Quantum corrections to the entropy of a Reissner-Nordström black hole due to spin fields

Li Zhong-heng
Department of Physics, Zhanjiang Normal College, Guangdong Zhanjiang 524048, China

(Received 7 December 1999; published 1 June 2000)

The quantum corrections to the entropy of the Reissner-Nordström black hole due to the gravitational, electromagnetic, and neutrino fields are calculated by using the brick-wall model. The appearance of logarithmically divergent terms is demonstrated. These terms not only depend on the characteristics of the black hole but also on the spin of the fields. The contribution of any spin field is not proportional to the scalar one and is compatible from the results obtained earlier. For some quasiextreme black holes, the spin field gives no contribution to the entropy.

PACS number(s): 04.70.Dy, 04.62.+v, 97.60.Lf

To find the statistical origin of black hole entropy, by employing the so-called brick-wall model, 't Hooft [1] studied first the contribution to the entropy of the Schwarzschild black hole due to the scalar field. After this, the method was applied to scalar fields in various black hole backgrounds [2–7]. Recently, the method was extended to the electromagnetic field in a Reissner-Nordström background in Ref. [8], where, in particular, it has been shown that the leading term in the one-loop contribution to the entropy due to the electromagnetic field is exactly twice that due to the scalar field.

In this paper, we give the calculation of the entropy of the Reissner-Nordström black hole due to fields of arbitrary spin ($s=0$ for the scalar field, $s=\frac{1}{2}$ for the neutrino field, $s=1$ for the electromagnetic field, $s=2$ for the gravitational field), and show that the contribution of the electromagnetic field is not just twice the scalar one when logarithmically divergent terms are taken into account. We derive the master equation governing the spin fields and then calculate the contribution to the black hole entropy by using the brick-wall model.

The line element of the Reissner-Nordström spacetime is given by

$$ds^2=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2 -r^2(d\theta^2+\sin^2\theta\, d\varphi^2), \tag{1}$$

where M and Q are the mass and charge of the black hole, respectively. The horizon is located at

$$r_\pm=M\pm\sqrt{M^2-Q^2}. \tag{2}$$

We assume the nonextremal Reissner-Nordström black hole with $M>Q$, so that $r=r_+$ and r_- correspond to the positions of the outer event horizon and the inner Cauchy horizon, respectively.

Choosing the coordinates $x^\mu=(t,r,\theta,\varphi)$, the resulting tetrad is then given by [9]

$$l^\mu=\frac{1}{\Delta}[r^2,\Delta,0,0],$$

$$n^\mu=\frac{1}{2r^2}[r^2,-\Delta,0,0], \tag{3}$$

$$m^\mu=\frac{1}{\sqrt{2}r}[0,0,1,i\ \mathrm{cosec}\ \theta],$$

where

$$\Delta=r^2-2Mr+Q^2. \tag{4}$$

The nonvanishing spin coefficients are

$$\rho=-\frac{1}{r},\quad \gamma=\frac{Mr-Q^2}{2r^3},\quad \mu=-\frac{\Delta}{2r^3},$$

$$\alpha=-\frac{\cot\theta}{2\sqrt{2}r}=-\beta, \tag{5}$$

whereas the only nonvanishing component of the Weyl tensor is given by

$$\Psi_2=-\frac{Mr-Q^2}{r^4}. \tag{6}$$

Equations (5) and (6) tell us that the Reissner-Nordström metric is of Petrov-type D. Using the result of Teukolsky [10,11], the field equations of spin $s=\frac{1}{2}$, 1, and 2 for the source free case can be combined into

$$\{[D-(2s+1)\rho][\boldsymbol{\Delta}-2s\gamma+\mu)]-[\delta+(2s-2)\alpha][\bar{\delta}-2s\alpha] -(2s-1)(s-1)\Psi_2\}\Phi_{+s}=0,$$

$$\{[\boldsymbol{\Delta}+(2s-2)\gamma+(2s+1)\mu][D-\rho]-[\bar{\delta}+(2s-2)\alpha] \times[\delta-2s\alpha]-(2s-1)(s-1)\Psi_2\}\Phi_{-s}=0, \tag{7}$$

where

$$D=l^\mu\partial_\mu,\quad \boldsymbol{\Delta}=n^\mu\partial_\mu,\quad \delta=m^\mu\partial_\mu. \tag{8}$$

In Eq. (7) the first equation is for spin states $p=s$, while the other one is for $p=-s$.

