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Poincaré invariance in the ADM Hamiltonian approach to the general relativistic two-body problem

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A previously found momentum-dependent regularization ambiguity in the third post-Newtonian two point-mass Arnowitt-Deser-Misner Hamiltonian is shown to be uniquely determined by requiring global Poincaré invariance. The phase-space generators realizing the Poincaré algebra are explicitly constructed.

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The equations of motion of a gravitationally interacting two point-mass system have been derived some years ago up to the 5/2 post-Newtonian (2.5PN) approximation,[1] in harmonic coordinates [1–3]. Recently, it has been possible to derive the third post-Newtonian (3PN) Hamiltonian of a two point-mass system [4] within the canonical formalism of Arnowitt, Deser and Misner (ADM) [5]. It was found that, at the 3PN level, the use of Dirac-delta-function sources to model the two-body system causes the appearance of badly divergent integrals which (contrary to what happened at the 2.5PN [3,6] and 3.5PN [7] levels) cannot be unambiguously regularized [4,8,9]. The ambiguities in the regularization of the 3PN divergent integrals are parametrized by two quantities: $\omega_{\rm static}$ and $\omega_{\rm kinetic}$.

Prompted by a recent remark [10], the purpose of this work is to show that requiring the (global) Poincaré invariance of the 3PN ADM Hamiltonian dynamics uniquely determines one (and only one) of these regularization ambiguities: namely, the ''kinetic ambiguity'' parameter $\omega_{\rm kinetic}$. [The ''static ambiguity'' $\omega_{\rm static}$ remains unconstrained because it parametrizes a $\mathcal{O}(c^{-6})$ Galileo-invariant additional contribution to the 3PN Hamiltonian.] Parallel work in the harmonic-coordinates approach to 3PN dynamics has recently obtained similar results [18].

Note that general relativity admits (when considering isolated systems) the full Poincaré group as a *global* symmetry. Therefore, whatever the coordinate system used (as long as it respects asymptotic flatness), the general relativistic dynamics of N-body systems should embody some representation of this global Poincaré symmetry. When solving Einstein's equation by a weak-field, ''post-Minkowskian'' expansion, $\sqrt{g}\,g^{\mu\nu}-\eta^{\mu\nu}\equiv h^{\mu\nu}=G\,h^{\mu\nu}_{(1)}+G^2\,h^{\mu\nu}_{(2)}+\cdots$, and fixing the gauge by the ''harmonicity condition,'' $\partial_\nu h^{\mu\nu}=0$, the whole scheme stays manifestly invariant under the usual (linear) representation of the Poincaré group: $x'^{\mu}=\Lambda^\mu_\nu x^\nu+a^\mu$ (assuming that the regularization procedure used to deal with the point-mass divergencies is manifestly Poincaré invariant). In such a case the N-body dynamics will be invariant under the representation of the Poincaré group induced on the dynamical variables, say $x^i_a(t)$, $\dot{x}^i_a(t)$, $a=1,\ldots,N$, by the action of the usual linear Poincaré transformations. This global Poincaré symmetry has been explicitly checked at the 2PN level in Ref. [11] by proving that the 2PN (acceleration-dependent) two-body Lagrangian in harmonic coordinates [12,2] changed only by a total time derivative under a generic, infinitesimal Poincaré transformation. In this work we consider the 3PN two-body Hamiltonian derived by Ref. [4] within the ADM canonical formalism. This formalism is not manifestly Poincaré invariant because it splits space and time, and fixes the coordinates by the following gauge conditions: $\delta_{ij}\pi^{ij}=0$, $\partial_j(g_{ij}-\frac{1}{3}g_{ss}\delta_{ij})=0$. This lack of *manifest* Poincaré invariance is not problematic (though it introduces some technical complications). Indeed, we shall explicitly show in this paper that the global Poincaré symmetry of the two-body dynamics can be realized in phase space, albeit by a somewhat complicated, nonlinear action.

