Hamiltonian structure of the teleparallel formulation of general relativity

M. Blagojević* and I. A. Nikolić
Institute of Physics, P. O. Box 57, 11001 Belgrade, Yugoslavia

(Received 28 January 2000; published 26 June 2000)

We apply Dirac's Hamiltonian approach to study the canonical structure of the teleparallel form of general relativity without matter fields. It is shown, without any gauge fixing, that the Hamiltonian has the generalized Dirac-ADM form, and constraints satisfy all the consistency requirements. The set of constraints involves some extra first-class constraints, which are used to find additional gauge symmetries and clarify the gauge structure of the theory.

PACS number(s): 04.50.+h, 04.20.Cv

I. INTRODUCTION

Among various attempts to overcome the problems of quantization and the existence of singular solutions in Einstein's general relativity (GR), gauge theories of gravity are especially attractive, as they are based on the concept of gauge symmetry which has been very successful in the foundation of other fundamental interactions. The importance of the Poincaré symmetry in particle physics leads one to consider the Poincaré gauge theory (PGT) as a natural framework for a description of the gravitational phenomena [1–5] (for more general attempts, see [6]).

The basic gravitational variables in PGT are the tetrad field $b^k{}_\mu$ and the Lorentz connection $A^{ij}{}_\mu$, which are associated with the translation and Lorentz subgroups of the Poincaré group, respectively. These gauge fields are coupled to the energy-momentum and spin of matter fields, and their field strengths are geometrically identified with the torsion and curvature: $T^i{}_{\mu\nu}=\partial_\mu b^i{}_\nu+A^i{}_{s\mu}b^s{}_\nu-(\mu\leftrightarrow\nu)$, $R^{ij}{}_{\mu\nu}=\partial_\mu A^{ij}{}_\nu+A^i{}_{s\mu}A^{sj}{}_\nu-(\mu\leftrightarrow\nu)$. The spacetime of PGT turns out to be Riemann-Cartan space U_4, equipped with a metric and linear, metric compatible connection. The dynamical content of PGT is determined by the Lagrangian $\tilde{\mathcal{L}}\equiv b(\mathcal{L}_G+\mathcal{L}_M)$, where the gravitational part $\mathcal{L}_G$ is usually assumed to be at most quadratic in field strengths, and $\mathcal{L}_M$ describes minimally coupled matter fields.

The general geometric arena of PGT, the Riemann-Cartan space U_4, may be *a priori* restricted by imposing certain conditions on the curvature and the torsion. Thus, Einstein's GR is defined in Riemann space V_4, which is obtained from U_4 by the requirement of vanishing torsion. Another interesting limit of PGT is *teleparallel* or *Weitzenböck* geometry T_4, defined by the requirement of vanishing curvature:

$$R^{ij}{}_{\mu\nu}(A)=0. \tag{1.1}$$

The vanishing of curvature means that parallel transport is path independent; hence we have an absolute parallelism. The teleparallel geometry is, in a sense, complementary to Riemannian geometry: curvature vanishes, and torsion remains to characterize the parallel transport.

Of particular importance for the physical interpretation of the teleparallel geometry is the fact that there is a one-parameter family of teleparallel Lagrangians which is *empirically* equivalent to GR [5,7,8]. For the parameter value $B=1/2$ the Lagrangian of the theory coincides, modulo a four-divergence, with the Einstein-Hilbert Lagrangian, and defines the teleparallel form of GR, $\mathrm{GR}_\parallel$.

The teleparallel description of gravity has been one of the most promising alternatives to GR. However, analyzing this theory Kopczyński [9] found a hidden gauge symmetry, and concluded that the torsion evolution is not completely determined by the field equations. Assuming, then, that the torsion should be a measurable physical quantity, he argued that this theory is internally inconsistent. Hayashi and Shirafuji [10] tried to avoid this problem by interpreting certain different torsion configurations as physically equivalent, i.e., related to each other by a gauge transformation, but the consistency of this idea in the interacting theory seems to be questionable for nonscalar matter [9,11]. Various modifications of the one-parameter teleparallel theory are proposed in order to avoid the above problems [9,12,13]. Trying to reexamine the gauge structure of the one-parameter teleparallel geometry Nester [14] improved the arguments of Kopczyński [9]; the predictability problem was stated more precisely and bound to certain special solutions. This conclusion has been further verified by Cheng *et al.* [15], who recognized the importance of nonlinear constraint effects for the dynamical structure of the theory.

Hecht *et al.* [16] traced the appearance of nonphysical modes of torsion back to some symmetries which are necessarily present in the 3+1 decomposition of spacetime. Using certain geometric arguments they concluded that some components of the tetrad velocity are not suited to represent dynamical degrees of freedom. In other words, these velocities must not appear in the evolution equations; hence they should appear at most linear in the Lagrangian. The choice of parameters in the teleparallel Lagrangian that ensures this to happen is just the one corresponding to $\mathrm{GR}_\parallel$.

The teleparallel geometry possesses many salient features. Thus, Nester [17] succeeded in formulating a pure tensorial proof of the positivity of total energy for Einstein's theory in terms of the teleparallel geometry. He found that special gauge features of $\mathrm{GR}_\parallel$, which are usually considered to be problematic, are quite beneficial for this purpose. Mielke [18] used the teleparallel geometry of $\mathrm{GR}_\parallel$ to give a trans-

*Email address: mb@phy.bg.ac.yu

parent description of Ashtekar's complex variables, while de Andrade *et al.* [19] formulated a five-dimensional teleparallel equivalent of Kaluza-Klein theory. There are also some attempts to understand the role of torsion at the quantum level [20].

The purpose of this paper is to investigate the canonical structure of $\mathrm{GR}_\parallel$ using Dirac's Hamiltonian approach [21], as this is, in our opinion, the best way to clarify both the nature of somewhat mysterious extra gauge symmetries and the question of consistency of $\mathrm{GR}_\parallel$. We shall find that a specific choice of coupling constants in the teleparallel Lagrangian leads to the appearance of some additional first-class constraints and, consequently, to extra gauge symmetries, which clarify the meaning of nondynamical torsion components and give us a complete picture of the gauge structure of $\mathrm{GR}_\parallel$.

We remark here that Maluf [22] tried to analyze some aspects of the Hamiltonian structure of $\mathrm{GR}_\parallel$. However, his approach is based on some unnecessary gauge fixing conditions, adopted at the level of Lagrangian in order to simplify the calculations, so that many specific gauge features of the theory remained effectively hidden.

The layout of the paper is as follows. After recalling some basic elements of the Lagrangian teleparallel formulation of GR in Sec. II, we work out all the primary constraints and construct the corresponding Hamiltonian density in Sec. III. It is shown that a specific choice of parameters in the Lagrangian leads to additional primary constraints. Then, we study the consistency conditions in Sec. IV, and derive the algebra of constraints in Sec. V. These results are used in Sec. VI to construct extra gauge generators and clarify the nature of the related gauge symmetries. Section VII is devoted to concluding remarks, while some technical details are presented in the Appendixes.

Our conventions are the same as in Ref. [23]: the Latin indices refer to the local Lorentz frame, whereas the Greek indices refer to the coordinate frame; the first letters of both alphabets $(a,b,c,\dots;\alpha,\beta,\gamma,\dots)$ run over 1,2,3, and the middle alphabet letters $(i,j,k,\dots;\mu,\nu,\lambda,\dots)$ run over 0,1,2,3; $\eta_{ij}=\mathrm{diag}(+,-,-,-)$, $\varepsilon^{0123}=+1$, $\varepsilon^{\mu\nu\lambda\rho}_{ijkl}=\varepsilon^{\mu\nu\lambda\rho}\varepsilon_{ijkl}$ and $\delta=\delta(x-x')$.

II. TELEPARALLEL FORMULATION OF GR

Lagrangian. A gravitational field in the framework of the teleparallel geometry in PGT is described by the tetrad $b^k{}_\mu$ and Lorentz connection $A^{ij}{}_\mu$, subject to the condition of vanishing curvature (1.1). We shall consider here the gravitational dynamics determined by a class of Lagrangians quadratic in the torsion [5,7,8]

$$\begin{aligned}\tilde{\mathcal{L}}&=b(\mathcal{L}_T+\lambda_{ij}{}^{\mu\nu}R^{ij}{}_{\mu\nu}+\mathcal{L}_M),\\ \mathcal{L}_T&=a(AT_{ijk}T^{ijk}+BT_{ijk}T^{jik}+CT_kT^k)\\ &\equiv\beta_{ijk}(T)T^{ijk},\end{aligned}\tag{2.1}$$

where $\lambda_{ij}{}^{\mu\nu}$ are Lagrange multipliers introduced to ensure the teleparallelism condition (1.1) in the variational formalism, $a=1/2\kappa$ ($\kappa=$ Einstein's gravitational constant), $T_k=T^m{}_{mk}$, and $\mathcal{L}_M$ is the Lagrangian of matter fields. The explicit form of β_{ijk} is

$$\beta_{ijk}=a(AT_{ijk}+BT_{[jik]}+C\eta_{i[j}T_{k]}).$$

The parameters A,B,C in the Lagrangian should be determined on physical grounds, so as to obtain a consistent theory which could describe all the known gravitational experiments. If we require that the theory (2.1) gives the same results as GR in the linear, weak–field approximation, we can restrict our considerations to the one–parameter family of Lagrangians, defined by the conditions [5,7,8]

(i) $2A+B+C=0$, $C=-1$.

