O}(g^3_{\Lambda'}). \tag{28}$$

This implies that

$$g_\Lambda=g_{\Lambda'}+\sum_{s=3}^{\infty} g^s_{\Lambda'}C_s(\Lambda,\Lambda'), \tag{29}$$

where the C_s's are functions of Λ and Λ'. For an integer $t\geqslant 1$, Eq. (29) implies that

$$g^t_\Lambda=g^t_{\Lambda'}+\sum_{s=2}^{\infty} g^{t+s}_{\Lambda'}B_{t,s}(\Lambda,\Lambda'), \tag{30}$$

where the $B_{t,s}$'s are functions of Λ and Λ', and can be calculated in terms of the C_s's by raising Eq. (29) to the tth power.

We substitute Eq. (30) into Eq. (25) and demand that it hold order-by-order in $g_{\Lambda'}$. At $\mathcal{O}(g^r_{\Lambda'})$ $(r\geqslant 1)$, this implies that

$$V^{(r)}(\Lambda)-V^{(r)}(\Lambda')=\delta V^{(r)}-\sum_{s=2}^{r-1} B_{r-s,s}V^{(r-s)}(\Lambda), \tag{31}$$

where $\delta V^{(1)}=0$, and we define any sum to be zero if its upper limit is less than its lower limit. The cutoff-independent parts of $V^{(r)}(\Lambda)$ and $V^{(r)}(\Lambda')$ cancel on the left-hand side (LHS), leaving

$$V^{(r)}_{\rm CD}(\Lambda)-V^{(r)}_{\rm CD}(\Lambda')=\delta V^{(r)}-\sum_{s=2}^{r-1} B_{r-s,s}V^{(r-s)}(\Lambda). \tag{32}$$

This equation can be used to derive the desired recursion relations.

In the remainder of this section, we summarize the results of Appendix B, which contains a derivation of the recursion relations that define the IMO. Appendix B is an extension of Appendix D of Ref. [1] to the case of pure-glue QCD.

Recall that momentum conservation implies that any matrix element $\langle F|V(\Lambda)|I\rangle$ can be written as an expansion in unique products of momentum-conserving delta functions. This means that an arbitrary matrix element of Eq. (32) can be expanded in products of delta functions and thus is equivalent to a set of equations, one for each possible product of delta functions. Given approximate transverse locality, each of the resulting equations can be expanded in powers of transverse momenta. Matching the coefficients of the powers of transverse momenta on either side of these equations allows us to rigorously derive the following results (see Appendix B for details).

First, the cutoff-dependent part of the $\mathcal{O}(g_\Lambda^r)$ reduced interaction is given in terms of lower-order reduced interactions by

$$\langle F|V_{\mathrm{CD}}^{(r)}(\Lambda)|I\rangle=\Bigg[\langle F|\delta V^{(r)}|I\rangle-\sum_{s=2}^{r-1}B_{r-s,s}\times\langle F|V^{(r-s)}(\Lambda)|I\rangle\Bigg]_{\Lambda\ \text{terms}},\tag{33}$$

where ''Λ terms'' means that the RHS is to be expanded in powers of transverse momenta and only the terms in the expansion that are proportional to powers or inverse powers of Λ contribute. Recall that $\delta V^{(r)}$ is defined by Eqs. (10) and (24).

Second, the cutoff-independent part of $V^{(r)}(\Lambda)$ has two contributions: a four-point interaction with no transverse-momentum dependence, and a three-point interaction that is linear in transverse momenta. If there are no such contributions to $V^{(r)}(\Lambda)$, then it is completely determined by Eq. (33). Third, the coupling runs at odd orders; i.e., C_s is zero if s is even [see Eq. (29)]. Fourth, there is no wave-function renormalization at any order in perturbation theory in our approach because this would violate the restrictions that we have placed on the IMO.

The fifth and final result from Appendix B is that the cutoff-independent parts of the $\mathcal{O}(g_\Lambda^r)$ and $\mathcal{O}(g_\Lambda^{r+1})$ reduced interactions for odd $r\geqslant 3$ are determined by the coupled integral equations[11]

$$\langle F|V_{\mathrm{CI}}^{(r)}|I\rangle=\frac{1}{B_{r,2}}\Bigg[\langle F|\delta V^{(r+2)}|I\rangle-\sum_{s=3}^{r+1}B_{r+2-s,s}\times\langle F|V^{(r+2-s)}(\Lambda)|I\rangle\Bigg]_{\vec{p}_\perp^{\,1}\ \text{term}}^{\text{3-point}},\tag{34}$$

$$\langle F|V_{\mathrm{CI}}^{(r+1)}|I\rangle=\frac{1}{B_{r+1,2}}\Bigg[\langle F|\delta V^{(r+3)}|I\rangle-\sum_{s=3}^{r+2}B_{r+3-s,s}\langle F|V^{(r+3-s)}(\Lambda)|I\rangle\Bigg]_{\vec{p}_\perp^{\,0}\ \text{term}}^{\text{4-point}}.\tag{35}$$

To use these equations, the right-hand sides have to be expanded in powers of transverse momenta. Only three-point interactions that are linear in transverse momenta contribute to $V_{\mathrm{CI}}^{(r)}$, and only four-point interactions that are independent of all transverse momenta contribute to $V_{\mathrm{CI}}^{(r+1)}$.

These equations are coupled integral equations because both $V_{\mathrm{CI}}^{(r)}$ and $V_{\mathrm{CI}}^{(r+1)}$ appear on the RHS of Eq. (34) inside integrals in $\delta V^{(r+2)}$, and $V_{\mathrm{CI}}^{(r+1)}$ appears on the RHS of Eq. (35) inside integrals in $\delta V^{(r+3)}$. It would seem that $V_{\mathrm{CI}}^{(r+2)}$ also appears on the RHS of Eq. (35) inside integrals in $\delta V^{(r+3)}$, but $V_{\mathrm{CI}}^{(r+2)}$ cannot couple to $V^{(1)}$ to produce a transverse-momentum-independent four-point contribution to $\delta V^{(r+3)}$. This is because the cutoff function $T_2^{(\Lambda,\Lambda')}$ vanishes when the intermediate state is massless and all external transverse momenta are zero. This means that since we specified $V^{(1)}$ and $V_{\mathrm{CI}}^{(2)}$ in Sec. II C 4 we can use Eqs. (34) and (35) to solve for $V_{\mathrm{CI}}^{(3)}$ and $V_{\mathrm{CI}}^{(4)}$ simultaneously, and $V_{\mathrm{CI}}^{(5)}$ and $V_{\mathrm{CI}}^{(6)}$ simultaneously, and so on. Note that before we can use these equations to solve for $V_{\mathrm{CI}}^{(r)}$ and $V_{\mathrm{CI}}^{(r+1)}$ simultaneously, we must first use Eq. (33) both to compute $V_{\mathrm{CD}}^{(r)}(\Lambda)$ in terms of lower-order interactions and to express $V_{\mathrm{CD}}^{(r+1)}(\Lambda)$ in terms of lower-order interactions and $V^{(r)}(\Lambda)$.

III. THE BASIS FOR THE EXPANSION OF PHYSICAL STATES

In the remainder of this paper, we use the results of our renormalization procedure to compute the physical states of pure-glue QCD and their masses. In this section we define a basis for the expansion of physical states. We begin by writing the states in terms of a Fock-space expansion that we truncate at two gluons. (This means that we are approximating all physical states as relatively simple single-glueball states.) Using momentum conservation and assuming that the states must be color singlets, we write each glueball state in terms of an undetermined momentum-spin wave function. We then show that boost covariance implies that this wave function can depend only on internal degrees of freedom, i.e., not on the total momentum of the glueball.

We expand the momentum-spin wave function in a complete orthonormal basis for each degree of freedom. The spin basis functions are the standard triplet and singlet functions. We divide the relative transverse momentum of the two gluons into magnitude and angular degrees of freedom. We represent the transverse-magnitude degree of freedom using orthogonal polynomials weighted by a Gaussian. The width of the Gaussian is a parameter that we can adjust to optimize this basis. For the transverse-angular degree of freedom we

[11] It is very difficult to prove that integral equations of this type have a unique solution; so we simply assume that it is true in this case.

use complex exponentials. For the degree of freedom representing the fraction of the total longitudinal momentum carried by one of the gluons, we use a type of Jacobi polynomial. The width of the weighting factor for this polynomial is a parameter that we can adjust to optimize this basis.

We then use the fact that we want our states to be eigenstates of the operator that generates rotations about the three-axis to place restrictions on the basis functions that can contribute to our physical states. We conclude this section by deriving the eigenvalue equation for the IMO in our basis.

A. Preliminaries

The physical states of pure-glue QCD will be eigenstates of $\mathcal{M}^2(\Lambda)$. Since our cutoff preserves translational covariance and covariance under rotations about the three-axis, we would like the states to also be simultaneous eigenstates of the generators of these symmetries, but this is impossible because translations do not commute with rotations. However, a rotation about the three-axis separates into a part that rotates the centers of mass of states and a part that rotates states' internal degrees of freedom, and translations do commute with these internal rotations.

To be precise, we define $\mathcal{J}_3$ to be the generator of rotations about the three-axis, and $\mathcal{J}_3^{\mathrm{R}}$ to be the part of $\mathcal{J}_3$ governing gluons' momenta in the center-of-mass frame and spin polarizations. $\mathcal{M}^2(\Lambda)$, $\mathcal{P}^+$, $\vec{\mathcal{P}}_\perp$, and $\mathcal{J}_3^{\mathrm{R}}$ are a set of commuting observables; so an eigenstate of $\mathcal{M}^2(\Lambda)$ can be labeled by their eigenvalues. We choose to write a physical state as $|\Psi^{jn}(P)\rangle$, where P is the three-momentum of the state, j is the eigenvalue of $\mathcal{J}_3^{\mathrm{R}}$ for the state, and n labels the mass eigenvalue of the state ($n=1$ has the smallest mass, $n=2$ has the second-smallest mass, etc.). Note that because $|\Psi^{jn}(P)\rangle$ will be determined by $\mathcal{M}^2(\Lambda)$, it will implicitly depend on Λ and g_Λ.

An examination of the matrix elements of $\mathcal{M}^2(\Lambda)$ leads us to believe that the light-front infrared divergences (see the discussion below) will not cancel unless the physical states are color singlets, although we do not have a rigorous proof of this. Therefore, we assume that the physical states are color singlets. Using the Fock-space expansion for the identity operator (see Appendix A for our light-front pure-glue QCD conventions), we can expand a physical state in terms of the number of gluons:

$$|\Psi^{jn}(P)\rangle = \mathbf{1}|\Psi^{jn}(P)\rangle \simeq \frac{1}{2!}\int D_1 D_2 \langle g_1 g_2|\Psi^{jn}(P)\rangle |g_1 g_2\rangle, \tag{36}$$

where there is no one-gluon component because there is no color-singlet gluon. We neglect contributions to the states with more than two gluons, which is a severe approximation. This means that we are approximating all physical states as relatively simple single-glueball states, and j is then the projection of the glueball's spin onto the three-axis. From now on, we refer to the approximate physical states simply as glueball states.

Since the IMO conserves momentum, the wave function $\langle g_1 g_2|\Psi^{jn}(P)\rangle$ must be proportional to a momentum-conserving delta function. Since it represents a color singlet state, it must be proportional to a color-conserving Kronecker delta [23]. This means that it has the form

$$\begin{aligned}\langle g_1 g_2|\Psi^{jn}(P)\rangle = 2!(16\pi^3)^{3/2} &\times \delta^{(3)}(P-p_1-p_2)\frac{1}{\sqrt{2(N_c^2-1)}}\delta_{c_1,c_2} \\ &\times \sqrt{p_1^+ p_2^+}\,\Phi^{jn}_{s_1 s_2}(p_1,p_2),\end{aligned} \tag{37}$$

where $\Phi^{jn}_{s_1 s_2}(p_1,p_2)$ is the momentum-spin wave function with the momentum-conserving delta function removed, and all the extra factors is this equation are present to simplify the normalization of $\Phi^{jn}_{s_1 s_2}(p_1,p_2)$. We must solve for $\Phi^{jn}_{s_1 s_2}(p_1,p_2)$, and this equation indicates that it is symmetric under exchange of particles 1 and 2.

B. Jacobi variables

Using Eq. (37), the Fock-state expansion for a glueball state in Eq. (36) becomes

$$\begin{aligned}|\Psi^{jn}(P)\rangle = \frac{1}{\sqrt{16\pi^3}}\frac{1}{\sqrt{2(N_c^2-1)}} &\times \sum_{s_1 s_2 c_1 c_2}\delta_{c_1,c_2}\int \frac{d^2p_{1\perp}dp_1^+}{\sqrt{p_1^+ p_2^+}}\,\theta(p_1^+ - \epsilon P^+) \\ &\times \theta(p_2^+ - \epsilon P^+)\Phi^{jn}_{s_1 s_2}(p_1,p_2)|g_1 g_2\rangle,\end{aligned} \tag{38}$$

where momentum conservation implies that $p_2 = P - p_1$. It is useful to separate the motion of the center of mass of the state from the internal motions of the gluons. To do this, we change variables from p_1 to the Jacobi variables x and $\vec{k}_\perp$:

$$\begin{aligned} p_1 &= (xP^+, x\vec{P}_\perp + \vec{k}_\perp), \\ p_2 &= ([1-x]P^+, [1-x]\vec{P}_\perp - \vec{k}_\perp). \end{aligned} \tag{39}$$

Here x is the fraction of the total longitudinal momentum that is carried by particle 1, and $\vec{k}_\perp$ is the transverse momentum of particle 1 in the center-of-mass frame. We only display the longitudinal and transverse components of the momenta. (Since the glueball state is a superposition of free-particle states, and since the momentum of a free gluon satisfies $p^2=0$, the minus components of the momenta of g_1 and g_2 are constrained to be given by $p_i^- = \vec{p}_{i\perp}^{\,2}/p_i^+$.)

In terms of the Jacobi variables, the glueball state is

$$|\Psi^{jn}(P)\rangle = \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2} \delta_{c_1,c_2} \int \frac{d^2k_\perp dx}{\sqrt{x(1-x)}} \theta(x-\epsilon)\,\theta(1-x-\epsilon) \times \Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp,P)|g(x,\vec{k}_\perp,P;s_1,c_1)g(1-x,-\vec{k}_\perp,P;s_2,c_2)\rangle, \tag{40}$$

where we explicitly show the dependence of the RHS ket on the Jacobi variables and the total momentum. We can use boost covariance to show that $\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp,P)$ is independent of P. To do this, we note that under a longitudinal boost, the longitudinal momentum of each particle in pure-glue QCD (whether the particle is point-like or composite) transforms according to

$$p^+ \to e^\nu p^+, \tag{41}$$

where ν is a boost parameter [24,25]. Under a transverse boost, each particle's transverse momentum transforms according to

$$\vec{p}_\perp \to \vec{p}_\perp + p^+ \vec{v}_\perp, \tag{42}$$

where $\vec{v}_\perp$ is a boost parameter. This means that if we apply a boost operator to both sides of Eq. (40), we find that

$$|\Psi^{jn}(P')\rangle = \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2} \delta_{c_1,c_2} \int \frac{d^2k_\perp dx}{\sqrt{x(1-x)}} \theta(x-\epsilon)\,\theta(1-x-\epsilon) \times \Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp,P)|g(x,\vec{k}_\perp,P';s_1,c_1)g(1-x,-\vec{k}_\perp,P';s_2,c_2)\rangle, \tag{43}$$

where the boost takes the glueball's momentum from P to P'. Note that the boost does not affect the wave function, only the kets. Since Eq. (40) holds for all P, it holds in particular for P':

$$|\Psi^{jn}(P')\rangle = \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2} \delta_{c_1,c_2} \int \frac{d^2k_\perp dx}{\sqrt{x(1-x)}} \theta(x-\epsilon)\,\theta(1-x-\epsilon) \times \Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp,P')|g(x,\vec{k}_\perp,P';s_1,c_1)g(1-x,-\vec{k}_\perp,P';s_2,c_2)\rangle. \tag{44}$$

Equations (43) and (44) contradict each other unless the wave function $\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp,P)$ is independent of P:

$$\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp,P) = \Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp). \tag{45}$$

Thus we can write $|\Psi^{jn}(P)\rangle$ as

$$|\Psi^{jn}(P)\rangle = \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2} \delta_{c_1,c_2} \int d^2k_\perp dx \frac{1}{\sqrt{x(1-x)}} \theta_\epsilon \Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp)|g_1 g_2\rangle, \tag{46}$$

where $\theta_\epsilon = \theta(x-\epsilon)\,\theta(1-x-\epsilon)$.

C. The momentum and spin wave-function bases

To solve for $\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp)$, we expand it in a complete orthonormal basis for each degree of freedom. Since $\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp)$ is symmetric under exchange of particles 1 and 2,

$$\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp) = \Phi^{jn}_{s_2 s_1}(1-x,-\vec{k}_\perp). \tag{47}$$

To use this, we define

$$\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp) = \sum_{q=1}^{4} \chi_q^{s_1 s_2} \Omega_q^{jn}(x,\vec{k}_\perp), \tag{48}$$

where the spin wave functions are

$$\chi_1^{s_1 s_2} = \delta_{s_1,1}\delta_{s_2,1},$$

$$\chi_2^{s_1 s_2} = \delta_{s_1,1}\delta_{s_2,-1},$$

$$\chi_3^{s_1 s_2} = \frac{1}{\sqrt{2}}[\,\delta_{s_1,1}\delta_{\bar{s}_2,1} + \delta_{\bar{s}_1,1}\delta_{s_2,1}],$$

$$\chi_4^{s_1 s_2} = \frac{1}{\sqrt{2}}[\,\delta_{s_1,1}\delta_{\bar{s}_2,1} - \delta_{\bar{s}_1,1}\delta_{s_2,1}], \tag{49}$$

where $\bar{s} = -s$, and the momentum wave functions satisfy

$$\Omega_1^{jn}(x,\vec{k}_\perp) = \Omega_1^{jn}(1-x,-\vec{k}_\perp),$$

$$\Omega_2^{jn}(x,\vec{k}_\perp) = \Omega_2^{jn}(1-x,-\vec{k}_\perp),$$

$$\Omega_3^{jn}(x,\vec{k}_\perp) = \Omega_3^{jn}(1-x,-\vec{k}_\perp),$$

$$\Omega_4^{jn}(x,\vec{k}_\perp) = -\Omega_4^{jn}(1-x,-\vec{k}_\perp). \tag{50}$$

Note that

$$\sum_{s_1 s_2} \chi_q^{s_1 s_2}\chi_{q'}^{s_1 s_2} = \delta_{q,q'}\,. \tag{51}$$

We define $k = |\vec{k}_\perp|$, and we define the angle ϕ by

$$\vec{k}_\perp = k\cos\phi\,\hat{x} + k\sin\phi\,\hat{y}. \tag{52}$$

We expand the momentum wave function in complete orthonormal bases:

$$\Omega_q^{jn}(x,\vec{k}_\perp) = \sum_{l=0}^{\infty}\sum_{t=0}^{\infty}\sum_{a=-\infty}^{\infty} \overline{R}_{qlta}^{jn} L_l^{(e)}(x) T_t^{(d)}(k) A_a(\phi), \tag{53}$$

where $L_l^{(e)}(x)$, $T_t^{(d)}(k)$, and $A_a(\phi)$ are basis functions for the longitudinal, transverse-magnitude, and transverse-angular degrees of freedom. e and d are parameters that govern the widths of the longitudinal and transverse-magnitude basis functions, respectively. We can adjust these widths to optimize the bases. Note that if we do not truncate the sums in Eq. (53), then the $\overline{R}_{qlta}^{jn}$'s depend on e and d such that $\Omega_q^{jn}(x,\vec{k}_\perp)$ is independent of e and d, although we do not indicate this dependence explicitly.

We define the transverse-magnitude basis functions $T_t^{(d)}(k)$ by

$$T_t^{(d)}(k) = d e^{-k^2 d^2}\sum_{s=0}^{t} \sigma_{t,s} k^s d^s, \tag{54}$$

where the $\sigma_{t,s}$'s are constants that we have computed numerically using the Gram-Schmidt orthogonalization procedure and are such that

$$\int_0^\infty dk k T_t^{(d)}(k) T_{t'}^{(d)}(k) = \delta_{t,t'}\,. \tag{55}$$

Adjusting d allows us to adjust the width of the Gaussian weight function in Eq. (54). d cannot be allowed to pass through zero, because when d is zero, $T_t^{(d)}(k)$ is zero. We choose d to be positive without loss of generality. When doing numerical computations, we work with a dimensionless form of these basis functions:

$$\overline{T}_t(kd) = \frac{1}{d} T_t^{(d)}(k). \tag{56}$$

Under exchange of the two particles, k is unaffected; so $T_t^{(d)}(k)$ is unaffected.

We define the longitudinal basis functions $L_l^{(e)}(x)$ by

$$L_l^{(e)}(x) = [x(1-x)]^e \sum_{m=0}^{l} \lambda_{l,m}^{(e)} x^m, \tag{57}$$

where

$$\lambda_{l,m}^{(e)} = (-1)^{l-m}\frac{1}{m!(l-m)!} \times \frac{\Gamma(1+4e+l+m)}{\Gamma(1+2e+m)}\sqrt{\frac{l!(1+4e+2l)}{\Gamma(1+4e+l)}}. \tag{58}$$

These definitions imply that

$$\int_0^1 dx L_l^{(e)}(x) L_{l'}^{(e)}(x) = \delta_{l,l'}\,, \tag{59}$$

as long as $e > -1/2$. (If $e \leqslant -1/2$, then the state is not normalizable.) We can adjust the width of the weighting function in Eq. (57) by adjusting e. It is often more convenient to work with

$$\overline{L}_l^{(e)}(x) = \frac{1}{\sqrt{x(1-x)}} L_l^{(e)}(x). \tag{60}$$

Under exchange of the two particles, $x \to 1-x$, which means that $L_l^{(e)}(x) \to (-1)^l L_l^{(e)}(x)$.