Using the spin coefficients, the components of the Weyl tensor and the directional derivatives written down in Eqs. (5), (6), and (8) and making the transformations [9,12]

$$\Phi_{+s},\Phi_{-s}=r^{p-s}\ {}_pR_{lE}(r)\ {}_pY_l^m(\theta,\varphi)e^{-iEt}, \tag{9}$$

we can separate the variables of Eq. (7) to write

$$\left[\Delta^{-p}\frac{d}{dr}\left(\Delta^{p+1}\frac{d}{dr}\right)+\frac{r^4E^2+2ipEr(r^2-3Mr+2Q^2)}{\Delta}-\lambda^2\right]{}_pR_{lE}(r)=0, \tag{10}$$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{2ip\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\theta}-p^2\cot^2\theta+p+\lambda^2\right]{}_pY_l^m(\theta,\varphi)=0. \tag{11}$$

Equation (11) shows that ${}_pY_l^m(\theta,\varphi)$ is the spin-weighted spherical harmonic [13,14], and the separation constant λ satisfies

$$\lambda=\sqrt{(l-p)(l+p+1)}. \tag{12}$$

Here l and m are integers satisfying the inequalities

$$l\geqslant|p| \quad \text{and} \quad -l\leqslant m\leqslant l. \tag{13}$$

It is remarkable that Eqs. (10) and (11) can also be shown to describe the behavior of a massless scalar field ($s=0$) on the Reissner-Nordström background.

In the WKB approximation one writes ${}_pR_{lE}(r)\sim\exp[iS(r,p,l,E)]$. Then Eq. (9) yields the radial wave number as

$$k^2\equiv(\partial_rS)^2=\frac{1}{\Delta}\left[\frac{r^4}{\Delta}E^2-(l-p)(l+p+1)\right]. \tag{14}$$

According to the semiclassical quantization rule, the radial wave number is quantized as

$$\int_{r_++\varepsilon}^{L}dr\,k(r,p,l,E)=n\pi, \tag{15}$$

under the brick-wall boundary conditions $\Phi_{+s}=\Phi_{-s}=0$ at $r=r_++\varepsilon$, $r=L$. Note that n is assumed to be a non-negative integer, and ε and L are ultraviolet and infrared regulators, respectively, where $0<\varepsilon\ll r_+$ and $L\gg r_+$. In this range, the energy E is always positive and the wave number k is real. Then, the number of wave solutions with energy not exceeding E is given by [using Eq. (13)]

$$\begin{aligned}g(E)&=\sum_p\sum_l(2l+1)n\\&=\frac{1}{\pi}\sum_p\int_{r_++\varepsilon}^{L}dr\int_{|p|}^{l_{\max}}dl(2l+1)\left[\frac{r^4}{\Delta^2}E^2-\frac{(l-p)(l+p+1)}{\Delta}\right]^{1/2}\\&=\frac{2}{3\pi}\sum_p\int_{r_++\varepsilon}^{L}\frac{r^6}{\Delta^2}\left[E^2-\frac{\Delta}{r^4}(|p|-p)\right]^{3/2}dr.\end{aligned} \tag{16}$$