This family represents a viable gravitational theory for macroscopic, spinless matter, empirically indistinguishable from GR. Von der Heyde [24] and Hehl *et al.* [5] have given certain theoretical arguments in favor of the choice $B=0$. There is, however, another, particularly interesting choice determined by the requirement

(ii) $2A-B=0$.

It leads effectively to the Einstein–Hilbert Lagrangian $\mathcal{L}_{GR}=-abR(\Delta)$, defined in Riemann spacetime V_4 with Levi-Cività connection $A=\Delta$, via the geometric identity (A3):

$$\begin{aligned}bR(A)=&bR(\Delta)+b(\tfrac{1}{4}T_{ijk}T^{ijk}+\tfrac{1}{2}T_{ijk}T^{jik}-T_kT^k)\\&-2\partial_\nu(bT^\nu).\end{aligned}$$

Indeed, in Weitzenböck spacetime the above identity in conjunction with the condition (1.1) implies that the torsion Lagrangian in Eqs. (2.1) is equivalent to the Einstein-Hilbert Lagrangian, up to a four-divergence, provided that

$$A=\tfrac{1}{4},\quad B=\tfrac{1}{2},\quad C=-1,\tag{2.2}$$

which coincides with conditions (i) and (ii) given above.

The theory defined by Eqs. (2.1) and (2.2) is called the teleparallel formulation of GR ($\mathrm{GR}_\parallel$). Note that the equivalence with GR holds certainly for scalar matter, while the gravitational couplings to spinning matter fields in T_4 and V_4 are in general different.

Field equations. By varying the Lagrangian (2.1) with respect to $b^i{}_\mu, A^{ij}{}_\mu$, and $\lambda_{ij}{}^{\mu\nu}$ we obtain the gravitational field equations

$$-4\nabla_\mu(b\beta_i{}^{\mu\nu})-4b\beta^{nm\nu}T_{nmi}+h_i^\nu b\mathcal{L}_T=\tau^\nu{}_i,\tag{2.3a}$$

$$-8b\beta_{[ij]}{}^\nu+4\nabla_\mu(b\lambda_{ij}{}^{\nu\mu})=\sigma^\nu{}_{ij},\tag{2.3b}$$

$$R^{ij}{}_{\mu\nu}=0,\tag{2.3c}$$

where $\tau^\nu{}_i$ and $\sigma^\nu{}_{ij}$ are the energy-momentum and spin currents of matter fields, respectively.

The first field equation can be rewritten as

$$-4\nabla_\mu(b\beta_i{}^{\mu k})+2b\beta_{imn}T^{kmn}-4b\beta^{nmk}T_{nmi}+\delta_i^k b\mathcal{L}_T=\tau^k{}_i.$$

Then, combined with the identity (A5), it takes the form of Einstein's field equation:

$$R^{ik}(\Delta)-\tfrac{1}{2}\eta^{ik}R(\Delta)=\tau^{ki}/2ab. \tag{2.4a}$$

Here, on the left hand side we have Einstein's tensor of GR, which is a symmetric tensor. Therefore, the dynamical energy-momentum tensor must be also symmetric, $\tau^{ik}=\tau^{ki}$.

Using the identity (A1) the second field equation can be written in the form

$$\nabla_\mu(2aH^{\nu\mu}_{ij}+4b\lambda_{ij}{}^{\nu\mu})=\sigma^\nu{}_{ij}, \tag{2.4b}$$

where $H^{\nu\mu}_{ij}=b(h_i{}^\nu h_j{}^\mu-h_j{}^\nu h_i{}^\mu$. The integrability condition for this equation is identically satisfied, because the covariant divergence of the left hand side vanishes on account of $R^{ij}{}_{\mu\nu}=0$, whereas

$$\nabla_\nu\sigma^\nu{}_{ij}=\tau_{ij}-\tau_{ji}=0.$$

The first equality in this relation is a consequence of the covariant conservation of angular momentum for matter fields (which holds when matter field equation is satisfied), and the vanishing of $\tau_{[ij]}$ follows from the first field equation.

Simple counting shows that the number of independent field equations (2.4b) is $24-6=18$. The multipliers $\lambda_{ij}{}^{\alpha\beta}$ remain arbitrary functions of time, as will be shown in the forthcoming Hamiltonian analysis. For any specific choice of $\lambda_{ij}{}^{\alpha\beta}$ (gauge fixing), Eq. (2.4b) can be used to determine (at least locally) the remaining 18 multipliers $\lambda_{ij}{}^{0\alpha}$.

In what follows we shall investigate the Hamiltonian structure and gauge properties of $\mathrm{GR}_\parallel$ without matter fields ($\sigma^\nu{}_{ij}=\tau_{ij}=0$). We expect that the results obtained here will be also useful for the analysis of interacting $\mathrm{GR}_\parallel$.

III. PRIMARY CONSTRAINTS AND HAMILTONIAN

(1) The basic Lagrangian dynamical variables of our theory are $(b^i{}_\mu, A^{ij}{}_\mu, \lambda_{ij}{}^{\mu\nu})$, and the corresponding momenta are denoted by $(\pi_i{}^\mu, \pi_{ij}{}^\mu, \pi^{ij}{}_{\mu\nu})$. Because of the fact that the torsion and the curvature do not involve velocities $\dot b^k{}_0$ and $\dot A^{ij}{}_0$, one immediately obtains the following set of so-called *sure* primary constraints:

$$\phi_k{}^0\equiv\pi_k{}^0\approx0,\quad \phi_{ij}{}^0\equiv\pi_{ij}{}^0\approx0. \tag{3.1}$$

Similarly, the absence of the time derivative of $\lambda_{ij}{}^{\mu\nu}$ implies

$$\phi^{ij}{}_{\mu\nu}\equiv\pi^{ij}{}_{\mu\nu}\approx0. \tag{3.2}$$

The next set of constraints follows from the linearity of the curvature in $\dot A^{ij}{}_\alpha$:

$$\phi_{ij}{}^\alpha\equiv\pi_{ij}{}^\alpha-4b\lambda_{ij}{}^{0\alpha}\approx0. \tag{3.3}$$

Before we continue, it is convenient introduce the so-called 3+1 decomposition of spacetime [23]. If $\boldsymbol{n}$ is the unit normal to the hypersurface Σ_0: $x^0=$ const, with $n_k=h^0_k/\sqrt{g^{00}}$, the four vectors $(\boldsymbol{n},\boldsymbol{e}_\alpha)$ define the Arnowitt-Deser-Misner (ADM) basis of tangent vector fields. Introducing the projectors on $\boldsymbol{n}$ and Σ_0, $(P_\perp)^i_k=n^in_k$, $(P_\parallel)^i_k=\delta^i_k-n^in_k$, any tangent vector $\boldsymbol{V}$ can be decomposed in terms of its parallel and orthogonal components: $V_k=V_{\bar k}+n_kV_\perp$, where $V_\perp=b^kV_k$, $V_{\bar k}=V_k-n_kV_\perp$, and $n^kV_{\bar k}=0$. Using an analogous decomposition of the torsion and the curvature in the last two indices,

$$T^i{}_{mk}=T^i{}_{\bar m\bar k}+2n_{[m}T^i{}_{\perp\bar k]},\quad R^{ij}{}_{mk}=R^{ij}{}_{\bar m\bar k}+2n_{[m}R^{ij}{}_{\perp\bar k]},$$

we find that the parallel components $T^i{}_{\bar k\bar l}$ and $R^{ij}{}_{\bar k\bar l}$ are independent of velocities. The replacement in the gravitational Lagrangian yields $\mathcal{L}=\bar{\mathcal{L}}(T^i{}_{\bar k\bar l},R^{ij}{}_{\bar k\bar l};T^i{}_{\perp\bar l},R^{ij}{}_{\perp\bar l},n^k)$.

The decomposition of $\boldsymbol{e}_0$ in the ADM basis yields $\boldsymbol{e}_0=N\boldsymbol{n}+N^\alpha\boldsymbol{e}_\alpha$, where $N=n_kb^k{}_0$ and $N^\alpha=h_{\bar k}{}^\alpha b^k{}_0$ are lapse and shift functions, respectively. We note also that b satisfies the factorization property $b=NJ$, where J does not depend on $b^k{}_0$.