The basic principle that we shall follow to study Poincaré invariance of the 3PN two-body Hamiltonian $H(\mathbf{x}_a,\mathbf{p}_a)$, $a=1,2$, with its associated Poisson brackets structure,

$$\{A(\mathbf{x}_a,\mathbf{p}_a),B(\mathbf{x}_a,\mathbf{p}_a)\}\equiv\sum_a\sum_i\left(\frac{\partial A}{\partial x^i_a}\frac{\partial B}{\partial p_{ai}}-\frac{\partial A}{\partial p_{ai}}\frac{\partial B}{\partial x^i_a}\right),\tag{1}$$

is the following: the presence of a Poincaré symmetry is equivalent to requiring the existence of ''generators'' $P^\mu, J^{\mu\nu}$, realized as functions $P^\mu(\mathbf{x}_a,\mathbf{p}_a), J^{\mu\nu}(\mathbf{x}_a,\mathbf{p}_a)$ on the two-body phase-space $(\mathbf{x}_1,\mathbf{x}_2,\mathbf{p}_1,\mathbf{p}_2)$, whose Poisson brackets (1) satisfy the usual Poincaré algebra (here we set $c=1$):

$$\{P^\mu,P^\nu\}=0,\tag{2a}$$

$$\{P^\mu,J^{\rho\sigma}\}=-\eta^{\mu\rho}P^\sigma+\eta^{\mu\sigma}P^\rho,\tag{2b}$$

$$\{J^{\mu\nu},J^{\rho\sigma}\}=-\eta^{\nu\rho}J^{\mu\sigma}+\eta^{\mu\rho}J^{\nu\sigma}+\eta^{\sigma\mu}J^{\rho\nu}-\eta^{\sigma\nu}J^{\rho\mu},\tag{2c}$$

where $\eta_{\mu\nu}=\mathrm{diag}(-1,+1,+1,+1)$.

The functions $P^\mu(\mathbf{x}_a,\mathbf{p}_a), J^{\mu\nu}(\mathbf{x}_a,\mathbf{p}_a)$ generate (in phase space) the infinitesimal Poincaré transformations: $\delta_{\alpha,\omega}F=\{F,\alpha^\mu P_\mu+\frac{1}{2}\omega^{\mu\nu}J_{\mu\nu}\}$. Finite transformations are then (in principle) defined by exponentiating these infinitesimal ac-

[1] We recall that the ''nPN approximation'' refers to the terms of order $(v/c)^{2n}\sim(GM/(c^2r))^n$ in the equations of motion.

tions. The satisfaction of the algebra (2) ensures that one thereby generates a consistent Poincaré symmetry. The time component P^0 (i.e., the total energy) is realized as the Hamiltonian $H(\mathbf{x}_a,\mathbf{p}_a)$ (including the rest-mass contribution). The other generators can be decomposed as P^i (three momentum), $J^i \equiv \frac{1}{2}\varepsilon^{ikl}J_{kl}$ (angular momentum), and $K^i \equiv J^{i0}$ (boost vector). One further decomposes the boost vector K^i (which represents the constant of motion associated to the center of mass theorem) as $K^i(\mathbf{x}_a,\mathbf{p}_a;t) \equiv G^i(\mathbf{x}_a,\mathbf{p}_a) - t\,P^i(\mathbf{x}_a,\mathbf{p}_a)$ so that the total time derivative $dK^i/dt = \partial K^i/\partial t + \{K^i,H\} = -P^i + \{G^i,H\} = 0$. Finally, the Poincaré algebra explicitly reads

$$\{P_i,H\} = \{J_i,H\} = 0, \tag{3a}$$

$$\{J_i,P_j\} = \varepsilon_{ijk}P_k, \quad \{J_i,J_j\} = \varepsilon_{ijk}J_k, \tag{3b}$$