We define the transverse-angular basis functions $A_a(\phi)$ by

$$A_a(\phi) = \frac{1}{\sqrt{2\pi}} e^{ia\phi}, \tag{61}$$

and then

$$\int_0^{2\pi} d\phi A_{a'}^*(\phi) A_a(\phi) = \delta_{a,a'}\,. \tag{62}$$

These basis functions are useful because they are eigenfunctions of $\mathcal{L}_3^{\mathrm{R}}$, the part of the generator of rotations about the three-axis that governs gluons' momenta in the center-of-mass frame. Under exchange of the two particles, $\phi \to \phi + \pi$, which means that $A_a(\phi) \to (-1)^a A_a(\phi)$.

Using the above definitions of the bases, $|\Psi^{jn}(P)\rangle$ can be written

$$|\Psi^{jn}(P)\rangle = \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2 qlta} \delta_{c_1,c_2} \overline{R}^{jn}_{qlta} \chi_q^{s_1 s_2} \int d^2k_\perp dx\, \theta_\epsilon \overline{L}_l^{(e)}(x) T_t^{(d)}(k) A_a(\phi) |g_1 g_2\rangle. \tag{63}$$

Since the glueball state is symmetric under exchange of the two gluons, the behaviors of the spin and momentum wave functions under exchange of the two gluons imply that if $q=4$, then $l+a$ must be odd, and if $q\neq 4$, then $l+a$ must be even (so that the spin and momentum wave functions have the same symmetry under exchange).

D. Rotations about the three-axis

We want to ensure that $|\Psi^{jn}(P)\rangle$ is an eigenstate of $\mathcal{J}_3^{\mathrm{R}}$ with eigenvalue j. The action of $\mathcal{J}_3$ on the state is given by

$$\mathcal{J}_3|\Psi^{jn}(P)\rangle = \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2 qlta} \delta_{c_1,c_2} \overline{R}^{jn}_{qlta} \chi_q^{s_1 s_2} \int d^2k_\perp dx\, \theta_\epsilon$$
$$\times \left\{ \left[ip^2_{1\perp} \frac{\partial}{\partial p^1_{1\perp}} - ip^1_{1\perp} \frac{\partial}{\partial p^2_{1\perp}} + ip^2_{2\perp} \frac{\partial}{\partial p^1_{2\perp}} - ip^1_{2\perp} \frac{\partial}{\partial p^2_{2\perp}} + s_1 + s_2 \right] \overline{L}_l^{(e)}(x) T_t^{(d)}(k) A_a(\phi) \right\} |g_1 g_2\rangle. \tag{64}$$

Using the definitions of the Jacobi variables, we can separate the center-of-mass and internal degrees of freedom:

$$\mathcal{J}_3|\Psi^{jn}(P)\rangle = \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2 qlta} \delta_{c_1,c_2} \overline{R}^{jn}_{qlta} \chi_q^{s_1 s_2} \int d^2k_\perp dx\, \theta_\epsilon$$
$$\times \left\{ \left[iP^2_\perp \frac{\partial}{\partial P^1_\perp} - iP^1_\perp \frac{\partial}{\partial P^2_\perp} - i\frac{\partial}{\partial \phi} + s_1 + s_2 \right] \overline{L}_l^{(e)}(x) T_t^{(d)}(k) A_a(\phi) \right\} |g_1 g_2\rangle. \tag{65}$$

This implies that the action of $\mathcal{J}_3^{\mathrm{R}}$ on the glueball state is given by

$$\mathcal{J}_3^{\mathrm{R}}|\Psi^{jn}(P)\rangle = \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2 qlta} \delta_{c_1,c_2} \overline{R}^{jn}_{qlta} \chi_q^{s_1 s_2} \int d^2k_\perp dx\, \theta_\epsilon \left\{ \left[-i\frac{\partial}{\partial \phi} + s_1 + s_2 \right] \overline{L}_l^{(e)}(x) T_t^{(d)}(k) A_a(\phi) \right\} |g_1 g_2\rangle$$
$$= \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2 qlta} \delta_{c_1,c_2} \overline{R}^{jn}_{qlta} \chi_q^{s_1 s_2} \int d^2k_\perp dx\, \theta_\epsilon [a+s_1+s_2] \overline{L}_l^{(e)}(x) T_t^{(d)}(k) A_a(\phi) |g_1 g_2\rangle. \tag{66}$$

Since $|\Psi^{jn}(P)\rangle$ is an eigenstate of $\mathcal{J}_3^{\mathrm{R}}$ with eigenvalue j, it must be the case that $a+s_1+s_2=j$ for all values of a, s_1, and s_2 that contribute to the sums in this equation. This implies that for some set of coefficients R^{jn}_{qlt},

$$\overline{R}^{jn}_{qlta} = R^{jn}_{qlt} [\delta_{q,1}\delta_{a,j-2} + \delta_{q,2}\delta_{a,j+2} + \delta_{q,3}\delta_{a,j} + \delta_{q,4}\delta_{a,j}]. \tag{67}$$

This means that $|\Psi^{jn}(P)\rangle$ can be written

$$|\Psi^{jn}(P)\rangle = \sum_{qlt} R^{jn}_{qlt} |q,l,t,j\rangle, \tag{68}$$

where

$$|q,l,t,j\rangle = \frac{1}{\sqrt{16\pi^3}} \frac{1}{\sqrt{2(N_c^2-1)}} \sum_{s_1 s_2 c_1 c_2} \int d^2k_\perp dx\, \theta_\epsilon \delta_{c_1,c_2} \chi_q^{s_1 s_2} \overline{L}_l^{(e)}(x) T_t^{(d)}(k) A_{j-s_1-s_2}(\phi) |g_1 g_2\rangle. \tag{69}$$

We are going to calculate the matrix elements of the IMO in the $|q,l,t,j\rangle$ basis and diagonalize it. Since $\mathcal{M}^2(\Lambda)$ commutes with $\mathcal{J}_3^{\mathrm{R}}$, we can do this for each value of j separately. The diagonalization procedure will yield mass eigenvalues and the R^{jn}_{qlt} coefficients. As long as the coefficients satisfy

$$\delta_{j,j'} \sum_{qlt} R^{j'n'}_{qlt} {}^* R^{jn}_{qlt} = \delta_{j,j'} \delta_{n,n'}, \tag{70}$$

the glueball state will have a plane-wave normalization:

$$\langle \Psi^{j'n'}(P') | \Psi^{jn}(P) \rangle = 16\pi^3 P^+ \delta^{(3)}(P-P') \delta_{j,j'} \delta_{n,n'}. \tag{71}$$

Because of the symmetry of the glueball state under the exchange of particles 1 and 2, the basis state $|q,l,t,j\rangle$ is zero if $l+j$ is odd and $q\neq 4$, or if $l+j$ is even and $q=4$. To take

advantage of this, we consider only the subspace in which $l+j$ is even if $q\neq 4$, and $l+j$ is odd if $q=4$.

E. The eigenvalue equation

The IMO's eigenvalue equation is

$$\mathcal{M}^2(\Lambda)|\Psi^{jn}(P)\rangle = M_n^2|\Psi^{jn}(P)\rangle, \tag{72}$$

where the lower-case subscript on M_n^2 indicates that it is an eigenvalue of $\mathcal{M}^2(\Lambda)$ rather than an eigenvalue of $\mathcal{M}^2_{\rm free}$. (An example of an eigenvalue of $\mathcal{M}^2_{\rm free}$ is M_K^2.)

In the basis that we have defined, the eigenvalue equation takes the form

$$\mathcal{M}^2(\Lambda)\sum_{qlt} R^{jn}_{qlt}|q,l,t,j\rangle = M_n^2\sum_{qlt} R^{jn}_{qlt}|q,l,t,j\rangle. \tag{73}$$

If we project $\langle q',l',t',j|$ onto the left of this equation and use the identity

$$\langle q',l',t',j|q,l,t,j\rangle = 16\pi^3 P^+\delta^{(3)}(P-P')\,\delta_{q,q'}\delta_{l,l'}\delta_{t,t'}, \tag{74}$$

then we find that the eigenvalue equation in our basis is

$$\sum_{qlt}\frac{\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle}{16\pi^3P^+\delta^{(3)}(P-P')}R^{jn}_{qlt} = M_n^2 R^{jn}_{q'l't'}. \tag{75}$$

IV. CALCULATION OF THE FREE-STATE MATRIX ELEMENT OF THE INVARIANT-MASS OPERATOR

In this section we use the recursion relations for the IMO to compute its two-gluon to two-gluon matrix element to second order in the running coupling. This is the only free-state matrix element that we need to solve the eigenvalue equation.

There are three contributions to this matrix element: the free contribution, the contribution from the cutoff-independent part of the reduced interaction, and the contribution from the cutoff-dependent part of the reduced interaction. The free contribution is just the free mass of the states. The contribution from the cutoff-independent part of the reduced interaction consists of a contact interaction and an instantaneous-exchange interaction, and is trivially determined by the conditions on the IMO given in Sec. II C 4. The contribution from the cutoff-dependent part of the reduced interaction must be calculated using the recursion relations for the IMO. It has two parts: a self-energy part and an exchange part. At the end of this section, we combine the various contributions to the matrix element in a way that simplifies our effort to cancel the light-front infrared divergences in the next section.

A. Preliminaries

To solve the eigenvalue equation, we must calculate the matrix elements $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$. These can be written in terms of the free-state matrix element $\langle g_1'g_2'|\mathcal{M}^2(\Lambda)|g_1g_2\rangle$, which is specified by our renormalization procedure. We are going to calculate this matrix element to second order in perturbation theory.

Before continuing, we would like to point out three simplifications that we repeatedly use in this section. To make these simplifications clear, we note that the matrix elements $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ can be written as integrals of wave functions times the free-state matrix element $\langle g_1'g_2'|\mathcal{M}^2(\Lambda)|g_1g_2\rangle$. To make the first simplification, we observe that $\langle g_1'g_2'|\mathcal{M}^2(\Lambda)|g_1g_2\rangle$ has step functions on the particles' longitudinal momenta, and these step functions are redundant because they also appear in the integrals that define $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$. We drop these step functions in the formulas that we present for $\langle g_1'g_2'|\mathcal{M}^2(\Lambda)|g_1g_2\rangle$. To make the second simplification, we point out that the matrix elements $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ are symmetric under exchange of the two initial-state gluons and also under exchange of the two final-state gluons. Because of this, when we are computing $\langle g_1'g_2'|\mathcal{M}^2(\Lambda)|g_1g_2\rangle$, we combine terms that are related by exchange of the two initial-state or two final-state gluons. Finally, due to the color-singlet nature of the glueball states, certain parts of $\langle g_1'g_2'|\mathcal{M}^2(\Lambda)|g_1g_2\rangle$ do not contribute to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$, and we drop these terms.

We begin by defining Jacobi variables for the final state of the free matrix element:

$$p_1' = (x'P^+, x'\vec{P}_\perp + \vec{k}_\perp'),$$
$$p_2' = ([1-x']P^+, [1-x']\vec{P}_\perp - \vec{k}_\perp'). \tag{76}$$

Then using the definition of the IMO in terms of the reduced interaction in Eqs. (1), (2), and (14),

$$\begin{aligned}\langle g_1'g_2'|\mathcal{M}^2(\Lambda)|g_1g_2\rangle &= \langle g_1'g_2'|\mathcal{M}^2_{\rm free}|g_1g_2\rangle + \langle g_1'g_2'|\mathcal{M}^2_{\rm int}(\Lambda)|g_1g_2\rangle \\ &= M_I^2\langle g_1'g_2'|g_1g_2\rangle + g_\Lambda^2 e^{-\Lambda^{-4}\Delta_{FI}^2}\langle g_1'g_2'|V_{\rm CI}^{(2)}|g_1g_2\rangle \\ &\quad + g_\Lambda^2 e^{-\Lambda^{-4}\Delta_{FI}^2}\langle g_1'g_2'|V_{\rm CI}^{(2)}(\Lambda)|g_1g_2\rangle,\end{aligned} \tag{77}$$

where the free masses of the final and initial states are given by

$$M_F^2 = \frac{k'^2}{x'(1-x')},$$
$$M_I^2 = \frac{k^2}{x(1-x)}. \tag{78}$$

B. The cutoff-independent part of the reduced interaction

Based on the definition of $V_{\rm CI}^{(2)}$ in Eq. (19), we find that

$$\langle g_1'g_2'|V_{\rm CI}^{(2)}|g_1g_2\rangle = 16\pi^3P^+\delta^{(3)}(P-P') \times\sum_{i=1}^{4}\theta^{(i)}_{1,2;1',2'}\prec g_1'g_2'|v|g_1g_2\succ_i. \tag{79}$$

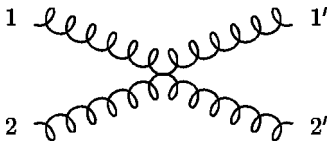

FIG. 1. A diagrammatic representation of $\langle g_1' g_2' | V_{\rm CI}^{(2)} | g_1 g_2 \rangle_{\rm CON}$, the two-gluon contact interaction. The numbers label the particles.

We can divide this matrix element into contact (momentum-independent) and instantaneous-exchange interactions:

$$\langle g_1' g_2' | V_{\rm CI}^{(2)} | g_1 g_2 \rangle = \langle g_1' g_2' | V_{\rm CI}^{(2)} | g_1 g_2 \rangle_{\rm CON} + \langle g_1' g_2' | V_{\rm CI}^{(2)} | g_1 g_2 \rangle_{\rm IN}, \tag{80}$$

where

$$\begin{aligned}
&\langle g_1' g_2' | V_{\rm CI}^{(2)} | g_1 g_2 \rangle_{\rm CON} \\
&\quad = 16\pi^3 P^+ \delta^{(3)}(P-P')\, \theta^{(1)}_{1,2;1',2'} \langle\!\langle g_1' g_2' | v | g_1 g_2 \rangle\!\rangle_1 \\
&\quad = 32\pi^3 P^+ \delta^{(3)}(P-P') f^{c_1 c_1' c} f^{c_2' c_2 c} \\
&\qquad \times (\delta_{s_2, s_1'} \delta_{s_1, s_2'} - \delta_{s_1', \bar{s}_2'} \delta_{s_1, \bar{s}_2}),
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
&\langle g_1' g_2' | V_{\rm CI}^{(2)} | g_1 g_2 \rangle_{\rm IN} \\
&\quad = 16\pi^3 P^+ \delta^{(3)}(P-P') \sum_{i=2}^{3} \theta^{(i)}_{1,2;1',2'} \langle\!\langle g_1' g_2' | v | g_1 g_2 \rangle\!\rangle_i \\
&\quad = 32\pi^3 P^+ \delta^{(3)}(P-P')\, \theta(|x-x'|-\epsilon) \\
&\qquad \times f^{c_1 c_1' c} f^{c_2' c_2 c} \delta_{s_1, s_1'} \delta_{s_2, s_2'} \\
&\qquad \times \frac{1}{(x-x')^2} (x+x')(1-x+1-x').
\end{aligned} \tag{82}$$

[The remaining term in Eq. (79) vanishes because the $|q,l,t,j\rangle$ states are color singlets.] The contact interaction is displayed in Fig. 1 and the instantaneous-exchange interaction is displayed in Fig. 2.

C. The cutoff-dependent part of the reduced interaction

1. Preliminaries

According to our recursion relation for the cutoff-dependent part of the reduced interaction, Eq. (33), $\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle$ is given by

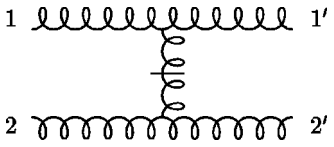

FIG. 2. A diagrammatic representation of $\langle g_1' g_2' | V_{\rm CI}^{(2)} | g_1 g_2 \rangle_{\rm IN}$, the two-gluon instantaneous-exchange interaction. The numbers label the particles.

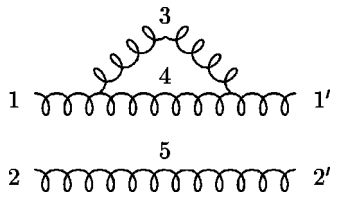

FIG. 3. A diagrammatic representation of $\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm SE}$, the self-energy interaction. The numbers label the particles.

$$\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle = \langle g_1' g_2' | \delta V^{(2)} | g_1 g_2 \rangle |_{\Lambda \text{ terms}}. \tag{83}$$

Using the definition of the $\mathcal{O}(g_{\Lambda'}^r)$ change to the reduced interaction in Eqs. (10) and (24), we find that

$$\begin{aligned}
\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle &= \frac{1}{2} \sum_K \langle g_1' g_2' | V^{(1)} | K \rangle \langle K | V^{(1)} | g_1 g_2 \rangle \\
&\quad \times T_2^{(\Lambda,\Lambda')}(F,K,I)|_{\Lambda \text{ terms}}.
\end{aligned} \tag{84}$$

The intermediate state can be either a one-particle or three-particle state. The contribution to the eigenvalue equation from the one-particle-intermediate-state part is zero because the $|q,l,t,j\rangle$ states are color singlets. This means that

$$\begin{aligned}
&\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle \\
&\quad = \frac{1}{12} \int D_3 D_4 D_5 \langle g_1' g_2' | V^{(1)} | g_3 g_4 g_5 \rangle \\
&\qquad \times \langle g_3 g_4 g_5 | V^{(1)} | g_1 g_2 \rangle T_2^{(\Lambda,\Lambda')}(F,K,I)|_{\Lambda \text{ terms}},
\end{aligned} \tag{85}$$

where M_K^2 is the mass of the state $|g_3 g_4 g_5\rangle$. Substituting the definition of $V^{(1)}$ in Eq. (18) into this equation, and simplifying, we find that

$$\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle = \langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm SE} + \langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm EX}, \tag{86}$$

where the self-energy interaction is given by

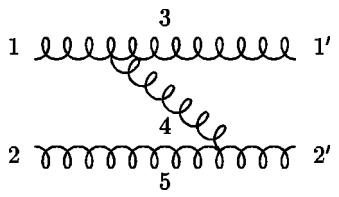

FIG. 4. A diagrammatic representation of $\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm EX}$, the exchange interaction. The numbers label the particles.

$$\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm SE} = P^{+2}(16\pi^3)^2 \int D_3 D_4 D_5 T_2^{(\Lambda,\Lambda')}(F,K,I) \times \delta_{2,5}\delta_{2',5}\beta_{134}\beta^*_{1'34}|_{\Lambda\ \rm terms}, \quad (87)$$

the exchange interaction is given by

$$\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm EX} = 2P^{+2}(16\pi^3)^2 \int D_3 D_4 D_5 T_2^{(\Lambda,\Lambda')}(F,K,I) \times \delta_{2,5}\delta_{1',3}\beta_{134}\beta^*_{2'45}|_{\Lambda\ \rm terms}, \quad (88)$$

and we have defined

$$\beta_{ijk} = \theta(p_i^+)\theta(p_j^+)\theta(p_k^+)\delta^{(3)}(p_i - p_j - p_k) \langle g_j g_k | v | g_i \rangle \quad (89)$$

and

$$\delta_{i,k} = 16\pi^3 p_i^+ \delta^{(3)}(p_i - p_k)\delta_{s_i,s_k}\delta_{c_i,c_k}. \quad (90)$$

The self-energy interaction is displayed in Fig. 3 and the exchange interaction is displayed in Fig. 4.

2. The self-energy interaction

After some simplification, the self-energy interaction takes the form

$$\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm SE} = P^{+2}\delta_{2,2'}\delta^{(3)}(p_1 - p_1') \sum_{s_3 s_4 c_3 c_4} \int \frac{d^2 p_{3\perp} dp_3^+}{p_3^+} \theta(p_3^+ - \epsilon P^+) \frac{1}{p_4^+} \theta(p_4^+ - \epsilon P^+) \times T_2^{(\Lambda,\Lambda')}(F,K,I) \langle g_3 g_4 | v | g_1 \rangle \langle g_1' | v | g_3 g_4 \rangle |_{\Lambda\ \rm terms}, \quad (91)$$

where $p_4 = p_1 - p_3$. At this point, it is useful to change variables from p_3 to Jacobi variables y and $\vec{r}_\perp$:

$$p_3 = (y p_1^+, y\vec{p}_{1\perp} + \vec{r}_\perp). \quad (92)$$

Then the free mass of the intermediate state is

$$M_K^2 = (p_3^- + p_4^- + p_5^-)P^+ - \vec{P}_\perp^2 = \frac{k^2}{x(1-x)} + \frac{r^2}{xy(1-y)}, \quad (93)$$

and after a lot of additional simplification,

$$\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm SE} = \frac{N_c}{2\pi^2}\sqrt{\frac{\pi}{2}}[\Lambda^2 - \Lambda'^2]\delta_{1,1'}\delta_{2,2'}\theta(x - 2\epsilon) \times \left[\log x - \log\epsilon - \frac{11}{12}\right]\Bigg|_{\Lambda\ \rm terms}. \quad (94)$$

Recalling that ''Λ terms'' means that we are to expand the RHS of this equation in powers of transverse momenta and keep only the terms that are proportional to powers or inverse powers of Λ, we find that

$$\langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm SE} = \frac{N_c}{2\pi^2}\sqrt{\frac{\pi}{2}}\Lambda^2\delta_{1,1'}\delta_{2,2'}\theta(x - 2\epsilon) \times \left[\log x - \log\epsilon - \frac{11}{12}\right]. \quad (95)$$

This result shows that our cutoff violates cluster decomposition. The evidence is that the self-energy depends on $x = p_1^+/(p_1^+ + p_2^+)$, even though p_2 is the momentum of a spectator (see Fig. 3).

3. The exchange interaction

For the exchange interaction, the free mass of the intermediate state is

$$M_K^2 = (p_3^- + p_4^- + p_5^-)P^+ - \vec{P}_\perp^2 = \frac{k'^2}{x'} + \frac{(\vec{k}_\perp - \vec{k}_\perp')^2}{x - x'} + \frac{k^2}{1-x}. \quad (96)$$

Then the changes in free mass are

$$\Delta_{FK} = -\frac{k'^2[1-x]^2 + k^2[1-x']^2 - 2kk'(1-x')(1-x)\cos\gamma}{(1-x')(x-x')(1-x)},$$
$$\Delta_{IK} = -\frac{k^2 x'^2 + k'^2 x^2 - 2kk' xx'\cos\gamma}{xx'(x-x')}, \quad (97)$$

where $\gamma = \phi - \phi'$.

