Nonsymmetric unified field theory

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A metric nonsymmetric unified theory of gravitation and electromagnetism is studied. By a suitable modification of the Einstein part of the Bonnor Lagrangian, it is shown that the antisymmetric part of the metric tensor can be made to describe a massless spin-1 field obeying Maxwell's equations in the flat space linear approximation and thus making its identification to the electromagnetic field strength tensor a possibly consistent procedure. The theory is shown to be free of unphysical ghost-negative energy radiative modes even when expanded about a curved Riemannian background. The Einstein-Maxwell theory is contained in the first approximation of the field equations about a curved general relativity background. The field equations contain only the symmetric part of the connection, making them as close as possible to those of general relativity. The equations of motion of charged particles are shown to contain the Coulomb force in the lowest nontrivial order of approximation.

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I. INTRODUCTION

Einstein [1,2] developed a unified theory of gravitation and electromagnetism based on a nonsymmetric metric tensor with its antisymmetric part $g_{[\alpha\beta]}$ being linked with the Maxwell field strength $F_{\alpha\beta}$. Later Bonnor [3] introduced an extra term into the Einstein Lagrangian in such a way that the Lorentz force, which could not arise from previous calculations [4] on the Einstein theory, could be obtained. Yet later Moffat and Boal [5] proposed a new interpretation of the Bonnor theory, based on an exact central symmetric solution, suggesting the identification of the two fields within a constant which we shall call p, as in Eq. (2.4) below, the formal vanishing of which leads the theory to the usual Einstein-Maxwell theory. However, none of these theories is able to reproduce Maxwell's type of equations for $g_{[\alpha\beta]}$ in the weak field linear approximation for normal nonzero p, as it really is, to sustain such an identification. As pointed out by Einstein himself [2], the linearized equations obeyed by $g_{[\alpha\beta]}$ in his theory are weaker than Maxwell's equations. In the Bonnor theory one sees [3] that the linearized $g_{[\alpha\beta]}$ does not obey Maxwell's equations either: more precisely, the first equation is obeyed but the second one, involving the curl of the field, does not. The same occurs in the Moffat-Boal theory [5] because here Bonnor's field equations are kept and the vanishing of p as by them considered is only formal, p being a fixed nonzero quantity always present in the theory. The fact that the complete Maxwell's equations are missing then damages the identification of the two fields. This is the kind of problem that this paper tries to solve. Here we show that by a suitable modification of the Einstein part of the Bonnor Lagrangian we can obtain both the desired Maxwellian type of equations for $g_{[\alpha\beta]}$ in the weak field linear approximation, making then its identification with $F_{\alpha\beta}$ a possibly consistent procedure.

Moreover, it has been pointed out by Damour, Deser, and McCarthy [6] (DDM) that, when expanded about a Riemannian background, the Einstein unified theory and a whole class of nonsymmetric theories of gravitation based on the Einstein Lagrangian exhibits curvature-coupled negative-energy (ghost) excitations and unacceptable asymptotic behavior. By making use of their type of analysis, it will be shown that the present theory is free of these unphysical features, and therefore on a rather safe ground. It is shown that with the aforementioned modification of the Einstein part of the Bonnor Lagrangian it is possible to construct an alternative theory which, besides having the proper Maxwellian behavior in the weak field limit, it is free of radiative ghosts and bad asymptotic behavior even when expanded about a Riemannian background space, becoming thereby a candidate for a physically consistent geometrical unified theory. We follow the procedure that we have adopted recently [7] to develop an alternative nonsymmetric theory of gravitation to cope with the aforementioned problems that DDM pointed out in previous nonsymmetric theories.

We show first that an extension of the Einstein Lagrangian to a more general form is possible, satisfying Einstein's condition of Hermiticity [1]. This means invariance under transposition, which is defined as the transformation that exchanges the indices of the metric tensor and the lower ones of the connection, followed by an exchange of the two indices of any second-order tensor that depends on the metric and connection. This symmetry property has the physical meaning [1] that the same field equations are satisfied for positive and negative charges, the transformation taking one into the other. With this condition and the requirement of having $g_{[\alpha\beta]}$ to obey a first Maxwellian type of equation in the flat space linear approximation we will be led to an extended form of the Bonnor-Moffat-Boal (BMB) theory which contains, besides the universal constant p, only two of an initial seven parameters. The BMB theory appears for particular values of these two parameters. Then we show how the requirement of having $g_{[\alpha\beta]}$ to obey also the second Maxwell equation in the same linear approximation forces their values, leading to a consistent unified theory.

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The equation of charged particles are shown to follow by the method of Einstein, Infeld and Hoffmann [8]. This is done along the same line of Bonnor [3] and Moffat-Boal [5] calculations. To lowest nontrivial order the result here obtained coincide with these previous calculations.

The paper is organized as follows. We start, in Sec. II, by looking for the most general Hermitian form of the second-order tensor, containing at most first-order derivatives and quadratic products of the affine connection, which will play the role of the Einstein tensor, which is the counterpart of the Ricci tensor of general relativity (GR). In Sec. III we write the corresponding field equations. In Sec. IV we focus our attention on the requirement of having Maxwell's equations for $g_{[\alpha\beta]}$ in the flat space linear approximation and analyze the physical implications for the values of the last parameters. The resulting final field equations are displayed in Sec. V. In Sec. VI we study the expansion about a Riemannian background. In Sec. VII we discuss the equation of motion of electric particles and in Sec. VIII we draw our conclusions and highlight future works.

II. HERMITICITY

We write the field Lagrangian density as

$$\mathbf{L}=-\mathbf{g}^{\alpha\beta}Q_{\alpha\beta}+\frac{1}{p^2}\mathbf{g}^{[\alpha\beta]}g_{[\alpha\beta]}, \qquad (2.1)$$

with the notation $\mathbf{X}=\sqrt{-g}X$, g being the determinant of $g_{\alpha\beta}$ whose inverse $g^{\alpha\beta}$ is defined by

$$g^{\alpha\beta}g_{\alpha\gamma}=\delta^{\beta}_{\gamma}. \qquad (2.2)$$

The first term on the right of Eq. (2.1) becomes the Einstein Lagrangian when $Q_{\alpha\beta}$ is the Einstein Hermitian tensor $P_{\alpha\beta}$ [2], defined by

$$P_{\alpha\beta}=\Gamma^{\sigma}_{\alpha\beta,\sigma}-\tfrac{1}{2}(\Gamma^{\sigma}_{\alpha\sigma,\beta}+\Gamma^{\sigma}_{\sigma\beta,\alpha})+\Gamma^{\sigma}_{\alpha\beta}\Gamma^{\gamma}_{(\sigma\gamma)}-\Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\gamma}_{\sigma\beta}, \qquad (2.3)$$

plus two Lagrange multipliers terms. The notation $(\alpha\beta)$ and $[\alpha\beta]$ will be used to designate symmetric and antisymmetric parts of the corresponding quantity. The second term on the right of Eq. (2.1) is the term introduced by Bonnor [3] (p being here the inverse of his p) in the Einstein Lagrangian in such way that the Lorentz force, which could not arise from previous calculations [4] from the Einstein Lagrangian, could be obtained. The problem we pose ourselves here is the following: to find the simplest form of $Q_{\alpha\beta}$ in order to have $g_{[\alpha\beta]}$ obeying the usual Maxwell's equations in the linear flat space approximation and to have the resultant theory free of ghost-negative energy radiative modes even when expanded about a Riemannian background, in such a way that the identification of $g_{[\alpha\beta]}$ to the Maxwell field strength $F_{\alpha\beta}$ defined by

$$g_{[\alpha\beta]}=pF_{\alpha\beta}, \qquad (2.4)$$

as suggested by Moffat and Boal [5], constitutes a consistent procedure. As we shall see, by having Maxwell's equations for $g_{[\alpha\beta]}$ in the flat space linear approximation, the theory will be automatically free of unphysical ghost modes even when expanded about a Riemannian background. Reasons will be given to have in Eq. (2.4) the proportionality constant equal to the same p of Eq. (2.1).

As showed in [7], the most general Hermitian form of $Q_{\alpha\beta}$, containing at most first-order derivatives and quadratic products of the connection only, is

$$Q_{\alpha\beta}=P_{(\alpha\beta)}+aP_{[\alpha\beta]}+b\Gamma^{\mu}_{[\alpha\nu]}\Gamma^{\nu}_{[\mu\beta]}+c\Gamma_{\alpha}\Gamma_{\beta}+d\Gamma_{[\alpha,\beta]}, \qquad (2.5)$$

with four arbitrary parameters, where $\Gamma_{\alpha}=\Gamma^{\mu}_{[\alpha\mu]}$ is the torsion vector. We shall sketch now the reasoning behind this expression. One starts from the fact that in an affine space the only tensors are [9] the non-Riemannian curvature tensor $R^{\mu}_{\alpha\nu\beta}$, the antisymmetric part of the connection $\Gamma^{\mu}_{[\alpha\beta]}$ and their contractions and covariant (;) derivatives. The most general form of the second-order tensor $Q_{\alpha\beta}$, containing at most first-order derivatives and quadratic products of the connection, is found to be a linear combination of eight second-order tensors [10], among which one could include, for instance, the two independent contractions of the curvature tensor, $R^{\mu}_{\alpha\mu\beta}$ and $R^{\mu}_{\mu\alpha\beta}$, or $\Gamma^{\mu}_{[\alpha\beta];\mu}$, which is related [7] to $P_{[\alpha\beta]}$, or $\Gamma_{\alpha;\beta}$. However, a most convenient set of eight candidates, all of which with definite Hermitian property, Hermitian or anti-Hermitian, can be constructed [7]. They are the five Hermitian tensors on the right of Eq. (2.5) and the following other three tensors: $\Gamma_{(\alpha;\beta)}$, $\Gamma_{\mu}\Gamma^{\mu}_{[\alpha\beta]}$ and $\Gamma^{\mu}_{(\alpha\mu),\beta}-\Gamma^{\mu}_{(\beta\mu),\alpha}$, all of which change sign under transposition. As this last three tensors are then anti-Hermitian, the general Hermitian form of $Q_{\alpha\beta}$ will include only the first five, as written in Eq. (2.5). We are then left with four arbitrary parameters since one of them can be taken equal to one.

III. FIELD EQUATIONS

Variations of the action $\int\mathbf{L}d^4x$ with respect to $g^{\alpha\beta}$ and $\Gamma^{\gamma}_{\alpha\beta}$ yields the field equations. The former gives

$$Q_{\alpha\beta}-K_{\alpha\beta}=0, \qquad (3.1)$$

where [3]

$$K_{\alpha\beta}=\frac{1}{p^2}\left(g_{[\alpha\beta]}+g_{\alpha\mu}g^{[\mu\nu]}g_{\nu\beta}+\frac{1}{2}g_{\alpha\beta}g^{[\mu\nu]}g_{[\mu\nu]}\right). \qquad (3.2)$$

The variation with respect to $\Gamma^{\gamma}_{\alpha\beta}$ is best accomplished by writing

$$\mathbf{g}^{\alpha\beta}(P_{(\alpha\beta)}+aP_{[\alpha\beta]})=\mathbf{f}^{\alpha\beta}P_{\alpha\beta}, \qquad (3.3)$$

where

$$\mathbf{f}^{\alpha\beta}=\mathbf{g}^{(\alpha\beta)}+a\mathbf{g}^{[\alpha\beta]}. \qquad (3.4)$$

Performing the variation, we get [7]

$$\mathbf{f}^{\alpha\beta}{}_{,\gamma}+\mathbf{f}^{\alpha\sigma}\Gamma^{\beta}_{\gamma\sigma}+\mathbf{f}^{\sigma\beta}\Gamma^{\alpha}_{\sigma\gamma}-\mathbf{f}^{\alpha\beta}\Gamma^{\sigma}_{(\sigma\gamma)} -\tfrac{1}{2}\delta^{\alpha}_{\gamma}(\mathbf{f}^{\sigma\beta}{}_{,\sigma}+\mathbf{f}^{\rho\sigma}\Gamma^{\beta}_{\rho\sigma}-2c\mathbf{g}^{(\beta\sigma)}\Gamma_{\sigma}+d\mathbf{g}^{[\beta\sigma]}{}_{,\sigma}) -\tfrac{1}{2}\delta^{\beta}_{\gamma}(\mathbf{f}^{\alpha\sigma}{}_{,\sigma}+\mathbf{f}^{\rho\sigma}\Gamma^{\alpha}_{\rho\sigma}+2c\mathbf{g}^{(\alpha\sigma)}\Gamma_{\sigma}-d\mathbf{g}^{[\alpha\sigma]}{}_{,\sigma}) -b(\mathbf{g}^{(\sigma\alpha)}\Gamma^{\beta}_{[\gamma\sigma]}-\mathbf{g}^{(\beta\sigma)}\Gamma^{\alpha}_{[\gamma\sigma]})=0. \tag{3.5}$$