$$\{J_i,G_j\} = \varepsilon_{ijk}G_k, \tag{3c}$$

$$\{G_i,H\} = P_i, \tag{3d}$$

$$\{G_i,P_j\} = c^{-2}H\delta_{ij}, \tag{3e}$$

$$\{G_i,G_j\} = -c^{-2}\varepsilon_{ijk}J_k. \tag{3f}$$

As the gauge fixing used in the ADM formalism manifestly respects the Euclidean group [which means that $H(\mathbf{x}_a,\mathbf{p}_a)$ is translationally and rotationally invariant], the generators P_i and J_i are simply realized as

$$P_i(\mathbf{x}_a,\mathbf{p}_a) = \sum_a p_{ai}, \tag{4a}$$

$$J_i(\mathbf{x}_a,\mathbf{p}_a) = \sum_a \varepsilon_{ikl}x_a^k p_{al}, \tag{4b}$$

and exactly satisfy Eqs. (3a) and (3b). The condition (3c) will also be exactly satisfied if G_i is constructed as a three-vector from $\mathbf{x}_a$ and $\mathbf{p}_a$. Finally, the condition for full Poincaré invariance boils down to the existence of a vector $G_i(\mathbf{x}_a,\mathbf{p}_a)$ satisfying the three non-trivial relations (3d), (3e) and (3f), in which enters, besides P_i and J_i given in Eqs. (4a) and (4b), the full (3PN-accurate) Hamiltonian:

$$H(\mathbf{x}_a,\mathbf{p}_a) = \sum_a m_a c^2 + H_{\mathrm{N}}(\mathbf{x}_a,\mathbf{p}_a) + \frac{1}{c^2}H_{\mathrm{1PN}}(\mathbf{x}_a,\mathbf{p}_a) + \frac{1}{c^4}H_{\mathrm{2PN}}(\mathbf{x}_a,\mathbf{p}_a) + \frac{1}{c^6}H_{\mathrm{3PN}}(\mathbf{x}_a,\mathbf{p}_a) + \mathcal{O}\left(\frac{1}{c^8}\right). \tag{5}$$

At the Newtonian order, i.e., when keeping the rest-mass term $\Sigma_a m_a c^2$ and the Newtonian-level Hamiltonian,

$$H_{\mathrm{N}}(\mathbf{x}_a,\mathbf{p}_a) = \sum_a \frac{\mathbf{p}_a^2}{2m_a} - \frac{1}{2}\sum_a \sum_{b\neq a} \frac{G m_a m_b}{r_{ab}}, \tag{6}$$

($r_{ab} \equiv |\mathbf{x}_a - \mathbf{x}_b|$), it is easily checked that the usual Newtonian center-of-mass vector

$$G_N^i(\mathbf{x}_a,\mathbf{p}_a) \equiv \sum_a m_a x_a^i \tag{7}$$

satisfies Eqs. (3d)–(3f). [Note that, in this approximation, the right-hand side of Eq. (3e) yields $(\Sigma_a m_a)\delta_{ij}$ from the rest-mass contribution to H.]

To study the existence of G^i beyond the Newtonian approximation, we need the explicit expressions of the 1PN, 2PN and 3PN contributions to the Hamiltonian (5) in an arbitrary reference frame. The 1PN contribution,

$$H_{\mathrm{1PN}}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{8}\frac{G m_1 m_2}{r_{12}}\left[-12\frac{\mathbf{p}_1^2}{m_1^2} + 14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + 2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right] + \frac{1}{4}\frac{G m_1 m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}} + (1\leftrightarrow 2), \tag{8}$$