Using the identity

$$\vec{\epsilon}_{\perp s}\cdot\vec{k}_{\perp}=\frac{-1}{\sqrt{2}}ks e^{is\phi}, \tag{98}$$

the exchange interaction becomes

$$\langle g_1' g_2'|V_{\mathrm{CD}}^{(2)}(\Lambda)|g_1 g_2\rangle_{\mathrm{EX}}=64\pi^3 P^+\delta^{(3)}(P-P')f^{c_1 c c_1'}f^{c c_2 c_2'}\frac{\theta(x-x'-\epsilon)}{x-x'}$$
$$\times\left(\frac{1}{\Delta_{FK}}+\frac{1}{\Delta_{IK}}\right)(e^{-2\Lambda'^{-4}\Delta_{FK}\Delta_{IK}}-e^{-2\Lambda^{-4}\Delta_{FK}\Delta_{IK}})\sum_{i,m=1}^{3}Q^{(i,m)}\Big|_{\Lambda\ \text{terms}}, \tag{99}$$

where

$$Q^{(1,1)}=\frac{1}{x-x'}s_1 s_1'\delta_{s_2,s_2'}\left\{\frac{x'}{x}ke^{is_1\phi}-k'e^{is_1\phi'}\right\}\{k'e^{-is_1'\phi'}[1-x]-ke^{-is_1'\phi}[1-x']\},$$

$$Q^{(1,2)}=s_1 s_2'\delta_{s_2,s_1'}\left\{\frac{x'}{x}ke^{is_1\phi}-k'e^{is_1\phi'}\right\}\left\{ke^{-is_2'\phi}-\frac{1-x}{1-x'}k'e^{-is_2'\phi'}\right\},$$

$$Q^{(1,3)}=-s_1 s_2\delta_{\bar{s}_1',s_2'}\left\{\frac{x'}{x}ke^{is_1\phi}-k'e^{is_1\phi'}\right\}\left\{k'e^{is_2\phi'}-\frac{1-x'}{1-x}ke^{is_2\phi}\right\},$$

$$Q^{(2,1)}=2\delta_{s_1,s_1'}\delta_{s_2,s_2'}\frac{1}{(x-x')^2}\{x(1-x)k'^2+x'(1-x')k^2-kk'[x'(1-x)+x(1-x')]\cos\gamma\},$$

$$Q^{(2,2)}=\delta_{s_1,s_1'}s_2 s_2'\frac{1}{x-x'}\{xk'e^{is_2\phi'}-x'ke^{is_2\phi}\}\left\{ke^{-is_2'\phi}-\frac{1-x}{1-x'}k'e^{-is_2'\phi'}\right\},$$

$$Q^{(2,3)}=s_2 s_2'\delta_{s_1,s_1'}\frac{1}{x-x'}\{xk'e^{-is_2'\phi'}-x'ke^{-is_2'\phi}\}\left\{k'e^{is_2\phi'}-\frac{1-x'}{1-x}ke^{is_2\phi}\right\},$$

$$Q^{(3,1)}=s_1 s_1'\delta_{s_2,s_2'}\frac{1}{x-x'}\left\{\frac{x}{x'}k'e^{-is_1'\phi'}-ke^{-is_1'\phi}\right\}\{k'e^{is_1\phi'}[1-x]-ke^{is_1\phi}[1-x']\},$$

$$Q^{(3,2)}=-s_1' s_2'\delta_{s_1,\bar{s}_2}\left\{\frac{x}{x'}k'e^{-is_1'\phi'}-ke^{-is_1'\phi}\right\}\left\{ke^{-is_2'\phi}-\frac{1-x}{1-x'}k'e^{-is_2'\phi'}\right\},$$

$$Q^{(3,3)}=s_2 s_1'\delta_{s_1,s_2'}\left\{\frac{x}{x'}k'e^{-is_1'\phi'}-ke^{-is_1'\phi}\right\}\left\{k'e^{is_2\phi'}-\frac{1-x'}{1-x}ke^{is_2\phi}\right\}. \tag{100}$$

In Eq. (99), if we expand everything multiplying the delta function in powers of transverse momenta, the lowest-order terms from the two exponentials cancel and leave two types of terms: those proportional to inverse powers of Λ and those proportional to inverse powers of Λ'. We can isolate the terms that are proportional to inverse powers of Λ without altering the cancellation of the lowest term by replacing the first exponential with a 1:

$$\langle g_1' g_2'|V_{\mathrm{CD}}^{(2)}(\Lambda)|g_1 g_2\rangle_{\mathrm{EX}}=64\pi^3 P^+\delta^{(3)}(P-P')f^{c_1 c c_1'}f^{c c_2 c_2'}\frac{1}{(x-x')}\theta(x-x'-\epsilon)$$
$$\times\left(\frac{1}{\Delta_{FK}}+\frac{1}{\Delta_{IK}}\right)(1-e^{-2\Lambda^{-4}\Delta_{FK}\Delta_{IK}})\sum_{i,m=1}^{3}Q^{(i,m)}. \tag{101}$$

D. Combining the interactions

In order to get the infrared ($\epsilon\to 0$) divergences to cancel, it is useful to combine the interactions in a particular manner. From Eqs. (77), (80), and (86),

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle = \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm KE} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm CON} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm SE} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm EX}, \tag{102}$$

where

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm KE} = M_I^2 \langle g_1' g_2' | g_1 g_2 \rangle,$$

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm CON} = g_\Lambda^2 e^{-\Lambda^{-4} \Delta_{FI}^2} \langle g_1' g_2' | V_{\rm CI}^{(2)} | g_1 g_2 \rangle_{\rm CON},$$

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN} = g_\Lambda^2 e^{-\Lambda^{-4} \Delta_{FI}^2} \langle g_1' g_2' | V_{\rm CI}^{(2)} | g_1 g_2 \rangle_{\rm IN},$$

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm SE} = g_\Lambda^2 e^{-\Lambda^{-4} \Delta_{FI}^2} \langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm SE},$$

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm EX} = g_\Lambda^2 e^{-\Lambda^{-4} \Delta_{FI}^2} \langle g_1' g_2' | V_{\rm CD}^{(2)}(\Lambda) | g_1 g_2 \rangle_{\rm EX}. \tag{103}$$

We break the instantaneous interaction into two parts: a part that is ''above'' the cutoff, i.e., a part that would vanish if we took $\Lambda \to \infty$, and a part that is ''below'' the cutoff, i.e., a part that would survive if we took $\Lambda \to \infty$:

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN} = \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN}^{\rm A} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN}^{\rm B}, \tag{104}$$

where

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN}^{\rm A} = (1 - e^{-2\Lambda^{-4} \Delta_{FK} \Delta_{IK}}) \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN},$$

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN}^{\rm B} = e^{-2\Lambda^{-4} \Delta_{FK} \Delta_{IK}} \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN}. \tag{105}$$

Next, we break the self-energy and exchange interactions into finite and divergent parts:

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm SE} = \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm SE}^{\rm F} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm SE}^{\rm D},$$

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm EX} = \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm EX}^{\rm F} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm EX}^{\rm D}, \tag{106}$$

where the divergent part of the self-energy interaction consists solely of the term containing the $\log \epsilon$, and the divergent part of the exchange interaction consists solely of the term containing $Q^{(2,1)}$. Finally, we define an interaction that is a combination of the instantaneous interaction ''above'' the cutoff and the divergent part of the exchange interaction:

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN+EX} = \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN}^{\rm A} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm EX}^{\rm D}. \tag{107}$$

Then

$$\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle = \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm KE} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm SE}^{\rm F} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm CON} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm EX}^{\rm F} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN+EX} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN}^{\rm B} + \langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm SE}^{\rm D}. \tag{108}$$

Perry showed that with a suitable definition of long-range interactions, a renormalization method that is similar to ours yields a logarithmically confining potential for quark-antiquark bound states at $\mathcal{O}(g_\Lambda^2)$ [7]. His calculation uses sharp step-function cutoffs and is based on an analysis of the part of the two-body interaction that is most singular in the limit in which the exchanged gluon has infinitesimal longitudinal momentum. The corresponding part of our interaction, which is contained in $\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN+EX}$ and $\langle g_1' g_2' | \mathcal{M}^2(\Lambda) | g_1 g_2 \rangle_{\rm IN}^{\rm B}$, is similar to what Perry found in the quark-antiquark case. However, to determine whether or not our interaction is truly confining, we would have to do a careful analysis of the complete two-body potential, not just the most singular part. This analysis would be complicated by the smooth cutoff that we employ, and we leave it for future consideration.

V. CALCULATION OF THE MATRIX ELEMENTS OF THE INVARIANT-MASS OPERATOR IN THE BASIS FOR PHYSICAL STATES

In this section we compute the matrix elements of the IMO in the basis that we have defined, in terms of integrals that must be evaluated numerically. To avoid roundoff error, we evaluate the integrals for the kinetic energy and the two-point interaction (the self-energy) by writing them as sums of gamma functions. The remaining integrals are five-dimensional, and we wish to compute them with Monte Carlo methods. However, before we can do this, it is necessary for us to make manifest the cancellation of the infrared divergences from exchanged gluons with infinitesimal longitudinal momentum. To do this, we first integrate one of the longitudinal momentum integrals by parts to extract the divergence. We show that this divergence exactly cancels the infrared divergence that appears in the self-energy. The remainder of the five-dimensional integral is finite. However, it converges poorly when it is calculated numerically using the natural Jacobi variables. This is because it peaks very strongly in the region where the divergence lay, even after the divergence is removed. To improve the convergence, we use a change of variables and subtract from the integrand a contribution that appears to be divergent but that integrates to zero. We present some of these convergence details and other technical issues in Appendix C.

A. Preliminaries

The matrix elements $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$, which appear in the eigenvalue equation, can be divided into contributions corresponding to the different terms in Eq. (108):

$$\begin{aligned}\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle =& \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{KE}} + \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{SE}}^{\mathrm{F}} \\ &+ \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{CON}} + \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{EX}}^{\mathrm{F}} \\ &+ \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{IN+EX}} + \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{IN}}^{\mathrm{B}} \\ &+ \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{SE}}^{\mathrm{D}}. \end{aligned} \tag{109}$$

In this section we express these different contributions in terms of integrals that can be computed numerically. These integrals fall into two classes: two-dimensional and five-dimensional. We treat each class of integral separately.

Each of the terms in Eq. (109) is proportional to the plane-wave normalization factor $16\pi^3 P^+ \delta^{(3)}(P-P')$. To make the remaining equations that we present simpler, we suppress this factor. We also take $\epsilon \to 0$ in any contribution to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ that is finite in this limit.

B. The two-dimensional integrals

1. The kinetic energy

Using the definition of our basis in Eq. (69) and the expression for the free-state matrix element of the kinetic energy in Eq. (103), we find that

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{KE}} = \delta_{q,q'} \frac{1}{d^2} \int_0^1 dx \overline{L}_{l'}^{(e)}(x) \overline{L}_{l}^{(e)}(x) \int_0^\infty dr r^3 \overline{T}_{t'}(r) \overline{T}_{t}(r). \tag{110}$$

Note that the kinetic energy is infinite unless $e>0$, whereas normalizability requires only $e>-1/2$.

These integrals can be computed with standard numerical integration routines, but the results can have large errors for large values of the function indices l, l', t, and t', due to the oscillating nature of the basis functions. It is better to rewrite the integrals as sums that can be computed numerically with MATHEMATICA [26] to any desired precision. Using the definitions of the basis functions

$$\begin{aligned}\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{KE}} =& \delta_{q,q'} \frac{\Gamma(2e)}{d^2} \left[\sum_{m=0}^{l} \lambda_{l,m}^{(e)} \sum_{m'=0}^{l'} \lambda_{l',m'}^{(e)} \frac{\Gamma(2e+m+m')}{\Gamma(4e+m+m')} \right] \\ &\times \left[\sum_{s=0}^{t} \sigma_{t,s} \sum_{s'=0}^{t'} \sigma_{t',s'} 2^{-3-(s+s')/2} \Gamma\left(2+\frac{s+s'}{2}\right) \right]. \end{aligned} \tag{111}$$

2. The Finite Part of the Self-Energy Interaction

The self-energy interaction conserves each particle's momentum, thus $M_I^2 = M_F^2$ for this contribution. This means that the Gaussian cutoff factor in Eq. (103) has no effect on the self-energy. For the finite part of the self-energy, we use the same method for evaluating integrals that we use for the kinetic energy. This yields

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm SE}^{\rm F}=\delta_{q,q'}\delta_{t,t'}\frac{N_c g_\Lambda^2}{4\pi^2}\sqrt{\frac{\pi}{2}}\Lambda^2\int_0^1 dx\overline{L}_{l'}^{(e)}(x)\overline{L}_l^{(e)}(x)x(1-x)\left[\log x-\frac{11}{12}\right] \tag{112}$$

$$=-\delta_{q,q'}\delta_{t,t'}\frac{N_c g_\Lambda^2}{4\pi^2}\sqrt{\frac{\pi}{2}}\Lambda^2\Bigg(\delta_{l,l'}\frac{11}{12}-\Gamma(1+2e)\sum_{m=0}^{l}\lambda_{l,m}^{(e)}\sum_{m'=0}^{l'}\lambda_{l',m'}^{(e)}$$

$$\times\frac{\Gamma(1+2e+m+m')}{\Gamma(2+4e+m+m')}[\psi(1+2e+m+m')-\psi(2+4e+m+m')]\Bigg), \tag{113}$$

where the digamma function $\psi(z)$ is given by

$$\psi(z)=\frac{\dfrac{d\Gamma(z)}{dz}}{\Gamma(z)}. \tag{114}$$

C. The five-dimensional integrals

1. The contact interaction

Using the definition of our basis in Eq. (69) and the expression for the free-state matrix element of the contact interaction in Eqs. (81) and (103), we find that

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm CON}$$

$$=-\frac{N_c g_\Lambda^2}{8\pi^2}[\delta_{j,-2}\delta_{q',2}\delta_{q,2}+\delta_{j,2}\delta_{q',1}\delta_{q,1}$$

$$-\delta_{j,0}\delta_{q',3}\delta_{q,3}]\int dkdk'kk'\theta(k)\theta(k')T_{t'}^{(d)}(k')$$

$$\times T_t^{(d)}(k)\int dxdx'\theta(x)\theta(1-x)\theta(x')\theta(1-x')$$

$$\times\overline{L}_{l'}^{(e)}(x')\overline{L}_l^{(e)}(x)e^{-\Lambda^{-4}\Delta_{FI}^2}. \tag{115}$$

The reader may have noticed that this integral is not five dimensional as we have implied, but rather four dimensional. However, when we numerically compute the integrals that have more than two dimensions, it is most efficient if we combine them into one integral; so we want their integration variables and their ranges of integration to be identical. Thus we increase the number of dimensions of this integral by one by introducing an extra integral over $\gamma=\phi-\phi'$ using the identity

$$1=\frac{1}{2\pi}\int d\gamma\theta(\gamma)\theta(2\pi-\gamma). \tag{116}$$

This will help us to combine this integral with others that contain integrals over γ that cannot be done analytically.

Since the integration domain of the exchange interaction is restricted so that $x>x'$ when $\epsilon\to 0$ [note the step function in Eq. (101)], we would like to enforce this restriction in the other contributions. Using the identity

$$1=\theta(x-x')+\theta(x'-x), \tag{117}$$

we break the longitudinal integral in Eq. (115) into two parts:

$$\int dxdx'\theta(x)\theta(1-x)\theta(x')\theta(1-x')\overline{L}_{l'}^{(e)}(x')$$

$$\times\overline{L}_l^{(e)}(x)e^{-\Lambda^{-4}\Delta_{FI}^2}[\theta(x-x')+\theta(x'-x)]. \tag{118}$$

In the second term, we let $x\to 1-x$ and $x'\to 1-x'$, and then the longitudinal integral becomes

$$\int dxdx'\theta(x)\theta(1-x)\theta(x')\theta(1-x')\overline{L}_{l'}^{(e)}(x')$$

$$\times\overline{L}_l^{(e)}(x)e^{-\Lambda^{-4}\Delta_{FI}^2}\theta(x-x')[1+(-1)^{l+l'}]. \tag{119}$$

Recall that we are restricting ourselves to the subspace of the states $|q,l,t,j\rangle$ in which $l+j$ is even if $q\neq 4$, and $l+j$ is odd if $q=4$. Then since $q=q'$ for the contact interaction, l and l' must both be even or both be odd for this interaction. This means that the two terms in Eq. (119) are equal. Thus we can write the contact interaction contribution as follows:

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm CON}$$

$$=-\frac{N_c g_\Lambda^2}{8\pi^3}[\delta_{j,-2}\delta_{q',2}\delta_{q,2}+\delta_{j,2}\delta_{q',1}\delta_{q,1}-\delta_{j,0}\delta_{q',3}\delta_{q,3}]$$

$$\times\int D\xi e^{-\Lambda^{-4}\Delta_{FI}^2}, \tag{120}$$

where

$$D=dxdx'dkdk'd\gamma kk'\theta(x)\theta(1-x)\theta(x')$$

$$\times\theta(x-x')\theta(k)\theta(k')\theta(\gamma)\theta(2\pi-\gamma), \tag{121}$$

and

$$\xi=\overline{L}_{l'}^{(e)}(x')\overline{L}_l^{(e)}(x)T_{t'}^{(d)}(k')T_t^{(d)}(k). \tag{122}$$

2. The finite part of the exchange interaction

In order to simplify the contribution to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ from the finite part of the exchange interaction, we wish to change variables from ϕ to $\gamma=\phi-\phi'$:

$$d^2k_\perp d^2k'_\perp = dkdk'd\phi d\phi' kk'\,\theta(k)\theta(k')\theta(\phi) \times\theta(2\pi-\phi)\theta(\phi')\theta(2\pi-\phi') = dkdk'd\gamma d\phi' kk'\,\theta(k)\theta(k')\theta(\gamma+\phi') \times\theta(2\pi-\gamma-\phi')\theta(\phi')\theta(2\pi-\phi'). \tag{123}$$

All the contributions to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ depend on γ only through dependence on $\cos\gamma$ and $\sin\gamma$. This means that we can use the identity

$$\int_{-\phi'}^{2\pi-\phi'} d\gamma f(\cos\gamma,\sin\gamma) = \int_0^{2\pi} d\gamma f(\cos\gamma,\sin\gamma) \tag{124}$$

to write

$$d^2k_\perp d^2k'_\perp = dkdk'd\gamma d\phi' kk'\,\theta(k)\theta(k')\theta(\gamma) \times\theta(2\pi-\gamma)\theta(\phi')\theta(2\pi-\phi'). \tag{125}$$

Inspection of Eq. (99) implies that the integrals in $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{F}}_{\mathrm{EX}}$ depend on complex exponentials of ϕ' and γ. However, using the identity

$$\int_0^{2\pi} d\gamma f(\cos\gamma)\sin a\gamma = 0, \tag{126}$$

where a is an integer, it is possible to trivially do the ϕ' integral in $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{F}}_{\mathrm{EX}}$ and write the remainder as a real quantity with integrals that depend on γ only through $\cos\gamma$ and $\sin\gamma$. The result is

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{F}}_{\mathrm{EX}} = -\frac{N_c g_\Lambda^2}{8\pi^3}\int D\xi e^{-\Lambda^{-4}\Delta_{FI}^2}\frac{1}{x-x'}\left(\frac{1}{\Delta_{FK}}+\frac{1}{\Delta_{IK}}\right) \times(1-e^{-2\Lambda^{-4}\Delta_{FK}\Delta_{IK}})\left[M_I^2 S^{(1)}_{q,q'}+M_F^2 S^{(2)}_{q,q'} + \frac{kk'}{x(1-x)x'(1-x')}S^{(3)}_{q,q'}\right], \tag{127}$$

where some of the $S^{(1)}_{q,q'}$'s and $S^{(3)}_{q,q'}$'s are given by

$$S^{(1)}_{1,1}=\cos([j-2]\gamma),$$

$$S^{(3)}_{1,1}=-\cos([j-1]\gamma)[x(1-x')+x'(1-x)],$$

$$S^{(1)}_{1,3}=\frac{-1}{\sqrt{2}}\cos(j\gamma)[x'^2+(1-x')^2],$$

$$S^{(3)}_{1,3}=\frac{1}{\sqrt{2}}\cos([j-1]\gamma)(x[1-x']+x'[1-x])\times(x'^2+[1-x']^2),$$

$$S^{(1)}_{1,4}=\frac{1}{\sqrt{2}}\cos(j\gamma)(1-2x'),$$

$$S^{(3)}_{1,4}=\frac{-1}{\sqrt{2}}\cos([j-1]\gamma)(1-2x')(x[1-x']+x'[1-x]),$$

$$S^{(1)}_{3,1}=\frac{-1}{\sqrt{2}}\cos([j-2]\gamma)[x^2+(1-x)^2],$$

$$S^{(1)}_{3,3}=\cos(j\gamma)[x^2+(1-x)^2-2x'(1-x')],$$

$$S^{(3)}_{3,3,}=-\cos(j\gamma)\cos\gamma(x[1-x']+x'[1-x])\times(1-2x[1-x]-2x'[1-x']),$$

$$S^{(1)}_{3,4}=0,$$

$$S^{(3)}_{3,4}=\sin\gamma\sin(j\gamma)(2x^3-2x^2[1+x']+x'[1-2x']+x[1-2x'+4x'^2]),$$

$$S^{(1)}_{4,1}=\frac{1}{\sqrt{2}}\cos([j-2]\gamma)(1-2x),$$

$$S^{(1)}_{4,3}=0,$$

$$S^{(1)}_{4,4}=\cos(j\gamma)(1-2x-2x'+4xx'),$$

$$S^{(3)}_{4,4}=-\cos\gamma\cos(j\gamma)(1-2x)(1-2x')(x+x'-2xx'), \tag{128}$$

and

$$S^{(3)}_{q,q'}=S^{(3)}_{q',q}\big|_{x\leftrightarrow x'}. \tag{129}$$

The rest of the $S^{(1)}_{q,q'}$'s and $S^{(3)}_{q,q'}$'s are given by

$$S^{(i)}_{1,2}=S^{(i)}_{2,1}=0,$$

$$S^{(i)}_{2,2}=S^{(i)}_{1,1}\big|_{j\to-j},$$

$$S^{(i)}_{2,3}=S^{(i)}_{1,3}\big|_{j\to-j},$$

$$S^{(i)}_{2,4}=-S^{(i)}_{1,4}\big|_{j\to-j},$$

$$S^{(i)}_{3,2}=S^{(i)}_{3,1}\big|_{j\to-j},$$

$$S^{(i)}_{4,2}=-S^{(i)}_{4,1}\big|_{j\to-j}, \tag{130}$$

where $i=1,3$. The $S^{(2)}_{q,q'}$'s are given by

$$S^{(2)}_{q,q'} = S^{(1)}_{q',q}\big|_{x\leftrightarrow x'}. \tag{131}$$

3. The instantaneous and exchange interactions combination

Using similar methods for the contribution to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ from the combination of the instantaneous interaction above the cutoff and the divergent part of the exchange interaction, we find that

$$\begin{aligned}
&\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{IN+EX}} \\
&\quad = -\frac{N_c g_\Lambda^2}{8\pi^3}\int D\xi W_{q,q'} e^{-\Lambda^{-4}\Delta_{FI}^2} \\
&\qquad \times \frac{1}{(x-x')^2}(1-e^{-2\Lambda^{-4}\Delta_{FK}\Delta_{IK}}) \\
&\qquad \times\Bigg[(x+x')(1-x+1-x')+\frac{2}{x-x'}\left(\frac{1}{\Delta_{FK}}+\frac{1}{\Delta_{IK}}\right) \\
&\qquad \times(x(1-x)k'^2+x'(1-x')k^2-kk'[x'(1-x) \\
&\qquad +x(1-x')]\cos\gamma)\Bigg],
\end{aligned} \tag{132}$$

where

$$\begin{aligned}
W_{q,q'} &= \delta_{q,l}\delta_{q',1}\cos(\gamma[j-2])+\delta_{q,2}\delta_{q',2}\cos(\gamma[j+2]) \\
&\quad +\delta_{q,3}\delta_{q',3}\cos(\gamma j)+\delta_{q,4}\delta_{q',4}\cos(\gamma j).
\end{aligned} \tag{133}$$

Note that the divergences from the two interactions that comprise $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{IN+EX}}$ cancel, allowing us to take $\epsilon\to 0$ in this contribution.