Contracting the pairs β,γ and α,γ and adding and subtracting the resulting equations, yields

$$\mathbf{f}^{(\alpha\sigma)}{}_{,\sigma}+\tfrac{2}{3}\mathbf{f}^{[\alpha\sigma]}\Gamma_{\sigma}+\mathbf{f}^{\rho\sigma}\Gamma^{\alpha}_{\rho\sigma}=0, \tag{3.6}$$

and

$$\mathbf{f}^{[\alpha\sigma]}{}_{,\sigma}+2\mathbf{f}^{(\alpha\sigma)}\Gamma_{\sigma}+2(1+3c-b)\mathbf{g}^{(\alpha\sigma)}\Gamma_{\sigma}-3d\mathbf{g}^{[\alpha\sigma]}{}_{,\sigma}=0. \tag{3.7}$$

Using $\mathbf{f}^{[\alpha\sigma]}=a\mathbf{g}^{[\alpha\sigma]}$, from Eq. (3.4), we get

$$(a-3d)\mathbf{g}^{[\alpha\sigma]}{}_{,\sigma}+2(1+3c-b)\mathbf{g}^{(\alpha\sigma)}\Gamma_{\sigma}=0. \tag{3.8}$$

This is the one equation with which the usual first Maxwell equation will have to be related to. Taking the symmetric and antisymmetric parts of Eq. (3.1) and recalling the expression of $Q_{\alpha\beta}$ in Eq. (2.5), we get

$$P_{(\alpha\beta)}+b\Gamma^{\mu}_{[\alpha\nu]}\Gamma^{\nu}_{[\mu\beta]}+c\Gamma_{\alpha}\Gamma_{\beta}-K_{(\alpha\beta)}=0, \tag{3.9}$$

and

$$aP_{[\alpha\beta]}+d\Gamma_{[\alpha,\beta]}-K_{[\alpha\beta]}=0, \tag{3.10}$$

which, upon taking its curl, gives

$$aP_{[\alpha\beta,\gamma]}-K_{[\alpha\beta,\gamma]}=0. \tag{3.11}$$

Here, we have used the indication $X_{[\alpha\beta,\gamma]}=X_{[\alpha\beta],\gamma}+X_{[\gamma\alpha],\beta}+X_{[\beta\gamma],\alpha}$ for the curl of $X_{[\alpha\beta]}$. Of course, the curl of $\Gamma_{[\alpha,\beta]}$ is zero. Equation (3.11) is the one equation with which the second Maxwell equation will have to do with.

In the next section we shall study the flat space linear approximation of the field equations, Eqs. (3.5), (3.8) and (3.11), and analyze the physical implications for the parameters when we require that the usual Maxwell's equations should hold in that flat space approximation.

IV. LINEAR APPROXIMATION ABOUT A FLAT SPACE: PHYSICAL IMPLICATION FOR THE PARAMETERS

We shall examine now the linear form of the field equations and analyze the spin content of $g_{(\alpha\beta)}$ and of $g_{[\alpha\beta]}$. Linearization about a Minkowski flat space with metric $\eta_{\alpha\beta}=(1,-1,-1,-1)$ is achieved by the expansion

$$g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta}, \tag{4.1a}$$

where $|h_{\alpha\beta}|\ll 1$. The inverse of this equation is

$$g^{\alpha\beta}=\eta^{\alpha\beta}-h^{\beta\alpha}, \tag{4.1b}$$

where the sub- and superscripts are moved by the metric $\eta_{\alpha\beta}$, that is, $h^{\beta\alpha}=\eta^{\beta\mu}\eta^{\alpha\nu}h_{\mu\nu}$. Notice that $g^{[\alpha\beta]}=+h^{[\alpha\beta]}=\eta^{\alpha\mu}\eta^{\beta\nu}h_{[\mu\nu]}$.

Let us first focus Eq. (3.8). Its first-order part is then

$$(a-3d)h^{[\alpha\sigma]}{}_{,\sigma}+2(1+3c-b)\eta^{\alpha\sigma}\Gamma_{\sigma}=0. \tag{4.2}$$

Therefore, to have a first Maxwellian type of equation for $h_{\alpha\beta}$, that is,

$$h_{[\alpha\sigma],}{}^{\sigma}=0, \tag{4.3}$$

we must have $a-3d\neq 0$ and the relation

$$1+3c-b=0, \tag{4.4}$$

with which Eq. (3.8) reduces to

$$\mathbf{g}^{[\alpha\sigma]}{}_{,\sigma}=0. \tag{4.5}$$

This is our first field equation. When divided by p it gives the generalized Maxwell first equation. With Eq. (4.4) we can then eliminate one more parameter. Equation (3.9) becomes

$$P_{(\alpha\beta)}+b\Gamma^{\mu}_{[\alpha\nu]}\Gamma^{\nu}_{[\mu\beta]}+\frac{1}{3}(b-1)\Gamma_{\alpha}\Gamma_{\beta}-K_{(\alpha\beta)}=0, \tag{4.6}$$

and, with the additional help of Eqs. (3.6) and (4.5), Eq. (3.5) becomes

$$\mathbf{f}^{\alpha\beta}{}_{,\gamma}+\mathbf{f}^{\alpha\sigma}\Gamma^{\beta}_{\gamma\sigma}+\mathbf{f}^{\sigma\beta}\Gamma^{\alpha}_{\sigma\gamma}-\mathbf{f}^{\alpha\beta}\Gamma^{\sigma}_{(\sigma\gamma)} -b(\mathbf{g}^{(\alpha\sigma)}\Gamma^{\beta}_{[\gamma\sigma]}-\mathbf{g}^{(\beta\sigma)}\Gamma^{\alpha}_{[\gamma\sigma]}) +\tfrac{1}{3}\delta^{\alpha}_{\gamma}(a\mathbf{g}^{[\beta\sigma]}+(b-1)\mathbf{g}^{(\beta\rho)})\Gamma_{\sigma} +\tfrac{1}{3}\delta^{\beta}_{\gamma}(a\mathbf{g}^{[\alpha\sigma]}-(b-1)\mathbf{g}^{(\alpha\rho)})\Gamma_{\sigma}=0. \tag{4.7}$$

We then see that up to here our field equations, Eqs. (3.11), (4.5), (4.6) and (4.7) depend on two parameters only, a and b. As we shall show in the sequel, by demanding that the second Maxwell equation should arise in this same flat space approximation the first parameter will be forced to be null, $a=0$, and, as a consequence of this, b will just disappear from the final field equations.

Before we go on we mention at this point that the field equations Eqs. (3.11), (4.6) and (4.7) can be reduced to simpler forms if, on account of the results in Eqs. (4.4) and (4.5), we make use of the invariance of the action $\int\mathbf{L}\ d^4x$ under Einstein's λ transformation [11], which is defined as a transformation to a new connection $\Delta^{\gamma}_{\alpha\beta}$ involving an arbitrary vector field λ_{α}. Then one can go to a new connection with zero torsion leading to the simplification of the field equations. However, we really do not need to go into these simpler forms of the equations to continue our discussion because they will lead exactly to the same final results. Notwithstanding, for completeness and to make a close contact with the BMB theory we discuss the λ transformation in Appendix A. There we show that by going to a new torsionless connection, as did Einstein in his theory, the BMB theory appears for the particular values $a=1$ and $b=0$ of our two free parameters. Then, following the same procedure to be discussed below, it is there shown that by demanding that the second Maxwell should hold in the flat space approximation, the same conditions mentioned at the end of the last paragraph will result for these two parameters leading to the same final field equations and showing then that they are independent of the transformation.

For future use we write here the symmetric and antisymmetric parts of Eq. (4.7):

$$\mathbf{g}^{(\alpha\beta)}{}_{,\gamma}+\mathbf{g}^{(\alpha\sigma)}\Gamma^{\beta}_{(\gamma\sigma)}+\mathbf{g}^{(\sigma\beta)}\Gamma^{\alpha}_{(\sigma\gamma)}-\mathbf{g}^{(\alpha\beta)}\Gamma^{\sigma}_{(\sigma\gamma)}+a\big(\mathbf{g}^{[\alpha\sigma]}\Gamma^{\beta}_{[\gamma\sigma]}+\mathbf{g}^{[\beta\sigma]}\Gamma^{\alpha}_{[\gamma\sigma]}+\tfrac{1}{3}(\delta^{\alpha}_{\gamma}\mathbf{g}^{[\beta\sigma]}+\delta^{\beta}_{\gamma}\mathbf{g}^{[\alpha\sigma]})\Gamma_{\sigma}\big)=0 \tag{4.7a}$$

and

$$a(\mathbf{g}^{[\alpha\beta]}{}_{,\gamma}+\mathbf{g}^{[\alpha\sigma]}\Gamma^{\beta}_{(\gamma\sigma)}+\mathbf{g}^{[\sigma\beta]}\Gamma^{\alpha}_{(\sigma\gamma)}-\mathbf{g}^{[\alpha\beta]}\Gamma^{\rho}_{(\rho\gamma)})+(1-b)\big(\mathbf{g}^{(\alpha\sigma)}\Gamma^{\beta}_{[\gamma\sigma]}-\mathbf{g}^{(\beta\sigma)}\Gamma^{\alpha}_{[\gamma\sigma]}-\tfrac{1}{3}(\delta^{\alpha}_{\gamma}\mathbf{g}^{(\beta\sigma)}-\delta^{\beta}_{\gamma}\mathbf{g}^{(\alpha\sigma)})\Gamma_{\sigma}\big)=0. \tag{4.7b}$$

Let us now focus Eq. (3.11). The first-order part of $K_{[\alpha\beta]}$ is, from Eq. (3.2),

$$K^{(1)}_{[\alpha\beta]}=\frac{2}{p^2}h_{[\alpha\beta]}. \tag{4.8}$$

Therefore to first-order Eq. (3.11) reads

$$aP^{(1)}_{[\alpha\beta,\gamma]}-\frac{2}{p^2}h_{[\alpha\beta,\gamma]}=0. \tag{4.9}$$

From here we immediately see that the second Maxwell equation will result if we restrict the parameter a to vanish,

$$a=0, \tag{4.10}$$

giving then

$$h_{[\alpha\beta,\gamma]}=0, \tag{4.11}$$

which is the desired result. This, together with Eq. (4.3) guarantees that $h_{[\alpha\beta]}$ has now a Maxwellian type of behavior describing a massless spin-1 particle. This is actually what we need for a consistent identification of $h_{[\alpha\beta]}$ to the electromagnetic field strength tensor in flat space, that is,

$$h_{[\alpha\beta]}=p\bar{F}_{\alpha\beta}, \tag{4.12}$$

where $\bar{F}_{\alpha\beta}$ here is the usual Maxwell field strength satisfying $\bar{F}_{\alpha\beta,}{}^{\beta}=0$ and $\bar{F}_{[\alpha\beta,\gamma]}=0$. Another, equivalent, way of putting things is to realize that with Eq. (4.8), Eq. (3.10) tell us that the condition to have $h_{[\alpha\beta]}$ derivable from a potential is to have $a=0$ because then the linearized form of Eq. (3.10) will give

$$h_{[\alpha\beta]}=\frac{1}{4}p^2d(\Gamma^{(1)}_{\alpha,\beta}-\Gamma^{(1)}_{\beta,\alpha}). \tag{4.13}$$

As a bonus we can conclude from this expression that the torsion vector is related to the vector potential. In fact, from Eq. (4.12) we must have

$$A_{\alpha}=-\frac{1}{4}pd\Gamma_{\alpha} \tag{4.14}$$

to obtain, to first order, $\bar{F}_{\alpha\beta}=A^{(1)}_{\beta,\alpha}-A^{(1)}_{\alpha,\beta}$.

Before we go on notice that upon multiplication of Eq. (4.9) by p we see clearly why in Moffat-Boal's theory which, as mentioned before is the case when $a=1$ and $b=0$ and with the identification in Eq. (2.4), the second Maxwell equation will result for a formally vanishing p. This is so because that formal limit will erase accidentally the first term, the one which actually shows that the second Maxwell equation is not present for fixed nonzero normal p always present in the theory.