has been known for a long time. The operation "$+(1\leftrightarrow 2)$" in Eq. (8) denotes the addition for each term in Eq. (8) (including the ones which are symmetric under label exchange) of another term obtained by the label permutation $1\leftrightarrow 2$. The 2PN-accurate explicit expression of $H(\mathbf{x}_a,\mathbf{p}_a)$, in the ADM formalism, was derived in Ref. [13] [Eq. (2.5) there]. [The corresponding explicit Lagrangian $L_{\mathrm{2PN}}^{\mathrm{ADM}}(\mathbf{x}_a,\dot{\mathbf{x}}_a)$ is given in Ref. [14].] These results corrected earlier results by Ohta *et al.* [15]. The final result reads

$$H_{\mathrm{2PN}}(\mathbf{x}_a,\mathbf{p}_a) = \frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5} + \frac{1}{8}\frac{G m_1 m_2}{r_{12}}\left[5\frac{(\mathbf{p}_1^2)^2}{m_1^4} - \frac{11}{2}\frac{\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2 m_2^2} - \frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + 5\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} - 6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} - \frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right] + \frac{1}{4}\frac{G^2 m_1 m_2}{r_{12}^2}\left[m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2} + 19\frac{\mathbf{p}_2^2}{m_2^2}\right) - \frac{1}{2}(m_1+m_2)\frac{27\,(\mathbf{p}_1\cdot\mathbf{p}_2) + 6\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right] - \frac{1}{8}\frac{G m_1 m_2}{r_{12}}\frac{G^2(m_1^2 + 5m_1 m_2 + m_2^2)}{r_{12}^2} + (1\leftrightarrow 2). \tag{9}$$

Reference [4] derived the 3PN-accurate ADM Hamiltonian *restricted to the center-of-mass reference frame*: $\mathbf{p}_1+\mathbf{p}_2=0$. For the present work we have generalized Ref. [4] in deriving $H_{3\mathrm{PN}}$ in an arbitrary reference frame. Our starting point for doing this calculation is the improved form of the 3PN Hamiltonian, $\widetilde{H}_{3\mathrm{PN}}$, given in Appendix A of Ref. [9] [Eqs. (A8)–(A10) there]. Note first that $\widetilde{H}_{3\mathrm{PN}}$ defined there denotes the *higher-order* Hamiltonian $\widetilde{H}_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a,\dot{\mathbf{x}}_a,\dot{\mathbf{p}}_a)$ defined by eliminating the field variables h_{ij}^{TT}, $\dot{h}_{ij}^{\mathrm{TT}}$ in the ''Routh functional'' $R(\mathbf{x}_a,\mathbf{p}_a,h_{ij}^{\mathrm{TT}},\dot{h}_{ij}^{\mathrm{TT}})$ introduced in Eq. (33) of Ref. [4]. However, it was shown in Ref. [9] that one could *reduce* the higher-order Hamiltonian $\widetilde{H}_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a,\dot{\mathbf{x}}_a,\dot{\mathbf{p}}_a)$ to an *ordinary* Hamiltonian $H_{3\mathrm{PN}}(\mathbf{x}_a',\mathbf{p}_a')$, at the price of the following (3PN-level) shift of phase-space coordinates:

$$\mathbf{x}_a'=\mathbf{x}_a+\partial\widetilde{H}/\partial\dot{\mathbf{p}}_a,\qquad \mathbf{p}_a'=\mathbf{p}_a-\partial\widetilde{H}/\partial\dot{\mathbf{x}}_a. \tag{10}$$

After performing the shift (10) with respect to the original ADM coordinates $\mathbf{x}_a$, $\mathbf{p}_a$ (we henceforth drop the primes for notational simplicity), the calculation of the 3PN (order-reduced) Hamiltonian consists in evaluating three very complicated integrals:

$$H_{3\mathrm{PN}}=-\frac{5}{128}\sum_a(\mathbf{p}_a^2)^4+\int d^3x\,(h_1^{\mathrm{red}}+h_2+h_3). \tag{11}$$