4. The instantaneous interaction below the cutoff

The contribution to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ from the instantaneous interaction below the cutoff is divergent. In this subsection, we extract the divergence, show that it cancels the divergent part of the self-energy, and compute the remainder of $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{B}}_{\mathrm{IN}}$.

After simplification, the complete contribution to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ from the instantaneous interaction below the cutoff is

$$\begin{aligned}
&\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{B}}_{\mathrm{IN}} \\
&\quad = -\frac{N_c g_\Lambda^2}{16\pi^4}\int_{2\epsilon}^{1-\epsilon} dx\int_{\epsilon}^{x-\epsilon} dx'\frac{1}{x-x'}F(x,x'),
\end{aligned} \tag{134}$$

where

$$\begin{aligned}
F(x,x') &= \int d^2k_\perp d^2k'_\perp \frac{\xi}{x-x'} W_{q,q'} \\
&\quad \times e^{-\Lambda^{-4}\Delta_{FI}^2} e^{-2\Lambda^{-4}\Delta_{FK}\Delta_{IK}(x+x')(1-x+1-x')}.
\end{aligned} \tag{135}$$

To extract the divergence, we integrate Eq. (134) by parts with respect to x':

$$\begin{aligned}
&\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{B}}_{\mathrm{IN}} \\
&\quad = -\frac{N_c g_\Lambda^2}{16\pi^4}\int_{2\epsilon}^{1-\epsilon} dx\Bigg[-\log(x-x')F(x,x')\big|_{x'=x-\epsilon} \\
&\qquad +\log(x-x')F(x,x')\big|_{x'=\epsilon}+\int_{\epsilon}^{x-\epsilon} dx' \\
&\qquad \times\log(x-x')\frac{dF(x,x')}{dx'}\Bigg] \equiv B_1+B_2+B_3.
\end{aligned} \tag{136}$$

The first contribution to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{B}}_{\mathrm{IN}}$ is

$$\begin{aligned}
B_1 &= \frac{N_c g_\Lambda^2}{16\pi^4}\frac{\log\epsilon}{\epsilon}\int_{2\epsilon}^{1-\epsilon} dx\int d^2k_\perp d^2k'_\perp \xi W_{q,q'} \\
&\quad \times e^{-\Lambda^{-4}\Delta_{FI}^2} e^{-2\Lambda^{-4}\Delta_{FK}\Delta_{IK}(x+x')} \\
&\quad \times(1-x+1-x')\big|_{x'=x-\epsilon}.
\end{aligned} \tag{137}$$

To simplify this, we change variables from x to y:

$$x = y(1-3\epsilon)+2\epsilon, \tag{138}$$

and from $\vec{k}_\perp$ and $\vec{k}'_\perp$ to $\vec{Q}_\perp$ and $\vec{N}_\perp$:

$$\vec{k}_\perp = \frac{\vec{Q}_\perp+\sqrt{\epsilon}\vec{N}_\perp}{2},$$

$$\vec{k}'_\perp = \frac{\vec{Q}_\perp-\sqrt{\epsilon}\vec{N}_\perp}{2}. \tag{139}$$

Then

$$\begin{aligned}
B_1 &= \frac{N_c g_\Lambda^2}{64\pi^4}(1-3\epsilon)\log\epsilon\int_0^1 dy\int d^2Q_\perp d^2N_\perp \xi W_{q,q'} e^{-\Lambda^{-4}\Delta_{FI}^2} e^{-2\Lambda^{-4}\Delta_{FK}\Delta_{IK}(2y[1-3\epsilon]+3\epsilon)} \\
&\quad \times(2-2y[1-3\epsilon]-3\epsilon)\big|_{x'=x-\epsilon}.
\end{aligned} \tag{140}$$

As $\epsilon\to 0$, the only contribution that survives is

$$B_1=\frac{N_c g_\Lambda^2}{16\pi^4}\delta_{q,q'}\log\epsilon\int_0^1 dyy(1-y)\overline{L}_{l'}^{(e)}(y)\overline{L}_l^{(e)}(y)$$
$$\times\int d^2Q_\perp T_{t'}^{(d)}(Q/2)T_t^{(d)}(Q/2)\int d^2N_\perp e^{-2\Lambda^{-4}N^4}$$
$$=\delta_{q,q'}\delta_{t,t'}\frac{N_c g_\Lambda^2}{4\pi^2}\sqrt{\frac{\pi}{2}}\Lambda^2\int_0^1 dx\overline{L}_{l'}^{(e)}(x)\overline{L}_l^{(e)}(x)$$
$$\times x(1-x)\log\epsilon. \tag{141}$$

Since $\langle g_1' g_2'|\mathcal{M}^2(\Lambda)|g_1 g_2\rangle_{\rm SE}^{\rm D}$ is the part of $\langle g_1' g_2'|\mathcal{M}^2(\Lambda)|g_1 g_2\rangle_{\rm SE}$ with the $\log\epsilon$ [see Eq. (95)], from Eq. (112) we see that $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm SE}^{\rm D}$ is just $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm SE}^{\rm F}$ with the $[\log x-11/12]$ factor replaced with $-\log\epsilon$. This means that

$$B_1=-\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm SE}^{\rm D}. \tag{142}$$

Thus the divergence in $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm IN}^{\rm B}$ cancels $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm SE}^{\rm D}$.

The second contribution to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm IN}^{\rm B}$ is

$$B_2=-\frac{N_c g_\Lambda^2}{8\pi^3}\int_{2\epsilon}^{1-\epsilon}dx\log(x-x')$$
$$\times\int dkdk'd\gamma kk'\theta(k)\theta(k')\theta(\gamma)\theta(2\pi-\gamma)$$
$$\times\frac{\xi}{x-x'}W_{q,q'}e^{-\Lambda^{-4}\Delta_{FI}^2}e^{-2\Lambda^{-4}\Delta_{FK}\Delta_{IK}}$$
$$\times(x+x')(1-x+1-x')|_{x'=\epsilon}. \tag{143}$$

To evaluate this, we change variables from k' to $s=k'/\sqrt{\epsilon}$. Then the leading term as $\epsilon\to 0$ is

$$B_2=-\epsilon^{e+1/2}\frac{N_c g_\Lambda^2}{8\pi^3}d\sigma_{t',0}\lambda_{l,0}^{(e)}\int_0^1 dx(2-x)\log x$$
$$\times\int dkdsd\gamma ks\,\theta(k)\theta(s)\theta(\gamma)\theta(2\pi-\gamma)\overline{L}_l^{(e)}(x)$$
$$\times T_t^{(d)}(k)W_{q,q'}e^{-\Lambda^{-4}[s^4+k^4/x^2(1-x)^2]}$$
$$=0, \tag{144}$$

since $e>-1/2$.

To simplify the third contribution to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm IN}^{\rm B}$, we take the derivative of $F(x,x')$ and take $\epsilon\to 0$. Then

$$B_3=-\frac{N_c g_\Lambda^2}{8\pi^3}\int D\log(x-x')W_{q,q'}T_{t'}^{(d)}(k')\overline{L}_l^{(e)}(x)T_t^{(d)}(k)$$
$$\times e^{-\Lambda^{-4}(\Delta_{FK}^2+\Delta_{IK}^2)}\sum_{i=1}^5 E_i'\prod_{m=1;m\neq i}^5 E_m, \tag{145}$$

where

$$E_1=\frac{1}{x-x'},$$
$$E_2=1,$$
$$E_3=\overline{L}_{l'}^{(e)}(x'),$$
$$E_4=x+x',$$
$$E_5=1-x+1-x', \tag{146}$$

and

$$E_1'=\frac{1}{(x-x')^2},$$
$$E_2'=-2\Lambda^{-4}(\Delta_{FK}\Delta_{FK}'+\Delta_{IK}\Delta_{IK}'),$$
$$E_3'=\overline{L}_{l'}'^{(e)}(x'),$$
$$E_4'=1,$$
$$E_5'=-1 \tag{147}$$

($E_i'=dE_i/dx'$, except for $i=2$), and

$$\overline{L}_{l'}'^{(e)}(x')=\frac{d\overline{L}_{l'}^{(e)}(x')}{dx'}$$
$$=\left(e-\frac{1}{2}\right)[x'(1-x')]^{e-1/2-1}$$
$$\times[1-2x']\sum_{m'=0}^{l'}\lambda_{l',m'}^{(e)}x'^{m'}+[x'(1-x')]^{e-1/2}$$
$$\times\sum_{m'=1}^{l'}m'\lambda_{l',m'}^{(e)}x'^{m'-1}, \tag{148}$$

and

$$\Delta_{FK}'=\frac{d\Delta_{FK}}{dx'}=\frac{k'^2}{(1-x')^2}-\frac{(\vec{k}_\perp-\vec{k}_\perp')^2}{(x-x')^2},$$
$$\Delta_{IK}'=\frac{d\Delta_{IK}}{dx'}=\frac{k'^2}{x'^2}-\frac{(\vec{k}_\perp-\vec{k}_\perp')^2}{(x-x')^2}. \tag{149}$$

This means that we can write the contribution to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ from the instantaneous interaction below the cutoff as a divergent part and a finite part:

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{B}}_{\mathrm{IN}} = -\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{D}}_{\mathrm{SE}} + \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{B,F}}_{\mathrm{IN}} \tag{150}$$

where

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{B,F}}_{\mathrm{IN}} = -\frac{N_c g_\Lambda^2}{8\pi^3}\int D\log(x-x')W_{q,q'}T^{(d)}_{t'}(k')\overline{L^{(e)}_l}(x)T^{(d)}_t(k) \times e^{-\Lambda^{-4}(\Delta^2_{FK}+\Delta^2_{IK})}\sum_{i=1}^{5}E'_i\prod_{m=1;m\neq i}^{5}E_m\,. \tag{151}$$

Using the results of this section, the expression in Eq. (109) for the matrix elements of the IMO becomes

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle = \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{KE}} + \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{F}}_{\mathrm{SE}} + \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{CON}} + \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{F}}_{\mathrm{EX}} + \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{IN+EX}} + \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle^{\mathrm{B,F}}_{\mathrm{IN}}\,. \tag{152}$$

Each of these terms is finite and we have taken $\epsilon\to 0$ everywhere. We have written the first two terms as sums that can be computed numerically, and the four remaining terms as five-dimensional integrals that can be grouped into one integral suitable for numerical calculation. (See Appendix C for a discussion of some of the technical issues involved in the numerical calculation of these matrix elements.) Once we have computed these matrix elements, we can diagonalize the matrix to obtain glueball states and masses.

VI. RESULTS AND ERROR ANALYSIS

In this section we diagonalize the IMO matrix, obtaining glueball states and masses, and then discuss the sources of error in the calculation. We begin by discussing how we assign quantum numbers to our numerical results for glueball states and proceed with a discussion of the procedure that we use to compute these results. We derive the nonperturbative cutoff dependence of the coupling and discuss the cutoff dependence of our glueball masses. We use this analysis to choose the value of the cutoff that minimizes our errors. We then present the spectrum that we find with this optimal cutoff and compare it to recent results from other approaches. The last results that we present are the probability densities for our five lightest glueballs. We conclude this section by discussing the sources of error in our calculation and estimating the sizes of these errors.

A. The procedure for calculating results

We represent a state using the notation J^{PC}_j, where J is our best guess for the spin of the state, P is our best guess for the parity of the state, C represents the charge-conjugation eigenvalue of the state (it is always + because we have two gluons), and j is the projection of the state's spin onto the three-axis. We need to distinguish states with identical J's and P's and different j's because we do not have manifest rotational symmetry. If $J=0$, we omit the subscript j in the state notation. We use an asterisk in the state notation next to the value of C to denote an excited state with the given quantum numbers. We will base our guesses for J and P on the numerical degeneracies of the states that have identical J's and P's and different j's, and on the ordering of the states according to lattice data (see the discussion below). We consider only the five lightest glueballs (not counting as distinct those states that differ only in their value of j). The five lightest glueballs have spins $J\leqslant 2$. This means that we need to consider only $|j|\leqslant 2$. For a given $|j|$, the states with $j=|j|$ and $j=-|j|$ are degenerate and simply related (see Appendix C); so we explicitly consider only $j=0,1,2$. We consider nine values of the coupling: $\alpha_\Lambda = g^2_\Lambda/(4\pi) = 0.1,0.2,0.3,\ldots,0.9$. To calculate our results, we implement the following four-step procedure.

We execute the first step for all pairs (j,α_Λ). In this step, we define $\Lambda=1$ and diagonalize $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ with all four spin basis functions ($q=1,2,3,4$), but with only the lowest transverse-magnitude basis function ($t=0$) and the two lowest longitudinal basis functions ($l=0,1$). (It is necessary to use an even number of longitudinal basis functions so that symmetry and antisymmetry of the wave function under $x\to 1-x$ are equally represented.) We perform this diagonalization as a function of the basis-function parameters d and e that determine the widths of the transverse-magnitude and longitudinal wave functions, respectively, and we find the values of d and e that minimize the ground-state mass. This yields what we consider to be the optimal wave-function widths for each pair (j,α_Λ).

We also execute the second step of the procedure for all pairs (j,α_Λ). In this step, we fix d and e to be their computed optimal values and again define $\Lambda=1$. We diagonalize the matrix with all four spin basis functions, N_t transverse-magnitude basis functions, and $N_1=2N_t$ longitudinal basis functions, for a total of $8N_t^2$ basis functions, with $N_t=1,2,3,\ldots,10$. We use twice as many longitudinal functions as transverse-magnitude functions because $|q,l,t,j\rangle$ is zero if

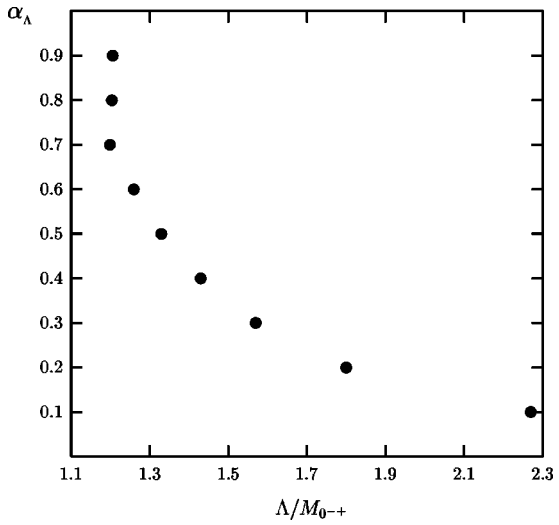

FIG. 5. The coupling as a function of the cutoff. We show the cutoff in units of the mass of the 0^{-+} state, and we use 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions. Using the recent anisotropic Euclidean lattice result of Morningstar and Peardon for the mass of the 0^{-+} state [27], we estimate that the cutoff is roughly varying from about 3.1–6.0 GeV in this figure.

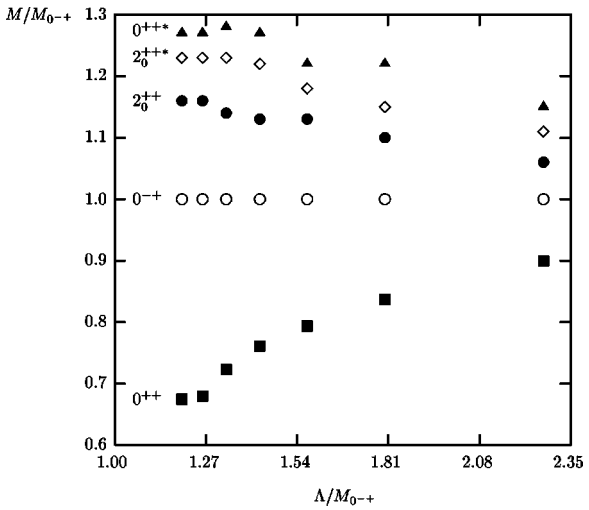

FIG. 6. The masses of the five lightest glueballs with $j=0$, as functions of the cutoff. The masses and the cutoff are displayed in units of the mass of the 0^{-+} state. The seven values of the cutoff at the points that we display correspond, from right to left, to $\alpha_\Lambda = 0.1,0.2,0.3,...,0.7$. We use 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

$l+j$ is even and $q=4$, or if $l+j$ is odd and $q\neq 4$. We want to use as many basis functions as possible, but we find for all pairs (j,α_Λ) that when $N_t>7$, the statistical errors from the Monte Carlo integrations of the matrix elements become overwhelming and the spectrum and wave functions become unreliable. The evidence of the breakdown is sudden contamination of the low-lying wave functions with high-order components. (See Appendix C for a more complete discussion of this topic.) Thus in the remainder of our procedure, we analyze the results that we find in this step when $N_t=7$, which corresponds to 392 basis functions.

In the third step of the procedure, we use the mass of the 0^{-+} state (our most numerically reliable state) to determine the value of Λ for each α_Λ. To do this, we note that Λ is the only mass scale in the problem. This means that the mass of the 0^{-+} state, $M_{0^{-+}}$, can be written

$$M_{0^{-+}} = b(\alpha_\Lambda)\Lambda, \tag{153}$$

where b is a dimensionless function of α_Λ. Since we defined $\Lambda=1$ in the second step of our procedure, the diagonalization of the IMO as a function of α_Λ yielded $b(\alpha_\Lambda)$. In this third step, we consider Λ to be a parameter and define $M_{0^{-+}}$ to be a constant. Then for a given coupling, we can use the results of the second step of our procedure to write the cutoff in units of $M_{0^{-+}}$:

$$\Lambda/M_{0^{-+}} = \frac{1}{b(\alpha_\Lambda)}. \tag{154}$$

Figure 5 shows the result for the third step of our procedure: a plot of the coupling as a function of the cutoff. When $\alpha_\Lambda>0.7$ it is not a single-valued function of the cutoff. This is an indication that the coupling is too large. For this reason, we consider only $\alpha_\Lambda \leqslant 0.7$ in the remainder of our procedure. When $\alpha_\Lambda \leqslant 0.7$, the coupling decreases as the cutoff increases, as expected. However, α_Λ depends on Λ more strongly than one may expect. We expect that perturbative pure-glue QCD would indicate that $\alpha_\Lambda \sim 1/\ln\Lambda$, but the result that we get is much closer to $\alpha_\Lambda \sim \exp(-a\Lambda)$, where a is a constant. The reason for this is that the truncation of the perturbative series for $\mathcal{M}^2(\Lambda)$ and the truncation of the free-sector expansion of the states introduce spurious cutoff dependence in our results for physical quantities. $M_{0^{-+}}$ is one such physical quantity. The spurious cutoff dependence of $M_{0^{-+}}$ is manifested through incorrect dependence of $b(\alpha_\Lambda)$ on α_Λ. This means that Λ has to compensate by depending on α_Λ incorrectly in order to keep $M_{0^{-+}}$ a constant function of α_Λ. This results in the strong dependence of α_Λ on Λ that is shown in Fig. 5.

Using the recent anisotropic Euclidean lattice results of Morningstar and Peardon [27], we can make a rough estimate of the range over which our cutoff is varying in Fig. 5. They found that the mass of the 0^{-+} state is $M_{0^{-+}}=2.590 \pm 0.040 \pm 0.130$ GeV. This means that our cutoff is varying from about 3.1–6.0 GeV in Fig. 5.