Now, we turn our attention to the remaining momenta $\pi_i{}^\alpha$ [23]. The relations defining $\pi_i{}^\alpha$ can be written in the form

$$\hat\pi_i{}^{\bar k}=J\frac{\partial\bar{\mathcal{L}}_T}{\partial T^i{}_{\perp\bar k}}=4J\beta_i{}^{\perp\bar k}(T),$$

where $\hat\pi_i{}^{\bar k}=\pi_i{}^\alpha b^k{}_\alpha$ are conveniently defined ''parallel'' gravitational momenta. Using now the fact that β is a linear function of T we can make the expansion $\beta(T)=\beta(0)+\beta(1)$, where $\beta(0)$ does not depend on ''velocities'' $T^i{}_{\perp\bar k}$ and $\beta(1)$ is linear in them, and rewrite the above equation in the form

$$P_{i\bar k}\equiv\hat\pi_{i\bar k}/J-4\beta_{i\perp\bar k}(0)=4\beta_{i\perp\bar k}(1).$$

Here, the so-called ''generalized momenta'' $P_{i\bar k}$ do not depend on velocities, which appear only on the right hand side of the equation. Explicit calculation leads to the result

$$\begin{aligned}P_{i\bar k}&\equiv\hat\pi_{i\bar k}/J-4a[\tfrac{1}{2}BT_{\perp\bar i\bar k}+\tfrac{1}{2}Cn_iT^{\bar m}{}_{\bar m\bar k}]\\&=4a[AT_{i\perp\bar k}+\tfrac{1}{2}BT_{\bar k\perp\bar i}+\tfrac{1}{2}C\eta_{\bar i\bar k}T^{\bar m}{}_{\perp\bar m}\\&\quad+\tfrac{1}{2}(B+C)n_iT_{\perp\perp\bar k}].\end{aligned}$$

This system of equations can be decomposed into irreducible parts with respect to the group of three-dimensional rotations in Σ_0:

$$P_{\perp\bar k}\equiv\hat\pi_{\perp\bar k}/J-2aCT^{\bar m}{}_{\bar m\bar k}=2a(2A+B+C)T_{\perp\perp\bar k},$$

$$P^A_{\bar i\bar k}\equiv\hat\pi^A_{\bar m\bar k}/J-2aBT_{\perp\bar i\bar k}=2a(2A-B)T^A_{\bar i\perp\bar k},$$

$$P^T_{\bar i\bar k}\equiv\hat\pi^T_{\bar i\bar k}/J=2a(2A+B)T^T_{\bar i\perp\bar k},$$

$$P^{\bar m}{}_{\bar m}\equiv\hat\pi^{\bar m}{}_{\bar m}/J=2a(2A+B+3C)T^{\bar m}{}_{\perp\bar m},$$

where $X^A_{\bar i\bar k}=X_{[\bar i\bar k]}$, $X^T_{\bar i\bar k}=X_{(\bar i\bar k)}-\eta_{\bar i\bar k}X^{\bar n}{}_{\bar n}/3$. Taking now into account the special choice of parameters adopted in Eq. (2.2), we recognize here two sets of relations: the first set represents *extra* primary constraints

$$P_{\perp\bar k}=\hat\pi_{\perp\bar k}/J+2aT^{\bar m}{}_{\bar m\bar k}\approx0,$$

$$P^A_{\overline{ik}} \equiv \hat{\pi}^A_{\overline{ik}}/J - aT_{\perp\overline{ik}} \approx 0, \tag{3.4a}$$

usually called *if constraints*, while the second set gives nonsingular equations

$$P^T_{\overline{ik}} \equiv \hat{\pi}^T_{\overline{ik}}/J = 2aT^T_{i\perp\bar{k}},$$
$$P^{\bar{m}}{}_{\bar{m}} \equiv \hat{\pi}^{\bar{m}}{}_{\bar{m}}/J = -4aT^{\bar{m}}{}_{\perp\bar{m}}, \tag{3.4b}$$

which can be solved for velocities.

Further calculations are greatly simplified by observing that both sets of extra constraints (3.4a) can be represented in a unified manner as

$$\phi_{ik} = \pi_{i\bar{k}} - \pi_{k\bar{i}} + a\nabla_\alpha B^{0\alpha}_{ik}, \quad B^{0\alpha}_{ik} \equiv \varepsilon^{0\alpha\beta\gamma}_{ikmn} b^m_\beta b^n_\gamma. \tag{3.5}$$

This is seen from the fact that relations (3.4a) can be equivalently written as

$$\pi_{i\bar{k}} - \pi_{k\bar{i}} \approx 2aJ(T_{\perp\overline{ik}} - n_i T^{\bar{m}}{}_{\overline{mk}} + n_k T^{\bar{m}}{}_{\overline{mi}}) = 2a\nabla_\alpha H^{0\alpha}_{ik},$$

where the last equality follows from Eq. (A1), and the identity $2H^{\mu\nu}_{ik} = -B^{\mu\nu}_{ik}$.

(2) Having found all the primary constraints, we now proceed to find the *canonical* Hamiltonian density [23]:

$$\mathcal{H}_c = \pi_i{}^\alpha \dot{b}^i{}_\alpha + \tfrac{1}{2}\pi^\alpha_{ij}\dot{A}^{ij}{}_\alpha - b\mathcal{L}.$$

The velocities $\dot{b}^i{}_\alpha$ and $\dot{A}^{ij}{}_\alpha$ can be calculated from the relations defining $T^i{}_{0\alpha}$ and $R^{ij}{}_{0\alpha}$:

$$T^i{}_{0\alpha} \equiv \partial_0 b^i{}_\alpha + A^i{}_{m0} b^m{}_\alpha - \partial_\alpha b^i{}_0 - A^i{}_{m\alpha} b^m{}_0$$
$$= NT^i{}_{\perp\alpha} + N^\beta T^i{}_{\beta\alpha},$$
$$R^{ij}{}_{0\alpha} \equiv \partial_0 A^{ij}{}_\alpha + A^i{}_{m0} A^{mj}{}_\alpha - \partial_\alpha A^{ij}{}_0 - A^i{}_{m\alpha} A^{mj}{}_0$$
$$= NR^{ij}{}_{\perp\alpha} + N^\beta R^{ij}{}_{\beta\alpha}.$$

After a simple algebra we find that the canonical Hamiltonian can be written as a linear function of unphysical variables $(b^k{}_0, A^{ij}{}_0)$, up to a three-divergence,

$$\mathcal{H}_c = N\mathcal{H}_\perp + N^\alpha \mathcal{H}_\alpha - \tfrac{1}{2}A^{ij}{}_0 \mathcal{H}_{ij} + \partial_\alpha D^\alpha, \tag{3.6a}$$

where

$$\mathcal{H}_{ij} = 2\pi_{[i}{}^\beta b_{j]\beta} + \nabla_\alpha \pi_{ij}{}^\alpha,$$
$$\mathcal{H}_\alpha = \pi_k{}^\beta T^k{}_{\alpha\beta} - b^k{}_\alpha \nabla_\beta \pi_k{}^\beta + \tfrac{1}{2}\pi_{ij}{}^\beta R^{ij}{}_{\alpha\beta},$$
$$\mathcal{H}_\perp = (\hat{\pi}_i{}^{\bar{k}} T^i{}_{\perp\bar{k}} - J\bar{\mathcal{L}}_T - n^k \nabla_\beta \pi^\beta_k)$$
$$- J\lambda_{ij}{}^{\overline{mn}} R^{ij}{}_{\overline{mn}},$$
$$D^\alpha = b^k{}_0 \pi_k{}^\alpha + \tfrac{1}{2}A^{ij}{}_0 \pi_{ij}{}^\alpha. \tag{3.6b}$$

Here, $\mathcal{H}_{ij}$ and $\mathcal{H}_\alpha$ are purely kinematical terms whose form does not depend on the choice of the Lagrangian, and $\mathcal{H}_\perp$ is the only dynamical part.

The explicit form of $\mathcal{H}_\perp$ can be obtained by eliminating "velocities" $T_{i\perp\bar{k}}$ with the help of the relations defining momenta $\pi_{i\bar{k}}$. To do that we first rewrite the first two terms of $\mathcal{H}_\perp$ in the form

$$\hat{\pi}^{i\bar{k}} T_{i\perp\bar{k}} - J\bar{\mathcal{L}}_T = \tfrac{1}{2}JP^{i\bar{k}} T_{i\perp\bar{k}} - J\bar{\mathcal{L}}_T(\bar{T}),$$

where $\bar{T}_{ikl} = T_{i\overline{kl}}$. Then, taking into account the constraints (3.4a) one finds that $T_{\perp\perp\bar{k}}$ and $T^A_{i\perp\bar{k}}$ are absent from $\mathcal{H}_\perp$, whereupon the relations (3.4b) can be used to eliminate the remaining "velocities" $T^T_{i\perp\bar{k}}$ and $T^m{}_{\perp\bar{m}}$, leading directly to

$$\mathcal{H}_\perp = [\tfrac{1}{2}P_T^2 - J\bar{\mathcal{L}}_T(\bar{T}) - n^k\nabla_\beta \pi_k{}^\beta] - J\lambda_{ij}{}^{\overline{mn}} R^{ij}{}_{\overline{mn}}, \tag{3.7a}$$

where

$$P_T^2 = \frac{1}{2aJ}\left(\pi_{\overline{(ik)}}\pi^{\overline{(ik)}} - \frac{1}{2}\pi^{\bar{m}}{}_{\bar{m}}\pi^{\bar{n}}{}_{\bar{n}}\right),$$
$$\bar{\mathcal{L}}_T(\bar{T}) = a(\tfrac{1}{4}T_{\overline{mnk}}T^{\overline{mnk}} + \tfrac{1}{2}T_{\overline{mnk}}T^{\overline{nmk}} - T^{\bar{m}}{}_{\overline{mk}}T_{\bar{n}}{}^{\overline{nk}}). \tag{3.7b}$$

The general Hamiltonian dynamics of the system is described by the *total* Hamiltonian, which is given as

$$\mathcal{H}_T = \mathcal{H}_c + u^i{}_0 \pi_i^0 + \tfrac{1}{2}u^{ij}{}_0 \pi_{ij}{}^0 + \tfrac{1}{4}u_{ij}{}^{\mu\nu}\pi^{ij}{}_{\mu\nu} + \tfrac{1}{2}u^{ik}\phi_{ik}$$
$$+ \tfrac{1}{2}u^{ij}{}_\alpha \phi_{ij}{}^\alpha, \tag{3.8}$$

where u's are, at this stage, arbitrary Hamiltonian multipliers.