To make a close contact with the Bonnor theory let us calculate the first term of Eq. (4.9) before the condition $a=0$. From Eq. (2.3) we get $P^{(1)}_{[\alpha\beta]}=\Gamma^{\sigma(1)}_{[\alpha\beta],\sigma}-\Gamma^{(1)}_{[\alpha,\beta]}$, and only the first term of this relation will contribute to Eq. (4.9). From Eqs. (4.7a) and (4.7b) we can easily solve for the connections to first order. This has been done before [7], but for completeness the calculation is delineated in Appendix B. The results are

$$\Gamma^{\sigma(1)}_{(\alpha\beta)}=\tfrac{1}{2}\eta^{\sigma\rho}(h_{(\alpha\rho),\beta}+h_{(\beta\rho),\alpha}-h_{(\alpha\beta),\rho}) \tag{4.15}$$

as in GR, which we shall be using shortly, and

$$(1-b)\Gamma^{\sigma(1)}_{[\alpha\beta]}=\tfrac{1}{2}a\,\eta^{\sigma\rho}(h_{[\alpha\rho],\beta}-h_{[\beta\rho],\alpha}+h_{[\alpha\beta],\rho})+\tfrac{1}{3}(1-b)(\Gamma^{(1)}_{\alpha}\delta^{\sigma}_{\beta}-\Gamma^{(1)}_{\beta}\delta^{\sigma}_{\alpha}). \tag{4.16}$$

Using Eq. (4.3) we then find

$$(1-b)P^{(1)}_{[\alpha\beta]}=\tfrac{1}{2}a\Box h_{[\alpha\beta]}-\tfrac{1}{3}(1-b)\Gamma^{(1)}_{[\alpha,\beta]}. \tag{4.17}$$

Thence Eq. (4.9) gives, for $b\neq 1$,

$$\frac{a^2}{2(1-b)}\Box h_{[\alpha\beta,\gamma]}-\frac{2}{p^2}h_{[\alpha\beta,\gamma]}=0. \tag{4.18}$$

Without the Bonnor, second, term and with $a=1$ and $b=0$, this relation reduces to the result of Einstein theory [2] which, as pointed out by Einstein himself, the resulting equation, $\Box h_{[\alpha\beta,\mu]}=0$, is weaker than Maxwell's second equation. Next, with the Bonnor term present in Eq. (4.18), and still with $a=1$ and $b=0$ we have the linearized field equation of Bonnor [3], with $h_{[\alpha\beta]}$ continuing then not to obey Maxwell's second equation. As discussed before, it will do so in the present scheme by choosing $a=0$, which eliminates the undesired contribution of $P_{[\alpha\beta]}$.

To complete the analysis of this linear approximation consider now Eq. (4.6). As the contributions of the b term and of the next one are at least of second order, as well as the term $I_{(\alpha\beta)}$ is from Eq. (3.2), that equation reduces to $P^{(1)}_{(\alpha\beta)}=0$, to first order. Then, from Eqs. (2.3) and (4.15), we get

$$\Box h_{(\alpha\beta)}=0, \tag{4.19}$$

where we have chosen harmonic coordinates, $h_{(\alpha\beta)},{}^{\beta} - \frac{1}{2}h_{,\alpha}=0$ with $h=\eta^{\alpha\beta}h_{(\alpha\beta)}$, which is identical to the GR result.

We end this section by analyzing the consequences of the value $a=0$ for the field equations. Equation (3.11) will lose the contribution of its first term containing the antisymmetric part of the connection. Consider now Eqs. (4.7a) and (4.7b). With $a=0$ the first one will also lose its Γ-antisymmetric contribution and the second one reduces to

$$(1-b)(\mathbf{g}^{(\alpha\sigma)}\Gamma^{\beta}_{[\gamma\sigma]}-\mathbf{g}^{(\beta\sigma)}\Gamma^{\alpha}_{[\gamma\sigma]} - \tfrac{1}{3}(\delta^{\alpha}_{\gamma}\mathbf{g}^{(\beta\sigma)}-\delta^{\beta}_{\gamma}\mathbf{g}^{(\alpha\sigma)})\Gamma_{\sigma})=0. \tag{4.20}$$

We shall show now that as a consequence of this relation the field equation Eq. (4.6) will lose the contribution of the antisymmetric part of the connection as well. For that purpose notice first that from Eq. (2.3) the first term of Eq. (4.6) is given by

$$P_{(\alpha\beta)}=\Gamma^{\sigma}_{(\alpha\beta),\sigma}-\tfrac{1}{2}[\Gamma^{\sigma}_{(\alpha\sigma),\beta}+\Gamma^{\sigma}_{(\beta\sigma),\alpha}]+\Gamma^{\sigma}_{(\alpha\beta)}\Gamma^{\gamma}_{(\sigma\gamma)} -\Gamma^{\sigma}_{(\alpha\gamma)}\Gamma^{\gamma}_{(\sigma\beta)}-\Gamma^{\sigma}_{[\alpha\gamma]}\Gamma^{\gamma}_{[\sigma\beta]}. \tag{4.21}$$

Therefore, the Γ-antisymmetric contribution to Eq. (4.6) can be easily localized and we shall show that it is also null, that is,

$$(b-1)\left(\Gamma^{\sigma}_{[\alpha\gamma]}\Gamma^{\gamma}_{[\sigma\beta]}+\frac{1}{3}\Gamma_{\alpha}\Gamma_{\beta}\right)=0, \tag{4.22}$$

independently of the value of b. We start by contracting Eq. (4.20) with $s_{\alpha\mu}s_{\beta\nu}$ where $s_{\alpha\beta}$, symmetric, is the inverse of $g^{(\alpha\beta)}$ as defined by $s_{\alpha\mu}g^{(\alpha\beta)}=\delta^{\beta}_{\mu}$. In this way we find

$$(1-b)\left(s_{\beta\nu}\Gamma^{\beta}_{[\gamma\mu]}-s_{\alpha\mu}\Gamma^{\alpha}_{[\gamma\nu]}+\frac{1}{3}(s_{\gamma\nu}\Gamma_{\mu}-s_{\gamma\mu}\Gamma_{\nu})\right)=0. \tag{4.23}$$

Adding to this relation those with γ and ν interchanged and then with γ and μ interchanged we get

$$(1-b)\left[s_{\alpha\gamma}\Gamma^{\alpha}_{[\nu\mu]}+\frac{1}{3}(s_{\gamma\nu}\Gamma_{\mu}-s_{\gamma\mu}\Gamma_{\nu})\right]=0, \tag{4.24}$$

which, upon contraction with $g^{(\lambda\gamma)}$ gives

$$(1-b)\left(\Gamma^{\lambda}_{[\nu\mu]}+\frac{1}{3}(\delta^{\lambda}_{\nu}\Gamma_{\mu}-\delta^{\lambda}_{\mu}\Gamma_{\nu})\right)=0. \tag{4.25}$$

Contracting this equation first with Γ_{λ} yields $(1-b)\Gamma^{\lambda}_{[\nu\mu]}\Gamma_{\lambda}=0$. Next, by making use of this last relation, contraction of that same equation with $\Gamma^{\mu}_{[\lambda\rho]}$ yields immediately Eq. (4.22). It is rather remarkable that the simple demand that $a=0$ to have the Maxwellian content of $g_{[\alpha\beta]}$ guaranteed in the linear approximation, should produce such a grand simplification of the field equations: the antisymmetric part of the connection has disappeared completely from sight, and together with it the parameter b too. Let us then go to the resultant field equations.

V. AN ALTERNATIVE UNIFIED THEORY

In the last section we saw that by restricting the parameters of the extended theory to suitable values we were able to achieve a consistent behavior of the fields upon linearization in flat space. The symmetric sector, becoming the one of GR, describes the spin-2 graviton and the antisymmetric sector describes, for $g_{[\alpha\beta]}$, a spin-1 Maxwellian type massless field, making then the identification in Eq. (2.4) a consistent procedure. Let us now see the form acquired by the field equations. As already said, with $a=0$ the field equation Eq. (3.11) loses completely its Γ-antisymmetric contribution so does Eq. (4.7a) and, as we have shown at the end of the last section, the same will happen to the field equation Eq. (4.6) when use is made of Eqs. (4.21), and (4.22). As it will be shown in a moment, see Eq. (5.9) below, the two terms inside the square brackets of Eq. (4.21) are equal. Consequently, Eq. (4.6) becomes

$$U_{\alpha\beta}-K_{(\alpha\beta)}=0, \tag{5.1}$$

where

$$U_{\alpha\beta}=\Gamma^{\sigma}_{(\alpha\beta),\sigma}-\Gamma^{\sigma}_{(\sigma\alpha),\beta}+\Gamma^{\sigma}_{(\alpha\beta)}\Gamma^{\gamma}_{(\sigma\gamma)}-\Gamma^{\sigma}_{(\alpha\gamma)}\Gamma^{\gamma}_{(\sigma\beta)}, \tag{5.2}$$

symmetric and containing only the symmetric part of the connection, is the analogue of the usual Ricci tensor. It is that piece of the Einstein tensor which contains only the symmetric part of the connection. Next, Eq. (3.11) with $a=0$, becomes just

$$K_{[\alpha\beta,\gamma]}=0, \tag{5.3}$$

and Eq. (4.7a) becomes

$$\mathbf{g}^{(\alpha\beta)}{}_{,\gamma}+\mathbf{g}^{(\alpha\sigma)}\Gamma^{\beta}_{(\gamma\sigma)}+\mathbf{g}^{(\sigma\beta)}\Gamma^{\alpha}_{(\sigma\gamma)}-\mathbf{g}^{(\alpha\beta)}\Gamma^{\sigma}_{(\sigma\gamma)}=0, \tag{5.4}$$

which has the same form of the equation for the contravariant metric density of GR. To these three field equations we have to add Eq. (4.5), which we repeat here for convenience,

$$\mathbf{g}^{[\alpha\beta]}{}_{,\beta}=0. \tag{5.5}$$

These are the final field equations of the theory. With Eq. (2.4), Eqs. (5.3) and (5.5) are the generalized Maxwell's equations.

Equation (5.4) can be solved for $\Gamma^{\sigma}_{(\alpha\beta)}$. We get [7], reproduced in Appendix C,

$$\Gamma^{\sigma}_{(\alpha\beta)}=\tfrac{1}{2}g^{(\sigma\lambda)}(s_{\alpha\lambda,\beta}+s_{\lambda\beta,\alpha}-s_{\alpha\beta,\lambda})+\Omega^{\sigma}_{\alpha\beta}, \tag{5.6}$$

where

$$\Omega^{\sigma}_{\alpha\beta}=\tfrac{1}{4}(g^{(\sigma\lambda)}s_{\alpha\beta}-\delta^{\sigma}_{\alpha}\delta^{\lambda}_{\beta}-\delta^{\lambda}_{\alpha}\delta^{\sigma}_{\beta})\left(\ln\frac{s}{g}\right)_{,\lambda} \tag{5.7}$$

and $s_{\alpha\beta}$, symmetric, and with determinant s is the inverse of $g^{(\alpha\beta)}$, as defined by

$$s_{\alpha\beta}g^{(\alpha\sigma)}=\delta^{\sigma}_{\beta}. \tag{5.8}$$

When the antisymmetric part of $g_{\alpha\beta}$ vanishes, $s_{\alpha\beta}$ will be equal to $g_{\alpha\beta}$ and the right-hand side of Eq. (5.6) becomes the usual Christoffel symbol, as it should. In the course of the derivation of Eq. (5.6) we come across the relation

$$\Gamma^{\sigma}_{(\sigma\alpha)}=\left(\ln\frac{-g}{\sqrt{-s}}\right)_{,\alpha}, \tag{5.9}$$

which can be reobtained from that equation. From this result, we see that the two terms inside the square brackets in Eq. (4.21) are equal, as promised.

The symmetric and antisymmetric parts of $K_{\alpha\beta}$ are, from Eq. (3.2),

$$K_{(\alpha\beta)}=\frac{1}{p^2}(g_{(\alpha\mu)}g^{[\mu\nu]}g_{[\nu\beta]}+g_{(\beta\mu)}g^{[\mu\nu]}g_{[\nu\alpha]}+\tfrac{1}{2}g_{(\alpha\beta)}g^{[\mu\nu]}g_{[\mu\nu]}) \tag{5.10}$$

and

$$K_{[\alpha\beta]}=\frac{1}{p^2}(g_{[\alpha\beta]}+g_{(\alpha\mu)}g^{[\mu\nu]}g_{(\nu\beta)}+g_{[\alpha\mu]}g^{[\mu\nu]}g_{[\nu\beta]}+\tfrac{1}{2}g_{[\alpha\beta]}g^{[\mu\nu]}g_{[\mu\nu]}). \tag{5.11}$$

This completes the discussion of the field equations. As it is shown in Appendix A the same field equations will result if we go first to the torsionless connection Δ.