The fourth step of our procedure is to determine the optimal value of the cutoff, or equivalently, the optimal value of the coupling. We use two criteria to determine this. First we determine the value of the cutoff for which the computed masses are most independent of the cutoff. Figures 6, 7, and 8 show the masses of our states with $j=0,1,2$, respectively, as functions of the cutoff. The masses and the cutoff are displayed in units of $M_{0^{-+}}$. (Recall that $M_{0^{-+}}$ was defined to be independent of the cutoff in the process of defining the cutoff as a function of the coupling.) The seven values of the cutoff at the points that we display correspond, from right to left, to $\alpha_\Lambda = 0.1,0.2,0.3,...,0.7$. It is difficult to tell from these

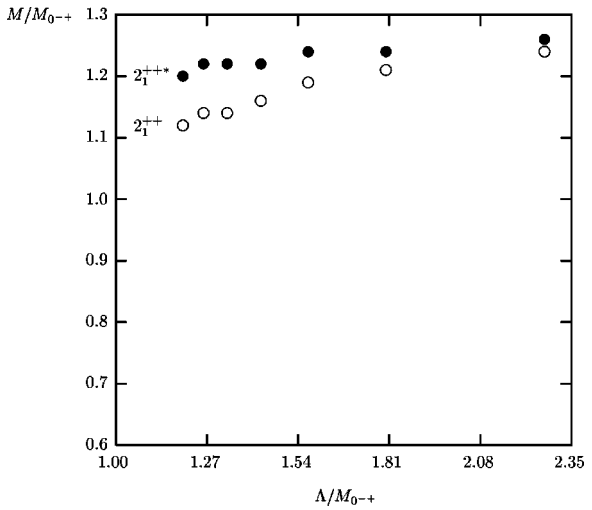

FIG. 7. The masses of the two lightest glueballs with $j=1$, as functions of the cutoff. The masses and the cutoff are displayed in units of the mass of the 0^{-+} state. The seven values of the cutoff at the points that we display correspond, from right to left, to $\alpha_\Lambda=0.1,0.2,0.3,...,0.7$. We use 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

plots where the cutoff dependence is weakest (more points are needed), but we see that the dependence is relatively weak from $\alpha_\Lambda=0.5$ to $\alpha_\Lambda=0.7$, which corresponds to $\Lambda/M_{0^{-+}}=1.33$ to $\Lambda/M_{0^{-+}}=1.20$.

The second criterion that we use to determine the optimal cutoff is the degree to which the states with a given J and P and different j's are degenerate. This determines the cutoff that minimizes the violation of rotational symmetry. We find that these degeneracies are best when $\alpha_\Lambda=0.5$. Given this

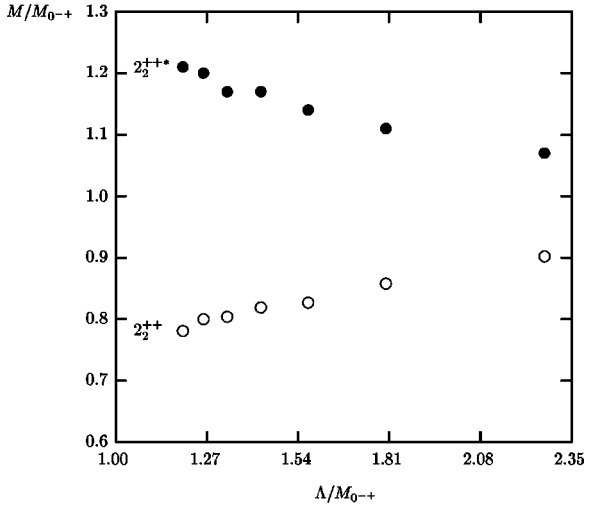

FIG. 8. The masses of the two lightest glueballs with $j=2$, as functions of the cutoff. The masses and the cutoff are displayed in units of the mass of the 0^{-+} state. The seven values of the cutoff at the points that we display correspond, from right to left, to $\alpha_\Lambda=0.1,0.2,0.3,...,0.7$. We use 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

TABLE I. The glueball masses from our calculation compared to those from a few other approaches: an average of lattice calculations [28], an effective-Hamiltonian approach [29], and a method that analyzes the asymptotic behavior of Wilson loops [30]. We display the masses in units of the mass of the 0^{++} state. The uncertainties for our results are only the statistical uncertainties associated with the Monte Carlo evaluation of the matrix elements of $\mathcal{M}^2(\Lambda)$. The three values of the masses for the 2^{++} and $2^{++}*$ states for our calculation correspond to $j=0,1,2$. We use the optimal coupling $\alpha_\Lambda=0.5$, with 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

State	$M/M_{0^{++}}$	Lattice [28]	Effective Hamiltonian [29]	Wilson loops [30]
0^{-+}	1.38 ± 0.02	1.34 ± 0.18	1.36	1.62
2^{++}	1.58 ± 0.01 1.58 ± 0.02 1.11 ± 0.01	1.42 ± 0.06	1.32	1.64
$2^{++}*$	1.70 ± 0.01 1.68 ± 0.02 1.62 ± 0.02	1.85 ± 0.20	1.88	2.36
$0^{++}*$	1.77 ± 0.02	1.78 ± 0.12	1.88	1.72

and the fact that Figs. 6–8 indicate that the cutoff dependence of the masses is weak when $\alpha_\Lambda=0.5$ to $\alpha_\Lambda=0.7$, we determine that $\alpha_\Lambda=0.5$ is the optimal coupling, and thus the optimal cutoff is $\Lambda/M_{0^{-+}}=1.33$. Using the result of Morningstar and Peardon for the mass of the 0^{-+} state, we estimate that this cutoff is about 3.4 GeV.

B. Results

Now we present our main results. Our glueball masses for $\alpha_\Lambda=0.5$ are summarized in Table I, in units of the mass of the ground state (the 0^{++} state). For the sake of comparison, Table I also shows results for the masses from a few other approaches: an average of lattice calculations [28], an effective-Hamiltonian approach [29], and a method that analyzes the asymptotic behavior of Wilson loops [30]. The uncertainties in our results that we report in Table I are only the statistical uncertainties associated with the Monte Carlo evaluation of the matrix elements of $\mathcal{M}^2(\Lambda)$. The full errors are much larger (see the discussion of sources of error below). We list three values of the masses for the 2^{++} and $2^{++}*$ states for our calculation, corresponding to $j=0,1,2$. In each case the three masses would be degenerate if our calculation were exact. Our results agree with the lattice results quite well, reasonably well with the effective-Hamiltonian results, and less well with the results from the Wilson-loop approach.

We display our spectrum graphically in Fig. 9. The masses are plotted in units of the mass of the 0^{++} state and the vertical widths of the levels represent the statistical uncertainties in the masses. The black lines connect the states that we believe should be degenerate. We see that the 2_0^{++} and 2_0^{++*} glueballs are relatively degenerate with their 2_1^{++} and 2_1^{++*} counterparts, and the 2_2^{++*} is not too bad, but the

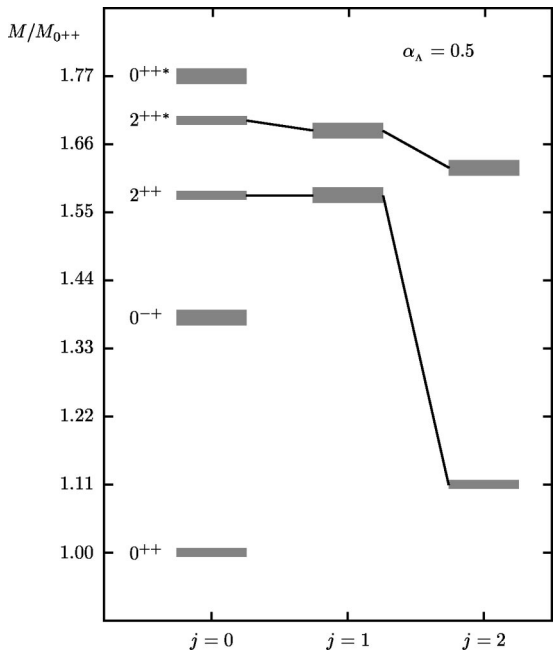

FIG. 9. Our glueball spectrum. The masses are plotted in units of the mass of the 0^{++} state and the vertical widths of the levels represent the statistical uncertainties in the masses. The black lines connect the states that we believe should be degenerate. We use the optimal coupling $\alpha_\Lambda = 0.5$, with 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

2_2^{++} glueball is much too light. Our labeling of the states and subsequent assignment of the expected degeneracies are based on the ordering of the lattice states (see Table I) and the apparent degeneracies of the 2_0^{++} and $2_0^{++}*$ states with the 2_1^{++} and $2_1^{++}*$ states, respectively.

We want to show some of the features of the glueball wave functions. Rather than presenting the spin-dependent wave functions themselves, we present more illuminating spin-independent probability densities. A glueball state has the plane-wave normalization shown in Eq. (71) as long as the wave function $\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp)$ satisfies

$$\int d^2k_\perp dx\, \theta(x)\,\theta(1-x) \sum_{s_1 s_2} |\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp)|^2 = 1. \quad (155)$$

This implies that

$$\int_0^\infty d\left(\frac{k}{\Lambda}\right) \int_0^1 dx \Pi(x,k/\Lambda) = 1, \quad (156)$$

where we define the dimensionless probability density $\Pi(x,k/\Lambda)$ by

$$\Pi(x,k/\Lambda) = 2\pi\Lambda k \sum_{s_1 s_2} |\Phi^{jn}_{s_1 s_2}(x,\vec{k}_\perp)|^2. \quad (157)$$

We show the probability densities for some of our glueballs in Figs. 10–14. The masses of the states tend to increase as the probability densities move away from the region $x \sim \frac{1}{2}$ and more towards the edges. This is what we expect based on the form of the kinetic energy of a free state. Notice that the probability density for the 2_2^{++} glueball is peaked around the region $x \sim \frac{1}{2}$ and looks similar to the probability density for the 0^{++} glueball. This is consistent with its small mass.

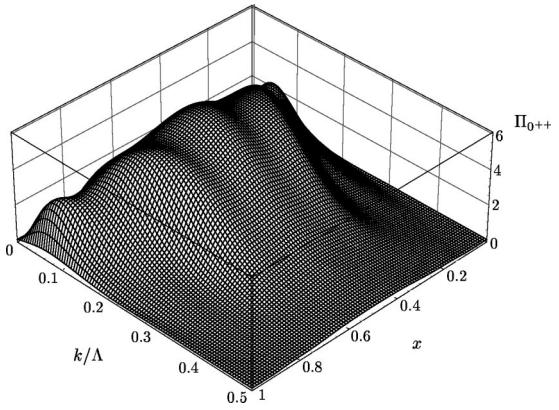

FIG. 10. The probability density of the 0^{++} glueball. We use the optimal coupling $\alpha_\Lambda = 0.5$, with 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

C. Error analysis

We now turn to a discussion of the sources of error in our calculation. The sources of error are truncation of the renormalized IMO at $\mathcal{O}(g_\Lambda^2)$, truncation of the free-sector expansion of physical states at two gluons, truncation of the basis-function expansion of wave functions, and numerical approximation of the matrix elements of $\mathcal{M}^2(\Lambda)$.

We do not know how to estimate the size of any physical effects that require nonperturbative renormalization. However, we can naively estimate the size of the effects of higher-order perturbative renormalization. We have calculated the matrix elements of $\mathcal{M}^2(\Lambda)$ through $\mathcal{O}(\alpha_\Lambda)$; so the corrections to these matrix elements should be $\mathcal{O}(\alpha_\Lambda^2)$. This

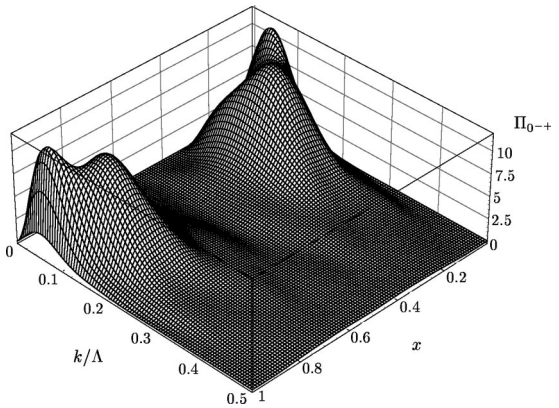

FIG. 11. The probability density of the 0^{-+} glueball. We use the optimal coupling $\alpha_\Lambda = 0.5$, with 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

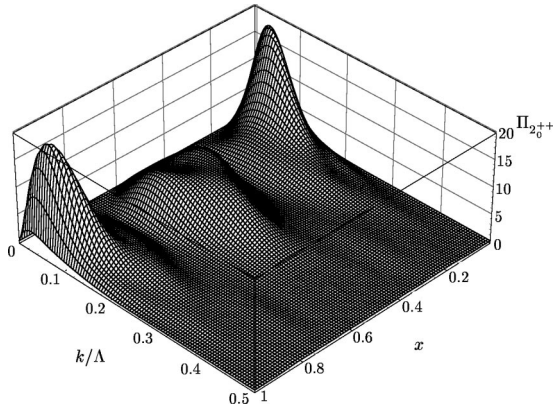

FIG. 12. The probability density of the 2_0^{++} glueball. We use the optimal coupling $\alpha_\Lambda = 0.5$, with 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

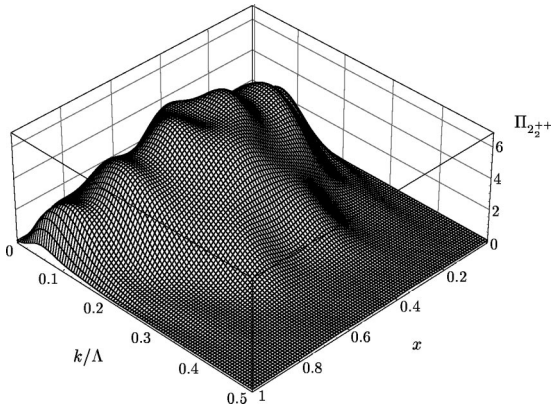

FIG. 14. The probability density of the 2_2^{++} glueball. We use the optimal coupling $\alpha_\Lambda = 0.5$, with 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

translates to corrections to the mass spectrum of $\mathcal{O}(\alpha_\Lambda^2/2)$. Since we have used $\alpha_\Lambda = 0.5$, we estimate that the uncertainty in the mass spectrum from the effects of higher-order perturbative renormalization is about 13%.

We do not know how to estimate the size of any physical effects that require an infinite number of particles. In fact, until we include at least two free sectors, it is impossible to directly estimate the size of corrections from higher free sectors. However, we can use the lack of degeneracy of the 2_2^{++} state with the 2_0^{++} and 2_1^{++} states to estimate these corrections. According to Table I, the discrepancy in the various 2^{++} states is about 33%, if we believe the quoted lattice result. Since the uncertainty in the mass spectrum from effects of higher-order perturbative renormalization is around 13%, an uncertainty of 30% due to the truncation of the free-sector expansion is necessary to explain the lack of rotational symmetry in the spectrum (neglecting the other sources of error, which we expect to be small).

As we mentioned, when we increase the number of basis functions that we use to represent the wave functions, we find that we reach a point where the statistical errors from the Monte Carlo evaluations of the matrix elements are overwhelming and cause our results to become completely unreliable (see Appendix C). For this reason, we have to truncate our basis-function expansion for the wave functions at $N_t = 7$ transverse-magnitude functions. ($N_1 = 2N_t = 14$, and there are 4 spin basis functions, for a total of 392 basis functions.) This truncation results in additional errors in our results. In Figs. 15, 16, and 17, we show the convergence of the masses of the states with $j = 0,1,2$ respectively, as functions of N_t. The masses do not decrease as rapidly as functions of the number of states as one might expect. This is primarily because we have already optimized the states quite a bit by determining the widths of the transverse-magnitude and longitudinal basis functions that minimize the mass of the 0^{++} state (using $N_t = 1$). Our best guess for the uncertainty that we introduce into the spectrum when we truncate the basis-function expansion, based on Figs. 15–17, is a few percent.[12]

We can estimate the uncertainty in our results associated with the Monte Carlo evaluation of our matrix elements. To do this, we compute our results with $\alpha_\Lambda = 0.5$ four times, obtaining statistically independent results, and we compute the standard deviations of the masses that we obtain. This leads us to estimate that the uncertainty in the spectrum from the Monte Carlo routine is 1–2 %. This is the uncertainty that we report in Table I and Fig. 9. Combining this uncertainty with the others leads us to estimate that the total uncertainty in our results is about 33%.

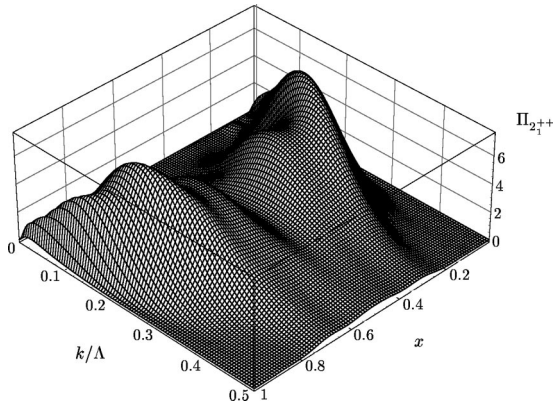

FIG. 13. The probability density of the 2_1^{++} glueball. We use the optimal coupling $\alpha_\Lambda = 0.5$, with 14 longitudinal basis functions, 7 transverse-magnitude basis functions, and 4 spin basis functions, for a total of 392 basis functions.

VII. CONCLUSION

We have presented a formalism for pure-glue QCD that allows the physical states of the theory to rapidly converge in a free-sector expansion. In this approach, we force the free-

[12]Technically, this is not an uncertainty because improving the states can only reduce their masses, according to the variational principle. However, our discussion of errors is not meant to be rigorous.

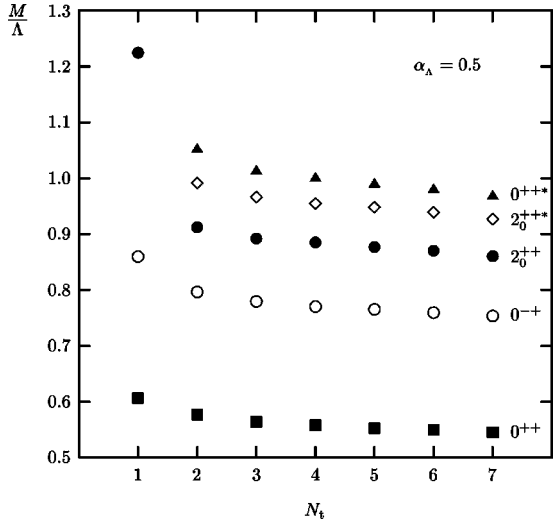

FIG. 15. The masses of the five lightest glueballs with $j=0$, in units of the cutoff, as functions of the number of transverse-magnitude basis functions, N_t. We use the optimal coupling $\alpha_\Lambda = 0.5$, with four spin basis functions and $N_l = 2N_t$ longitudinal basis functions, for a total of $8N_t^2$ basis functions.

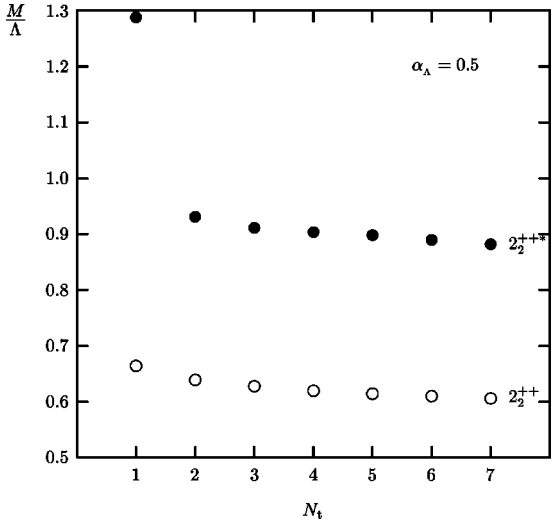

FIG. 17. The masses of the two lightest glueballs with $j=2$, in units of the cutoff, as functions of the number of transverse-magnitude basis functions, N_t. We use the optimal coupling $\alpha_\Lambda = 0.5$, with four spin basis functions and $N_l = 2N_t$ longitudinal basis functions, for a total of $8N_t^2$ basis functions.

state matrix elements of the IMO to satisfy three conditions to make the desired expansion possible. First, the diagonal matrix elements of the IMO must be dominated by the free part of the IMO. Second, the off-diagonal matrix elements of the IMO must quickly decrease as the difference of the free masses of the states increases. Third, the free mass of a free state must quickly increase as the number of particles in the state increases.

We assume that we can use perturbation theory to derive the operators of the theory, and if this is valid, then the first condition is automatically satisfied. To satisfy the second condition, we place a smooth cutoff on the IMO. We use

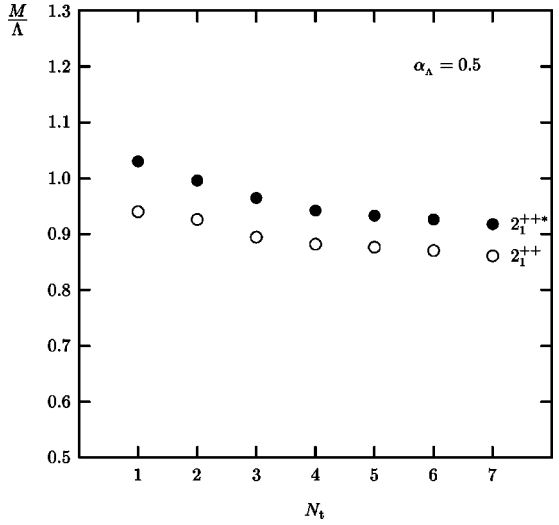

FIG. 16. The masses of the two lightest glueballs with $j=1$, in units of the cutoff, as functions of the number of transverse-magnitude basis functions, N_t. We use the optimal coupling $\alpha_\Lambda = 0.5$, with four spin basis functions and $N_l = 2N_t$ longitudinal basis functions, for a total of $8N_t^2$ basis functions.

LFFT so that the effects of the vacuum are isolated in particles with zero longitudinal momentum, and we remove these particles from the theory with the intent of replacing their physical effects with interactions. This makes it reasonable to expect that the third condition on the IMO is satisfied automatically due to the free-particle dispersion relation of LFFT.

The cutoff that we use violates a number of physical principles of light-front pure-glue QCD. However, by requiring the IMO to produce cutoff-independent physical quantities and by requiring it to respect the unviolated physical principles of the theory, we are able to derive recursion relations that uniquely determine the IMO to all orders in perturbation theory.

We have applied our method to the calculation of physical states and masses. For this calculation, we approximated all physical states as two-gluon states. We calculated the color parts of the states analytically, and we expanded the states' momentum and spin degrees of freedom in terms of basis functions. We designed the states to be simultaneous eigenstates of the IMO, the three-momentum operator, and the projection of the internal rotation generator onto the three-axis.