Although the torsion components $T_{\perp\perp\bar{k}}$ and $T^A_{i\perp\bar{k}}$ are absent from the canonical Hamiltonian, they reappear in the total Hamiltonian as the nondynamical Hamiltonian multipliers. Indeed, the Hamiltonian field equations for $b^k{}_\alpha$ imply (Appendix B)

$$NT^{\perp\perp\bar{k}} = u^{\perp\bar{k}}, \quad NT_A^{\bar{i}\perp\bar{k}} = u^{\overline{ik}}. \tag{3.9}$$

The presence of nondynamical torsion components does not imply that $\mathrm{GR}_\parallel$ is an inconsistent theory [9], as it has a very clear interpretation via the gauge structure of the theory: it is related to the existence of additional first-class constraints ϕ_{ik}, as we shall see in Sec. V.

IV. CONSISTENCY CONDITIONS

The consistency of the theory requires that the constraints do not change during the time evolution of the system governed by the total Hamiltonian:

$$\chi(x) \equiv \frac{d}{dt}\phi(x) = \int d^3x' \{\phi, \mathcal{H}'_T\} \approx 0,$$

where $\{A, B'\}$ denotes the Poisson bracket (PB) of two variables $A(x)$ and $B(x')$, and $x^0 = (x')^0$. The integration sign will be often omitted for simplicity.

A. Consistency conditions of primary constraints

Having found the form of primary constraints in $\mathrm{GR}_\parallel$, displayed in Eqs. (3.1), (3.2), (3.3), and (3.5), we now consider the requirements for their consistency.

Since the canonical Hamiltonian is linear in unphysical variables $(b^i{}_0, A^{ij}{}_0)$, the consistency conditions of the sure primary constraints (3.1) are given by

$$\chi_\perp \equiv \mathcal{H}_\perp \approx 0, \quad \chi_\alpha \equiv \mathcal{H}_\alpha \approx 0, \quad \chi_{ij} \equiv \mathcal{H}_{ij} \approx 0. \tag{4.1}$$

By noting that the components $\pi^{ij}{}_{\alpha\beta}$ in Eq. (3.2) have vanishing PBs with all primary constraints, we easily obtain

$$\chi^{ij}{}_{\alpha\beta} \equiv R^{ij}{}_{\alpha\beta} \approx 0. \tag{4.2a}$$

On the other hand, the consistency of $\pi^{ij}{}_{0\alpha}$ implies

$$\chi^{ij}{}_{0\beta} \equiv u^{ij}{}_\beta - N^\alpha R^{ij}{}_{\alpha\beta} \approx 0 \Rightarrow u^{ij}{}_\beta \approx 0. \tag{4.2b}$$

The dynamical meaning of the last condition can be seen more clearly if we note that the equation of motion for $A^{ij}{}_\beta$, $\partial_0 A^{ij}{}_\beta = \{A^{ij}{}_\beta, H_c\} + u^{ij}{}_\beta$, can be transformed into the form

$$R^{ij}{}_{0\beta} \approx u^{ij}{}_\beta. \tag{4.3}$$

Hence, Eqs. (4.2) tell us that all components of the curvature tensor weakly vanish, as one could have expected.

Using the PB relation

$$\{\phi_{ij}{}^\alpha, \phi_{kl}\} = a(\eta_{ik} B^{0\alpha}_{lj} + \eta_{jk} B^{0\alpha}_{il})\delta - (k \leftrightarrow l),$$

the consistency condition for $\phi_{ij}{}^\alpha$ takes the form

$$\chi^\alpha_{ij} = \{\phi_{ij}{}^\alpha, \mathcal{H}_c\} + a(u_i^s B^{0\alpha}_{sj} + u_j^s B^{0\alpha}_{is}) - 4b u_{ij}{}^{0\alpha} \approx 0.$$

It can be used to determine $u_{ij}{}^{0\alpha}$:

$$u_{ij}{}^{0\alpha} = \frac{a}{4b}(u_i{}^s B^{0\alpha}_{sj} + u_j{}^s B^{0\alpha}_{is}) + \bar{u}_{ij}{}^{0\alpha}, \quad \bar{u}_{ij}{}^{0\alpha} \equiv \frac{1}{4b}\{\pi^\alpha_{ij}, \mathcal{H}_c\}, \tag{4.4}$$

where $\{\pi_{ij}{}^\alpha, \mathcal{H}_c\}$ is calculated in Appendix C. The first part of $u_{ij}{}^{0\alpha}$ contains u_{kl} and gives an additional contribution to $u^{kl}\phi_{kl}$, so that the replacement of this result into $\mathcal{H}_T$ leads effectively to

$$u_{ij}{}^{0\alpha} \to \bar{u}_{ij}{}^{0\alpha}, \quad u^{kl}\phi_{kl} \to u^{kl}\tilde{\phi}_{kl},$$

$$\tilde{\phi}_{kl} \equiv \phi_{kl} - \frac{a}{4b}(\pi^s_{k\,0\alpha} B^{0\alpha}_{sl} + \pi^s_{l\,0\alpha} B^{0\alpha}_{ks}). \tag{4.5}$$

Note that $\{\phi_{ij}{}^\alpha, \tilde{\phi}'_{kl}\} = 0$.

The most complicated consistency conditions are those for the tetrad constraints ϕ_{ij} (or, equivalently, $\tilde{\phi}_{ij}$). First we note that their PB algebra has the form

$$\{\phi_{ij}, \phi'_{mn}\} = (\eta_{im}\phi_{nj} + \eta_{jm}\phi_{in})\delta - (m \leftrightarrow n), \tag{4.6}$$

and that the term $\frac{1}{2}u^{ij}{}_\alpha \phi_{ij}{}^\alpha$ in $\mathcal{H}_T$ can be discarded according to Eq. (4.2b). Then, after showing that the Poisson brackets $\{\phi_{ij}, \mathcal{H}'_c\}$ vanish weakly, we will be able to conclude that the consistency condition for ϕ_{ij} is automatically satisfied:

$$\chi_{ij} \equiv \{\phi_{ij}, \mathcal{H}'_T\} \approx \{\phi_{ij}, \mathcal{H}'_c\} \approx 0. \tag{4.7}$$

B. Consistency condition of ϕ_{ij}

In order to simplify the derivation of the consistency condition for ϕ_{ij} we rewrite this constraint in the form

$$\phi_{ij} = \mathcal{H}_{ij} - F_{ij},$$

$$F_{ij} = \nabla_\alpha(\pi_{ij}{}^\alpha - a B^{0\alpha}_{ij}) \equiv \nabla_\alpha \Pi_{ij}{}^\alpha. \tag{4.8}$$

General arguments in PGT, related to the local Lorentz symmetry of the theory, imply that the constraint $\mathcal{H}_{ij}$ is of the first class [23]. This is also clear from the PB algebra of constraints, discussed in the next section. As a consequence, the consistency of ϕ_{ij} follows from the consistency of F_{ij}.

We are now going to show that $\{F_{ij}, \mathcal{H}'_T\} \approx 0$. First, we note that

$$\{F_{ij}, \phi'_{kl}\} = 0. \tag{4.9}$$

Then, using the results (C.4a) we obtain

$$\{F_{ij}, \mathcal{H}'_{kl}\} = (\eta_{ik}F_{lj} + \eta_{jk}F_{il})\delta - (k \leftrightarrow l),$$

$$\{F_{ij}, \mathcal{H}'_\beta\} = -\nabla_\beta(\phi_{ij}\delta) + [\nabla_\alpha, \nabla_\beta](\Pi_{ij}{}^\alpha \delta) \tag{4.10a}$$

while Eq. (C.4b) leads to

$$\{F_{ij}, \mathcal{H}'_\perp\} = \tfrac{1}{2}\nabla_\beta[(n_i\phi_{jk} - n_j\phi_{ik})h^{\bar{k}\beta}\delta] + \tfrac{1}{2}[\nabla_\alpha, \nabla_\beta] \times (M^{\alpha\beta}_{ij}\delta),$$

$$NM^{\alpha\beta}_{ij} \equiv 2aH^{\alpha\beta}_{ij} - 4b\lambda_{ij}{}^{\beta\alpha} + N^\alpha(\Pi_{ij}{}^\beta - \phi_{ij}{}^\beta) - N^\beta(\Pi_{ij}{}^\alpha - \phi_{ij}{}^\alpha). \tag{4.10b}$$

Hence, the consistency condition of F_{ij}, and consequently of ϕ_{ij}, is automatically satisfied.

C. Consistency conditions of secondary constraints

In the process of investigating the consistency of primary constraints in $\mathrm{GR}_\parallel$, we obtained secondary constraints (4.1) and (4.2a).

Consider, first, the consistency condition of the secondary constraint $R^{ij}{}_{\alpha\beta}$. Since $R^{ij}{}_{\alpha\beta}$ depends only on $A^{ij}{}_\alpha$, one can express $dR^{ij}{}_{\alpha\beta}/dt$ in terms of $dA^{ij}{}_\alpha/dt$, use the equation of motion (4.3) for $A^{ij}{}_\alpha$, and rewrite the result in the form

$$\nabla_0 R^{ij}{}_{\alpha\beta} \approx \nabla_\alpha u^{ij}{}_\beta - \nabla_\beta u^{ij}{}_\alpha.$$

Hence, the consistency condition for $R^{ij}{}_{\alpha\beta}$ is identically satisfied.

The above relation has a very interesting geometric interpretation. Indeed, using Eq. (4.3) we see that it is a weak consequence of the second Bianchi identity.

General arguments in PGT show that the secondary constraints $\mathcal{H}_{ij}, \mathcal{H}_\alpha$, and $\mathcal{H}_\perp$ are related to Poincaré gauge symmetry [23,25]. Consequently, they are of the first class, and their consistency conditions are automatically satisfied. This will be explicitly seen in the next section, from the form of their PB algebra.