Let us write now the Lagrangian in Eq. (2.1) in terms of $U_{\alpha\beta}$, which will lead directly to the field equations. With $c=(b-1)/3$ from Eq. (4.4), and with $a=0$ together with Eqs. (4.21) and (4.22), $Q_{\alpha\beta}$ in Eq. (2.5) reduces to

$$Q_{\alpha\beta}=U_{\alpha\beta}+d\Gamma_{[\alpha,\beta]}. \tag{5.12}$$

From here we see that Eq. (3.1) becomes

$$U_{\alpha\beta}+d\Gamma_{[\alpha,\beta]}-K_{\alpha\beta}=0, \tag{5.13}$$

whose symmetric and antisymmetric parts are, respectively, Eqs. (5.1) and

$$d\Gamma_{[\alpha,\beta]}-K_{[\alpha\beta]}=0, \tag{5.14}$$

which leads to Eq. (5.3). Now, from Eq. (5.12) the Lagrangian in Eq. (2.1) becomes

$$\mathbf{L}=-\mathbf{g}^{\alpha\beta}(U_{\alpha\beta}+d\Gamma_{[\alpha,\beta]})+\frac{1}{p^2}\mathbf{g}^{\alpha\beta}g_{[\alpha\beta]}. \tag{5.15}$$

If we vary this Lagrangian with respect to $g_{\alpha\beta}$, $\Gamma^{\sigma}_{(\alpha\beta)}$ and Γ_{α} we get, respectively, Eqs. (5.13), (5.4) and (5.5), the first one of these leading to Eqs. (5.1) and (5.3). From Eq. (5.15) we then see that the modification of the Einstein part [2] of the Bonnor Lagrangian [3] that we end up with turns out to be the replacement of the Einstein tensor $P_{\alpha\beta}$ by its piece $U_{\alpha\beta}$ which contains the symmetric part of the connection only, plus a single Lagrange multiplier term which can be written $d\Gamma_{\alpha}\mathbf{g}^{[\alpha\beta]}{}_{,\beta}$, up to a total derivative.

We end this section by remarking that the situation that we have here for the electromagnetic related part of Eq. (5.15) is analogous to the first-order form of the action of Maxwell theory as mentioned by Arnowitt, Deser and Misner [12], Maxwell's equation being derived from the Lagrangian

$$L_{em}=\overline{F}^{\alpha\beta}\overline{F}_{\alpha\beta}-2\overline{F}^{\alpha\beta}(\overline{A}_{\beta,\alpha}-\overline{A}_{\alpha,\beta}), \tag{5.16}$$

by independent variations of $\overline{F}_{\alpha\beta}$ and $\overline{A}$. One finds, respectively, $\overline{F}_{\alpha\beta}=\overline{A}_{\beta,\alpha}-\overline{A}_{\alpha,\beta}$ and $\overline{F}^{\alpha\beta}{}_{,\beta}=0$. Actually, recalling Eq. (4.14), the electromagnetic related part of Eq. (5.15) can be written $-(2/p)\mathbf{g}^{[\alpha\beta]}(A_{\beta,\alpha}-A_{\alpha,\beta})+(1/p^2)\mathbf{g}^{\alpha\beta}g_{[\alpha\beta]}$, which to first-order reduces to Eq. (5.16), as it should.

VI. LINEARIZATION ABOUT A GR BACKGROUND: A GHOST-FREE THEORY

Linearization about a Riemannian background with metric $g^{(0)}_{\alpha\beta}$ can be achieved by the expansion

$$g_{\alpha\beta}=g^{(0)}_{\alpha\beta}+g^{(1)}_{\alpha\beta}, \tag{6.1}$$

where $g^{(1)}_{\alpha\beta}$ is the perturbation. The inverse of $g_{\alpha\beta}$, as defined in Eq. (2.2), is then

$$g^{\alpha\beta}=g^{(0)\alpha\beta}-g^{(1)\beta\alpha}, \tag{6.2}$$

where the sub- and superscripts are moved by the initial metric tensor $g^{(0)}_{\alpha\beta}$, that is, $g^{(1)\alpha\beta}=g^{(0)\alpha\mu}g^{(0)\beta\nu}g^{(1)}_{\mu\nu}$. We then have $g^{(\alpha\beta)}=g^{(0)\alpha\beta}-g^{(1)(\alpha\beta)}$ and $g^{[\alpha\beta]}=g^{(1)[\alpha\beta]}$. Thence, to first order Eq. (5.5) reads

$$(\sqrt{-g^{(0)}}g^{(0)\alpha\mu}g^{(0)\beta\nu}g^{(1)}_{[\mu\nu]})_{,\beta}=0. \tag{6.3}$$

On the other hand, as Eq. (5.11) gives

$$K^{(1)}_{[\alpha\beta]}=\frac{2}{p^2}g^{(1)}_{[\alpha\beta]}, \tag{6.4}$$

we find that Eq. (5.3) yields

$$g^{(1)}_{[\alpha\beta,\gamma]}=0. \tag{6.5}$$

The conclusion from Eqs. (6.3) and (6.5) is then that $g^{(1)}_{[\alpha\beta]}$ satisfies the Maxwell's equations of the Einstein-Maxwell theory, making then its identification with the field strength $\tilde{F}_{\alpha\beta}$ of that theory now,

$$g^{(1)}_{[\alpha\beta]}=p\tilde{F}_{\alpha\beta}, \tag{6.6}$$

a consistent procedure. Let us look now at Eq. (5.1). First, from Eq. (5.10) we see that to lowest order

$$K_{(\alpha\beta)}=\frac{2}{p^2}\left(\frac{1}{4}g^{(0)}_{(\alpha\beta)}g^{(1)[\mu\nu]}g^{(1)}_{[\mu\nu]}-g^{(1)}{}_{[\alpha}{}^{\nu]}g^{(1)}_{[\beta\nu]}\right). \tag{6.7}$$

Therefore, with the identification in Eq. (6.6) we see that this quantity is proportional to the electromagnetic energy-momentum-stress tensor of the Einstein-Maxwell field:

$$K_{(\alpha\beta)}=8\pi E_{\alpha\beta}, \tag{6.8a}$$

where

$$E_{\alpha\beta}=\frac{1}{4\pi}\left(\frac{1}{4}g^{(0)}_{(\alpha\beta)}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}-\tilde{F}_{\alpha}{}^{\nu}\tilde{F}_{\beta\nu}\right). \tag{6.8b}$$

As $U_{\alpha\beta}$ to zeroth order becomes the usual Ricci tensor, Eq. (6.8a) is exactly what we need to have Eq. (5.1) going into the Einstein-Maxwell field equation. By the way, this condition fixes the sign of the Bonnor term as written in Eq. (2.1). Also, this is why we have chosen the proportionality constant in Eq. (2.4) equal to the same p of Eq. (2.1) so as to have the proper cancelation of constants in deriving Eq. (6.8a). Finally, consider the Lagrangian in Eq. (5.15) in the same approximation. With Eqs. (5.14), (6.4), and Eq. (6.6) we get

$$d\Gamma^{(1)}_{[\alpha,\beta]}=\frac{2}{p}\tilde{F}_{\alpha\beta}, \tag{6.9}$$

with which the Γ term inside the square brackets of Eq. (5.15) will be equal to $-2\sqrt{-g^{(0)}}\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}$. This then combines with the one that comes out from the Bonnor term, $\sqrt{-g^{(0)}}\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}$, reversing its sign to give finally $-\sqrt{-g^{(0)}}\tilde{F}^{\alpha\beta}\tilde{F}_{\alpha\beta}$ for the total electromagnetic contribution. This is exactly the Maxwell field Lagrangian of GR, with the correct minus sign, in the second-order form of the action.

From all these considerations it is clear that the theory is free from ghost-negative energy radiative excitations and bad asymptotic behavior. However, one might argue that there is a small piece of $g_{[\alpha\beta]}$ second-order dependence of $U_{\alpha\beta}$ to be analyzed. We shall do so following the work of DDM [6]. Using their notation we write

$$g_{\alpha\beta}=G_{\alpha\beta}+B_{\alpha\beta}, \tag{6.10}$$

where $G_{\alpha\beta}$ is the metric of the Riemannian background field and $B_{\alpha\beta}=g_{[\alpha\beta]}$ is the antisymmetric part of $g_{\alpha\beta}$, acting as a perturbation. The inverse of the metric is then, to second order,

$$g^{\alpha\beta}=G^{\alpha\beta}+B^{\alpha\beta}+B^{\alpha\gamma}B_{\gamma}{}^{\beta}, \tag{6.11}$$

where the sub- and superscripts are now moved by the metric $G_{\alpha\beta}$, that is, $B^{\beta\alpha}=G^{\beta\mu}G^{\alpha\nu}B_{\mu\nu}$, and its determinant is given by

$$\sqrt{-g}=\sqrt{-G}\left(1+\frac{1}{4}B^{\alpha\beta}B_{\alpha\beta}\right). \tag{6.12}$$

To zeroth and first order, the field equations Eqs. (5.1), (5.5) and (5.14) become

$$R_{\alpha\beta}(G)=0, \tag{6.13}$$

where $R_{\alpha\beta}(G)=U^{(0)}_{\alpha\beta}$ is the Ricci tensor of the background field,

$$B_{\alpha\beta|}{}^{\beta}=0, \tag{6.14}$$

where the vertical bar denotes covariant differentiation with respect to the background metric $G_{\alpha\beta}$, and

$$-\frac{2}{p^2}B_{\alpha\beta}+d\Gamma^{(1)}_{[\alpha,\beta]}=0, \tag{6.15}$$

leading to

$$B_{[\alpha\beta,\gamma]}=0. \tag{6.16}$$

Continuing with the arguments of DDM let us calculate the energy in the theory by their method. We give the details of the calculation in Appendix E and quote here only the final results. Using the expansions in Eqs. (6.10)–(6.12), the Lagrangian in Eq. (5.15), through order B^2, turns out to be

$$\mathbf{L}=-\sqrt{-G}R(G)+\mathbf{L}_B, \tag{6.17}$$

where $R(G)$ is the Ricci scalar of the background field and the second-order B part is, up to total derivatives,

$$\mathbf{L}_B=-\sqrt{-G}\left[M^{\alpha\beta}R_{\alpha\beta}+B^{\alpha\beta}\left(d\Gamma^{(1)}_{[\alpha,\beta]}-\frac{1}{p^2}B_{\alpha\beta}\right)\right], \tag{6.18}$$

where

$$M^{\alpha\beta}=\frac{1}{4}G^{\alpha\beta}B^{\mu\nu}B_{\mu\nu}-B^{\alpha\mu}B^{\beta}{}_{\mu}, \tag{6.19}$$

having the same structure of the Maxwell electromagnetic stress tensor. Actually, by identifying $B_{\alpha\beta}$ to the electromagnetic tensor $F^{G}_{\alpha\beta}$ of the G-field, $B_{\alpha\beta}=pF^{G}_{\alpha\beta}$, $M^{\alpha\beta}$ is proportional to the electromagnetic stress tensor $E^{\alpha\beta}(G)$ of the background G-field, $M^{\alpha\beta}=4\pi p^2E^{\alpha\beta}(G)$. In this formulation the B stress tensor $T^{\mu\nu}$ is defined through the variation of the B action according to [13]

$$\delta I_B=\int \delta\mathbf{L}_B d^4x=-8\pi\int T^{\mu\nu}\delta G_{\mu\nu}\sqrt{-G}d^4x. \tag{6.20}$$

Performing the variation we get, after using the field equations Eqs. (6.13)–(6.15),

$$T^{\mu\nu}=\frac{1}{4\pi p^2}\left(\frac{1}{4}G^{\mu\nu}B^{\alpha\beta}B_{\alpha\beta}-B^{\mu\alpha}B^{\nu}{}_{\alpha}\right)+\frac{1}{8\pi}\left[M^{\rho(\mu}{}_{|}{}^{\nu)}{}_{\rho}-\frac{1}{2}M^{\mu\nu}{}_{|}{}^{\rho}{}_{\rho}\right]. \tag{6.21}$$