Using our recursion relations for the IMO, we calculated to second order in perturbation theory the two-gluon to two-gluon matrix element of the IMO, which is required for the calculation of physical states. We then used it to calculate the IMO matrix in terms of the basis functions. We showed that the infrared divergences in the matrix from exchanged gluons with infinitesimal longitudinal momentum cancel when treated properly.

In order to diagonalize the IMO matrix, we computed the five-dimensional integrals in the matrix elements using Monte Carlo methods. We calculated the glueball spectrum for a range of couplings and found that we could not use more than about 400 basis functions without the statistical

errors becoming overwhelming. We used the mass of our 0^{-+} glueball to compute the nonperturbative cutoff dependence of the coupling, and we analyzed the cutoff dependence of the spectrum. We found that the cutoff that minimizes our errors is $\Lambda/M_{0^{-+}}=1.33$. The corresponding coupling is $\alpha_\Lambda=0.5$. We presented the probability densities for some of our glueballs and found that our results for the spectrum compare favorably with some recent results from other approaches. The largest discrepancy seems to be the 2_2^{++} state, which is much too light. Finally, we analyzed the errors in our calculation from the various possible sources, and estimated the total uncertainty in our spectrum to be 33%.

There are two main paths that we can take for future work with our approach. The first path is to further test our method with the theories that we have considered so far. Since the scalar theory that we considered in Ref. [1] is relatively simple, it would be interesting to use it compute the IMO to higher orders in perturbation theory. This would require us to solve the integral equation for the cutoff-independent reduced interaction and could be used to further check our conjecture that our IMO leads to correct scattering amplitudes order-by-order in perturbation theory.

In pure-glue QCD, we can further test our approach for calculating physical states by computing the IMO to higher orders in perturbation theory and by keeping more free sectors in the expansion of the states. However, to keep more free sectors in the expansion of the states, we have to be able to calculate IMO matrices that have more degrees of freedom, while controlling the statistical errors in the spectrum. This means that we need a better algorithm for determining how accurately individual matrix elements of the IMO have to be computed in order to get a desired uncertainty in the spectrum. We could also use a better basis that requires fewer momentum functions to represent a wave function. Overcoming these problems will be challenging, but it is important to test our method by studying the rate of convergence of the free-sector expansion of states as a function of both the cutoff and the masses of the states.

Another test of our method that we can do with pure-glue QCD is to analyze the interaction that we have derived in this paper to test the conjecture that it is logarithmically confining. It would also be interesting to analyze the long-range parts of higher-order interactions to see if the perturbative series for the interaction is building towards a linearly confining potential. Analyzing the long-range parts of higher-order interactions may be much easier than computing these interactions in their entirety.

The second main path that we can take in the future is to extend our method to other theories and operators. In order to compute quantities that can be compared with experiment, we wish to extend our method to full QCD.[13] This is complicated for two reasons. First, there is additional algebraic and numerical complexity from the vertices involving quarks. Second, quark masses complicate the method for determining the IMO because they increase the number of reduced interactions that can be cutoff-independent [18]. In addition, if large and small quark masses are considered simultaneously, then efficient numerical representation of the states and accurate calculation of the IMO's matrix elements become more difficult. Masses also quickly enlarge the parameter space that must be explored to compare to experimental data.

We can also extend our method by applying it to the computation of operators other than the IMO, such as the rotation generators, the parity operator, and currents. The rotation generators and the parity operator are of particular interest because they may aid in the classification of the physical states of a theory.

In summary, there are many avenues of research that must be explored, and some of them are quite complex. However, all the improvements that we have discussed are necessary if we are to accurately represent the physical states of quantum field theories as rapidly convergent expansions in free sectors.

APPENDIX A: CONVENTIONS FOR LIGHT-FRONT PURE-GLUE QCD

The purpose of this appendix is to state our conventions. With these conventions, any four-vector a is written in the form

$$a=(a^+,a^-,\vec{a}_\perp), \tag{A1}$$

where in terms of equal-time vector components

$$a^\pm=a^0\pm a^3 \tag{A2}$$

and

$$\vec{a}_\perp=\sum_{i=1}^{2} a_\perp^i \hat{e}_i=\sum_{i=1}^{2} a^i \hat{e}_i, \tag{A3}$$

where $\hat{e}_i$ is the unit vector pointing along the i-axis. The inner product is

$$a\cdot b=\frac{1}{2}a^+b^-+\frac{1}{2}a^-b^+-\vec{a}_\perp\cdot\vec{b}_\perp \tag{A4}$$

and

$$\vec{a}_\perp^{\,2}=\vec{a}_\perp\cdot\vec{a}_\perp . \tag{A5}$$

A spacetime coordinate is a four-vector, and according to Eq. (A1), it is written

$$x=(x^+,x^-,\vec{x}_\perp). \tag{A6}$$

The time component is chosen to be x^+. x^- is referred to as the longitudinal component, and $\vec{x}_\perp$ contains the transverse components.

The gradient operator is treated just like any other four-vector. Its components are

[13]We are not thinking of including QCD effects that require nonperturbative renormalization or an infinite number of particles because these would require a new method rather than an extension of our current approach.

$$\partial^{\pm} = 2\frac{\partial}{\partial x^{\mp}} \tag{A7}$$

and

$$\partial^{i}_{\perp} = \frac{-\partial}{\partial x^{i}_{\perp}}. \tag{A8}$$

The canonical Lagrangian density for pure-glue QCD is

$$\mathcal{L} = -\frac{1}{4} F_{c\mu\nu} F_{c}^{\mu\nu}, \tag{A9}$$

where

$$F_{c}^{\mu\nu} = \partial^{\mu} A_{c}^{\nu} - \partial^{\nu} A_{c}^{\mu} - g A_{c_1}^{\mu} A_{c_2}^{\nu} f^{c_1 c_2 c}. \tag{A10}$$

Greek indices are Lorentz indices, c's are color indices, repeated indices are summed over, and the f's are the SU(N_c) structure constants.

We derive the canonical Hamiltonian from $\mathcal{L}$ by the following procedure.

(1) We choose the light-cone gauge, $A_c^+ = 0$.

(2) We derive the Euler-Lagrange equation that determines A_c^- in terms of $\vec{A}_{\perp c}$.

(3) Using the canonical procedure and treating the field classically (i.e., letting it and its derivatives commute), we derive the Hamiltonian in terms of $\vec{A}_{\perp c}$, dropping terms that are zero if the gluon field is zero at spacetime infinity.

(4) We quantize the gluon field by expanding it in terms of free-particle creation and annihilation operators. (We define the field expansion and its inverse longitudinal derivative below.)

(5) In each term in the Hamiltonian, we treat the creation and annihilation operators as if they commute and move all the creation operators to the left of all the annihilation operators. (This ''normal ordering'' drops the so-called ''self-inertias,'' as well as some constants.)

(6) We drop the terms in the Hamiltonian that have no effect if there are no particles with $p^+ = 0$.

We work in the Schrödinger representation, where operators are time-independent and states are time-dependent. Thus we quantize the field by defining it to be a superposition of solutions to the Klein-Gordon equation (since gluons are bosons), with the quantization surface $x^+ = 0$:

$$\vec{A}_{\perp c}(x^-, \vec{x}_\perp) = \int D_1 \delta_{c,c_1} [a_1 \vec{\varepsilon}_{\perp s_1} e^{-ip_1\cdot x} + a_1^\dagger \vec{\varepsilon}^{\,*}_{\perp s_1} e^{ip_1\cdot x}]|_{x^+=0}, \tag{A11}$$

where

$$D_i = \sum_{c_i=1}^{N_c} \sum_{s_i=-1,1} \frac{d^2 p_{i\perp} dp_i^+}{16\pi^3 p_i^+} \theta(p_i^+ - \epsilon \mathcal{P}^+). \tag{A12}$$

Here s_i is the spin polarization of particle i, $\mathcal{P}$ is the four-momentum operator, ϵ is a positive infinitesimal, and the gluon polarization vector is defined by

$$\vec{\varepsilon}_{\perp s} = \frac{-1}{\sqrt{2}}(s, i). \tag{A13}$$

p^+ and $\vec{p}_\perp$ are the momenta conjugate to x^- and $\vec{x}_\perp$; so they are referred to as the longitudinal and transverse momenta, respectively. The purpose of ϵ is to regulate divergent effects from exchanged gluons (either instantaneous or real) with infinitesimal longitudinal momentum. We take $\epsilon \to 0$ before we calculate physical quantities (see Sec. V).

In the process of deriving the canonical Hamiltonian, we need to take the inverse longitudinal derivative of the gluon field. We do this by moving the derivative inside the expansion:

$$\frac{1}{\partial^+}\vec{A}_{\perp c}(x^-, \vec{x}_\perp) = \int D_1 \delta_{c,c_1} \left[-\frac{1}{ip_1^+} a_1 \vec{\varepsilon}_{\perp s_1} e^{-ip_1\cdot x} + \frac{1}{ip_1^+} a_1^\dagger \vec{\varepsilon}^{\,*}_{\perp s_1} e^{ip_1\cdot x} \right]\Bigg|_{x^+=0}. \tag{A14}$$

The expansion coefficients $a_i^\dagger$ and a_i are identified as particle creation and annihilation operators. They follow the convention

$$a_i = a(p_i, s_i, c_i), \tag{A15}$$

and have the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij} \equiv 16\pi^3 p_i^+ \delta^{(3)}(p_i - p_j)\delta_{s_i,s_j}\delta_{c_i,c_j} \tag{A16}$$

and

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \tag{A17}$$

where

$$\delta^{(3)}(p_i - p_j) = \delta(p_i^+ - p_j^+)\delta^{(2)}(\vec{p}_{i\perp} - \vec{p}_{j\perp}). \tag{A18}$$

Let $\mathcal{M}^2$ be the invariant-mass operator. The momentum conjugate to x^+ is p^-; so the Hamiltonian is identified as $\mathcal{P}^-$ and it follows from

$$\mathcal{P}^2 = \mathcal{M}^2 \tag{A19}$$

that

$$\mathcal{P}^- = \frac{\vec{\mathcal{P}}_\perp^2 + \mathcal{M}^2}{\mathcal{P}^+}. \tag{A20}$$

The canonical Hamiltonian that results from the above procedure can be written as the sum of a free part and an interacting part:

$$\mathcal{P}^- = \mathcal{P}^-_{\text{free}} + v, \tag{A21}$$

where

$$\mathcal{P}^-_{\text{free}} = \int D_1 \frac{\vec{p}_{1\perp}^2}{p_1^+} a_1^\dagger a_1. \tag{A22}$$

We give the interaction in terms of its ''modified'' matrix elements:

$$
\begin{aligned}
v = g\frac{16\pi^3}{2!}\int D_1D_2D_3 a_2^\dagger a_3^\dagger a_1 \delta^{(3)}(p_1-p_2-p_3)\prec g_2g_3|v|g_1\succ + g\frac{16\pi^3}{2!}\int D_1D_2D_3 a_3^\dagger a_1 a_2 \delta^{(3)}(p_1+p_2-p_3)\\
\times \prec g_3|v|g_1g_2\succ + g^2\frac{16\pi^3}{2!2!}\int D_1D_2D_3D_4 a_3^\dagger a_4^\dagger a_1 a_2 \delta^{(3)}(p_1+p_2-p_3-p_4)\\
\times \sum_{i=1}^{4} \prec g_3g_4|v|g_1g_2\succ_i + g^2\frac{16\pi^3}{3!}\int D_1D_2D_3D_4 a_2^\dagger a_3^\dagger a_4^\dagger a_1 \delta^{(3)}(p_1-p_2-p_3-p_4)\\
\times \sum_{i=1}^{4} \prec g_2g_3g_4|v|g_1\succ_i + g^2\frac{16\pi^3}{3!}\int D_1D_2D_3D_4 a_4^\dagger a_1 a_2 a_3 \delta^{(3)}(p_1+p_2+p_3-p_4)\\
\times \sum_{i=1}^{4} \prec g_4|v|g_1g_2g_3\succ_i,
\end{aligned}
\tag{A23}
$$

where a modified matrix element is defined by

$$
\prec i|v|j\succ = \left.\frac{\langle i|v|j\rangle}{16\pi^3\delta^{(3)}(p_i-p_j)}\right|_{g=1}. \tag{A24}
$$

The modified matrix elements are

$$
\begin{aligned}
\prec g_2g_3|v|g_1\succ = if^{c_1c_2c_3}\Bigg[&-\delta_{s_2,\bar{s}_3}\vec{\varepsilon}_{\perp s_1}\cdot\left\{(\vec{p}_{2\perp}-\vec{p}_{3\perp})-\frac{\vec{p}_{1\perp}}{p_1^+}(p_2^+-p_3^+)\right\}+\delta_{s_1,s_3}\vec{\varepsilon}^{\,*}_{\perp s_2}\cdot\left\{(\vec{p}_{1\perp}+\vec{p}_{3\perp})-\frac{\vec{p}_{2\perp}}{p_2^+}(p_1^++p_3^+)\right\}\\
&+\delta_{s_1,s_2}\vec{\varepsilon}^{\,*}_{\perp s_3}\cdot\left\{-(\vec{p}_{1\perp}+\vec{p}_{2\perp})+\frac{\vec{p}_{3\perp}}{p_3^+}(p_1^++p_2^+)\right\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\prec g_3|v|g_1g_2\succ = -if^{c_1c_2c_3}\Bigg[&\delta_{s_2,s_3}\vec{\varepsilon}_{\perp s_1}\cdot\left\{(\vec{p}_{3\perp}+\vec{p}_{2\perp})-\frac{\vec{p}_{1\perp}}{p_1^+}(p_3^++p_2^+)\right\}\\
&-\delta_{s_1,\bar{s}_2}\vec{\varepsilon}^{\,*}_{\perp s_3}\cdot\left\{(\vec{p}_{1\perp}-\vec{p}_{2\perp})-\frac{\vec{p}_{3\perp}}{p_3^+}(p_1^+-p_2^+)\right\}\\
&+\delta_{s_1,s_3}\vec{\varepsilon}_{\perp s_2}\cdot\left\{-(\vec{p}_{1\perp}+\vec{p}_{3\perp})+\frac{\vec{p}_{2\perp}}{p_2^+}(p_1^++p_3^+)\right\}\Bigg],
\end{aligned}
\tag{A25}
$$

and

$$
\begin{aligned}
\prec g_3g_4|v|g_1g_2\succ_1 &= [f^{c_1c_3c}f^{c_4c_2c}(\delta_{s_2,s_3}\delta_{s_1,s_4}-\delta_{s_3,\bar{s}_4}\delta_{s_1,\bar{s}_2})+f^{c_1c_4c}f^{c_3c_2c}(\delta_{s_2,s_4}\delta_{s_1,s_3}-\delta_{s_3,\bar{s}_4}\delta_{s_1,\bar{s}_2})\\
&\quad +f^{c_1c_2c}f^{c_3c_4c}(\delta_{s_2,s_4}\delta_{s_1,s_3}-\delta_{s_1,s_4}\delta_{s_2,s_3})],\\
\prec g_3g_4|v|g_1g_2\succ_2 &= f^{c_1c_3c}f^{c_4c_2c}\delta_{s_1,s_3}\delta_{s_2,s_4}\frac{p_1^++p_3^+}{p_1^+-p_3^+}\frac{p_4^++p_2^+}{p_4^+-p_2^+},\\
\prec g_3g_4|v|g_1g_2\succ_3 &= f^{c_1c_4c}f^{c_3c_2c}\delta_{s_1,s_4}\delta_{s_2,s_3}\frac{p_1^++p_4^+}{p_1^+-p_4^+}\frac{p_3^++p_2^+}{p_3^+-p_2^+},\\
\prec g_3g_4|v|g_1g_2\succ_4 &= f^{c_1c_2c}f^{c_4c_3c}\delta_{s_1,\bar{s}_2}\delta_{s_3,\bar{s}_4}\frac{p_1^+-p_2^+}{p_1^++p_2^+}\frac{p_4^+-p_3^+}{p_4^++p_3^+},
\end{aligned}
\tag{A26}
$$

and

$$\begin{aligned}
\prec g_2 g_3 g_4|v|g_1 \succ_1 &= [f^{c_1 c_3 c} f^{c_4 c_2 c}(-\delta_{\bar{s}_2,s_3}\delta_{s_1,s_4}+\delta_{s_3,\bar{s}_4}\delta_{s_1,s_2}) + f^{c_1 c_4 c} f^{c_3 c_2 c}(-\delta_{\bar{s}_2,s_4}\delta_{s_1,s_3}+\delta_{s_3,\bar{s}_4}\delta_{s_1,s_2}) \\
&\quad + f^{c_1 c_2 c} f^{c_3 c_4 c}(-\delta_{\bar{s}_2,s_4}\delta_{s_1,s_3}+\delta_{s_1,s_4}\delta_{\bar{s}_2,s_3})], \\
\prec g_2 g_3 g_4|v|g_1 \succ_2 &= -f^{c_1 c_2 c} f^{c_4 c_3 c}\delta_{s_1,s_2}\delta_{\bar{s}_3,s_4}\frac{p_1^+ + p_2^+}{p_1^+ - p_2^+}\frac{p_4^+ - p_3^+}{p_4^+ + p_3^+}, \\
\prec g_2 g_3 g_4|v|g_1 \succ_3 &= -f^{c_1 c_3 c} f^{c_4 c_2 c}\delta_{s_1,s_3}\delta_{\bar{s}_2,s_4}\frac{p_1^+ + p_3^+}{p_1^+ - p_3^+}\frac{p_4^+ - p_2^+}{p_4^+ + p_2^+}, \\
\prec g_2 g_3 g_4|v|g_1 \succ_4 &= -f^{c_1 c_4 c} f^{c_2 c_3 c}\delta_{s_1,s_4}\delta_{\bar{s}_3,s_2}\frac{p_1^+ + p_4^+}{p_1^+ - p_4^+}\frac{p_2^+ - p_3^+}{p_2^+ + p_3^+},
\end{aligned} \tag{A27}$$

and

$$\begin{aligned}
\prec g_4|v|g_1 g_2 g_3 \succ_1 &= [f^{c_1 c_3 c} f^{c_4 c_2 c}(-\delta_{s_2,\bar{s}_3}\delta_{s_1,s_4}+\delta_{s_3,s_4}\delta_{s_1,\bar{s}_2}) + f^{c_1 c_4 c} f^{c_3 c_2 c}(-\delta_{s_2,s_4}\delta_{s_1,\bar{s}_3}+\delta_{s_3,s_4}\delta_{s_1,\bar{s}_2}) \\
&\quad + f^{c_1 c_2 c} f^{c_3 c_4 c}(-\delta_{s_2,s_4}\delta_{s_1,\bar{s}_3}+\delta_{s_1,s_4}\delta_{s_2,\bar{s}_3})], \\
\prec g_4|v|g_1 g_2 g_3 \succ_2 &= -f^{c_1 c_4 c} f^{c_3 c_2 c}\delta_{s_1,s_4}\delta_{s_2,\bar{s}_3}\frac{p_1^+ + p_4^+}{p_1^+ - p_4^+}\frac{p_3^+ - p_2^+}{p_3^+ + p_2^+}, \\
\prec g_4|v|g_1 g_2 g_3 \succ_3 &= -f^{c_2 c_4 c} f^{c_3 c_1 c}\delta_{s_2,s_4}\delta_{s_1,\bar{s}_3}\frac{p_2^+ + p_4^+}{p_2^+ - p_4^+}\frac{p_3^+ - p_1^+}{p_3^+ + p_1^+}, \\
\prec g_4|v|g_1 g_2 g_3 \succ_4 &= -f^{c_3 c_4 c} f^{c_1 c_2 c}\delta_{s_3,s_4}\delta_{s_2,\bar{s}_1}\frac{p_3^+ + p_4^+}{p_3^+ - p_4^+}\frac{p_1^+ - p_2^+}{p_1^+ + p_2^+},
\end{aligned} \tag{A28}$$

where $\overline{s_n} = -s_n$.

We work in the free basis, the basis of eigenstates of $\mathcal{P}^-_{\rm free}$. They are given by

$$|g_1 g_2 \cdots g_n\rangle = a_1^\dagger a_2^\dagger \cdots a_n^\dagger |0\rangle, \tag{A29}$$

for any integer $n \geqslant 0$. The associated eigenvalue equation is

$$\mathcal{P}^-_{\rm free}|g_1 g_2 \cdots g_n\rangle = \sum_{i=1}^n p_i^-|g_1 g_2 \cdots g_n\rangle, \tag{A30}$$

where

$$p_i^- = \frac{\vec{p}_{i\perp}^{\,2}}{p_i^+} \tag{A31}$$

(since $p_i^2 = 0$) and the sum is zero if $n = 0$.

The noninteracting limit of Eq. (A20) is

$$\mathcal{P}^-_{\rm free} = \frac{\vec{\mathcal{P}}_\perp^{\,2} + \mathcal{M}^2_{\rm free}}{\mathcal{P}^+}, \tag{A32}$$

where $\mathcal{M}^2_{\rm free}$ is the free invariant-mass operator. It has the eigenvalue equation

$$\mathcal{M}^2_{\rm free}|g_1 g_2 \cdots g_n\rangle = M^2|g_1 g_2 \cdots g_n\rangle, \tag{A33}$$

where

$$M^2 = P^+ \sum_{i=1}^n p_i^- - \vec{P}_\perp^{\,2}, \tag{A34}$$

and P is the total momentum of the state. Finally, in terms of the free states, the completeness relation is

$$1 = |0\rangle\langle 0| + \int D_1 |g_1\rangle\langle g_1| + \frac{1}{2!}\int D_1 D_2 |g_1 g_2\rangle\langle g_1 g_2| + \cdots. \tag{A35}$$

APPENDIX B: THE DERIVATION OF THE RECURSION RELATIONS FOR THE REDUCED INTERACTION

This appendix is an extension of Appendix D of Ref. [1] to the case of pure-glue QCD. In Sec. II D, we derived a constraint on the $\mathcal{O}(g_\Lambda^r)$ reduced interaction for $r \geqslant 1$:

$$V^{(r)}_{\rm CD}(\Lambda) - V^{(r)}_{\rm CD}(\Lambda') = \delta V^{(r)} - \sum_{s=2}^{r-1} B_{r-s,s} V^{(r-s)}(\Lambda). \tag{B1}$$

Since we already know the first-order reduced interaction [see Eq. (18)] and the cutoff-independent part of the second-order reduced interaction [see Eq. (19)], we wish to use this

equation to compute $V_{\rm CD}^{(r)}(\Lambda)$ for $r \geqslant 2$ and $V_{\rm CI}^{(r)}$ for $r \geqslant 3$, in terms of lower-order reduced interactions.