Finally, at the end of the consistency procedure, we give the final expression for the total Hamiltonian:

$$\mathcal{H}_T=\mathcal{H}'+u^i{}_0\pi_i{}^0+\tfrac{1}{2}u^{ij}{}_0\pi_{ij}{}^0+\tfrac{1}{4}u_{ij}{}^{\alpha\beta}\pi^{ij}{}_{\alpha\beta}+\tfrac{1}{2}u^{ik}\tilde{\phi}_{ik},$$

$$\mathcal{H}'=\mathcal{H}_c+\tfrac{1}{2}\overline{u_{ij}}{}^{0\beta}\pi^{ij}{}_{0\beta}. \tag{4.11}$$

The multipliers $u^i{}_0, u^{ij}{}_0, u_{ij}{}^{\alpha\beta}$, and u^{ik} remained arbitrary functions of time; hence we expect that the related constraints $\pi_i{}^0, \pi_{ij}{}^0, \pi^{ij}{}_{\alpha\beta}$, and $\tilde{\phi}_{ik}$ are of the first class.

V. ALGEBRA OF CONSTRAINTS

In the previous analysis we found that $\mathrm{GR}_\parallel$ is characterized by the following set of constraints:

primary: $\pi_i{}^0, \pi_{ij}{}^0, \tilde{\phi}_{ij}, \pi^{ij}{}_{\alpha\beta}, \phi_{ij}{}^\alpha, \pi^{ij}{}_{0\beta}$;

secondary: $\mathcal{H}_\perp, \mathcal{H}_\alpha, \mathcal{H}_{ij}, R^{ij}{}_{\alpha\beta}$.

It is simple to see that $\phi_{ij}{}^\alpha$ and $\pi^{ij}{}_{0\beta}$ are *second-class* constraints. They can and will be used as strong equalities to eliminate $\lambda_{ij}{}^{0\alpha}$ and $\pi^{ij}{}_{0\beta}$ from the theory and simplify further exposition. In particular, the term with the determined multiplier $\overline{u^{ij}}_\alpha$ in the total Hamiltonian can be now neglected, and $\tilde{\phi}_{ij}$ reduces to ϕ_{ij}, Eq. (4.5). Because of a simple form of these second-class constraints, the related Dirac brackets have the form of PBs in the phase space of the remaining variables. All the remaining constraints are of the *first class*, as follows from their PB algebra.

Since the kinematical constraints $\mathcal{H}_{ij}, \mathcal{H}_\alpha, \mathcal{H}_\perp$ have the same general form as in [23,25], their algebra remains the same:

$$\{\mathcal{H}_{ij},\mathcal{H}'_{kl}\}=(\eta_{ik}\mathcal{H}_{lj}+\eta_{jk}\mathcal{H}_{il})\delta-(k\leftrightarrow l),$$

$$\{\mathcal{H}_{ij},\mathcal{H}'_\alpha\}=0,$$

$$\{\mathcal{H}_\alpha,\mathcal{H}'_\beta\}=(\mathcal{H}'_\alpha\partial_\beta+\mathcal{H}_\beta\partial_\alpha-\tfrac{1}{2}R^{ij}{}_{\alpha\beta}\mathcal{H}_{ij})\delta. \tag{5.1a}$$

As a consequence of $R^{ij}{}_{\alpha\beta}\approx 0$, the last term in $\{\mathcal{H}_\alpha,\mathcal{H}'_\beta\}$ is quadratic in constraints.

The brackets involving $\mathcal{H}_\perp$ are found to have the form

$$\{\mathcal{H}_{ij},\mathcal{H}'_\perp\}=0,$$

$$\{\mathcal{H}_\alpha,\mathcal{H}'_\perp\}=\mathcal{H}_\perp\partial_\alpha\delta,$$

$$\{\mathcal{H}_\perp,\mathcal{H}'_\perp\}=-({}^3g^{\alpha\beta}\mathcal{H}_\alpha+{}^3g'^{\alpha\beta}\mathcal{H}'_\alpha)\partial_\beta\delta. \tag{5.1b}$$

The first two sets of brackets are most easily verified by taking into account that $\mathcal{H}_\perp$ can be written in the form $\mathcal{H}_\perp=Jf(\xi^A)-n^k\nabla_\alpha\pi_k{}^\alpha$, where f is a Lorentz scalar formed from variables $\xi^A=(T^i{}_{\overline{mn}}, R^{ij}{}_{\overline{m},\overline{n}}, \hat{\pi}_i{}^{\bar{k}}/J, \hat{\pi}_{ij}{}^{\bar{k}}/J, n^k)$, as shown in [25]. The second set of brackets is obtained from the general formula based on the chain rule for PBs,

$$\{\mathcal{H}_\alpha,\mathcal{H}'_\perp\}=\frac{\partial\mathcal{H}_\alpha}{\partial\xi^A}\{\xi^A,\xi'^B\}\frac{\partial\mathcal{H}'_\perp}{\partial\xi'^B}=\left(\mathcal{H}_\perp\partial_\alpha+\tfrac{1}{2}\frac{\partial\mathcal{H}_\perp}{\partial\pi_{ij}{}^\alpha}\mathcal{H}_{ij}\right)\delta,$$

which explains why the second term is absent in Eq. (5.1b).

The last and most important set of brackets $\{\mathcal{H}_\perp,\mathcal{H}'_\perp\}$ is evaluated using the chain rule and keeping only those terms that contain $\partial_\alpha\delta$ (terms proportional to δ do not have the correct symmetry under $\boldsymbol{x}\leftrightarrow\boldsymbol{x}'$, hence they cancel each other).

In the next step we want to extend the above algebra by adding $\tilde{\phi}_{ij}=\phi_{ij}$. The relevant PBs involving ϕ_{ij} are given by

$$\{\phi_{ij},\phi_{kl}\}=(\eta_{ik}\phi_{lj}+\eta_{jk}\phi_{il})-(k\leftrightarrow l),$$

$$\{\phi_{ij},\mathcal{H}'_{kl}\}=(\eta_{ik}\phi_{lj}+\eta_{jk}\phi_{il})\delta-(k\leftrightarrow l),$$

$$\{\phi_{ij},\mathcal{H}'_\beta\}=\nabla_\beta(\phi_{ij}\delta)-[\nabla_\alpha,\nabla_\beta](\Pi_{ij}{}^\alpha\delta),$$

$$\{\phi_{ij},\mathcal{H}'_\perp\}=-\tfrac{1}{2}\nabla_\beta[(n_i\phi_{jk}-n_j\phi_{ik})h^{\bar{k}\beta}\delta]-\tfrac{1}{2}[\nabla_\alpha,\nabla_\beta](M^{\alpha\beta}_{ij}\delta). \tag{5.2}$$

Finally, we display the nonvanishing PBs involving $R^{ij}{}_{\alpha\beta}$ and $\pi^{ij}{}_{\alpha\beta}$:

$$\{R^{ij}{}_{\alpha\beta},\mathcal{H}'_{kl}\}=(\delta^i_k R_l{}^j{}_{\alpha\beta}+\delta^j_k R^i{}_{l\,\alpha\beta})\delta-(k\leftrightarrow l),$$

$$\{R^{ij}{}_{\alpha\beta},\mathcal{H}'_\gamma\}=\nabla_\alpha(R^{ij}{}_{\gamma\beta}\delta)-(\alpha\leftrightarrow\beta),$$

$$\{\pi^{ij}{}_{\alpha\beta},\mathcal{H}'_\perp\}=4JR^{ij}{}_{\alpha\beta}. \tag{5.3}$$

Thus, all constraints except $\phi_{ij}{}^\alpha$ and $\pi^{ij}{}_{0\beta}$ are of the first class. The fact that ϕ_{ij} is first class is of particular importance for the consistent interpretation of the nondynamical torsion components, as noted at the end of Sec. III.

VI. EXTRA GAUGE SYMMETRIES

The presence of arbitrary multipliers in the total Hamiltonian is related to the existence of gauge symmetries in the theory. The general method of constructing the generators of such symmetries has been given by Castellani [26]. If we limit ourselves to gauge transformations given in terms of arbitrary parameters $\varepsilon(t)$ and their first time derivative $\dot{\varepsilon}(t)$, which is sufficient for the present analysis, the gauge generators take the form

$$G=\int d^3x[\varepsilon(t)G^{(0)}+\dot{\varepsilon}(t)G^{(1)}], \tag{6.1a}$$

where $G^{(0)}$ and $G^{(1)}$ are phase space functions determined by the conditions

$$G^{(1)}=C_{PFC},$$

$$G^{(0)}+\{G^{(1)},H_T\}=C_{PFC},$$
$$\{G^{(0)},H_T\}=C_{PFC}, \tag{6.1b}$$

and C_{PFC} denotes primary first-class (PFC) constraint.

The Poincaré gauge symmetry is present in our formulation of $\mathrm{GR}_\parallel$ by construction, and the related gauge generator is based on the sure constraints $\pi_i{}^0, \pi_{ij}{}^0$ and $\mathcal{H}_\perp, \mathcal{H}_\alpha, \mathcal{H}_{ij}$ [27]. Here, we shall focus our attention on extra gauge symmetries based on $\pi^{ij}{}_{\alpha\beta}, \tilde{\phi}_{ij}$, and $R^{ij}{}_{\alpha\beta}$.