Now, the first, Maxwellian type term on the right-hand side leads to positive energy. All the other terms inside the square brackets, involving total G-covariant derivatives, can give no contribution to the average over wavelengths of radiation, this average being the measurable observable which leads to the physically well defined effective energy-momentum-stress tensor, as shown by Isaacson [14] for high frequency radiation after the work of Brill and Hartle [15], and extended to lower frequencies by Efroimsky [16]. We are then left effectively with only the Maxwellian term of Eq. (6.21),

$$T^{\mu\nu}_{eff}=\frac{1}{4\pi p^2}\left(\frac{1}{4}G^{\mu\nu}B^{\alpha\beta}B_{\alpha\beta}-B^{\mu\alpha}B^{\nu}{}_{\alpha}\right), \qquad (6.22)$$

which, with the identification $B_{\alpha\beta}=pF^{G}_{\alpha\beta}$, becomes the electromagnetic stress tensor of the background G-field

$$T^{\mu\nu}_{eff}=E^{\mu\nu}(G). \qquad (6.23)$$

Therefore, the theory is, in fact, free of ghost radiative modes. We close this section with a few comments regarding the comparison of the present theory with the standard first-order formulation of the Einstein-Maxwell (EM) theory. Both theories share the same symmetries, general covariance and Maxwell gauge invariance, here for Γ_α and there for the EM vector potential $\tilde{A}_\alpha$, which appear both as Lagrange multipliers, but all this on completely different grounds. The variations with respect to these two quantities lead, respectively, to the divergence equations for $\mathbf{g}^{[\alpha\beta]}$ and for the EM $\mathbf{F}^{\alpha\beta}$. The variation with respect to $\tilde{F}^{\alpha\beta}$ leads to the curl equation for $\tilde{F}_{\alpha\beta}$ and, here, the variation of the electromagnetic part of the Lagrangian with respect to $g^{\alpha\beta}$ (and, by the way, not $g^{[\alpha\beta]}$) leads to the curl equation for $K_{\alpha\beta}$. Now, $\tilde{F}^{\alpha\beta}$ is related to the electromagnetic field strength $\tilde{F}_{\alpha\beta}$ through the EM metric, $\tilde{F}^{\alpha\beta}=g^{(0)\alpha\mu}g^{(0)\beta\nu}\tilde{F}_{\mu\nu}$, but no such a simple kind of relation holds between $g^{[\alpha\beta]}$ and $g_{[\alpha\beta]}$. Introducing the determinants of the symmetric and antisymmetric parts of the metric, $g_S=\det(g_{(\alpha\beta)})$ and $g_A=\det(g_{[\alpha\beta]})$, and calling $a^{\alpha\beta}$ and $m^{\alpha\beta}$ the inverses of the respective parts, as defined by $a^{\alpha\beta}g_{(\alpha\gamma)}=\delta^{\beta}_{\gamma}$ and $m^{\alpha\beta}g_{[\alpha\gamma]}=\delta^{\beta}_{\gamma}$, one has [17] first the relation $g^{[\alpha\beta]}=g^{-1}(g_S a^{\alpha\mu}a^{\beta\nu}g_{[\mu\nu]}+\frac{1}{2}\sqrt{g_A}\epsilon^{\alpha\beta\mu\nu}g_{[\mu\nu]})$. Here, $\sqrt{g_A}=8^{-1}\epsilon^{\alpha\beta\mu\nu}g_{[\alpha\beta]}g_{[\mu\nu]}$ and if g_A is not equal to zero, $m^{\alpha\beta}=(2\sqrt{g_A})^{-1}\epsilon^{\alpha\beta\mu\nu}g_{[\mu\nu]}$. As the determinants are related by [17] $g=g_S(1+2^{-1}a^{\mu\nu}a^{\alpha\beta}g_{[\mu\alpha]}g_{[\nu\beta]})+g_A$, we see that if the $g_{[\alpha\beta]}$ are small $g^{[\alpha\beta]}$ will contain only odd powers of $g_{[\alpha\beta]}$. To first order we have $g^{(1)[\alpha\beta]}=g^{(0)\alpha\mu}g^{(0)\beta\nu}g^{(1)}_{[\mu\nu]}$, as stated before, and with the identification of $g^{(1)}_{[\alpha\beta]}$ to $\tilde{F}_{\alpha\beta}$ the electromagnetic field related equations of the theory will reproduce the Maxwell equations of the EM theory. At higher orders we shall have, of course, corrections to the EM field equations. Second, focusing now the symmetric parts of the metric and of its inverse, one has the relation $g^{(\alpha\beta)}=g^{-1}(g_S a^{\alpha\beta}+g_A m^{\alpha\mu}m^{\beta\nu}g_{(\mu\nu)})$. This shows that $g^{(\alpha\beta)}$ contains even powers of $g_{[\alpha\beta]}$ only. Therefore, to lowest order, the EM theory is reobtained. At higher orders the present theory will then give corrections to it.

VII. EQUATIONS OF MOTION

In this section we shall derive the equations of motion of charged point particles by the method of Einstein, Infeld and Hoffmann [8]. This was already applied by Bonnor [3] showing that with his extra term in the Einstein Lagrangian the equations of motion do contain the Coulomb force in the lowest nontrivial order of approximation considered. The identification was made of $g_{[\alpha\beta]}$ to the dual $F^{*}_{\alpha\beta}$ of the electromagnetic field tensor. The calculation was repeated by Moffat and Boal [5] in their new interpretation of the theory but with this same identification, although they suggested themselves the identification of $g_{[\alpha\beta]}$ to $F_{\alpha\beta}$ itself. This was pointed out by Johnson [18] in a footnote, mentioning also that the two identifications lead to the same result in the lowest order considered.

Here we shall derive the equations with the identification in Eq. (2.4). It will be shown that, although it will appear in the course of the calculation an extra term induced by the present field equations, that term will not contribute to the last step of the calculation leaving only the Coulomb force term: all calculations coincide to the lowest order.

We then put $g_{(\alpha\beta)}=a_{\alpha\beta}$ and $g_{[\alpha\beta]}=f_{\alpha\beta}$ and assume that they can be expanded in powers of a parameter λ as follows:

$$a_{00}=1+\lambda^2\,{}_2a_{00}+\lambda^4\,{}_4a_{00}+\cdots,$$
$$a_{0i}=\lambda^3\,{}_3a_{0i}+\lambda^5\,{}_5a_{0i}+\cdots,$$
$$a_{ij}=-\delta_{ij}+\lambda^2\,{}_2a_{ij}+\lambda^4\,{}_4a_{ij}+\cdots, \qquad (7.1)$$

as in GR, and, compatible with the identification in Eq. (2.4), where f_{oi} is related to the electric field E_i and f_{ij} to the magnetic field B_k,

$$f_{0i}=\lambda^2\,{}_2f_{0i}+\lambda^4\,{}_4f_{0i}+\cdots \qquad (7.2a)$$

and

$$f_{ij}=\lambda^3\,{}_3f_{ij}+\lambda^5\,{}_5f_{ij}+\cdots. \qquad (7.2b)$$

We shall need $f_{\alpha\beta}$ to order λ^2 only. Then as in GR [8], we shall be able to construct from the field equation Eq. (5.1) a nonlinear quantity of order λ^4, the surface integral of which around each one of the N particle system, will give the equation of motion of that particular particle, in the lowest order of approximation. Starting then with Eq. (5.5) we find to lowest order,

$${}_2f_{0i,i}=0. \qquad (7.3)$$

Next Eq. (5.3) gives, using Eq. (6.4) and keeping in mind that time derivatives are one order higher than space derivatives,

$${}_2f_{0i,j}-{}_2f_{0j,i}=0. \qquad (7.4)$$

From here we see that ${}_2f_{0i}$ derives from a potential. We write

$${}_2f_{0i}=-p\,\phi_{,i} \qquad (7.5)$$

with which, according to Eq. (2.4), ϕ is then the second-order electric potential. From Eq. (7.3) it satisfies the Laplace equation,

$$\nabla^2\phi=0. \tag{7.6}$$

Equation (7.2b) refers to the magnetic field, which we can then neglect in the approximation we are working on. For simplicity we shall consider the case in which only two particles are present. The appropriate solution of Eq. (7.6) is then

$$\phi(\mathbf{x})=\frac{{}_2\overset{1}{e}}{\overset{1}{r}}+\frac{{}_2\overset{2}{e}}{\overset{2}{r}}, \tag{7.7}$$

where, with $A=1,2$,

$$\overset{A}{r}=|\mathbf{x}-\overset{A}{\mathbf{z}}(t)|, \tag{7.8}$$

$\overset{A}{\mathbf{z}}(t)$ being the position vector of particle A at time t, and charges are expanded as $\overset{A}{e}=\lambda^2{}_2\overset{A}{e}+\cdots$.

Finally we consider Eq. (5.1) to order λ^4, from which the equations of motion should follow in lowest order. To quadratic order in $h_{\alpha\beta}=g_{\alpha\beta}-\eta_{\alpha\beta}$ Eq. (5.6) gives, derived in Appendix D,

$$\Gamma^{\sigma}_{(\alpha\beta)}=(\eta^{\sigma\lambda}-h^{(\sigma\lambda)})(h_{(\alpha\lambda)},_{\beta}+h_{(\beta\lambda),\alpha}-h_{(\alpha\beta),\lambda})+\frac{1}{2}(m_{\alpha\beta},^{\sigma}-m_{\alpha}{}^{\sigma},_{\beta}-m_{\beta}{}^{\sigma},_{\alpha}), \tag{7.9}$$

where

$$m_{\alpha\beta}=\frac{1}{4}\eta_{\alpha\beta}h^{[\mu\nu]}h_{[\mu\nu]}-h_{[\alpha}{}^{\mu]}h_{[\beta\mu]}, \tag{7.10}$$

having the same structure of the flat space electromagnetic energy-momentum tensor. Therefore, to quadratic terms in $h_{\alpha\beta}$ we shall have outside the singularities, where $m_{\alpha\beta},^{\beta}=0$ by Eqs. (4.3) and (4.11),

$$U_{\alpha\beta}=R_{\alpha\beta}+V_{\alpha\beta}, \tag{7.11}$$

$R_{\alpha\beta}$ being the usual Ricci tensor of GR to order λ^4 and, as $m^{\alpha}{}_{\alpha}=0$,

$$V_{\alpha\beta}=\frac{1}{2}\Box m_{\alpha\beta}. \tag{7.12}$$

As in GR [8], the equations of motion to the lowest order considered here come from the following fourth-order relation:

$${}_4\overset{A}{C'}_i-\frac{1}{4\pi}\int^{A}2({}_4V^{*}_{ij}-{}_4K^{*}_{(ij)})n_j dS=0. \tag{7.13}$$

The first term, constructed with R_{ij}, is the result of GR and given by [8], for particle 1,

$${}_4\overset{1}{C'}_i=4\,{}_2\overset{1}{m}\left(\frac{d^2}{d\tau^2}\overset{1}{z}_i+{}_2m_2(\overset{1}{z}_i-\overset{2}{z}_i)r^{-3}\right), \tag{7.14}$$

where $r=|\overset{1}{\mathbf{z}}-\overset{2}{\mathbf{z}}|$ is the distance between the two particles. The masses are expanded as $\overset{A}{m}=\lambda^2\,{}_2\overset{A}{m}+\cdots$ and $\tau=\lambda t$ is the auxiliary time variable. In the second term of Eq. (7.13), n_i is the normal to the spherical surface of integration enclosing particle 1 and no other, then

$${}_4K^{*}_{(ij)}={}_4K_{(ij)}+\frac{1}{2}\delta_{ij}({}_4K_{00}-{}_4K_{kk}), \tag{7.15}$$

and

$${}_4V^{*}_{(ij)}={}_4V_{(ij)}+\frac{1}{2}\delta_{ij}({}_4V_{00}-{}_4V_{kk}). \tag{7.16}$$

From Eq. (5.10) we find, using Eq. (7.5),

$$-{}_4K^{*}_{(ij)}=2\phi_{,i}\phi_{,j}-\delta_{ij}\phi_{,k}\phi_{,k}, \tag{7.17}$$

which coincides with the result of Moffat and Boal [5]. The corresponding surface integral is found to be

$$-\frac{1}{4\pi}\int^{1}2\,{}_4K^{*}_{(ij)}n_j dS=4\,{}_2\overset{1}{e}\,{}_2\overset{2}{e}(\overset{1}{z}_i-\overset{2}{z}_i)r^{-3}. \tag{7.18}$$

For the second integral in Eq. (7.13) one finds a null result,

$$\int^{1}{}_4V^{*}_{ij}n_j dS=0, \tag{7.19}$$

as we show in Appendix F, not contributing then to the last step of the calculation. Thus, with the above two results and Eq. (7.14) we find from Eq. (7.13), after multiplication by λ^4,

$$\overset{1}{m}\frac{d^2\mathbf{r}}{dt^2}=-\frac{\overset{1}{m}\overset{2}{m}\mathbf{r}}{r^3}+\frac{\overset{1}{e}\overset{2}{e}\mathbf{r}}{r^3}, \tag{7.20}$$

where $\mathbf{r}=\overset{1}{\mathbf{z}}-\overset{2}{\mathbf{z}}$ is the instantaneous position vector of particle 1 relative to 2. Equation (7.20) gives the equation of motion of the first particle in the field of the second. A similar relation holds for particle 2. Therefore, the Coulomb force is present in the lowest nontrivial order. We notice that because of the cancelation of p^2 in deriving Eq. (7.17), there is no extra factor in the Coulomb term in Eq. (7.20). This is the second reason for the choice of the proportionality constant in Eq. (2.4) equals to the same p which is present in Eq. (2.1).