1. The recursion relation for the cutoff-dependent part

We begin by computing the recursion relation for the cutoff-dependent part of the reduced interaction. Recall that momentum conservation implies that any matrix element of $V(\Lambda)$ can be written as an expansion in unique products of momentum-conserving delta functions. This means that an arbitrary matrix element of Eq. (B1) can be expanded in products of delta functions:

$$\sum_i \langle F|V_{\rm CD}^{(r)}(\Lambda)|I\rangle^{(i)} - \sum_i \langle F|V_{\rm CD}^{(r)}(\Lambda')|I\rangle^{(i)} = \sum_i \left[\langle F|\delta V^{(r)}|I\rangle - \sum_{s=2}^{r-1} B_{r-s,s}\langle F|V^{(r-s)}(\Lambda)|I\rangle\right]^{(i)}, \tag{B2}$$

where the (i) superscripts denote that we are considering the ith product of delta functions that can occur in a delta-function expansion of $\langle F|V^{(r)}(\Lambda)|I\rangle$. This equation is equivalent to a set of equations, one for each possible product of delta functions:

$$\langle F|V_{\rm CD}^{(r)}(\Lambda)|I\rangle^{(i)} - \langle F|V_{\rm CD}^{(r)}(\Lambda')|I\rangle^{(i)} = \left[\langle F|\delta V^{(r)}|I\rangle - \sum_{s=2}^{r-1} B_{r-s,s}\langle F|V^{(r-s)}(\Lambda)|I\rangle\right]^{(i)}. \tag{B3}$$

Cluster decomposition implies that we can write

$$\langle F|V_{\rm CD}^{(r)}(\Lambda)|I\rangle^{(i)} = \left[\prod_{j=1}^{N_\delta^{(i)}} \delta_j^{(i)}\right] F_{\rm CD}^{(i)}(\{p_n\},\{s_n\},\{c_n\},\Lambda), \tag{B4}$$

where $\delta_j^{(i)}$ is the jth momentum-conserving delta function in the ith product of delta functions ($\delta_j^{(i)}$includes a longitudinal-momentum factor), $N_\delta^{(i)}$ is the number of delta functions in the ith product, and $F_{\rm CD}^{(i)}$ is a function of the cutoff and the quantum numbers of the particles in the matrix element, but does not contain delta functions that fix momenta. We define $N_{\rm part}$ to be the number of particles in state $|I\rangle$ plus the number of particles in state $|F\rangle$, and $n = 1,2,3,...,N_{\rm part}$. The momentum, spin polarization, and color of particle n are given by p_n, s_n, and c_n. We define $N_{\rm int}^{(i)}$ to be the number of particles in the matrix element that participate in an interaction for the ith product of delta functions. In order for the IMO to have the dimensions (mass)^2, $F_{\rm CD}^{(i)}$ must have the dimensions (mass)$^{4-N_{\rm int}^{(i)}}$. Note that we are suppressing the dependence of the RHS of this equation on r.

We have assumed that any matrix element of the IMO can be expanded in powers of transverse momenta, not including the momentum-conserving delta functions; so

$$F_{\rm CD}^{(i)}(\{p_n\},\{s_n\},\{c_n\},\Lambda) = \Lambda^{4-N_{\rm int}^{(i)}} \sum_{\{m_{nt}\}} z_i^{\{m_{nt}\}}(\{p_n^+\},\{s_n\},\{c_n\}) \times \prod_{n=1}^{N_{\rm part}} \prod_{t=1}^{2} \left(\frac{p_{n\perp}^t}{\Lambda}\right)^{m_{nt}}, \tag{B5}$$

where t denotes a component of transverse momentum and m_{nt} is a non-negative integer index associated with transverse-momentum component t of particle n. The sum is over all values of each of the m_{nt}'s, subject to the constraint that

$$4 - N_{\rm int}^{(i)} - \sum_{n,t} m_{nt} \neq 0, \tag{B6}$$

which is necessary to avoid terms in the momentum expansion that are cutoff-independent. The $z_i^{\{m_{nt}\}}$'s are the coefficients for the momentum expansion. They depend on i and the m_{nt}'s and are functions of the particles' longitudinal momenta, spin polarizations, and colors.

Since the RHS of Eq. (B3) has the same product of delta functions as the LHS, we can write

$$\left[\langle F|\delta V^{(r)}|I\rangle - \sum_{s=2}^{r-1} B_{r-s,s}\langle F|V^{(r-s)}(\Lambda)|I\rangle\right]^{(i)} = \left[\prod_{j=1}^{N_\delta^{(i)}} \delta_j^{(i)}\right] G^{(i)}(\{p_n\},\{s_n\},\{c_n\},\Lambda,\Lambda'), \tag{B7}$$

where $G^{(i)}$ has dimensions (mass)$^{4\text{-}N_{\rm int}^{(i)}}$, and by inspection of the LHS of Eq. (B3) and Eq. (B4), is a function of the quantum numbers of the particles, Λ, and Λ'. Substitution of Eqs. (B7) and (B4) into Eq. (B3) yields

$$F_{\rm CD}^{(i)}(\{p_n\},\{s_n\},\{c_n\},\Lambda) - F_{\rm CD}^{(i)}(\{p_n\},\{s_n\},\{c_n\},\Lambda') = G^{(i)}(\{p_n\},\{s_n\},\{c_n\},\Lambda,\Lambda'), \tag{B8}$$

where the momenta in this equation are constrained by the delta-function conditions.

Since the LHS of Eq. (B8) is the difference of a function of Λ and the same function with $\Lambda \to \Lambda'$, $G^{(i)}$ must be as well. Since the LHS of Eq. (B8) can be expanded in powers of transverse momenta, $G^{(i)}$ must have the form

$$G^{(i)}(\{p_n\},\{s_n\},\{c_n\},\Lambda,\Lambda') = \sum_{\{m_{nt}\}} Z_i^{\{m_{nt}\}}(\{p_n^+\},\{s_n\},\{c_n\}) \left[\Lambda^{4-N_{\rm int}^{(i)}} \prod_{n,t} \left(\frac{p_{n\perp}^t}{\Lambda}\right)^{m_{nt}} - \Lambda'^{4-N_{\rm int}^{(i)}} \prod_{n,t} \left(\frac{p_{n\perp}^t}{\Lambda'}\right)^{m_{nt}}\right], \tag{B9}$$

where the sum is restricted by Eq. (B6).

Substituting Eqs. (B9) and (B5) into Eq. (B8), we find that

$$\Lambda^{4\text{-}N_{\rm int}^{(i)}}\sum_{\{m_{nt}\}} z_i^{\{m_{nt}\}}(\{p_n^+\},\{s_n\},\{c_n\})\prod_{n,t}\left(\frac{p_{n\perp}^t}{\Lambda}\right)^{m_{nt}} - \Lambda'^{4\text{-}N_{\rm int}^{(i)}}\sum_{\{m_{nt}\}} z_i^{\{m_{nt}\}}(\{p_n^+\},\{s_n\},\{c_n\})\prod_{n,t}\left(\frac{p_{n\perp}^t}{\Lambda'}\right)^{m_{nt}}$$
$$= \sum_{\{m_{nt}\}} Z_i^{\{m_{nt}\}}(\{p_n^+\},\{s_n\},\{c_n\})\left[\Lambda^{4\text{-}N_{\rm int}^{(i)}}\prod_{n,t}\left(\frac{p_{n\perp}^t}{\Lambda}\right)^{m_{nt}} - \Lambda'^{4\text{-}N_{\rm int}^{(i)}}\prod_{n,t}\left(\frac{p_{n\perp}^t}{\Lambda'}\right)^{m_{nt}}\right]. \tag{B10}$$

Matching powers of transverse momenta on both sides of this equation gives

$$z_i^{\{m_{nt}\}}(\{p_n^+\},\{s_n\},\{c_n\})[\Lambda^{4\text{-}N_{\rm int}^{(i)}-\Sigma_{n,t}m_{nt}} - \Lambda'^{4\text{-}N_{\rm int}^{(i)}-\Sigma_{n,t}m_{nt}}]$$
$$= Z_i^{\{m_{nt}\}}(\{p_n^+\},\{s_n\}),\{c_n\})[\Lambda^{4\text{-}N_{\rm int}^{(i)}-\Sigma_{n,t}m_{nt}} - \Lambda'^{4\text{-}N_{\rm int}^{(i)}-\Sigma_{n,t}m_{nt}}]. \tag{B11}$$

The factor in brackets cannot be zero because $\Lambda\neq\Lambda'$ and Eq. (B6) holds. Thus Eq. (B11) implies that

$$z_i^{\{m_{nt}\}} = Z_i^{\{m_{nt}\}}. \tag{B12}$$

Then Eqs. (B5), (B9), and (B12) imply that

$$F_{\rm CD}^{(i)}(\{p_n\},\{s_n\},\{c_n\},\Lambda) = G^{(i)}(\{p_n\},\{s_n\},\{c_n\},\Lambda,\Lambda')|_{\Lambda\ \rm terms}, \tag{B13}$$

where ''Λ terms'' means that $G^{(i)}$ is to be expanded in powers of transverse momenta and only the terms in the expansion that are proportional to powers or inverse powers of Λ contribute. From Eqs. (B4), (B7), and (B13),

$$\langle F|V_{\rm CD}^{(r)}(\Lambda)|I\rangle^{(i)} = \left[\langle F|\delta V^{(r)}|I\rangle - \sum_{s=2}^{r-1} B_{r-s,s}\langle F|V^{(r-s)}(\Lambda)|I\rangle\right]^{(i)}_{\Lambda\ \rm terms}, \tag{B14}$$

where it is understood that the momentum-conserving delta functions are ignored for the purposes of transverse-momentum expansions. Since a matrix element is the sum of the contributions to it from different products of delta functions, both sides of this equation can be summed over i to obtain

$$\langle F|V_{\rm CD}^{(r)}(\Lambda)|I\rangle = \left[\langle F|\delta V^{(r)}|I\rangle - \sum_{s=2}^{r-1} B_{r-s,s}\langle F|V^{(r-s)}(\Lambda)|I\rangle\right]_{\Lambda\ \rm terms}. \tag{B15}$$

This equation tells us how to calculate the cutoff-dependent part of the $\mathcal{O}(g_\Lambda^r)$ reduced interaction in terms of lower-order contributions.

2. The recursion relation for the cutoff-independent part

Since we have specified $V^{(1)}$ and $V_{\rm CI}^{(2)}$, we need to determine $V_{\rm CI}^{(r)}$ for $r\geqslant 3$. It is useful to first consider which contributions to $V^{(r)}(\Lambda)$ can be cutoff independent. A matrix element of the cutoff-independent part of $V^{(r)}(\Lambda)$ can be expanded in products of delta functions and in powers of transverse momenta just as was done for the cutoff-dependent part. Thus we can write

$$\langle F|V_{\rm CI}^{(r)}|I\rangle = \sum_i \langle F|V_{\rm CI}^{(r)}|I\rangle^{(i)}, \tag{B16}$$

where

$$\langle F|V_{\rm CI}^{(r)}|I\rangle^{(i)} = \left[\prod_{j=1}^{N_\delta^{(i)}}\delta_j^{(i)}\right]F_{\rm CI}^{(i)}(\{p_n\},\{s_n\},\{c_n\}) \tag{B17}$$

and

$$F_{\rm CI}^{(i)}(\{p_n\},\{s_n\},\{c_n\}) = \Lambda^{4\text{-}N_{\rm int}^{(i)}}\sum_{\{m_{nt}\}} w_i^{\{m_{nt}\}}(\{p_n^+\},\{s_n\},\{c_n\}) \times\prod_{n=1}^{N_{\rm part}}\prod_{t=1}^{2}\left(\frac{p_{n\perp}^t}{\Lambda}\right)^{m_{nt}}. \tag{B18}$$

The sum is over all values of each of the m_{nt}'s, subject to the constraint that

$$4\text{-}N_{\rm int}^{(i)} - \sum_{n,t} m_{nt} = 0, \tag{B19}$$

which ensures that all the terms in the expansion of $F_{\rm CI}^{(i)}$ are cutoff-independent.

Eq. (B19) places constraints on the possible cutoff-independent contributions to the reduced interaction. Any contribution to a matrix element of $V^{(r)}(\Lambda)$ has an $N_{\rm int}^{(i)}\geqslant 2$, but Eq. (B19) can only hold if $N_{\rm int}^{(i)}\leqslant 4$.

Suppose that $N_{\rm int}^{(i)}=2$. In this case, Eq. (B19) implies that $F_{\rm CI}^{(i)}$ is quadratic in transverse momenta. Due to approximate cluster decomposition, only interacting particles' transverse momenta can appear in $F_{\rm CI}^{(i)}$. So any contribution to $F_{\rm CI}^{(i)}$ can depend on the transverse momenta of two interacting particles. Thus $F_{\rm CI}^{(i)}$ can be written as a sum of terms, where each term corresponds to a distinct pair of interacting particles. The momentum dependence of each term in $F_{\rm CI}^{(i)}$ is limited to dependence on the momenta of the interacting particles and the total longitudinal momentum:

$$F_{\rm CI}^{(i)}(\{p_n\},\{s_n\},\{c_n\})=\sum_m F_{\rm CI}^{(i,m)}(k_m,k_m',P^+,\{s_n\},\{c_n\})$$

$$=\sum_m F_{\rm CI}^{(i,m)}(k_m,P^+,\{s_n\},\{c_n\}), \quad \text{(B20)}$$

where k_m and k_m' are the momenta for the initial and final particles in the mth interacting pair, and where we have used the fact that for $N_{\rm int}^{(i)}=2$, momentum conservation implies that $k_m=k_m'$. $F_{\rm CI}^{(i,m)}$ must be quadratic in $\vec{k}_{m\perp}$ or it must be zero. The matrix elements of the IMO are boost invariant, as is the delta-function product in Eq. (B17). This means that $F_{\rm CI}^{(i)}$ must be boost-invariant, but it cannot be if $F_{\rm CI}^{(i,m)}$ is quadratic in $\vec{k}_{m\perp}$; so $F_{\rm CI}^{(i,m)}$ must be zero. Thus the reduced interaction does not contain any cutoff-independent two-point interactions.

Note that two-point interactions are self-energies, and they change the particle dispersion relation. If they change the dispersion relation such that the coefficients of the free relation become modified by interactions, then this can be viewed as renormalization of the field operators, i.e., wave-function renormalization. This effect is absent unless either $F_{\rm CI}^{(i)}$ or $F_{\rm CD}^{(i)}$ can be quadratic in transverse momenta for $N_{\rm int}^{(i)}=2$. We have just shown that boost invariance prevents this for $F_{\rm CI}^{(i)}$, and according to Eq. (B6), $F_{\rm CD}^{(i)}$ cannot be quadratic in transverse momenta for $N_{\rm int}^{(i)}=2$; so there is no wave-function renormalization at any order in g_Λ in our approach.

According to Eq. (B19), if $N_{\rm int}^{(i)}=3$, then $F_{\rm CI}^{(i)}$ has to be linear in transverse momenta, and if $N_{\rm int}^{(i)}=4$, then $F_{\rm CI}^{(i)}$ has to be independent of all transverse momenta. According to assumptions that we made in Sec. II C 4, if r is odd then $V_{\rm CI}^{(r)}$ has no $N_{\rm int}^{(i)}=4$ part, and if r is even then $V_{\rm CI}^{(r)}$ has no $N_{\rm int}^{(i)}=3$ part. This means that if r is odd, then $V_{\rm CI}^{(r)}$ can contain only three-point interactions that are linear in transverse momenta, and if r is even, then $V_{\rm CI}^{(r)}$ can contain only four-point interactions that are independent of all transverse momenta.

To calculate $V_{\rm CI}^{(r)}$, we consider Eq. (B3) with $r\to r+2$:

$$\langle F|V_{\rm CD}^{(r+2)}(\Lambda)|I\rangle^{(i)}-\langle F|V_{\rm CD}^{(r+2)}(\Lambda')|I\rangle^{(i)}$$
$$=\left[\langle F|\,\delta V^{(r+2)}|I\rangle-\sum_{s=2}^{r+1}B_{r+2-s,s}\langle F|V^{(r+2-s)}(\Lambda)|I\rangle\right]^{(i)}. \quad \text{(B21)}$$

In the remainder of this appendix, we assume that r is odd. Then we need to consider only $N_{\rm int}^{(i)}=3$ initially. We expand Eq. (B21) in powers of transverse momenta and keep only the linear term:

$$0=\left[\langle F|\,\delta V^{(r+2)}|I\rangle-\sum_{s=2}^{r+1}B_{r+2-s,s}\times\langle F|V^{(r+2-s)}(\Lambda)|I\rangle\right]^{(i)}_{\vec{p}_\perp^{\,1}\ \text{term}}, \quad \text{(B22)}$$

where we have used the fact that $\langle F|V_{\rm CD}^{(r+2)}(\Lambda)|I\rangle^{(i)}$ has no linear part when $N_{\rm int}^{(i)}=3$ [see Eq. (B6)]. We move the first term in the sum on the RHS to the LHS:

$$B_{r,2}\langle F|V_{\rm CI}^{(r)}|I\rangle^{(i)}$$
$$=\left[\langle F|\,\delta V^{(r+2)}|I\rangle-\sum_{s=3}^{r+1}B_{r+2-s,s}\times\langle F|V^{(r+2-s)}(\Lambda)|I\rangle\right]^{(i)}_{\vec{p}_\perp^{\,1}\ \text{term}}. \quad \text{(B23)}$$

Now we can sum over all values of i corresponding to three-point interactions:

$$\langle F|V_{\rm CI}^{(r)}|I\rangle=\frac{1}{B_{r,2}}\left[\langle F|\,\delta V^{(r+2)}|I\rangle-\sum_{s=3}^{r+1}B_{r+2-s,s}\times\langle F|V^{(r+2-s)}(\Lambda)|I\rangle\right]^{\text{3-point}}_{\vec{p}_\perp^{\,1}\ \text{term}}. \quad \text{(B24)}$$

To use this equation, we also need to use Eq. (B21) to solve for $V_{\rm CI}^{(r+1)}$. Since r is odd, $r+1$ is even. Thus $V_{\rm CI}^{(r+1)}$ will contain only transverse-momentum-independent four-point interactions. Using steps analogous to those that led to Eq. (B24), we find that

$$\langle F|V_{\rm CI}^{(r+1)}|I\rangle=\frac{1}{B_{r+1,2}}\left[\langle F|\,\delta V^{(r+3)}|I\rangle-\sum_{s=3}^{r+2}B_{r+3-s,s}\langle F|V^{(r+3-s)}(\Lambda)|I\rangle\right]^{\text{4-point}}_{\vec{p}_\perp^{\,0}\ \text{term}}. \quad \text{(B25)}$$

To use these equations, the right-hand sides have to be expanded in powers of transverse momenta. Only three-point interactions that are linear in transverse momenta contribute to $V_{\rm CI}^{(r)}$, and only four-point interactions that are independent of all transverse momenta contribute to $V_{\rm CI}^{(r+1)}$.

These equations are coupled integral equations[14] because both $V_{\rm CI}^{(r)}$ and $V_{\rm CI}^{(r+1)}$ appear on the RHS of Eq. (B24) inside integrals in $\delta V^{(r+2)}$, and $V_{\rm CI}^{(r+1)}$ appears on the RHS of Eq. (B25) inside integrals in $\delta V^{(r+3)}$. It would seem that $V_{\rm CI}^{(r+2)}$ also appears on the RHS of Eq. (B25) inside integrals in $\delta V^{(r+3)}$, but $V_{\rm CI}^{(r+2)}$ cannot couple to $V^{(1)}$ to produce a transverse-momentum-independent four-point contribution to $\delta V^{(r+3)}$. This is because the cutoff function $T_2^{(\Lambda,\Lambda')}$ vanishes when the intermediate state is massless and all external transverse momenta are zero. This means that since we specified $V^{(1)}$ and $V_{\rm CI}^{(2)}$ in Sec. II C 4, we can use Eqs. (B24)

[14]It is very difficult to prove that integral equations of this type have a unique solution; so we simply assume that it is true in this case.

and (B25) to solve for $V_{\rm CI}^{(3)}$ and $V_{\rm CI}^{(4)}$ simultaneously, and $V_{\rm CI}^{(5)}$ and $V_{\rm CI}^{(6)}$ simultaneously, and so on. Note that before we can use these equations to solve for $V_{\rm CI}^{(r)}$ and $V_{\rm CI}^{(r+1)}$ simultaneously, we must first use Eq. (B15) both to compute $V_{\rm CD}^{(r)}(\Lambda)$ in terms of lower-order interactions and to express $V_{\rm CD}^{(r+1)}(\Lambda)$ in terms of lower-order interactions and $V^{(r)}(\Lambda)$.

Before concluding this appendix, we would like to deduce a bit more about the relationship of g_Λ to $g_{\Lambda'}$. According to Eq. (27) and the surrounding discussion, this relationship is determined by the matrix element $\langle g_2 g_3|\delta V|g_1\rangle$, which can be expanded in powers of $g_{\Lambda'}$:

$$\langle g_2 g_3|\delta V|g_1\rangle=\sum_{t=3}^{\infty} g_{\Lambda'}^t \langle g_2 g_3|\delta V^{(t)}|g_1\rangle. \tag{B26}$$

Recall that $\delta V^{(t)}$ is built from products of $V^{(r)}(\Lambda')$'s. This implies that $\delta V^{(t)}$ can change particle number by 1 only if t is odd, and thus Eq. (B26) implies that the coupling runs at odd orders; i.e., C_s is zero if s is even [see Eq. (29)].