The integrands h_1,h_2,h_3 are given in Eqs. (A9) of [9]. The order-reduced integrand h_1^{red} is defined (as shown in [9]) by using the Newtonian equations of motion to eliminate $\dot{\mathbf{x}}_a$ and $\dot{\mathbf{p}}_a$ when computing the time derivative $\dot{h}_{ij}^{\mathrm{TT}}$ (which enters the last two terms of h_1). As explained in [9], this new form of the 3PN Hamiltonian is free of ''contact term'' ambiguities, and the integrals it contains can all be uniquely defined by using the *Riesz-type regularization* procedure explained in [4]. We have recomputed from scratch all the integrals by using the generalized Riesz formula given in [4]. This *Riesz-regularized* 3PN Hamiltonian reads explicitly (in an arbitrary reference frame)

$$\begin{aligned}
H_{3\mathrm{PN}}^{\mathrm{reg}}(\mathbf{x}_a,\mathbf{p}_a)=&-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7}+\frac{1}{32}\frac{Gm_1m_2}{r_{12}}\Bigg[-14\frac{(\mathbf{p}_1^2)^3}{m_1^6}+4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\,\mathbf{p}_1^2\,\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2}+\frac{(\mathbf{p}_1^2\,\mathbf{p}_2^2-2\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}\\
&-10\frac{(\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2}+24\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}+2\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\
&+\frac{(7\,\mathbf{p}_1^2\,\mathbf{p}_2^2-10\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}+6\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}\\
&-18\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}+5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\Bigg]+\frac{G^2m_1m_2}{r_{12}^2}\Bigg[\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{115}{16}m_1\frac{\mathbf{p}_1^2\,(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}\\
&+\frac{1}{48}m_2\frac{25\,(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\,\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2}+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3}-\frac{1}{8}m_1\frac{(15\,\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11\,(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2}\\
&+\frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\Bigg]\\
&+\frac{G^3m_1m_2}{r_{12}^3}\Bigg[-\frac{1}{48}\left(466\,m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}+\frac{1}{16}\Bigg(77(m_1^2+m_2^2)\\
&+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\Bigg)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(61\,m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}+\frac{1}{16}\Bigg(21(m_1^2+m_2^2)\\
&+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\Bigg)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\Bigg]+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left[\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right]+(1\leftrightarrow2).
\end{aligned} \tag{12}$$

However, it was emphasized in Refs. [4,8,9] that the nature of the divergent integrals which had to be regularized to compute $H_{3\mathrm{PN}}^{\mathrm{reg}}$, Eq. (12), was such that the result should be considered as being partly ambiguous. These *regularization ambiguities* have been discussed in Refs. [4,8], and, in more detail, in the Appendix A of [9]. We have recomputed, in an arbitrary reference frame, the various regularized versions of all the momentum-dependent formal ''exact divergences''

$\Delta_{31}, \Delta_{32}, \ldots, \Delta_{38}$ defined in Appendix A of [9]. These contributions should formally vanish, but their regularized values do not vanish and thereby exhibit the regularization ambiguities present at 3PN. We find (in confirmation of the result given in the Introduction section of Ref. [4]) that all the momentum-dependent regularization ambiguities are equivalent to adding to Eq. (12) a term of the (specific[2]) form

$$H_{3\mathrm{PN}}^{\mathrm{kinetic}}(\mathbf{x}_a,\mathbf{p}_a) = +\tfrac{1}{2}\omega_{\mathrm{kinetic}}(G^3 m_1 m_2/r_{12}^3) \times[\mathbf{p}_1^2 - 3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2 + \mathbf{p}_2^2 - 3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2], \tag{13}$$

where $\omega_{\mathrm{kinetic}}$ is an arbitrary parameter. In addition to this ''kinetic'' regularization ambiguity, it was pointed out in [8] and [9] that there is also a ''static'' (i.e., momentum-independent) regularization ambiguity of the form

$$H_{3\mathrm{PN}}^{\mathrm{static}}(\mathbf{x}_a,\mathbf{p}_a) = +\omega_{\mathrm{static}}[G^4 m_1^2 m_2^2 (m_1+m_2)/r_{12}^4], \tag{14}$$