VIII. CONCLUSIONS

We have developed a metric nonsymmetric unified theory of gravitation and electromagnetism. The formulation is based on a modification of the Einstein [2] part of the Bonnor Lagrangian [3], in such a way that the metric $g_{[\alpha\beta]}$ describes a spin-1 massless particle obeying Maxwell's equations in the flat space linear approximation, and thus making the identification of $g_{[\alpha\beta]}$ to the electromagnetic field tensor $F_{\alpha\beta}$ as in Eq. (2.4), a consistent procedure. The modification of the Einstein part of the Bonnor Lagrangian that we end up with turns out to be the replacement of the Einstein tensor by its piece that contains the symmetric part of the connection only, plus a single Lagrange multiplier term. The field equations, then containing only the symmetric part of the connection, came out to be as close as possible to those of general relativity. The Einstein-Maxwell theory is contained in the first approximation of the field equations about a curved general relativity background . The theory is shown to be free of ghost-negative energy radiative modes even when expanded about a Riemannian background.

The equation of motion of a charged test particle was obtained in the lowest nontrivial approximation by the method of Einstein *et al.* [8], showing the appearance of the Coulomb force.

We end by making a few comments on future works. In a following paper we shall analyze the particle content of the theory showing by a study of the corresponding propagator that the theory is free of ghost-negative energy particles, in the sense of elementary particle theory, and tachyons in the linear approximation.

Another topic of interest would be the analysis of the equations of motion to the next orders of approximation where, besides the appearance of the second, velocity dependent, term of the Lorentz force, a velocity independent but p-dependent term modifying the Coulomb force is expected to be present. This Coulomb correction term might then be used to determine or to put an upper limit on the universal constant p.

In a forthcoming paper we shall study the solution of the field equations for a pointlike charged source with a spherically symmetric field.

APPENDIX A: λ TRANSFORMATION

Here we shall show that the same final field equations will result if we go first to a torsionless connection by means of Einstein's λ transformation [11]. We start from the form of the field equations after the requirement of having the first Maxwell equation in the flat space linear approximation limit, as established at the end of Sec. III and in the beginning of Sec. IV. They are Eqs. (3.11), (4.5), (4.6) and (4.7). Following Einstein we shall show now that the last three field equations can be brought to a simpler form, which will permit a close contact with the Bonnor [3] theory. Then we show that the final field equations, that is after the requirement of having also the second Maxwell equation in the weak field limit, will be the same. Einstein has shown [11] that his action is invariant under a suitable transformation of the connections involving an arbitrary vector field λ_α (λ transformation), as are also the field equations resulting from it. In its Hermitian form the λ transformation is defined by the following relation between Γ and a new connection Δ,

$$\Gamma^\sigma_{\alpha\beta}=\Delta^\sigma_{\alpha\beta}+\lambda_\alpha\delta^\sigma_\beta-\lambda_\beta\delta^\sigma_\alpha. \tag{A1}$$

Notice that the transformation affects only the antisymmetric part of the connection, that is $\Gamma^\sigma_{(\alpha\beta)}=\Delta^\sigma_{(\alpha\beta)}$. It is easy to see that with Eqs. (4.4) and (4.5), our field action is invariant under this transformation. In fact, a simple calculation will show that $Q_{\alpha\beta}$ transforms as [7]

$$Q_{\alpha\beta}(\Gamma)=Q_{\alpha\beta}(\Delta)+(3d-a)\lambda_{[\alpha,\beta]}+(1-b+3c)\times(\Delta_\alpha\lambda_\beta+\Delta_\beta\lambda_\alpha+3\lambda_\alpha\lambda_\beta), \tag{A2}$$

which is to be substituted in Eq. (2.1). Then, after an integration by parts, we see that the action transforms as

$$\int \mathbf{L}(\Gamma)d^4x=\int \mathbf{L}(\Delta)d^4x+\int\{(3d-a)\mathbf{g}^{[\alpha\beta]}{}_{,\beta}\lambda_\alpha-(1-b+3c)\mathbf{g}^{(\alpha\beta)}\times(\Delta_\alpha\lambda_\beta+\Delta_\beta\lambda_\alpha+3\lambda_\alpha\lambda_\beta)\}d^4x. \tag{A3}$$

Therefore, with Eqs. (4.4) and (4.5) the invariance of the action is established. It is interesting to note that, conversely, if we demand the invariance of the action under the λ transformation, we get the abovementioned two equations. From this viewpoint Eq. (4.5) appears as the physical content of the transformation. With Eq. (4.4), Eq. (A2) reduces to

$$Q_{\alpha\beta}(\Gamma)=Q_{\alpha\beta}(\Delta)+(3d-a)\lambda_{[\alpha,\beta]}. \tag{A4}$$

One can easily check that the field equations are also invariant under the λ transformation as they should. For this, notice that when taking Eq. (A4) into Eq. (3.1) its last, λ, term will be eliminated from the antisymmetric part of the resulting equation by taking the curl of it. Then, following Einstein, we can use the invariance property to simplify the field equations by going to that particular connection Δ with zero torsion. As the torsions are related by $\Gamma_\alpha=\Delta_\alpha+3\lambda_\alpha$, from Eq. (A1), we can achieve

$$\Delta_\alpha=0 \tag{A5a}$$

by making a convenient choice for λ_α, that is,

$$\lambda_\alpha=\tfrac{1}{3}\Gamma_\alpha. \tag{A5b}$$

Therefore, the λ transformation leading to a torsionless connection Δ is

$$\Gamma^\sigma_{\alpha\beta}=\Delta^\sigma_{\alpha\beta}+\tfrac{1}{3}(\Gamma_\alpha\delta^\sigma_\beta-\Gamma_\beta\delta^\sigma_\alpha). \tag{A6}$$

Thence, Eq. (A4) becomes, using Eq. (2.5) for a torsionless connection Δ,

$$Q_{\alpha\beta}(\Gamma)=P_{(\alpha\beta)}(\Delta)+aP_{[\alpha\beta]}(\Delta)+b\Delta^\mu_{[\alpha\nu]}\Delta^\nu_{[\mu\beta]}+(d-\tfrac{1}{3}a)\Gamma_{[\alpha,\beta]}, \tag{A7}$$

which is to be substituted into Eq. (3.1), to give

$$P_{(\alpha\beta)}(\Delta)+aP_{[\alpha\beta]}(\Delta)+b\Delta^\mu_{[\alpha\nu]}\Delta^\nu_{[\mu\beta]}+(d-\tfrac{1}{3}a)\Gamma_{[\alpha,\beta]}-K_{\alpha\beta}=0. \tag{A8}$$

Taking the symmetric and antisymmetric parts of this equation we get

$$P_{(\alpha\beta)}(\Delta)+b\Delta^{\mu}_{[\alpha\nu]}\Delta^{\nu}_{[\mu\beta]}-K_{(\alpha\beta)}=0 \tag{A9}$$

and

$$aP_{[\alpha\beta]}(\Delta)+\left(d-\frac{1}{3}a\right)\Gamma_{[\alpha,\beta]}-K_{[\alpha\beta]}=0, \tag{A10}$$

which gives

$$aP_{[\alpha\beta,\gamma]}(\Delta)-K_{[\alpha\beta,\gamma]}=0. \tag{A11}$$

On the other hand, taking Eq. (A6) into Eq. (4.7) we find

$$\begin{aligned}\mathbf{f}^{\alpha\beta}{}_{,\gamma}+\mathbf{f}^{\alpha\rho}\Delta^{\beta}_{\gamma\rho}+\mathbf{f}^{\rho\beta}\Delta^{\alpha}_{\rho\gamma}-\mathbf{f}^{\alpha\beta}\Delta^{\rho}_{(\rho\gamma)}\\ -b(\mathbf{g}^{(\alpha\sigma)}\Delta^{\beta}_{[\gamma\sigma]}-\mathbf{g}^{(\beta\sigma)}\Delta^{\alpha}_{[\gamma\sigma]})=0.\end{aligned} \tag{A12}$$

Equations (A9), (A11), (A12) and Eq. (4.5), which we reproduce here for completeness

$$\mathbf{g}^{[\alpha\beta]}{}_{,\beta}=0, \tag{A13}$$

are the new field equations, in terms of the torsionless connection Δ. The first three have simpler forms than before, this being true for Eq. (A11) as well because the Einstein tensor in Eq. (2.3) becomes simpler now,

$$P_{\alpha\beta}=\Delta^{\sigma}_{\alpha\beta,\sigma}-\tfrac{1}{2}(\Delta^{\sigma}_{(\alpha\sigma),\beta}+\Delta^{\sigma}_{(\sigma\beta),\alpha})+\Delta^{\sigma}_{\alpha\beta}\Delta^{\gamma}_{(\sigma\gamma)}-\Delta^{\sigma}_{\alpha\gamma}\Delta^{\gamma}_{\sigma\beta}, \tag{A14}$$

containing only the symmetric part of the connection in the second term. The field equations are seen to depend on two parameters only a and b as before. In the particular case in which $a=1$ and $b=0$ they reduce to the Bonnor field equations [3]. Notice that when $a=1$, Eq. (3.4) tell us that $\mathbf{f}^{\alpha\sigma}=\mathbf{g}^{\alpha\sigma}$ and, therefore, when we also have $b=0$, Eq. (A12) reduces to the corresponding Einstein's field equation for $g_{\alpha\beta}$, by using Eq. (2.2).

The symmetric and antisymmetric parts of Eq. (A12) are

$$\begin{aligned}\mathbf{g}^{(\alpha\beta)}{}_{,\gamma}+\mathbf{g}^{(\alpha\sigma)}\Delta^{\beta}_{(\gamma\sigma)}+\mathbf{g}^{(\sigma\beta)}\Delta^{\alpha}_{(\sigma\gamma)}-\mathbf{g}^{(\alpha\beta)}\Delta^{\sigma}_{(\sigma\gamma)}\\ +a(\mathbf{g}^{[\alpha\sigma]}\Delta^{\beta}_{[\gamma\sigma]}+\mathbf{g}^{[\beta\sigma]}\Delta^{\alpha}_{[\gamma\sigma]})=0\end{aligned} \tag{A15a}$$

and

$$\begin{aligned}a(\mathbf{g}^{[\alpha\beta]}{}_{,\gamma}+\mathbf{g}^{[\alpha\sigma]}\Delta^{\beta}_{(\gamma\sigma)}+\mathbf{g}^{[\sigma\beta]}\Delta^{\alpha}_{(\sigma\gamma)}-\mathbf{g}^{[\alpha\beta]}\Delta^{\sigma}_{(\sigma\gamma)})\\ +(1-b)(\mathbf{g}^{(\alpha\sigma)}\Delta^{\beta}_{[\gamma\sigma]}-\mathbf{g}^{(\beta\sigma)}\Delta^{\alpha}_{[\gamma\sigma]})=0.\end{aligned} \tag{A15b}$$