A. Extra gauge symmetry 1

Starting with $\pi^{ij}{}_{\alpha\beta}$ as $G^{(1)}$ in Eqs. (6.1b) we find that the related gauge generator is given by

$$G=\int d^3x[\tfrac{1}{4}(\nabla_0\varepsilon_{ij}{}^{\alpha\beta})\pi^{ij}{}_{\alpha\beta} + \tfrac{1}{4}\varepsilon_{ij}{}^{\alpha\beta}(-4bR^{ij}{}_{\alpha\beta}+(\dot{b}/b)\pi^{ij}{}_{\alpha\beta})]. \tag{6.2}$$

The only nontrivial gauge transformations $\delta_0 X=\{X,G\}$ are

$$\delta_0(b\lambda_{ij}{}^{\alpha\beta})=\nabla_0(b\varepsilon_{ij}{}^{\alpha\beta}),$$
$$\delta_0\pi_{ij}{}^{\alpha}=4\nabla_\beta(b\varepsilon_{ij}{}^{\alpha\beta}). \tag{6.3}$$

To see the meaning of these transformations, consider the Hamiltonian equation for the variable $\Pi_{ij}{}^\alpha=\pi_{ij}{}^\alpha-aB^{0\alpha}_{ij}$. Introducing $K^{\alpha\beta}_{ij}=4b\lambda_{ij}{}^{\alpha\beta}-aB^{\alpha\beta}_{ij}$ and using the results of Appendix C we obtain the equation

$$\nabla_0\Pi_{ij}{}^\alpha-\nabla_\beta K^{\alpha\beta}_{ij}=0, \tag{6.4}$$

which is the Hamiltonian analogue of Eq. (2.4b). The application of the above gauge transformation to this equation yields

$$(\nabla_0\nabla_\beta-\nabla_\beta\nabla_0)(4b\varepsilon_{ij}{}^{\alpha\beta})=0.$$

The invariance follows from the fact that the left hand side vanishes in Weitzenböck space, where $R^{ij}{}_{0\beta}=0$.

B. Extra gauge symmetry 2

Starting with $G^{(1)}_{ij}=\phi_{ij}$ in Eqs. (6.1b), one finds that the gauge generator has the form

$$G_{ij}=\int d^3x[\tfrac{1}{2}\dot{\varepsilon}^{ij}G^{(1)}_{ij}+\tfrac{1}{2}\varepsilon^{ij}G^{(0)}_{ij}], \tag{6.5a}$$

where (Appendix D)

$$G^{(0)}_{ij}=\frac{1}{2}R_i{}^s{}_{\alpha\beta}K^{\alpha\beta}_{sj}+\frac{1}{2}\frac{1}{4b}[(A_i{}^n{}_0\pi_n{}^s{}_{\alpha\beta}+A^s{}_{n0}\pi_i{}^n{}_{\alpha\beta}) \times K^{\alpha\beta}_{sj}-\pi_i{}^s{}_{\alpha\beta}\dot{K}^{\alpha\beta}_{sj}]-(i\leftrightarrow j). \tag{6.5b}$$

The corresponding gauge transformations are

$$\delta_0 b^k{}_\alpha=\dot{\varepsilon}^k{}_s b^s{}_\alpha,\quad \delta_0 A^{ij}{}_\alpha=0,$$
$$4b\,\delta_0\lambda_{ij}{}^{\alpha\beta}=[\varepsilon_i{}^n\dot{K}^{\alpha\beta}_{nj}+A_i{}^m{}_0(\varepsilon_m{}^nK^{\alpha\beta}_{nj}+\varepsilon_j{}^nK^{\alpha\beta}_{mn})] -(i\leftrightarrow j),$$
$$\delta_0\pi_{ij}{}^\alpha=[a(\dot{\varepsilon}_i{}^sB^{0\alpha}_{sj})+\nabla_\beta(\varepsilon_i{}^nK^{\alpha\beta}_{nj})] -(i\leftrightarrow j), \tag{6.6}$$

and similarly for other variables.

Consider, again, Eq. (6.4). Using $\delta_0\Pi_{ij}{}^\alpha=\nabla_\beta(\varepsilon_i{}^nK^{\alpha\beta}_{nj})-(i\leftrightarrow j)$, we easily obtain

$$\begin{aligned}\delta_0(\nabla_0\Pi_{ij}{}^\alpha)&=\nabla_0(\delta_0\Pi_{ij}{}^\alpha)\approx\nabla_\beta\nabla_0(\varepsilon_i{}^nK^{\alpha\beta}_{nj})-(i\leftrightarrow j)\\&=\nabla_\beta[\dot{\varepsilon}_i{}^nK^{\alpha\beta}_{nj}+\varepsilon_i{}^n\dot{K}^{\alpha\beta}_{nj}+A_i{}^s{}_0(\varepsilon_s{}^nK^{\alpha\beta}_{nj}\\&\quad+\varepsilon_j{}^nK^{\alpha\beta}_{sn})]-(i\leftrightarrow j),\end{aligned}$$

where we made use of $R^{ij}{}_{0\beta}=0$. On the other hand,

$$\delta_0K^{\alpha\beta}_{ij}=[\dot{\varepsilon}_i{}^nK^{\alpha\beta}_{nj}+\varepsilon_i{}^n\dot{K}^{\alpha\beta}_{nj}+A_i{}^s{}_0(\varepsilon_s{}^nK^{\alpha\beta}_{nj}+\varepsilon_j{}^nK^{\alpha\beta}_{sn})] -(i\leftrightarrow j),$$

and we see that Eq. (6.4) is gauge invariant.

VII. CONCLUDING REMARKS

The investigation of the Hamiltonian structure of the teleparallel formulation of GR presented here is based on Dirac's general method for constrained dynamical systems [21].

To complete our results, we now discuss how the physical degrees of freedom of $\mathrm{GR}_\parallel$ are counted. After the elimination of $\lambda_{ij}{}^{0\alpha}$ and $\pi^{ij}{}_{0\alpha}$, the reduced phase space is spanned by the $40+18$ field components $(b^i{}_\mu, A^{ij}{}_\mu, \lambda_{ij}{}^{\alpha\beta})$ and the same number of momenta. The primary first-class constraints $\pi^{ij}{}_{\alpha\beta}$ diminish the number of independent variables for 2×18, leaving us with the phase space containing effectively 2×40 components. Before going on, we wish to clarify the counting of constraints $R^{ij}{}_{\alpha\beta}\approx0$. Note that here we have formally 18 equations, but they represent only 12 independent conditions on $A^{ij}{}_\alpha$. Indeed, starting with the simplest solution $\overline{A^{ij}}_\alpha=0$ of $R^{ij}{}_{\alpha\beta}(A)=0$, one can construct a new, Lorentz-rotated solution $\overline{A^{ij}}_\alpha(\Lambda)=\Lambda^i{}_k\partial_\alpha\Lambda^{jk}$, containing six arbitrary parameters Λ^{ik} [8], so that the number of independent conditions on $A^{ij}{}_\alpha$ is $18-6=12$. Continuing now the counting, we find 20 sure first-class constraints [ten primary $(\pi_i{}^0,\pi_{ij}{}^0)$ and ten secondary $(\mathcal{H}_\perp,\mathcal{H}_\alpha,\mathcal{H}_{ij})$] and $6+12=18$ additional first-class constraints ϕ_{ij} and $R^{ij}{}_{\alpha\beta}$, which leaves us with $2\times40-2\times38=4$ physical degrees of freedom, corresponding to the massless graviton.

We found two types of extra gauge symmetries in the PGT formulation of $\mathrm{GR}_\parallel$. The first type is related to the primary constraints $\pi^{ij}{}_{\alpha\beta}$. The related gauge transformations do not act on $b^i{}_\mu$; hence they are irrelevant for the structure of the first field equation (2.4a). On the other hand, the gauge symmetry acts nontrivially on Lagrange multipliers. If we recall that the only role of the second field equation (2.4b) is to determine these multipliers [9], it becomes clear that this cannot be done uniquely without fixing the gauge.

The second type of extra gauge symmetry originates from the tetrad constraints ϕ_{ij}. We note that Nester [14] derived these constraints in the form (3.4a), in his analysis of the positivity of energy in the teleparallel form of $\mathrm{GR}_\parallel$. Their existence may be interpreted as a consequence of the fact that the velocities contained in $T_{\perp\perp\bar{k}}$ and $T^A_{\bar{i}\perp\bar{k}}$ appear at most linear in the Lagrangian [16] and, consequently, remain arbitrary functions of time. The phenomenon that some velocities are dynamically undetermined is quite usual for constrained dynamical systems [21]. Heht *et al.* [16] concluded that the initial-value problem for $\mathrm{GR}_\parallel$ becomes well defined if these undetermined velocities are simply gauged away, ensuring the new kinetic Hessian matrix to be nondegenerate. However, according to the results of Refs. [15,28], this conclusion should be revised by taking into account nonlinear constraint effects.

The role of this symmetry is very clearly seen if we observe that the teleparallel geometry can be also formulated as the translational gauge theory, where local Lorentz symmetry is in general absent [5,8]. However, for the special choice of parameters corresponding to $\mathrm{GR}_\parallel$ one finds that ϕ_{ij} is an additional first-class constraint, which generates local Lorentz symmetry as an extra gauge symmetry [14]. This also clarifies the form (3.5) of ϕ_{ij}, which is seen to ''imitate'' $\mathcal{H}_{ij}$ in the tetrad sector.