where ω_{static} is a second arbitrary parameter. Finally, the 3PN (order-reduced) Hamiltonian is of the form

$$H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = H_{3\mathrm{PN}}^{\mathrm{reg}} + H_{3\mathrm{PN}}^{\mathrm{kinetic}} + H_{3\mathrm{PN}}^{\mathrm{static}}, \tag{15}$$

and depends on two, up to now undetermined, real parameters $\omega_{\mathrm{kinetic}}$ and ω_{static}.

The problem to solve is now the following: does there exist a (3PN-accurate) center-of-mass vector, of the generic form,

$$G^i(\mathbf{x}_a,\mathbf{p}_a) = \sum_a [M_a(\mathbf{x}_b,\mathbf{p}_b) x_a^i + N_a(\mathbf{x}_b,\mathbf{p}_b) p_a^i], \tag{16}$$

where M_a and N_a are scalars that reduce to m_a and 0, respectively, in the Newtonian approximation, such that Eqs. (3d)–(3f) are fulfilled (within the 3PN accuracy) when the Hamiltonian is given by inserting Eqs. (6), (8), (9) and (15) in Eq. (5)? We have tackled this problem by the method of undetermined coefficients, i.e., by writing the most general expressions for the successive PN approximations to the functions $M_a(\mathbf{x}_b,\mathbf{p}_b)$ and $N_a(\mathbf{x}_b,\mathbf{p}_b)$,

$$M_a = m_a + c^{-2} M_a^{1\mathrm{PN}} + c^{-4} M_a^{2\mathrm{PN}} + c^{-6} M_a^{3\mathrm{PN}};$$

$$N_a = c^{-4} N_a^{2\mathrm{PN}} + c^{-6} N_a^{3\mathrm{PN}}, \tag{17}$$

as sums of scalar monomials of the form

$$c_{n_0 n_1 n_2 n_3 n_4 n_5} r_{12}^{-n_0} (\mathbf{p}_1^2)^{n_1} (\mathbf{p}_2^2)^{n_2} (\mathbf{p}_1\cdot\mathbf{p}_2)^{n_3} (\mathbf{n}_{12}\cdot\mathbf{p}_1)^{n_4} \times(\mathbf{n}_{12}\cdot\mathbf{p}_2)^{n_5},$$

with positive integers $n_0, \ldots, n_5$. In addition to dimensional analysis (which constrains the possible values of $n_0, \ldots, n_5$ at each given PN order), and Euclidean covariance, including parity symmetry, we only required time reversal symmetry (which imposes that M_a be even, and N_a odd, under $\mathbf{p}_a \to -\mathbf{p}_a$). We did not impose any *a priori* constraints on the mass dependence of the coefficients $c_n(m_1,m_2)$, nor did we use the $1\leftrightarrow 2$ relabeling symmetry.

The 1PN approximation to G^i being well known (see, e.g., [16]),

$$M_1^{1\mathrm{PN}} = \tfrac{1}{2}(\mathbf{p}_1^2/m_1) - \tfrac{1}{2}(G m_1 m_2/r_{12}), \tag{18a}$$

$$N_1^{1\mathrm{PN}} = 0, \tag{18b}$$

with $M_2^{1\mathrm{PN}}$ obtained by a $1\leftrightarrow 2$ relabeling, we started looking for the most general G^i at the 2PN level. At this level, there are 20 unknown coefficients c_n, and Eq. (3d) yields 40 equations to be satisfied. We found that there is a unique solution[3] to these redundant equations, namely

$$M_1^{2\mathrm{PN}} = -\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3} + \frac{1}{4}\frac{G m_1 m_2}{r_{12}}\left[-5\frac{\mathbf{p}_1^2}{m_1^2} - \frac{\mathbf{p}_2^2}{m_2^2} + 7\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right] + \frac{1}{4}\frac{G m_1 m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}, \tag{19a}$$

$$N_1^{2\mathrm{PN}} = -\tfrac{5}{4} G (\mathbf{n}_{12}\cdot\mathbf{p}_2), \tag{19b}$$

with $M_2^{2\mathrm{PN}}$ and $N_2^{2\mathrm{PN}}$ obtained by a $1\leftrightarrow 2$ relabeling.