The free energy at inverse temperature β is given by

$$-\beta F=\pm\sum_j\ln(1\pm e^{-\beta E_j}), \tag{17}$$

where j represents the set of quantum numbers. The plus sign in Eq. (17) corresponds to the Fermi case, while the minus sign corresponds to the Bose case. Using Eq. (16) to determine the density of states, we find the leading behavior of the free energy:

$$F=\mp\frac{1}{\beta}\int_0^\infty dE\,\frac{dg(E)}{dE}\ln(1\pm e^{-\beta E})$$

$$\approx\begin{cases}-\dfrac{2\omega\pi^3L^3}{135\beta^4}-\dfrac{2\omega\pi^3r_+^6}{45\beta^4(r_+-r_-)^2}\dfrac{1}{\varepsilon}-\dfrac{2\omega\pi^3r_+^5(4r_+-6r_-)}{45\beta^4(r_+-r_-)^3}\ln\dfrac{L}{\varepsilon}+\dfrac{s\pi r_+^2}{3\beta^2(r_+-r_-)}\ln\dfrac{L}{\varepsilon} & \text{(bosons)}\\ -\dfrac{7\omega\pi^3L^3}{540\beta^4}-\dfrac{7\omega\pi^3r_+^6}{180\beta^4(r_+-r_-)^2}\dfrac{1}{\varepsilon}-\dfrac{7\omega\pi^3r_+^5(4r_+-6r_-)}{180\beta^4(r_+-r_-)^3}\ln\dfrac{L}{\varepsilon}+\dfrac{s\pi r_+^2}{6\beta^2(r_+-r_-)}\ln\dfrac{L}{\varepsilon} & \text{(fermions)},\end{cases} \tag{18}$$

where the factor ω is the degeneracy due to the spin. For the gravitational and electromagnetic fields we have $\omega=2$; for the neutrino and scalar fields we have $\omega=1$. The first term on the right-hand side of the latter equation is the usual one proportional to the volume, while the second and following terms give the quantum corrections. Note that the last term is due to the fields of spin state $p=-s$. From Eq. (18) we obtain, for the quantum corrections to the entropy,

$$S^q=\begin{cases}\dfrac{8\omega\pi^3r_+^6}{45\beta^3(r_+-r_-)^2}\dfrac{1}{\varepsilon}+\dfrac{8\omega\pi^3r_+^5(4r_+-6r_-)}{45\beta^3(r_+-r_-)^3}\ln\dfrac{L}{\varepsilon}-\dfrac{2s\pi r_+^2}{3\beta(r_+-r_-)}\ln\dfrac{L}{\varepsilon} & \text{(bosons)}\\ \dfrac{7\omega\pi^3r_+^6}{45\beta^3(r_+-r_-)^2}\dfrac{1}{\varepsilon}+\dfrac{7\omega\pi^3r_+^5(4r_+-6r_-)}{45\beta^3(r_+-r_-)^3}\ln\dfrac{L}{\varepsilon}-\dfrac{s\pi r_+^2}{3\beta(r_+-r_-)}\ln\dfrac{L}{\varepsilon} & \text{(fermions)}.\end{cases} \tag{19}$$

Choosing the inverse temperature β to correspond to the Hawking temperature of the nonextremal Reissner-Nordström black hole, we set

$$\beta=\frac{4\pi r_+^2}{r_+-r_-}, \tag{20}$$

upon which the entropy (19) becomes

$$S^q=\begin{cases}\dfrac{\omega(r_+-r_-)}{360\varepsilon}+\omega\left[\dfrac{(r_+-r_-)}{60r_+}-\dfrac{1}{180}\right]\ln\dfrac{L}{\varepsilon}-\dfrac{s}{6}\ln\dfrac{L}{\varepsilon} & \text{(bosons)}\\ \dfrac{7\omega}{8}\dfrac{(r_+-r_-)}{360\varepsilon}+\dfrac{7\omega}{8}\left[\dfrac{(r_+-r_-)}{60r_+}-\dfrac{1}{180}\right]\ln\dfrac{L}{\varepsilon}-\dfrac{s}{12}\ln\dfrac{L}{\varepsilon} & \text{(fermions)}.\end{cases} \tag{21}$$

On the other hand, the distance of the brick wall from the horizon is related to the ultraviolet cutoff as

$$\varepsilon_p=\int_{r_+}^{r_++\varepsilon}\sqrt{-g_{rr}}dr\approx\sqrt{4r_+^2\varepsilon/(r_+-r_-)}. \tag{22}$$