To obtain the second Maxwell equation from Eq. (A11) in the flat space linear approximation, when Eq. (4.8) holds, we need again to have

$$a=0. \tag{A16}$$

To make the same close contact with the corresponding Bonnor linear field equation we calculate the first term of Eq. (A11) prior to the condition $a=0$. Following the same procedure as before, Eqs. (A15a) and (A15b) give, to first order,

$$\Delta^{\sigma(1)}_{(\alpha\beta)}=\tfrac{1}{2}\eta^{\sigma\rho}(h_{(\alpha\rho),\beta}+h_{(\beta\rho),\alpha}-h_{(\alpha\beta),\rho}) \tag{A17a}$$

and

$$(1-b)\Delta^{\sigma(1)}_{[\alpha\beta]}=\tfrac{1}{2}a\,\eta^{\sigma\rho}(h_{[\alpha\rho],\beta}-h_{[\beta\rho],\alpha}+h_{[\alpha\beta],\rho}). \tag{A17b}$$

As a check we can verify that these relations follows directly from Eqs. (4.15) and (4.16) when use is made of Eq. (A6). From Eq. (A14) we have now $P^{(1)}_{[\alpha\beta]}(\Delta)=\Delta^{\sigma(1)}_{[\alpha\beta],\sigma}$ only. Therefore, Eq. (4.18) holds again, the coincidence being do to the fact that the torsion in Eq. (4.17) does not contribute to the curl, and the discussion proceeds as before. With $a=0$ Eq. (A10) becomes

$$d\Gamma_{[\alpha,\beta]}-K_{\alpha\beta}=0, \tag{A18}$$

and Eq. (A11) becomes

$$K_{[\alpha\beta,\gamma]}=0, \tag{A19}$$

with which Eqs. (5.14) and (5.3) respectively coincide with. Next, Eq. (A15a) will lose its antisymmetric contribution, becoming

$$\mathbf{g}^{(\alpha\beta)}{}_{,\gamma}+\mathbf{g}^{(\alpha\sigma)}\Delta^{\beta}_{(\gamma\sigma)}+\mathbf{g}^{(\sigma\beta)}\Delta^{\alpha}_{(\sigma\gamma)}-\mathbf{g}^{(\alpha\beta)}\Delta^{\sigma}_{(\sigma\gamma)}=0, \tag{A20}$$

with which Eq. (5.4) coincides with because $\Gamma^{\beta}_{(\gamma\sigma)}=\Delta^{\beta}_{(\gamma\sigma)}$, leading then to the same solution as in Eq. (5.6). From Eq. (A15b) we shall have

$$(1-b)(\mathbf{g}^{(\alpha\sigma)}\Delta^{\beta}_{[\gamma\sigma]}-\mathbf{g}^{(\beta\sigma)}\Delta^{\alpha}_{[\gamma\sigma]})=0, \tag{A21}$$

which is simpler than Eq. (4.20). Repeating the reasoning after this equation, we get from Eq. (A21),

$$(1-b)\Delta^{\alpha}_{[\gamma\mu]}=0. \tag{A22}$$

With this result and from the symmetric part of Eq. (A14), the field equation Eq. (A9) will then also lose its Γ-antisymmetric contribution becoming

$$U_{\alpha\beta}(\Delta)-K_{(\alpha\beta)}=0, \tag{A23}$$

where $U_{\alpha\beta}(\Delta)$ is given by

$$U_{\alpha\beta}=\Delta^{\sigma}_{(\alpha\beta),\sigma}-\Delta^{\sigma}_{(\alpha\sigma),\beta}+\Delta^{\sigma}_{(\alpha\beta)}\Delta^{\gamma}_{(\sigma\gamma)}-\Delta^{\sigma}_{(\alpha\gamma)}\Delta^{\gamma}_{(\sigma\beta)}, \tag{A24}$$

which coincides with Eq. (5.2) because, again, $\Gamma^{\gamma}_{(\alpha\beta)}=\Delta^{\gamma}_{(\alpha\beta)}$. Consequently Eq. (5.1) coincides with Eq. (A23). Therefore, the final field equations that we got in Sec. V are the same as those that would be obtained by going first to the torsionless connection through the λ transformation, showing then that the field equations are independent of it.

Taking Eqs. (A16) and (A22), into Eq. (A7) we see that the Lagrangian in Eq. (2.1) is

$$\mathbf{L}=-\mathbf{g}^{\alpha\beta}(U_{\alpha\beta}+d\Gamma_{[\alpha,\beta]})+\frac{1}{p^2}\mathbf{g}^{\alpha\beta}g_{[\alpha\beta]}, \tag{A25}$$

with which Eq. (5.15), of course, coincides.

APPENDIX B: SOLUTION FOR Γ TO FIRST ORDER

We derive here Eqs. (4.15) and (4.16). Consider Eq. (4.7a). As the a term is of second order, we can write to first order,

$$g^{(\alpha\beta)}{}_{,\gamma}+g^{(\alpha\sigma)}\Gamma^{\beta}_{(\gamma\sigma)}+g^{(\sigma\beta)}\Gamma^{\alpha}_{(\sigma\gamma)}+g^{(\alpha\beta)}C_{\gamma}=0, \tag{B1}$$

where

$$C_{\gamma}=[\ln(\sqrt{-g})]_{,\gamma}-\Gamma^{\sigma}_{(\sigma\gamma)}. \tag{B2}$$

Introduce now the inverse of $g^{(\alpha\beta)}$, as defined by

$$s_{\sigma\beta}g^{(\alpha\beta)}=\delta^{\alpha}_{\sigma}, \tag{B3}$$

where $s_{\sigma\beta}=s_{\beta\sigma}$, symmetric. From here, if s designates the determinant of $s_{\sigma\beta}$, then

$$[\ln(\sqrt{-s})]_{,\gamma}=\tfrac{1}{2}s_{\alpha\beta,\gamma}g^{(\alpha\beta)}. \tag{B4}$$

Contraction of Eq. (B1) with $s_{\mu\beta}$ and using Eq. (B3) will give

$$-s_{\mu\beta,\gamma}g^{(\alpha\beta)}+s_{\mu\beta}g^{(\alpha\rho)}\Gamma^{\beta}_{(\gamma\rho)}+\Gamma^{\alpha}_{(\mu\gamma)}+\delta^{\alpha}_{\mu}C_{\gamma}=0. \tag{B5}$$

If we contract μ and α and use Eq. (B2) we get

$$-s_{\alpha\beta,\gamma}g^{(\alpha\beta)}+4[\ln(\sqrt{-g})]_{,\gamma}-2\Gamma^{\sigma}_{(\sigma\gamma)}=0. \tag{B6}$$

Using Eq. (B4) it follows that

$$\Gamma^{\sigma}_{(\sigma\gamma)}=\left[\ln\left(\frac{-g}{\sqrt{-s}}\right)\right]_{,\gamma}, \tag{B7}$$

which is Eq. (5.9). With this result, Eq. (B2) becomes

$$C_{\gamma}=\tfrac{1}{2}\left[\ln\left(\frac{s}{g}\right)\right]_{,\gamma}. \tag{B8}$$

Now let us go back to Eq. (B5). After contraction with $s_{\alpha\nu}$, we obtain

$$-s_{\mu\nu,\gamma}+s_{\mu\beta}\Gamma^{\beta}_{(\gamma\nu)}+s_{\alpha\nu}\Gamma^{\alpha}_{(\mu\gamma)}+s_{\mu\nu}C_{\gamma}=0. \tag{B9}$$

We show now that C_{γ} is of second order. To first order we have $g=-(1+h)$, where $h=\eta^{\alpha\beta}h_{\alpha\beta}$ and, from Eq. (B3),

$$s_{\sigma\beta}=\eta_{\sigma\beta}+h_{(\sigma\beta)}. \tag{B10}$$

From here, we get $s=-(1+h)$, to first order. Therefore, to first order, g and s are equal and, consequently, C_{γ} is at least of second order. Then, using Eq. (B10) in Eq. (B9), we get, to first order

$$-h_{(\mu\nu),\gamma}+\eta_{\mu\beta}\Gamma^{\beta}_{(\gamma\nu)}+\eta_{\alpha\nu}\Gamma^{\alpha}_{(\mu\gamma)}=0. \tag{B11}$$

Subtracting from this relation those obtained by exchanging first μ and γ, and then ν and γ yields, after contraction with $\eta^{\alpha\gamma}$

$$\Gamma^{\alpha}_{(\mu\nu)}=\tfrac{1}{2}\eta^{\alpha\gamma}(h_{(\mu\gamma),\nu}+h_{(\nu\gamma),\mu}-h_{(\mu\nu),\gamma}), \tag{B12}$$

which is Eq. (4.15). Next, let us consider Eq. (4.7b). To first order we have

$$ah^{[\alpha\beta]}{}_{,\gamma}+(1-b)(\eta^{\alpha\sigma}\Gamma^{\beta}_{[\gamma\sigma]}-\eta^{\beta\sigma}\Gamma^{\alpha}_{[\gamma\sigma]}-\tfrac{1}{3}(\delta^{\alpha}_{\gamma}\eta^{\beta\sigma}-\delta^{\beta}_{\gamma}\eta^{\alpha\sigma})\Gamma_{\sigma})=0. \tag{B13}$$

Contraction with $\eta_{\alpha\mu}\eta_{\beta\lambda}$ gives

$$ah_{[\mu\lambda],\gamma}+(1-b)(\eta_{\beta\lambda}\Gamma^{\beta}_{[\gamma\mu]}-\eta_{\alpha\mu}\Gamma^{\alpha}_{[\gamma\lambda]}-\tfrac{1}{3}(\eta_{\mu\gamma}\Gamma_{\lambda}-\eta_{\lambda\gamma}\Gamma_{\mu}))=0. \tag{B14}$$

Adding to this equation, those obtained by exchanging first μ and γ, and then λ and γ we get, after contraction with $\eta^{\nu\gamma}$,

$$\tfrac{1}{2}a\,\eta^{\nu\gamma}(h_{[\mu\lambda],\gamma}+h_{[\gamma\lambda],\mu}+h_{[\mu\gamma],\lambda})=(1-b)[\Gamma^{\nu}_{[\mu\lambda]}-\tfrac{1}{3}(\delta^{\nu}_{\lambda}\Gamma_{\mu}-\delta^{\nu}_{\mu}\Gamma_{\lambda})], \tag{B15}$$

which is Eq. (4.16).

APPENDIX C: SOLUTION FOR $\Gamma^{\sigma}_{(\alpha\beta)}$

Starting from Eq. (5.4), we see that Eq. (B1) holds as an exact relation. Therefore, all the results up to Eq. (B9) hold as exact results. Subtracting from this relation those obtained by exchanging first μ and γ, then by exchanging λ and γ, and contracting the final result with $g^{(\alpha\gamma)}$ we obtain Eq. (5.6):

$$\Gamma^{\alpha}_{(\mu\nu)}=\tfrac{1}{2}g^{(\alpha\gamma)}(s_{\mu\gamma,\nu}+s_{\nu\gamma,\mu}-s_{\mu\nu,\gamma})+\tfrac{1}{4}(g^{(\alpha\gamma)}s_{\mu\nu}-\delta^{\alpha}_{\nu}\delta^{\gamma}_{\mu}-\delta^{\gamma}_{\nu}\delta^{\alpha}_{\mu})\left(\ln\frac{s}{g}\right)_{,\gamma}. \tag{C1}$$

APPENDIX D: $\Gamma^{\sigma}_{(\alpha\beta)}$ TO QUADRATIC TERMS

Pushing Eq. (4.1b) to quadratic terms we find $g^{\alpha\beta}=\eta^{\alpha\beta}-h^{\beta\alpha}+h^{\sigma\alpha}h^{\beta}{}_{\sigma}$ and, therefore,

$$g^{(\alpha\beta)}=\eta^{\alpha\beta}-h^{(\alpha\beta)}+h^{\sigma(\alpha}h^{\beta)}{}_{\sigma}. \tag{D1}$$

Then, from Eq. (5.8) the inverse of this is

$$s_{\alpha\beta}=\eta_{\alpha\beta}+h_{(\alpha\beta)}+h_{[\alpha\rho]}h_{[\beta\sigma]}\eta^{\rho\sigma}, \tag{D2}$$

to quadratic terms in $h_{\alpha\beta}$. Next we need the ratio s/g. It is easier to work with the inverses of the determinants. We have, with $\varepsilon_{0123}=1$, $g^{-1}=\varepsilon_{\alpha\beta\gamma\delta}\,g^{0\alpha}g^{1\beta}g^{2\gamma}g^{3\delta}$ and $s^{-1}=\varepsilon_{\alpha\beta\gamma\delta}\,g^{(0\alpha)}g^{(1\beta)}g^{(2\gamma)}g^{(3\delta)}$. Writing $g^{\alpha\beta}=g^{(\alpha\beta)}+g^{[\alpha\beta]}$ we find $g^{-1}=s^{-1}-\tfrac{1}{2}h^{[\mu\nu]}h_{[\mu\nu]}$ or $s/g=1+\tfrac{1}{2}h^{[\mu\nu]}h_{[\mu\nu]}$. Therefore,

$$\ln\frac{s}{g}=\frac{1}{2}h^{[\mu\nu]}h_{[\mu\nu]}, \tag{D3}$$

to quadratic terms in $h_{\alpha\beta}$. Placing all these three results in Eq. (C1) we get Eq. (7.9).