Maluf [22] studied $\mathrm{GR}_\parallel$ by imposing the time gauge at the Lagrangian level. His arguments concerning the necessity of the time gauge in the canonical formalism are conceptually misleading: this gauge (as well as any other gauge) may be useful, but certainly not essential [21]. After fixing the time gauge, he found the Hamiltonian and derived the constraint corresponding to our $\phi_{\bar{i}\bar{k}}$ [Eq. (25) in his paper], while $\phi_{\perp\bar{k}}$ is missed. Moreover, Maluf was not able to calculate the constraint algebra unless imposing another gauge condition. His constraint algebra [Eqs. (30)–(34)] does not agree with our results, which might be a consequence of the adopted gauge conditions. All this makes his analysis of the gauge structure of $\mathrm{GR}_\parallel$ rather unclear.

The results obtained in this paper refer to noninteracting $\mathrm{GR}_\parallel$, and can be used to define and analyze the gravitational energy and other conserved quantities [29,17]. The interaction with matter fields may be included in a straightforward manner [7,30]. Studying consistency requirements imposed by extra gauge symmetries on the matter sector will tell us more about the existence and nature of consistent couplings [16].

ACKNOWLEDGMENTS

One of us (M.B.) appreciates a short visit to F. W. Hehl at the University of Cologne, where some interesting features of $\mathrm{GR}_\parallel$ were discussed in a stimulating atmosphere. This work was partially supported by the Serbian Science Foundation, Yugoslavia.

APPENDIX A: SOME GEOMETRIC IDENTITIES IN T_4

We begin with a simple but technically important identity

$$\nabla_\nu H^{\mu\nu}_{ij} = b h_k{}^\mu (T^k{}_{ij} - \delta^k_i T_j + \delta^k_j T_i) = -4b\beta_{[ij]}{}^\mu / a,$$

$$H^{\mu\nu}_{ij} \equiv b(h_i{}^\mu h_j{}^\nu - h_j{}^\mu h_i{}^\nu), \tag{A1}$$

which implies $\nabla_\mu(b\beta_{[ij]}{}^\mu) = 0$ for $R^{ij}{}_{\mu\nu} = 0$.

In Riemann-Cartan space U_4 the Lorentz connection can be expressed in the form $A = \Delta + K$, where Δ is Levi-Cività connection and K the contortion. Substituting this expression into the definition of the curvature tensor $R^{ij}{}_{\mu\nu}(A)$, we obtain the basic identity

$$R^{ij}{}_{\mu\nu}(A) = R^{ij}{}_{\mu\nu}(\Delta) + [\nabla'_\mu K^{ij}{}_\nu + K^i{}_{s\mu} K^{sj}{}_\nu - (\mu \leftrightarrow \nu)], \tag{A2}$$

where $\nabla' = \nabla(\Delta)$ is the Riemannian covariant derivative. Then, multiplying this relation by $H^{\mu\nu}_{ij}/2$ and using $\nabla'_\mu H^{\mu\nu}_{ij} = 0$, we find

$$bR(A) = bR(\Delta) + b(\tfrac{1}{4} T_{ijk} T^{ijk} + \tfrac{1}{2} T_{ijk} T^{jik} - T_k T^k) + 2\partial_\mu(bK^\mu), \tag{A3}$$

where $K^\mu = K^{\mu n}{}_n = -T^\mu$.

Now, if we write Eq. (A2) in an equivalent form

$$R^{ij}{}_{\mu\nu}(A) = R^{ij}{}_{\mu\nu}(\Delta) + [\nabla_\mu K^{ij}{}_\nu - K^i{}_{s\mu} K^{sj}{}_\nu - (\mu \leftrightarrow \nu)],$$

and multiply it by $H^{\mu\nu}_{kj}/2$, we obtain the result

$$bR^i{}_k(A) = bR^i{}_k(\Delta) + [\nabla_\mu K^{ij}{}_\nu - K^i{}_{s\mu} K^{sj}{}_\nu] H^{\nu\mu}_{jk},$$

which can be written as

$$abR^{ik}(A) = abR^{ik}(\Delta) + 2\nabla_\mu(b\beta^{i\mu k}) + 2b\beta_{mn}{}^k T^{mni} - b\beta^{imn} T^k{}_{mn} - \eta^{ik} a\partial_\mu(bT^\mu) - 4\nabla_\mu(b\beta^{[ik]\mu}). \tag{A4}$$

The last term on the right hand side vanishes for $R^{ij}{}_{\mu\nu}(A) = 0$. In that case we find

$$2ab[R^{ik}(\Delta) - \tfrac{1}{2}\eta^{ik} R(\Delta)] = -4\nabla_\mu(b\beta^{i\mu k}) - 4b\beta_{mn}{}^k T^{mni} + 2b\beta^{imn} T^k{}_{mn} + \eta^{ik} b\mathcal{L}_T. \tag{A5}$$

APPENDIX B: UNPHYSICAL TORSION COMPONENTS

In this appendix we show that the unphysical torsion components $T_{\perp\perp\bar{k}}$ and $T^A_{\bar{i}\perp\bar{k}}$ can be expressed in terms of the Hamiltonian multipliers u_{kl}.

Using the PB relations

$$\{b^i{}_\alpha, \mathcal{H}'_{kl}\} = \delta^i_k b_{l\alpha}\delta - (k \leftrightarrow l),$$

$$\{b^i{}_\alpha, \mathcal{H}'_\beta\} = (\nabla_\alpha b^i{}_\beta)\delta - b^i{}_\beta \partial'_\alpha \delta = T^i_{\beta\alpha}\delta + \nabla_\alpha(b^i{}_\beta \delta),$$

$$\{b^i{}_\alpha, \mathcal{H}'_\perp\} = \frac{1}{2a}\left(b_{m\alpha} P_T^{\overline{im}} - \frac{1}{6} b^i{}_\alpha P^{\bar{m}}{}_{\bar{m}}\right)\delta + \nabla_\alpha(n^i\delta), \tag{B1}$$

one easily finds that the Hamiltonian equation for $b^k{}_\alpha$ can be written in the form

$$\nabla_0 b^i{}_\alpha = \nabla_\alpha b^i{}_0 + N^\beta T^i{}_{\beta\alpha} + N b_{k\alpha} \frac{1}{2aJ} (\hat{\pi}^{(\bar{i}\bar{k})} - \tfrac{1}{2}\eta^{\bar{i}\bar{k}} \hat{\pi}^{\bar{m}}{}_{\bar{m}}) + b_{k\alpha}(n^i u^{\perp\bar{k}} + u^{\bar{i}\bar{k}}). \tag{B2}$$

As a consequence,

$$\hat{\pi}^{(\bar{i}\bar{k})} - \tfrac{1}{2}\eta^{\bar{i}\bar{k}}\hat{\pi}^{\bar{m}}{}_{\bar{m}} = 2aJT^{(\bar{i}\perp\bar{k})},$$
$$u^{\perp\bar{k}} = NT^{\perp\perp\bar{k}}, \quad u^{\bar{i}\bar{k}} = NT_A^{\bar{i}\perp\bar{k}}. \tag{B3}$$

APPENDIX C: CONSISTENCY CONDITIONS

We collect here several technical relations which simplify the derivation of the consistency conditions for $\phi_{ij}{}^\alpha$ and F_{ij}.

(1) The term $\{\pi_{ij}{}^\alpha, H_c\}$ in the consistency condition for the primary constraint $\phi_{ij}{}^\alpha$ is calculated using the relations

$$\{\pi_{ij}{}^\alpha, \mathcal{H}'_{kl}\} = (\eta_{ik}\pi_{lj}{}^\alpha + \eta_{jk}\pi_{il}{}^\alpha)\delta - (k \leftrightarrow l),$$
$$\{\pi_{ij}{}^\alpha, \mathcal{H}'_\beta\} = -\delta^\alpha_\beta(\pi_{i\bar{j}} - \pi_{j\bar{i}})\delta - \delta^\alpha_\beta \nabla_\gamma(\pi_{ij}{}^\gamma\delta) + \nabla_\beta(\pi_{ij}{}^\alpha\delta). \tag{C1a}$$

and

$$\{\pi_{ij}{}^\alpha, \mathcal{H}'_\perp\} = -4\nabla_\beta(J\lambda_{ij}{}^{\beta\alpha}\delta) + 4\nabla_\beta[J(N^\alpha\lambda_{ij}{}^{0\beta} - N^\beta\lambda_{ij}{}^{0\alpha})\delta] - [8J\beta_{[\bar{i}\bar{j}]\bar{k}}(0) + 2aJn_{[i}T_{\perp\bar{j}]\bar{k}}]h^{\bar{k}\alpha}\delta + (n_i\pi_{j\bar{k}} - n_j\pi_{i\bar{k}})h^{\bar{k}\alpha}\delta. \tag{C1b}$$

Using the identity $8J\beta_{[\bar{i}\bar{j}]\bar{k}}(0)h^{\bar{k}\alpha} = a\varepsilon^{0\alpha\beta\gamma}_{ijmn}T^m{}_{\gamma\beta}n^n$, we obtain

$$\{\pi_{ij}{}^\alpha, H_c\} = -(A_i{}^s{}_0\pi_{sj}{}^\alpha + A_j{}^s{}_0\pi_{is}{}^\alpha) - N^\alpha(\pi_{i\bar{j}} - \pi_{j\bar{i}}) - 4\nabla_\beta(b\lambda_{ij}{}^{\beta\alpha}) - aN\varepsilon^{0\alpha\beta\gamma}_{ijmn}T^m{}_{\gamma\beta}n^n + N(n_i\pi_{(\bar{j}\bar{k})} - n_j\pi_{(\bar{i}\bar{k})})h^{\bar{k}\alpha} - \nabla_\beta[(N^\alpha\phi_{ij}{}^\beta - N^\beta\phi_{ij}{}^\alpha)] + \tfrac{1}{2}N(n_i\phi_{\bar{j}\bar{k}} - n_j\phi_{\bar{i}\bar{k}})h^{\bar{k}\alpha}. \tag{C2}$$