Let

$$\varepsilon_p^2=\frac{2\epsilon^2}{15},\quad \Lambda^2=\frac{L\epsilon^2}{\varepsilon} \tag{23}$$

where ϵ and Λ are, respectively, the ultraviolet cutoff parameter and infrared cutoff parameter in Ref. [15]. The quantum corrections to the entropy (21) then reads

$$S^q=\begin{cases}\dfrac{\omega A}{48\pi\epsilon^2}+\omega\left(\dfrac{1}{18}-\dfrac{M}{15r_+}\right)\ln\dfrac{\Lambda}{\epsilon}-\dfrac{s}{3}\ln\dfrac{\Lambda}{\epsilon} & \text{(bosons)}\\ \dfrac{7\omega}{8}\dfrac{A}{48\pi\epsilon^2}+\dfrac{7\omega}{8}\left(\dfrac{1}{18}-\dfrac{M}{15r_+}\right)\ln\dfrac{\Lambda}{\epsilon}-\dfrac{s}{6}\ln\dfrac{\Lambda}{\epsilon} & \text{(fermions)},\end{cases} \tag{24}$$

where $A=4\pi r_+^2$ is the surface area of the event horizon.

From Eq. (24), it is clear that the first term has the geometric feature that when $\epsilon\to 0$, it is a quadratic divergence. The last two terms are logarithmic divergences, which not only depend on the characteristics of the black hole (mass, charge) but also the spin of fields, and therefore cannot be neglected as nonessential additive constants. In the calculation, we found the third term comes from the fields of spin state $p=-s$.

It should be noted that when $\omega=1$ and $s=0$, our results reduce to the case discussed by Solodukhin [15]. Obviously, the first two terms in Eq. (24) have completely the same form as that of the scalar field, except that the coefficient is different. However, the third term exists in Eq. (24) and therefore the whole expression does not take the form of the scalar field. This result is very different from that of Refs. [8, 16].

It is interesting to note that for some quasiextreme black holes ($r_\pm=M\pm\delta$, where $\delta\approx[(\omega+30s)/\omega]\varepsilon\ln(L/\varepsilon)$ for the Bose case, $\delta\approx[(7\omega+120s)/7\omega]\varepsilon\ln(L/\varepsilon)$ for the Fermi case) Eq. (21) becomes zero. This means that the spin fields do not contribute to the entropies of these black holes.

This work was supported by the Key Project Foundation of the Higher Education Bureau of Guangdong Province.

[1] G. 't Hooft, Nucl. Phys. **B256**, 727 (1985).
[2] A. Ghosh and P. Mitra, Phys. Rev. Lett. **73**, 2521 (1994).
[3] J. Demers, R. Lafrance, and R. C. Myers, Phys. Rev. D **52**, 2245 (1995).
[4] A. Ghosh and P. Mitra, Phys. Lett. B **357**, 295 (1995).
[5] M. H. Lee and J. K. Kim, Phys. Rev. D **54**, 3904 (1996).
[6] J. Ho, W. T. Kim, and Y. J. Park, Class. Quantum Grav. **14**, 2617 (1997).
[7] G. Cognola, Phys. Rev. D **57**, 6292 (1998).
[8] G. Cognola and P. Lecca, Phys. Rev. D **57**, 1108 (1998).
[9] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1983).
[10] S. A. Teukolsky and W. H. Press, Astrophys. J. **193**, 443 (1974).
[11] M. Carmeli, *Classical Fields: General Relativity and Gauge Theory* (Wiley, New York, 1982).
[12] Li Zhong-heng, Mod. Phys. Lett. A **14**, 1951 (1999).
[13] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, and E. C. G. Sudarshan, J. Math. Phys. **8**, 2155 (1967).
[14] B. P. Jensen, J. G. Mc Laughlin, and A. C. Ottewill, Phys. Rev. D **51**, 5676 (1995).
[15] S. N. Solodukhin, Phys. Rev. D **51**, 618 (1995).
[16] Y. G. Shen, D. M. Chen, and T. J. Zhang, Phys. Rev. D **56**, 6698 (1997).