We have *a posteriori* checked that this unique ADM-gauge, 2PN center-of-mass vector agrees (after taking into account the shift $\mathbf{x}_a^{\mathrm{ADM}} = \mathbf{z}_a - \delta^* \mathbf{z}_a(z,\dot{z})$ [13]) both with the harmonic-gauge 2PN $G^i(\mathbf{z}_a,\dot{\mathbf{z}}_a)$ first derived in Ref. [11], and with the Landau-Lifshitz-like [16], ADM-gauge calculation of $G^i(\mathbf{x}_a,\dot{\mathbf{x}}_a)$ performed in Ref. [17]. We have also checked that the remaining Poincaré-symmetry constraints, Eqs. (3e) and (3f), are also fulfilled. Concerning Eq. (3e), it is easy to see, in general, that it is equivalent to the constraint

$$\sum_a M_a(\mathbf{x}_b,\mathbf{p}_b) = \frac{1}{c^2} H(\mathbf{x}_b,\mathbf{p}_b). \tag{20}$$

At the 3PN level, the most general ansatz for $M_a^{3\mathrm{PN}}, N_a^{3\mathrm{PN}}$, involves 78 unknown coefficients c_n, while Eq. (3d) yields 138 equations to be satisfied. The quantity $\omega_{\mathrm{kinetic}}$ parametrizing the momentum-dependent regularization ambiguity (13) in the 3PN Hamiltonian enters the system of equations for the unknown c_n's. (Indeed, it was recently noticed that $H_{3\mathrm{PN}}^{\mathrm{kinetic}}$ is not separately boost-invariant [10].) By contrast, the other regularization ambiguity (14) drops out of the problem (because $H_{3\mathrm{PN}}^{\mathrm{static}}$ is Galileo invariant). We found that there was a *unique* value of $\omega_{\mathrm{kinetic}}$ for which the system of equations to be satisfied was compatible, namely, $\omega_{\mathrm{kinetic}} = 41/24$. If $\omega_{\mathrm{kinetic}} \neq 41/24$, the 3PN Hamiltonian does not admit a global Poincaré invariance. If $\omega_{\mathrm{kinetic}} = 41/24$, there is a *unique* solution to Eq. (3d), namely,

[2]Note in particular the absence of terms mixing $\mathbf{p}_1$ and $\mathbf{p}_2$.

[3]All the algebraic manipulations reported in this paper were done with the aid of MATHEMATICA.

$$M_1^{3PN}=\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{16}\frac{Gm_1m_2}{r_{12}}\left[9\frac{(\mathbf{p}_1^2)^2}{m_1^4}+\frac{(\mathbf{p}_2^2)^2}{m_2^4}-11\frac{\mathbf{p}_1^2\,\mathbf{p}_2^2}{m_1^2m_2^2}-2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+3\frac{\mathbf{p}_1^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+7\frac{\mathbf{p}_2^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2m_2^2}\right.$$
$$\left.-12\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-3\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right]+\frac{1}{24}\frac{G^2m_1m_2}{r_{12}^2}\left[(112m_1+45m_2)\frac{\mathbf{p}_1^2}{m_1^2}+(15m_1+2m_2)\frac{\mathbf{p}_2^2}{m_2^2}\right.$$
$$\left.-\frac{1}{2}(209m_1+115m_2)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}-(31m_1+5m_2)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}+\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1}-\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2}\right]$$
$$-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}, \tag{21a}$$

$$N_1^{3PN}=\tfrac{1}{8}(G/m_1m_2)[2\,(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)-\mathbf{p}_2^2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)+3\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2]$$
$$+\tfrac{1}{48}(G^2/r_{12})[19\,m_2\,(\mathbf{n}_{12}\cdot\mathbf{p}_1)+(130\,m_1+137\,m_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)]. \tag{21b}$$