APPENDIX E: THE SECOND-ORDER LAGRANGIAN

With the expansion in Eqs. (6.10)–(6.12), the $U_{\alpha\beta}$ term of Eq. (5.15) is, to second order in $B_{\mu\nu}$,

$$\mathbf{g}^{\alpha\beta}U_{\alpha\beta}=\sqrt{-G}G^{\alpha\beta}U_{\alpha\beta}+\sqrt{-G}\left(\frac{1}{4}G^{\alpha\beta}B^{\mu\nu}B_{\mu\nu}+B^{\alpha\mu}B_{\mu}{}^{\beta}\right)R_{\alpha\beta}(G), \tag{E1}$$

where $R_{\alpha\beta}(G)=U^{(0)}_{\alpha\beta}$ is the Ricci tensor of the background field. To the same order we shall have [7]

$$U_{\alpha\beta}=R_{\alpha\beta}(G)+U^{(1)}_{\alpha\beta}+U^{(2)}_{\alpha\beta}, \tag{E2}$$

$$U^{(1)}_{\alpha\beta}=\Gamma^{(1)\sigma}_{(\alpha\beta)|\sigma}-\Gamma^{(1)\sigma}_{(\alpha\sigma)|\beta}, \tag{E3}$$

and

$$U^{(2)}_{\alpha\beta}=\Gamma^{(2)\sigma}_{(\alpha\beta)|\sigma}-\Gamma^{(2)\sigma}_{(\alpha\sigma)|\beta}+\Gamma^{(1)\sigma}_{(\alpha\beta)}\Gamma^{(1)\gamma}_{(\sigma\gamma)}-\Gamma^{(1)\sigma}_{(\alpha\gamma)}\Gamma^{(1)\gamma}_{(\sigma\beta)}. \tag{E4}$$

Here, a vertical bar indicates the Riemannian covariant derivative with respect to the background Christoffel connection $\{^{\lambda}_{\alpha\beta}\}(G)\equiv\Sigma^{\lambda}_{\alpha\beta}$. Now, from Eq. (5.4) we see that $\Gamma^{\sigma}_{(\alpha\beta)}$ differs from its zeroth-order value $\Sigma^{\lambda}_{\alpha\beta}$ by terms of order B^2, because this is what happens to $g^{(\alpha\beta)}$ and $\sqrt{-G}$, from Eqs. (6.11) and (6.12). Consequently, $\Gamma^{(1)\sigma}_{(\alpha\beta)}=0$. Therefore, there will be no contribution from $U^{(1)}_{\alpha\beta}$ and, up to total derivatives, neither from $U^{(2)}_{\alpha\beta}$ to Eq. (E1). Consequently, Eq. (5.15) then reads as in Eq. (6.17) with $\mathbf{L}_B$ given in Eq. (6.18). The stress B tensor is defined through the variation of the action as in Eq. (6.20). Taking into account the first and second-order metric derivative of the Ricci tensor when calculating $\delta\mathbf{L}_B$ we get

$$T^{\mu\nu}=-\frac{1}{8\pi\sqrt{-G}}\left(\frac{\partial\mathbf{L}_B}{\partial G_{\mu\nu}}-\partial_\lambda\frac{\partial\mathbf{L}_B}{\partial G_{\mu\nu,\lambda}}+\partial_\kappa\partial_\lambda\frac{\partial\mathbf{L}_B}{\partial G_{\mu\nu,\kappa\lambda}}\right). \tag{E5}$$

We could proceed with the calculation from here. However, it is much easier to work directly with Eq. (6.20) by performing the explicit variation of Eq. (6.18) before integration. We get

$$\delta\mathbf{L}_B=-(\delta\mathbf{M}^{\alpha\beta})R_{\alpha\beta}-\mathbf{M}^{\alpha\beta}\delta R_{\alpha\beta}-\left(d\Gamma^{(1)}_{[\alpha,\beta]}-\frac{1}{p^2}B_{\alpha\beta}\right)\delta(\sqrt{-G}B^{\alpha\beta}). \tag{E6}$$

After using the field Eq. (6.13), $R_{\alpha\beta}=0$, the first term on the right of this expression drops out. For the second one we use the well-known relation

$$\delta R_{\alpha\beta}=(\delta\Sigma^{\lambda}_{\alpha\beta})_{|\lambda}-(\delta\Sigma^{\lambda}_{\lambda\beta})_{|\alpha}, \tag{E7}$$

with which it can be written

$$-\mathbf{M}^{\alpha\beta}\delta R_{\alpha\beta}=-(\mathbf{M}^{\alpha\beta}\delta\Sigma^{\lambda}_{\alpha\beta}-\mathbf{M}^{\lambda\beta}\delta\Sigma^{\sigma}_{\sigma\beta})_{|\lambda}+\mathbf{M}^{\alpha\beta}{}_{|\lambda}\delta\Sigma^{\lambda}_{\alpha\beta}-\mathbf{M}^{\lambda\beta}{}_{|\lambda}\delta\Sigma^{\sigma}_{\sigma\beta}. \tag{E8}$$

The quantity in the parentheses is a contravariant vector density and consequently the covariant derivative can be replaced by the ordinary derivative. Therefore, the first term gives no contribution to δI_B. In this way we have then eliminated second-order derivatives from the start. The last term of Eq. (E8) also drops out because $\mathbf{M}^{\lambda\beta}{}_{|\lambda}=0$, by making use of Eqs. (6.14) and (6.16). Using Eqs. (5.11) and (5.14) for $d\Gamma^{(1)}_{[\alpha,\beta]}$ in the last term of Eq. (E6) we then find

$$\delta I_B=\int d^4x\left(\mathbf{M}^{\alpha\beta}{}_{|\lambda}\delta\Sigma^{\lambda}_{\alpha\beta}-\frac{1}{p^2}B_{\alpha\beta}\delta(\sqrt{-G}B^{\alpha\beta})\right). \tag{E9}$$

To calculate the variations we need the following results:

$$\delta G^{\alpha\beta}=-G^{\alpha(\mu}G^{\nu)\beta}\delta G_{\mu\nu}, \tag{E10}$$

then, from $B^{\alpha\beta}=G^{\alpha\rho}G^{\beta\sigma}B_{\rho\sigma}$,

$$\delta B^{\alpha\beta}=(-G^{\alpha(\mu}B^{\nu)\beta}+G^{\beta(\mu}B^{\nu)\alpha})\delta G_{\mu\nu} \tag{E11}$$

and

$$\delta\Sigma^{\lambda}_{\alpha\beta}=-G^{\lambda(\mu}\Sigma^{\nu)}_{\alpha\beta}\delta G_{\mu\nu}+\frac{1}{2}(\delta^{(\mu}_{\alpha}G^{\nu)\lambda}\delta^{\rho}_{\beta}+\delta^{(\mu}_{\beta}G^{\nu)\lambda}\delta^{\rho}_{\alpha}-\delta^{(\mu}_{\alpha}\delta^{\nu)}_{\beta}G^{\lambda\rho})(\delta G_{\mu\nu})_{,\rho}. \tag{E12}$$

Thence, up to a ρ ordinary total derivative,

$$\mathbf{M}^{\alpha\beta}{}_{|\lambda}\delta\Sigma^{\lambda}_{\alpha\beta}=\left(-\mathbf{M}^{\alpha\beta}{}_{|}{}^{(\mu}\Sigma^{\nu)}_{\alpha\beta}-\left(\mathbf{M}^{\rho(\mu}{}_{|}{}^{\nu)}-\frac{1}{2}\mathbf{M}^{\mu\nu}{}_{|}{}^{\rho}\right)_{,\rho}\right)\delta G_{\mu\nu}. \tag{E13}$$

The expression inside the parentheses can be written in terms of a covariant derivative. Substitution into Eq. (E9) and using Eq. (E11) for its last term, one finds

$$\delta I_B=\int d^4x\left(\begin{array}{c}-\mathbf{M}^{\rho(\mu}{}_{|}{}^{\nu)}{}_{\rho}+\frac{1}{2}\mathbf{M}^{\mu\nu}{}_{|}{}^{\rho}{}_{\rho}\\ -\frac{1}{p^2}\sqrt{-G}\left(\frac{1}{2}G^{\mu\nu}B^{\alpha\beta}B_{\alpha\beta}-2B^{\mu\beta}B^{\nu}{}_{\beta}\right)\end{array}\right). \tag{E14}$$

From here we finally arrive at Eq. (6.21), recalling Eq. (6.20).

APPENDIX F: THE VANISHING OF THE ADDITIONAL SURFACE INTEGRAL

Keeping in mind that time derivatives are of one order higher than space derivatives, Eq. (7.12) gives, for the components we are interested in Eq. (7.19),

$$ {}_4V^*_{ij} = {}_4V_{ij} = -\frac{1}{2}\,{}_4m_{ij,kk}, \tag{F1} $$

the first equality following from the relation $m^{\alpha}_{\alpha}=0$, from Eq. (7.10). Thence, we see that we have to calculate the spherical surface integral of ${}_4m_{ij,kk}n_j$. Using Eq. (7.5) we get, from Eq. (7.10), ${}_4m_{ij}=p^2(\frac{1}{2}\delta_{ij}\phi,_m\phi,_m-\phi,_i\phi,_j)$ and, as ϕ is a harmonic function outside the singularities, we conclude from Eq. (F1) that, for instance around particle 1,

$$ \int^1 {}_4V^*_{ij}n_j dS = -p^2\int^1\left(\frac{1}{2}\delta_{ij}\phi,_{mk}\phi,_{mk}-\phi,_{ik}\phi,_{jk}\right)n_j dS, \tag{F2} $$

where for two particles only ϕ is given by Eq. (7.7), which we write as $\phi=\psi+\chi$, where $\psi={}_2\overset{1}{e}/\overset{1}{r}$ and $\chi={}_2\overset{2}{e}/\overset{2}{r}$. Consider first the second term inside the parentheses of Eq. (F2), which can then be written as

$$ \phi,_{ik}\phi,_{jk}=\psi,_{ik}\psi,_{jk}+\psi,_{ik}\chi,_{jk}+\chi,_{ik}\psi,_{jk}+\chi,_{ik}\chi,_{jk}. \tag{F3} $$

Now we contract this expression with the normal $n_j=(x_j-\overset{1}{z}_j)(\overset{1}{r})^{-1}$ to the spherical surface which is centered in particle 1. Consider the contribution of the first term. As $\psi,_{ik}$ is proportional to $3n_in_k-\delta_{ik}$, we shall end up by having the solid angle integral (s.a.i.) of n_i, which is equal to zero. For the second term we first expand $\chi,_{jk}$ around $\overset{1}{\mathbf{z}}$,

$$ \chi,_{jk}=\chi,_{jk}(1)+(x_m-\overset{1}{z}_m)\chi,_{jkm}(1)+\cdots. \tag{F4} $$

When taken in the second term of Eq. (F3) the first term of this expression will lead to the s.a.i. of an odd number of normal components, which is zero. This will also happen to all the other odd terms of Eq. (F4). On the other hand the second term of Eq. (F4) will lead to the s.a.i. of $(3n_in_k-\delta_{ij})n_mn_j,$ the first one being proportional to $\delta_{ik}\delta_{mj}+\delta_{im}\delta_{jk}+\delta_{ij}\delta_{km}$ and the second to $-\delta_{ik}\delta_{jm}$. This will then produce a sum of terms all equal to $\chi,_{jji}(1)$, which vanishes because χ is a harmonic function outside of the singularities. The same will occur for the even terms of the expansion in Eq. (F4). Similar considerations will show that the last two terms of Eq. (F3) leads also to vanishing results. Therefore, there is no contribution from the second term in the parentheses of Eq. (F2) and, similarly, none from the first one as well. Therefore, the surface integral in Eq. (F2) vanishes,

$$ \int^1 {}_4V^*_{ij}n_j dS = 0, \tag{F5} $$

as stated in Eq. (7.19).

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