APPENDIX C: TECHNICAL ISSUES IN THE NUMERICAL CALCULATION OF MATRIX ELEMENTS

In this appendix, we discuss some of the technical issues involved in the numerical calculation of the matrix elements $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$. In Appendix C 1 we discuss how we put the matrix elements into a form that is amenable to numerical calculation. In Appendix C 2 we briefly cover three topics: we show how the glueball state with $\mathcal{J}_3^{\rm R}$ eigenvalue $-j$ can be written in terms of the glueball state with $\mathcal{J}_3^{\rm R}$ eigenvalue j, we list a few tricks that allow us to reduce the number of matrix elements that we must compute, and we present our method for estimating how numerical uncertainties in the matrix elements translate into uncertainties in the spectrum.

1. Preparation of integrals for the Monte Carlo calculation

There are two types of contributions to $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$: finite sums and five-dimensional integrals. We use MATHEMATICA [26] to evaluate the finite sums to as many digits as we wish. To evaluate the integrals, we combine them into one integral and use the VEGAS Monte Carlo routine [31]. It takes a bit of work to put the integral into a form that will converge.

There are two main difficulties with getting the integral to converge. The first difficulty is that $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm IN}^{\rm B,F}$ looks divergent: as $x'\to x$, the sum in Eq. (151) diverges. This divergence is misleading because it actually contributes nothing to the integral (assuming that we calculate the integral carefully; see the discussion below). Left unchecked, this false divergence prevents the integral from converging with VEGAS. To rectify the problem, we want to subtract the false divergence from the integrand. Since it integrates to zero, this is allowed.

The second main difficulty with getting the integral to converge is roundoff error. Even after we subtract the false divergence from the integrand, the integrand peaks around $x'=x$, and in this region there are large cancellations in some of the quantities that we have defined. To prevent these cancellations from causing roundoff error, we rewrite these quantities so that the cancellations are explicit, before we turn to numerics.

a. Subtraction of the false divergence

We begin by defining a set of variables that is natural for dealing with the false divergence. We define

$$\eta=x-x'. \tag{C1}$$

We change variables from $\vec{k}_\perp$ and $\vec{k}'_\perp$ to the dimensionless transverse variables $\vec{r}_\perp$ and $\vec{w}_\perp$:

$$\vec{r}_\perp=\frac{d}{2}(\vec{k}_\perp+\vec{k}'_\perp),$$

$$\vec{w}_\perp=\frac{d}{2}\frac{\vec{k}_\perp-\vec{k}'_\perp}{\sqrt{\eta}}. \tag{C2}$$

We define the angle between $\vec{r}_\perp$ and $\vec{w}_\perp$ to be β:

$$\vec{r}_\perp\cdot\vec{w}_\perp=rw\cos\beta. \tag{C3}$$

We also define

$$r_\pm=\sqrt{r^2+\eta w^2\pm 2rw\sqrt{\eta}\cos\beta}, \tag{C4}$$

and then we can derive a host of useful relations:

$$\vec{k}_\perp=\frac{\vec{r}_\perp+\sqrt{\eta}\vec{w}_\perp}{d},$$

$$\vec{k}'_\perp=\frac{\vec{r}_\perp-\sqrt{\eta}\vec{w}_\perp}{d},$$

$$k=\frac{r_+}{d},$$

$$k'=\frac{r_-}{d},$$

$$\vec{k}_\perp\cdot\vec{k}'_\perp=\frac{1}{d^2}(r^2-\eta w^2),$$

$$\cos\gamma=\frac{r^2-\eta w^2}{r_+r_-},$$

$$\sin\gamma=\frac{-2rw\sqrt{\eta}}{r_+r_-}\sin\beta, \tag{C5}$$

and

$$dkdk'd\gamma kk'\theta(k)\theta(k')\theta(\gamma)\theta(2\pi-\gamma)$$

$$=\frac{4\eta}{d^4}drdwd\beta rw\theta(r)\theta(w)\theta(\beta)\theta(2\pi-\beta). \tag{C6}$$

In terms of these variables, $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\rm IN}^{\rm B,F}$ takes the form

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{IN}}^{\mathrm{B,F}} = -\frac{N_c g_\Lambda^2}{2\pi^3 d^2}\int D' \log\eta W_{q,q'}\overline{T}_{t'}(r_-)\overline{T}_t(r_+) \times e^{-(\Lambda d)^{-4}(\overline{\Delta}_{FK}^2+\overline{\Delta}_{IK}^2)}\sum_{i=1}^{5} E_i' \prod_{m=1;m\neq i}^{5} E_m, \tag{C7}$$

where

$$D' = dx\,dx'\,dr\,dw\,d\beta\, rw\,\eta\,\theta(x)\,\theta(1-x)\,\theta(x') \times \theta(x-x')\,\theta(r)\,\theta(w)\,\theta(\beta)\,\theta(2\pi-\beta)\overline{L}_l^{(e)}(x). \tag{C8}$$

To avoid roundoff error, we have defined simplified dimensionless versions of the differences of the free masses and the derivatives of these differences:

$$\overline{\Delta}_{FI} = \frac{r^2\eta(1-x-x')+w^2\eta^2(1-x-x')-2wr\sqrt{\eta}(x[1-x]+x'[1-x'])\cos\beta}{x(1-x)x'(1-x')},$$

$$\overline{\Delta}_{FK} = -\frac{r^2\eta+w^2(1-x+1-x')^2+2rw\sqrt{\eta}(1-x+1-x')\cos\beta}{(1-x)(1-x')},$$

$$\overline{\Delta}_{IK} = -\frac{r^2\eta+w^2(x+x')^2-2rw\sqrt{\eta}(x+x')\cos\beta}{xx'},$$

$$\overline{\Delta}'_{FK} = \frac{r_-^2}{(1-x')^2}-4\frac{w^2}{\eta},$$

$$\overline{\Delta}'_{IK} = \frac{r_-^2}{x'^2}-4\frac{w^2}{\eta}. \tag{C9}$$

Equation (C7) is now dimensionless, except for the factor in front of the integral, which is proportional to $1/d^2$.

As $x'\to x$ ($\eta\to 0$), the contributions to the integrand from the first and second terms in the sum diverge. In the limit $x'\to x$, the contribution to the integral from these terms can be written[15]

$$-\frac{N_c g_\Lambda^2}{2\pi^3 d^2}\delta_{q,q'}\int D' \log\eta\,\overline{T}_{t'}(r)\overline{T}_t(r)e^{-32(\Lambda d)^{-4}w^4}\left[\frac{1}{\eta^2}-64(\Lambda d)^{-4}\frac{w^4}{\eta^2}\right]\overline{L}_{l'}^{(e)}(x')(x+x')\times(1-x+1-x'). \tag{C10}$$

An examination of this integral reveals a problem: the transverse integrals are zero and the longitudinal integrals are infinite. To solve this problem, we consider what would have happened if we had not yet taken $\epsilon\to 0$. In this case, the transverse integrals would be zero and the longitudinal integrals would be finite. Thus the $\epsilon\to 0$ limit of this integral would be zero. This means that this integral is actually zero, and we can subtract it from the full integral in Eq. (C7):

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{IN}}^{\mathrm{B,F}} = -\frac{N_c g_\Lambda^2}{2\pi^3 d^2}\int D' \log\eta\left[W_{q,q'}\overline{T}_{t'}(r_-)\overline{T}_t(r_+)e^{-(\Lambda d)^{-4}(\overline{\Delta}_{FK}^2+\overline{\Delta}_{IK}^2)}\sum_{i=1}^{5} E_i' \prod_{m=1;m\neq i}^{5} E_m - \delta_{q,q'}\overline{T}_{t'}(r)\overline{T}_t(r)e^{-32(\Lambda d)^{-4}w^4}\left(\frac{1}{\eta^2}-64(\Lambda d)^{-4}\frac{w^4}{\eta^2}\right)\overline{L}_{l'}^{(e)}(x')(x+x')(1-x+1-x')\right]. \tag{C11}$$

Once we have performed this subtraction, there is no ambiguity about the value of the full integral, and it converges when computed numerically.

[15]We do not replace all the occurrences of x' with x because doing so hampers the convergence of the integral.

b. Combination of the integrals

We now combine all the five-dimensional integrals into one integral:

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{5-\mathrm{D}} = -\frac{N_c g_\Lambda^2}{2\pi^3 d^2}\int D'[I_{\mathrm{CON}}+I_{\mathrm{EX}}^{\mathrm{F}}+I_{\mathrm{IN+EX}}+I_{\mathrm{IN}}^{\mathrm{B,F}}], \tag{C12}$$

where

$$I_{\mathrm{CON}} = [\delta_{j,-2}\delta_{q',2}\delta_{q,2}+\delta_{j,2}\delta_{q',1}\delta_{q,1}-\delta_{j,0}\delta_{q',3}\delta_{q,3}]\overline{L}_{l'}^{(e)}(x')\overline{T}_{t'}(r_-)\overline{T}_t(r_+)e^{-(\Lambda d)^{-4}\overline{\Delta}_{FI}^2},$$

$$I_{\mathrm{EX}}^{\mathrm{F}} = \overline{L}_{l'}^{(e)}(x')\overline{T}_{t'}(r_-)\overline{T}_t(r_+)e^{-(\Lambda d)^{-4}\overline{\Delta}_{FI}^2}\frac{1}{\eta}\left(\frac{1}{\overline{\Delta}_{FK}}+\frac{1}{\overline{\Delta}_{IK}}\right)(1-e^{-2(\Delta d)^{-4}\overline{\Delta}_{FK}\overline{\Delta}_{IK}})$$
$$\times\left[\frac{r_+^2}{x(1-x)}S_{q,q'}^{(1)}+\frac{r_-^2}{x'(1-x')}S_{q,q'}^{(2)}+\frac{r_+r_-}{x(1-x)x'(1-x')}S_{q,q'}^{(3)}\right],$$

$$I_{\mathrm{IN+EX}} = W_{q,q'}\overline{L}_{l'}^{(e)}(x')\overline{T}_{t'}(r_-)\overline{T}_t(r_+)e^{-(\Lambda d)^{-4}\overline{\Delta}_{FI}^2}\frac{(1-e^{-2(\Lambda d)^{-4}\overline{\Delta}_{FK}\overline{\Delta}_{IK}})}{x(1-x)x'(1-x')}\frac{1}{\eta}\frac{1}{\overline{\Delta}_{FK}\overline{\Delta}_{IK}}$$
$$\times[-\eta^5w^4-2\eta^4w^2(r^2+2w^2[1-2x])-\eta^3(r^4+4r^2w^2[1-2x]+6w^4[1-2x]^2)-4\cos\beta\sqrt{\eta}rw([-1+\eta^2]r^2$$
$$+\eta w^2[1-x-x']^2)(1-x-x')+8r^2w^2x(1-x)-4\cos^2\beta r^2w^2(\eta^4+2\eta^3[1-2x]+4x[1-x]-4\eta^2x[1-x]$$
$$-2\eta[1-2x])-4\eta^2w^2(w^2[1-2x]^3+r^2[1-2x(1-x)])+2\eta(r^4-2r^2w^2[1-2x]$$
$$-4w^4x[-1+3x-4x^2+2x^3])],$$

$$I_{\mathrm{IN}}^{\mathrm{B,F}} = \log\eta\left[W_{q,q'}\overline{T}_{t'}(r_-)\overline{T}_t(r_+)e^{-(\Lambda d)^{-4}(\overline{\Delta}_{FK}^2+\overline{\Delta}_{IK}^2)}\sum_{i=1}^{5}E_i'\prod_{m=1;m\neq i}^{5}E_m-\delta_{q,q'}\overline{T}_{t'}(r)\overline{T}_t(r)\right.$$
$$\left.\times e^{-32(\Lambda d)^{-4}w^4}\left(\frac{1}{\eta^2}-64(\Lambda d)^{-4}\frac{w^4}{\eta^2}\right)\overline{L}_{l'}^{(e)}(x')(x+x')(1-x+1-x')\right]. \tag{C13}$$

We have rewritten the integrand of $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{\mathrm{IN+EX}}$ to eliminate large roundoff errors, at the expense of making it more complicated. To further avoid roundoff errors, we rewrite a few of the $S_{q,q'}^{(i)}$'s:

$$S_{3,3}^{(1)} = \cos(j\gamma)[2\eta^2+2\eta(1-2x)+(1-2x)^2],$$

$$S_{3,4}^{(3)} = -\sin\gamma\sin(j\gamma)(2\eta^2[1-2x]+\eta[1-6x+6x^2]$$
$$-2x[1-3x+2x^2]),$$

$$S_{4,4}^{(1)} = \cos(j\gamma)(1-2x)(1+2\eta-2x). \tag{C14}$$

Note that to compute some of the trigonometric functions that appear in this integral [such as $\cos(j\gamma)$] in terms of the integration variables, it is necessary to use recursion relations that define these functions in terms of $\cos\gamma$ and $\sin\gamma$ so that we can use Eq. (C5).

The integral in Eq. (C12) converges slowly. This is because it is strongly peaked when $x'\simeq x$, even though we have subtracted the false divergence. To spread out this region, we change variables from x' to p where

$$x' = x(1-e^{-p}). \tag{C15}$$

Now the integral is

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{5-\mathrm{D}}$$
$$= -\frac{N_c g_\Lambda^2}{2\pi^3 d^2}\int_0^1 dx\int_0^\infty dp\int_0^\infty dr\int_0^\infty dw$$
$$\times\int_0^{2\pi}d\beta rw\eta^2\overline{L}_l^{(e)}(x)[I_{\mathrm{CON}}+I_{\mathrm{EX}}^{\mathrm{F}}+I_{\mathrm{IN+EX}}+I_{\mathrm{IN}}^{\mathrm{B,F}}]. \tag{C16}$$

As a final step, we note that VEGAS requires the region of integration to be finite. Thus we change variables from p, r, and w to y_p, y_r, and y_w:

$$p = \frac{2}{1+y_p}-1,$$

$$r=\frac{2}{1+y_r}-1,$$

$$w=\frac{2}{1+y_w}-1, \tag{C17}$$

and then the final expression for the contribution to the matrix elements from the five-dimensional integral is

$$\begin{aligned}\langle q',l't',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle_{5-\mathrm{D}}&\\ =-\frac{4N_c g_\Lambda^2}{\pi^3 d^2}\int_0^1 dx\int_{-1}^1 dy_p\int_{-1}^1 dy_r\int_{-1}^1 dy_w\int_0^{2\pi} d\beta rw&\\ \times\frac{1}{(1+y_p)^2}\frac{1}{(1+y_r)^2}\frac{1}{(1+y_w)^2}\,\eta^2\overline{L_l^{(e)}}(x)&\\ \times[I_{\mathrm{CON}}+I_{\mathrm{EX}}^{\mathrm{F}}+I_{\mathrm{IN+EX}}+I_{\mathrm{IN}}^{\mathrm{B,F}}].&\end{aligned} \tag{C18}$$

The integral converges nicely in this form.

2. Miscellaneous issues

a. Equivalent bases: $j\to -j$

To compute the glueball spectrum, we compute the matrix $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ and diagonalize it for each value of j separately. For a given $|j|$, the matrices with $j=|j|$ and $j=-|j|$ are simply related, and we can use this fact to avoid computing and diagonalizing both of them. By inspection, we determine that

$$\begin{aligned}\langle q',l',t',-j|\mathcal{M}^2(\Lambda)|q,l,t,-j\rangle&\\ =\sum_{q'',q'''}\langle q''',l',t',j|\mathcal{M}^2(\Lambda)|q'',l,t,j\rangle f(q,q'')f(q',q'''),&\end{aligned} \tag{C19}$$

where

$$f(q,q')=\delta_{q,1}\delta_{q',2}+\delta_{q,2}\delta_{q',1}+\delta_{q,3}\delta_{q',3}-\delta_{q,4}\delta_{q',4}. \tag{C20}$$

This simply means that the basis $|q,l,t,-j\rangle$ is related to the basis $|q,l,t,j\rangle$ by swapping the states $|1,l,t,j\rangle$ and $|2,l,t,j\rangle$, and changing the sign of $|4,l,t,j\rangle$. Renaming basis states and changing their phases has no effect on the eigenvalues of the matrix; so $|\Psi^{(-j)n}(P)\rangle$ has the same mass as $|\Psi^{jn}(P)\rangle$. It also means that since

$$\begin{aligned}|\Psi^{jn}(P)\rangle&=\sum_{qlt}R_{qlt}^{jn}|q,l,t,j\rangle\\ &=\sum_{lt}[R_{1lt}^{jn}|1,l,t,j\rangle+R_{2lt}^{jn}|2,l,t,j\rangle\\ &\quad+R_{3lt}^{jn}|3,l,t,j\rangle+R_{4lt}^{jn}|4,l,t,j\rangle],\end{aligned} \tag{C21}$$

we have

$$\begin{aligned}|\Psi^{(-j)n}(P)\rangle&=\sum_{lt}[R_{1lt}^{jn}|2,l,t,j\rangle+R_{2lt}^{jn}|1,l,t,j\rangle+R_{3lt}^{jn}|3,l,t,j\rangle\\ &\quad-R_{4lt}^{jn}|4,l,t,j\rangle]\\ &=\sum_{q'qlt}R_{qlt}^{jn}f(q,q')|q',l,t,j\rangle.\end{aligned} \tag{C22}$$

Thus by diagonalizing $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$, we also obtain the eigenvalues and eigenstates of $\langle q',l',t',-j|\mathcal{M}^2(\Lambda)|q,l,t,-j\rangle$.

b. Reducing the number of matrix elements to compute

There are a few facts that allow us to reduce the number of matrix elements that we have to compute. First, because of gluon-exchange symmetry, the basis state $|q,l,t,j\rangle$ is zero if $l+j$ is even and $q=4$, or if $l+j$ is odd and $q\neq 4$. Second, $\mathcal{M}^2(\Lambda)$ is Hermitian, and its matrix elements in this basis are real; so it is symmetric in this basis. Third, by inspection, we see that

$$\langle 1,l',t',j|\mathcal{M}^2(\Lambda)|2,l,t,j\rangle=0. \tag{C23}$$

Finally, there are some redundancies and additional zeros in the matrix when $j=0$:

$$\begin{aligned}\langle 2,l',t',0|\mathcal{M}^2(\Lambda)|2,l,t,0\rangle&=\langle 1,l',t',0|\mathcal{M}^2(\Lambda)|1,l,t,0\rangle,\\ \langle 1,l',t',0|\mathcal{M}^2(\Lambda)|3,l,t,0\rangle&=\langle 2,l',t',0|\mathcal{M}^2(\Lambda)|3,l,t,0\rangle,\\ \langle 1,l',t',0|\mathcal{M}^2(\Lambda)|4,l,t,0\rangle&=-\langle 2,l',t',0|\mathcal{M}^2(\Lambda)|4,l,t,0\rangle,\\ \langle 3,l',t',0|\mathcal{M}^2(\Lambda)|4,l,t,0\rangle&=0.\end{aligned} \tag{C24}$$

C. Estimating uncertainties in the spectrum

When we use the VEGAS Monte Carlo routine to compute the matrix elements, the results for the matrix elements have statistical uncertainties. In order to control the resulting uncertainties in the spectrum, we would like to have a method that allows us to estimate how accurately we must calculate any given matrix element in order for the spectrum to have a desired uncertainty.

Suppose a diagonal matrix element $\langle q,l,t,j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ is given by

$$\langle q,l,t,j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle=Z\pm\delta, \tag{C25}$$

where Z is the Monte Carlo estimate of the matrix element and δ is the associated absolute uncertainty. Using MATHEMATICA with test matrices, it is straightforward to convince oneself that if δ is small, then it will yield a relative uncertainty $e_{M_n^2}\sim\delta/Z$ in the eigenvalues of the matrix.[16] This translates to a relative uncertainty of $e_{M_n}\sim\delta/(2Z)$ in the masses.

[16]In the development of this method, we are guided by the principles of quantum-mechanical perturbation theory, although we cannot legitimately use perturbation theory to analyze the uncertainties.

Estimating the uncertainty in the spectrum due to uncertainties in off-diagonal matrix elements is more difficult. Using MATHEMATICA with test matrices, the simplest method that we have found that is reasonably reliable is to use a type of degenerate perturbation theory. When we have an off-diagonal matrix element given by

$$\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle = Z \pm \delta, \tag{C26}$$

we diagonalize the two matrices

$$\begin{pmatrix} \langle q,l,t,j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle & Z+\delta \\ Z+\delta & \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q',l',t',j\rangle \end{pmatrix}, \tag{C27}$$

and

$$\begin{pmatrix} \langle q,l,t,j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle & Z-\delta \\ Z-\delta & \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q',l',t',j\rangle \end{pmatrix}, \tag{C28}$$

and we compare their eigenvalues to the eigenvalues of the matrix

$$\begin{pmatrix} \langle q,l,t,j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle & Z \\ Z & \langle q',l',t',j|\mathcal{M}^2(\Lambda)|q',l',t',j\rangle \end{pmatrix}. \tag{C29}$$

We then define $e_{M_n^2}$ to be the largest relative deviation that we have found in the eigenvalues, and we estimate the resulting relative uncertainty in the mass spectrum to be $e_{M_n} \sim e_{M_n^2}/2$. This estimate tends to work well unless there are too many diagonal matrix elements that are nearly degenerate with either $\langle q,l,t,j|\mathcal{M}^2(\Lambda)|q,l,t,j\rangle$ or $\langle q',l',t',j|\mathcal{M}^2(\Lambda)|q',l',t',j\rangle$.

To achieve a relative uncertainty of $\mathcal{O}(\varepsilon)$ in the glueball masses, we require $e_{M_n} < \varepsilon$ for each matrix element. This method tends to work reasonably well. The main difficulty is that we have neglected to consider the highly nonlinear couplings between the uncertainties in different matrix elements. For this reason, as we increase the size of the matrix, the error in our estimate of the uncertainty eventually becomes critical. At this point, the spectrum that we get when we diagonalize the matrix becomes completely unreliable. (The evidence of the breakdown is sudden contamination of the low-lying wave functions with high-order components.) It should be possible to develop more sophisticated methods of estimating uncertainties to suppress this problem.

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