We have then checked that this unique solution does satisfy the remaining Poincaré-symmetry constraints, Eqs. (3e) and (3f), or, equivalently, Eqs. (20) and (3f). It is to be noted that the last two momentum-dependent terms in M_1^{3PN}, proportional to $(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2/m_1-(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2/m_2$, are antisymmetric in the labels $1\leftrightarrow 2$ and therefore drop out in the constraint (20), which reads $M_1^{3PN}+M_2^{3PN}=H_{2PN}$. In fact, the corresponding monomials appear nowhere in H_{2PN}, but must crucially be included in M_a^{3PN}.

The main conclusion of this work is therefore that the necessary existence of a global Poincaré symmetry in the two-body problem *uniquely fixes* the regularization ambiguity parameter $\omega_{\rm kinetic}$ to the value 41/24. The explicit realization of this Poincaré invariance is then defined by the phase-space generator $G^i(\mathbf{x}_a,\mathbf{p}_a)$ defined by Eqs. (16), (17), (18), (19), and (21).

Within the ADM formalism it would be very difficult to implement a Poincaré-invariant regularization procedure. (The situation is different in harmonic coordinates, where one can conceive a Lorentz-invariant regularization [18].) It is very satisfying (and in keeping with the general lore about renormalization theory) that we were able to use a non-Poincaré-invariant regularization, but then, *a posteriori*, correct for it in a unique way. There remains, however, a last regularization ambiguity,[4] Eq. (14), which has all the needed global symmetries and cannot be fixed in this way.

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[4] As argued in Ref. [8] this ‘‘static’’ regularization ambiguity seems to be linked to the breakdown of the possibility to use Dirac-delta functions to model extended objects, such as neutron stars or black holes.

[1] T. Damour and N. Deruelle, Phys. Lett. **87A**, 81 (1981).
[2] T. Damour, C. R. Seances Acad. Sci., Ser. 2 **294**, 1355 (1982).
[3] T. Damour, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), pp. 59–144.
[4] P. Jaranowski and G. Schäfer, Phys. Rev. D **57**, 7274 (1998).
[5] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), pp. 227–265.
[6] G. Schäfer, Ann. Phys. (N.Y.) **161**, 81 (1985).
[7] P. Jaranowski and G. Schäfer, Phys. Rev. D **55**, 4712 (1997).
[8] P. Jaranowski and G. Schäfer, Phys. Rev. D **60**, 124003 (1999).
[9] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D (to be published), gr-qc/9912092
[10] P. Jaranowski and G. Schäfer, Ann. Phys. (Leipzig) **9**, 378 (2000).
[11] T. Damour and N. Deruelle, C. R. Seances Acad. Sci., Ser. 2 **293**, 877 (1981).
[12] T. Damour and N. Deruelle, C. R. Seances Acad. Sci., Ser. 2 **293**, 537 (1981).
[13] T. Damour and G. Schäfer, Nuovo Cimento Soc. Ital. Fis., B **101**, 127 (1988).
[14] T. Damour and G. Schäfer, Gen. Relativ. Gravit. **17**, 879 (1985).
[15] T. Ohta, H. Okamura, T. Kimura, and K. Hiida, Prog. Theor. Phys. **50**, 492 (1973); **51**, 1221 (1974); **51**, 1598 (1974).
[16] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1989).
[17] T. Ohta and T. Kimura, Phys. Lett. A **129**, 436 (1988); Prog. Theor. Phys. **81**, 662 (1989).
[18] L. Blanchet and G. Faye, gr-qc/0004009.