(2) In order to calculate the Poisson brackets between F_{ij} and the Hamiltonian constraints, we also need the following relations:

$$\{B^{o\alpha}_{ij}, \mathcal{H}'_{kl}\} = (\eta_{ik}B^{0\alpha}_{lj} + \eta_{jk}B^{0\alpha}_{il})\delta - (k \leftrightarrow l),$$
$$\{B^{o\alpha}_{ij}, \mathcal{H}'_\beta\} = \nabla_\beta(B^{0\alpha}_{ij}\delta) + \delta^\alpha_\beta B^{0\gamma}_{ij}\partial'_\gamma\delta, \tag{C3a}$$

and

$$\{B^{o\alpha}_{ij}, \mathcal{H}'_\perp\} = \frac{1}{a}(n_i\pi_{(\bar{j}\bar{k})} - n_j\pi_{(\bar{i}\bar{k})})h^{k\alpha}\delta + 2\varepsilon^{0\alpha\beta\gamma}_{ijmn}b^m_\beta\nabla_\gamma(n^n\delta). \tag{C3b}$$

Combining Eqs. (C1a) and (C3a) we find

$$\{\Pi_{ij}{}^\alpha, \mathcal{H}'_{kl}\} = (\eta_{ik}\Pi_{lj}{}^\alpha + \eta_{jk}\Pi_{il}{}^\alpha)\delta - (k \leftrightarrow l),$$
$$\{\Pi_{ij}{}^\alpha, \mathcal{H}'_\beta\} = -\delta^\alpha_\beta\phi_{ij}\delta - \delta^\alpha_\beta\nabla_\gamma(\Pi_{ij}{}^\gamma\delta) + \nabla_\beta(\Pi_{ij}{}^\alpha\delta), \tag{C4a}$$

which implies Eqs. (4.10a).

Similarly, combining Eqs. (C1b) and (C3b), and using the identity

$$B^{\alpha\beta}_{ij} = 2N\varepsilon^{\alpha\beta\gamma 0}_{ijmn}b^m_\gamma n^n - (N^\alpha B^{0\beta}_{ij} - N^\beta B^{0\alpha}_{ij}),$$

we obtain

$$\{\Pi_{ij}{}^\alpha, \mathcal{H}'_\perp\} = \tfrac{1}{2}(n_i\phi_{\bar{j}\bar{k}} - n_j\phi_{\bar{i}\bar{k}})h^{\bar{k}\alpha}\delta + \nabla_\beta(M^{\alpha\beta}_{ij}\delta),$$
$$NM^{\alpha\beta}_{ij} \equiv 2aH^{\alpha\beta}_{ij} + 4b\lambda_{ij}{}^{\alpha\beta} + N^\alpha(\Pi_{ij}{}^\beta - \phi_{ij}{}^\beta) - N^\beta(\Pi_{ij}{}^\alpha - \phi_{ij}{}^\alpha), \tag{C4b}$$

which implies Eq. (4.10b).

APPENDIX D: EXTRA GAUGE GENERATORS

In this appendix we derive the form of the gauge generator (6.5). We start with $G^{(1)}_{ij} = \phi_{ij}$ in Eqs. (6.1b). In order to find the form of the accompanying component $G^{(0)}_{ij}$, we use the PB algebra given in Eq. (5.2), and calculate

$$\{\phi_{ij}, H_T\} = -\tfrac{1}{2}[\nabla_\alpha, \nabla_\beta]K^{\alpha\beta}_{ij}, \quad K^{\alpha\beta}_{ij} \equiv 2aH^{\alpha\beta}_{ij} + 4b\lambda_{ij}{}^{\alpha\beta},$$

where terms proportional to $\phi_{ij}{}^\alpha$ are discarded. The second condition in Eqs. (6.1b) implies

$$G^{(0)}_{ij} = \tfrac{1}{2}[R_i{}^s{}_{\alpha\beta}K^{\alpha\beta}_{sj} - (i \leftrightarrow j)] + \theta_{ij}, \tag{D1a}$$

where θ_{ij} is a primary FC constraint. The third condition in Eqs. (6.1b) can be written in the form

$$\tfrac{1}{2}[\dot{R}_i{}^s{}_{\alpha\beta}K^{\alpha\beta}_{sj} + R_i{}^s{}_{\alpha\beta}\dot{K}^{\alpha\beta}_{sj} - (i \leftrightarrow j)] + \dot{\theta}_{ij} = C_{PFC},$$

where $\dot{X} = \{X, H_T\}$. Then, using the relations $\nabla_0 R_i{}^s{}_{\alpha\beta} \approx 0$ and $\dot{\pi}^{is}{}_{\alpha\beta} = 4bR^{is}{}_{\alpha\beta}$ we obtain

$$\theta_{ij} = \frac{1}{2}\frac{1}{4b}[(A_i{}^n{}_0\pi_n{}^s{}_{\alpha\beta} + A^s{}_{n0}\pi_i{}^n{}_{\alpha\beta})K^{\alpha\beta}_{sj} - \pi_i{}^s{}_{\alpha\beta}\dot{K}^{\alpha\beta}_{sj}] - (i \leftrightarrow j). \tag{D1b}$$

[1] T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961).

[2] F. W. Hehl, P. von der Heyde. D. Kerlick, and J. Nester, Rev. Mod. Phys. **48**, 393 (1976).

[3] F. W. Hehl, in *Cosmology and Gravitation: Spin, Torsion, Rotation and Supergravity*, edited by P. G. Bergman and V. de Sabbata (Plenum, New York, 1980), p. 5.

[4] K. Hayashi and T. Shirafuji, Prog. Theor. Phys. **64**, 866 (1980); **64**, 883 (1980); **64**, 1435 (1980); **64**, 2222 (1980).

[5] F. W. Hehl, J. Nietsch, and P. von der Heyde, in *General Relativity and Gravitation—One Hundred Years After the Birth of Albert Einstein*, edited by A. Held (Plenum, New York, 1980), Vol. 1, p. 329. Here, the reader can find further references concerning the history of the teleparallel approach to gravity.
[6] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, Phys. Rep. **258**, 1 (1995).
[7] K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979).
[8] J. Nitsch, in *Cosmology and Gravitation: Spin, Torsion, Rotation and Supergravity*, edited by P. G. Bergman and V. de Sabbata (Plenum, New York, 1980), p. 63.
[9] W. Kopczyński, J. Phys. A **15**, 493 (1982).
[10] K. Hayashi and T. Shirafuji, Phys. Rev. D **24**, 3312 (1981).
[11] H. Goenner and F. Müller-Hoisson, Class. Quantum Grav. **1**, 651 (1984).
[12] F. Müller-Hoisson and J. Nitsch, Phys. Rev. D **28**, 718 (1983); Gen. Relativ. Gravit. **17**, 747 (1985).
[13] R. Kuhfuss and J. Nitsch, Gen. Relativ. Gravit. **18**, 947 (1986).
[14] J. Nester, Class. Quantum Grav. **5**, 1003 (1988).
[15] Wen-Hao Cheng, De-Ching Chern, and James M. Nester, Phys. Rev. D **38**, 2656 (1988).
[16] R. D. Hecht, J. Lemke, and R. P. Walner, Phys. Rev. D **44**, 2442 (1991).
[17] J. Nester, Int. J. Mod. Phys. A **4**, 1755 (1989); Phys. Lett. A **139**, 112 (1989).
[18] E. W. Mielke, Ann. Phys. (N.Y.) **219**, 78 (1992).
[19] V. C. de Andrade, L. C. T. Guillen, and J. G. Pereira, Phys. Rev. D **61**, 084031 (2000).
[20] S. Okubo, Gen. Relativ. Gravit. **23**, 599 (1991); J. Math. Phys. **33**, 895 (1992); **33**, 2148 (1992); see also M. Yu. Kalmykov and P. I. Pronin, Nucl. Phys. **B450**, 267 (1995).
[21] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964); A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Dynamics* (Academia Nacionale dei Lincei, Rome, 1976); K. Sundermeyer, *Constrained Dynamics* (Springer, Berlin, 1982); M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).
[22] J. W. Maluf, J. Math. Phys. **35**, 335 (1994).
[23] I. A. Nikolić, Phys. Rev. D **30**, 2508 (1984). See also M. Blagojević, Acta Phys. Pol. B **29**, 881 (1998).
[24] P. von der Heyde, Phys. Lett. **58A**, 141 (1976); Z. Naturforsch. A **31**, 1725 (1976).
[25] I. A. Nikolić, Fiz. Suppl. **1**, 135 (1986); Gen. Relativ. Gravit. **24**, 159 (1992); M. Blagojević and M. Vasilić, Phys. Rev. D **36**, 1679 (1987).
[26] L. Castellani, Ann. Phys. (N.Y.) **143**, 357 (1982).
[27] M. Blagojević, I. Nikolić, and M. Vasilić, Nuovo Cimento B **101**, 439 (1988).
[28] Hsin Chen, James M. Nester, and Hwei-Jang Yo, Acta Phys. Pol. B **29**, 961 (1998).
[29] M. Blagojević and M. Vasilić, Class. Quantum Grav. **5**, 1241 (1988); J. W. Maluf, J. Math. Phys. **36**, 4242 (1995).
[30] V. C. de Andrade and J. G. Pereira, Gen. Relativ. Gravit. **30**, 263 (1998); Int. J. Mod. Phys. D **8**, 141 (1998).